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Mathematical structure of quantum superspace as a consequence of time asymmetry

Mario Castagnino*
Instituto de Astronomía y Física del Espacio, Casilla de Correos 67, Sucursal 28, 1428 Buenos Aires, Argentina
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It is demonstrated how a convenient choice of the mathematical structure of the quantum cosmology superspace of the wave functions of the universe, precisely the definition of a convenient regular state superspace and the restriction of the dynamics to this space, yields directly an irreversible evolution, in the classical (and semiclassical) phase of the universe, where decoherence and correlations take place and therefore give origin to a classical universe, the second law of thermodynamics is demonstrated, connection with the Reichenbach branch system idea can be implemented, some rough coincidences with observational data are obtained, the arrows of time can be correlated, and time asymmetry can be explained as a state space asymmetry (e.g., like a spontaneous symmetry breaking). All these facts solve the problem of time asymmetry and show that it is time asymmetry itself that defines the most important features of the mathematical structure of superspace. [S0556-2821(97)03624-2]

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I. INTRODUCTION

The role of physics is to explain nature in the *best possible way*. Therefore physicists consider a set of physical phenomena and choose the best axiomatic structure to imitate these phenomena. This axiomatic structure contains a mathematical structure and a set of axioms (or postulates, or principles, or hypotheses) stated using the language of the chosen mathematical structure. If the mathematical structure is the most naturally related to the set of phenomena and a minimal number of axioms is used, physicists say that they have explained nature in the *best possible way*. But frequently a better mathematical structure and a smaller set of axioms are found to explain a larger set of physical phenomena; then physicists say that they understand the problem even better because, in fact, they have found a better explanation, i.e., a *more economical one*. But the chosen mathematical structure and the chosen axioms cannot be explained by themselves, since the only motivation of the choice is to imitate nature in the best possible way.

Thus, physical phenomena are not a consequence of the chosen mathematical structure; quite the contrary, the choice of the mathematical structure is a consequence of the physical phenomena that we are trying to explain.

Gravitation was explained by Einstein choosing a Riemannian manifold as the mathematical structure and postulating that particle space-time paths were the geodesic of such a manifold, etc. The only explanation of this choice is that the theoretical motions, so described, mimic real motions better than the motions described using other curves or other kinds of manifolds (flat space-time, projective manifolds, etc.). But the Riemannian manifold and the geodesics cannot be explained by themselves. In fact, the choice of the Riemannian manifold, as the mathematical structure to explain gravity, is really inspired by the experimental physics of the phenomenon (precisely the method we use to measure time and distance [1]).

It also happens that, when a new or unfamiliar mathematical structure is introduced, some physicists think that the new structure is introduced *by hand*, because they do not realize that *every* mathematical structure was introduced by hand, in order to explain nature in the best possible way. In the present case it turns out that the time asymmetry of nature is explained in the simplest way if we choose a time-asymmetric mathematical structure (the space ϕ_- of Sec. I B) than a time-symmetric one (the space $\mathcal{H}$ of the same section). This is the essential argument of the paper.

In the perspective of this pedagogical (perhaps pedantic but necessary) introduction we will choose the following:

A. Set of phenomena

The set of phenomena considered in this paper will be those of the usual quantum cosmology (QC) based on the Wheeler-De Witt (WDW) equation [2,3], plus those of irreversible statistical quantum physics, such as the definitions of the various arrows of time, the final equilibrium state, decoherence, correlations, etc. (see [4] where almost all subjects, results, and problems can be found).

B. Mathematical structure

QC is based in the WDW equation

$$H\Psi=0, \tag{1}$$

where H is the Hamiltonian operator of the considered model for the universe and Ψ the wave function of the universe (which will also be called $|\Psi\rangle$) [3,5]. Usually H is well defined by the model we are studying, but the real mathematical nature of Ψ is not so well defined. In fact, let us call q the configuration variables and Q the configuration superspace; then $\Psi=\Psi(q)$ is a function of the configuration variables. We must define the space of wave functions Ψ. Let us suppose that this space has a discrete basis $\mathcal{P}$ of certain functions of the q (let us say polynomials of the q or polynomials multiplied by convenient dumping factors for those q's that go to infinity). This $\mathcal{P}$ is a very good model of the real

*Email: Castagni@iafe.uba.ar

physical space since, the number of measurement being always finite, we can only determine a finite number of polynomial coefficients. But this set is not enough because, in order to perform the mathematical operations we need to develop the theory, usually wave functions must belong to a complete space. Then we must complete the space expanded by $\mathcal{P}$ with some topology. The usual idea is to make this completion with the topology of the norm. Even if the choice of a norm contains a very important physical element—the definition of probability—we are now exclusively interested in the mathematical problem of how to complete $\mathcal{P}$ with an adequate topology. Then we have really two problems.

(i) We must define a norm in the space expanded by $\mathcal{P}$ something like [6]

$$\|\Psi\| = \int_Q \Psi^*(q)\Psi(q)dq. \tag{2}$$

But we can also foliate Q with hypersurfaces Σ and define [2]

$$\|\Psi\| = \int_\Sigma \Psi^*(q)\Psi(q)d\sigma_n, \tag{3}$$

where n is the normal to Σ and $d\sigma_n$ is the hypervolume of the hypersurface Σ differential element (this quantity will be positive definite in a convenient subspace of superspace). Or we can consider that really Ψ is an operator and then we can go to a third quantization [7], etc. So we have many possibilities to define the norm. Actually we are trying to copy the usual quantum mechanics and the essential property to define an adequate norm is that *it would be a constant under time evolution.* But in QC there is no time [8,9].[1] In fact, Eq. (1) is not a Schrödinger equation but an eigenequation that defines a ''stationary'' eigenfunction, with no time variable in it. Therefore, as the definition of norm is intimately intertwined with the problem of the definition of time in QC, and as this problem is not solved[2] [8,11], it is not possible to give a definitive definition of norm. Really in this paper we will adopt the conservative attitude that this problem cannot be solved and therefore QC is a timeless theory [8,9]. As a consequence it is very difficult to define a satisfactory norm.

(ii) But even if a norm would be chosen, say, Eq. (2), we have another problem. If we complete the space expanded by $\mathcal{P}$ with Eq. (2), we will find $\mathcal{H} = L^2(Q)$, the Hilbert space of square integrable functions over Q. But H is an operator with derivatives and the functions of $\mathcal{H}$ are, in general, not derivable; they are just square integrable, and so for an arbitrary function Ψ of $\mathcal{H}$, $H\Psi$ has no meaning (and we would have the same problem with the other norms). So we must choose another topology to complete the space expanded by $\mathcal{P}$. Let us consider the Schwarz class functions, namely, functions that can be derived an infinite number of times and that are well behaved in the eventual infinite of coordinates q (precisely they vanish faster than any polynomial). The set of Schwarz functions is a nuclear space. Then if we complete the space expanded by $\mathcal{P}$ with the corresponding topology, which is not a norm topology but a nuclear one [12], we obtain a space $\mathcal{S}$ where we can derive any number of times. If $\mathcal{S}^\times$ is the space of an (anti)linear functional over $\mathcal{S}$, our mathematical structure is really the Gel'fand triplet (or rigged Hilbert space):

$$\mathcal{S} \subset \mathcal{H} \subset \mathcal{S}^\times. \tag{4}$$

Then we can say that $\mathcal{S}$ is the quantum superspace of *regular states*[3] where we can do all our computations and $\mathcal{S}^\times$ is the quantum superspace of *generalized states*, where we will find the usual distributions such as Dirac's δ or plane (or curved) waves, which can be used to expand regular states. In ordinary quantum mechanics $\mathcal{H}$ would be the quantum space of *states*, but in QC this space loses almost all its importance because we do not know what norm we have to use for sure (even if in this paper, to compute probabilities, we will use norm (2) in spite of the fact that we have other possible choices[4]), while $\mathcal{S}$ and $\mathcal{S}^\times$ are well defined, since their definitions are norm independent. In fact, the ''inner product'' $\langle F|\Psi\rangle$ of an element of $\mathcal{S}^\times$, written as a ''bra,''

[1]In fact, the state of the universe, in this period, is described by a density matrix $\rho(q,q')$ where $q,q' \in Q$. Among all the configuration variables q we may choose a particular one as the hand of our clock, let us say the radius of the universe, a. Then $\rho(q,q') = \rho(a,x,a',x')$, where x symbolizes the rest of the variables. But to use a as the hand of our clock, it is necessary to have decoherence for $a \neq a'$, e.g., $\rho(a,x,a',x') = 0$ if $a \neq a'$ or $\rho(a,x,a',x') \sim \delta(a-a')\rho(a,x,x')$. Namely, at least one variable, e.g., a, must have classical properties; if not, we will be forced to make a theory with two ''times'' a and a'. But if we have just one a, we are not more in full QC, but in a semiclassical regime, as those of Sec. IV, where, in fact, we will find decoherence of the radius a (see Sec. IV A or Ref. [19]). We conclude that, even if some generalization of time may be defined in the full quantum gravity period, the usual time, with its known properties, cannot exist in such a regime.

[2]We have defined the time, for the semiclassical phase of the universe (even with a back reaction), in [10]. But it was impossible for us to extend this definition to the quantum period.

[3]As this space is defined over the superspace Q we will call it also a quantum superspace.

[4]If only spaces Φ_- and $\Phi_-^\times$ were defined, for every state $\rho \in \Phi_-$ and every observable $A \in \Phi_-^\times$ [cf. Eq. (18)] we can compute the ''mean value'' $\langle A\rangle_\rho = A[\rho]$. This number is well defined by the linear operator A acting on the state ρ. Really this is the maximal answer QC can give: a typical ''mean value'' for every observable and for every quantum state of the universe, since the ensemble of QC has just one system, the universe. Therefore the ''mean value'' is not a proper mean value but just a ''typical'' one. On the other hand, if we want to know the probability $p(q)$ to find the universe at a certain q and if Q is the position operator, $Q = \int q|q\rangle\langle q|dq$, and ρ is a pure state, $\rho = |\varphi\rangle\langle\varphi|$, we need to define a norm and the inner product of $\mathcal{H}$ to compute $p(q) = |\langle\varphi|q\rangle|^2$. But we know that the notion of probability in an ensemble of just one element in not a reasonable one. So really it is more logical to work just with the couple $\Phi_-, \Phi_-^\times$ and with the typical mean values like $A[\rho]$ than to use the notion of probability. Nevertheless, in this paper we will use the more familiar ''probabilistic approach'' of QC, where we need a norm like Eq. (2). We will study the ''typical approach'' of QC, where the norm is superfluous, in a forthcoming paper.

$\langle F|$, with an element of $\mathcal{S}$, written like a ''ket,'' $|\Psi\rangle$, is also well defined, since it is just the antilinear functional acting over a regular state (also the product in the inverted order is well defined if we set $\langle\Psi|F\rangle=\langle F|\Psi\rangle^*$). So $\mathcal{S}$ is the arena where timeless QC works: where time and norm have disappeared from the mathematical structure and are substituted by the nuclear topology of $\mathcal{S}$.

But if we use $\mathcal{S}$ as a regular space, we cannot encompass (in the most economical way) irreversible statistical quantum mechanics, which has an asymmetry that it is not contained in $\mathcal{S}$. There are only two causes for asymmetry in nature: Either the laws of nature are asymmetric or the solutions of the equation of the theory are asymmetric. E.g., the laws of nature are asymmetric in the case of the weak interaction. The solutions of the theory are asymmetric in the case of spontaneous symmetry breaking.

Time asymmetry is not an exception. Thus, if we want to retain the time-symmetric laws of nature [namely, the symmetry of Eq. (1)], the only way we have to explain the time asymmetry of the universe or its subsystems is to postulate that the space of solutions is not time symmetric; namely, we use the second cause of asymmetry. So the proper way to solve the problem is simply to define a realistic time-asymmetric space of physical admissible solutions ϕ_-, i.e., an adequate mathematical structure for superspace. ϕ_- will contain the states that evolve in an admissible way (e.g., Gibbs ink drop spreading in a glass of water, a sugar lump solving in a cup of coffee, etc.) and will not contain the nonadmissible evolutions (the ink or the sugar concentrating spontaneously and creating the drop or the lump). The problem is to choose ϕ_- in the *best possible way.* As we will see we will choose one with the required asymmetry.

Let us first follow a heuristic approach: We will suppose that H is endowed with all the properties necessary to define a reasonable universe. Obviously these ''realistic Hamiltonians'' are the only H that we must consider. Therefore, practically, in H there must be always some fields, such as the matter field, the electromagnetical field, and also the gravitational field (precisely only the graviton field). So we would write this Hamiltonian in the usual midisuperspace way as

$$H=h_g(g_j,\pi j)+h_f(\varphi,p_\varphi)+h_i(g_{j,}\varphi), \tag{5}$$

where $h_g(g_{j,}\pi_j)$ is the gravitational Hamiltonian (usually a function of a discrete set of modes of the gravitational field g_j and the corresponding momenta π_j), h_f is the ''field'' Hamiltonian (let us say the continuous set of all the modes of just one scalar field φ, which represents all the physical fields in our model, and the corresponding momenta p_φ), and h_i is the interaction Hamiltonian among fields and no fields (usually an interaction among the configuration variables only).

Let us sketch the panorama in the complete quantum gravity case (which we shall not study in this paper). From the Gel'fand-Maurin theorem [13] we know that, for the partial Hamiltonian h_g+h_f, a spectral decomposition exists [14]:

$$h_g+h_f=\sum_{i,\omega}(\omega_i+\omega)|i,\omega\rangle\langle i,\omega|, \tag{6}$$

where i symbolizes (in a shorthand notation) the set of quantum numbers of the discrete spectrum, ω_i their energy, ω those of the continuous spectrum and also their energy, and $\Sigma_{i,\omega}$ is a shorthand notation for a sum in i and an integral in ω. Some negative eigenvalues ω must appear because H is not bounded from below. Any state $|\Psi\rangle\in\mathcal{S}$ can be expanded as

$$|\Psi\rangle=\sum_{i,\omega}|i,\omega\rangle\langle i,\omega|\Psi\rangle; \tag{7}$$

analogously, h_i can be expanded as

$$h_i=\sum_{i,\omega;j,\omega'}h^{(i)}_{i,\omega;j,\omega'}|i,\omega\rangle\langle j,\omega'|. \tag{8}$$

These are the decomposition of $|\Psi\rangle$ and h_i in the basis $\{|i,\omega\rangle\}$. In order that $|\Psi\rangle$ would satisfy the WDW equation (1) it must be

$$H|\Psi\rangle=\sum_{i,\omega}(\omega_i+\omega)|i,\omega\rangle\langle i,\omega|\Psi\rangle + \sum_{i,\omega;j,\omega'}h^{(i)}_{i,\omega;j,\omega'}|i,\omega\rangle\langle j,\omega'|\Psi\rangle=0. \tag{9}$$

Most likely this equation can be solved.

But we shall work in the semiclassical case only. In fact, in every reasonable model, the universe geometry will end in a classical phase; some part of the matter and some fields will become also classical, while others will remain in the quantum regime, yielding a semiclassical model. Then, a variable a will exist (one of the g_j or a function of the g_j) such that a time η can be defined as a function of a. When $\eta\to\infty$ we will obtain a classical geometry $g^{\mathrm{out}}_{\mu\nu}$ for the universe. It will be the most probable geometry of the universe, i.e., the geometry that appears most frequently.[5] Using time η we can transform Eq. (1) in a Schrödinger equation, with the corresponding Hamiltonian $h=h(\mathrm{out})$ [5]. Then using h and the classical geometry $g^{\mathrm{out}}_{\mu\nu}$ we can find a semiclassical vacuum state $|0,\mathrm{out}\rangle$ for the fields and the interaction (the so-called adiabatic vacuum), which diagonalizes the Hamiltonian $h=h_f+h_i=h(g^{\mathrm{out}}_{\mu\nu})=h(\mathrm{out})$ (computed in the geometry $g^{\mathrm{out}}_{\mu\nu}$), the creation and annihilation operators related to this vacuum, and the corresponding Fock space. The only essential ingredient we need to implement the theory is the $h(\mathrm{out})$. Then, using these objects, we can find a set of eigenvectors $|\omega,\mathrm{out}\rangle$,[6] such that

$$h|\omega,\mathrm{out}\rangle=\omega|\omega,\mathrm{out}\rangle, \tag{10}$$

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[5] In every reasonable model of the universe a final equilibrium state seems to exist that acts like an attractor for every initial state of the theory. See, e.g., [15].

[6] As we will see in the model of Sec. II, we must restore variable a in $|\omega,\mathrm{out}\rangle$, which is really a function of $\eta=\eta(a)$, and then multiply it by an adequate prefactor, to get a solution of WDW equation. With these modifications $|\omega,\mathrm{out}\rangle$ becomes a vector of superspace.

where ω is a continuous eigenvalue of h (say, $0 \leqslant \omega < \infty$).[7] So

$$h=\int_0^\infty \omega|\omega,\text{out}\rangle\langle\omega,\text{out}|d\omega, \tag{11}$$

where $|\omega,\text{out}> \in \mathcal{S}^\times$; thus, if $|\Psi\rangle \in \mathcal{S}, \langle\omega,\text{out}|\Psi\rangle$ is well defined and so $h|\Psi\rangle$ (and the quantum number i disappears since now the geometry is a classic one). The existence of this kind of expansion can be also considered a consequence of Gel'fand-Maurin theorem [13]. But all these manipulations are just formal, and so we will be sure that what we are doing is correct only in concrete examples, as the one in the next section. In fact, we will find all these mathematical elements in the model presented there. Surely we will also find these elements in more complex models.[8]

Now, let us define our new regular state space $\mathcal{H}_-$. Precisely, we can promote ω [or some linearly related variable like ϖ, Eq. (36)] to a complex variable z and ask not only that $|\Psi\rangle \in \mathcal{S}$, but also that $\langle z,\text{out}|\Psi\rangle$ would be an *analytic function in the lower complex half plane* (precisely that $\langle\omega,\text{out}|\Psi> \in H^2_-$, H^2_- being the Hardy functions class from below[9]). If these functions $|\Psi\rangle$ belong also to $\mathcal{S}$, they belong to a space ϕ_- such that

$$\phi_- \subset \mathcal{S}, \tag{12}$$

and we have a new Gel'fand triplet

$$\phi_- \subset \mathcal{H}_- \subset \phi_-^\times; \tag{13}$$

then, we know that [12]

$$\mathcal{S}^\times \subset \phi_-^\times . \tag{14}$$

So we have restricted the regular state superspace and simultaneously we have enlarged the generalized state superspace so that we will have more general spectral expansions (this fact will be of utmost practical importance). But as we can as well choose the upper complex half plane (precisely $\langle\omega,\text{out}|\Psi\rangle \in H^2_+$, H^2_+ being the Hardy functions class from above), we also have another space ϕ_+ such that

$$\phi_+ \subset \mathcal{S} \tag{15}$$

and another Gel'fand triplet

$$\phi_+ \subset \mathcal{H} \subset \phi_+^\times . \tag{16}$$

We also know that [12]

$$\mathcal{S}^\times \subset \phi_+^\times . \tag{17}$$

Now we can also say that we have obtained the space ϕ_- completing the space expanded by $\mathcal{P}$ with the nuclear topology of ϕ_-, namely, the $\mathcal{S}$ topology restricted to ϕ_- (and the same thing can be said about ϕ_+). Clearly this topology is endowed with a new asymmetry which $\mathcal{S}$ does not have. Precisely, this asymmetry allows us to choose between ϕ_- or ϕ_+ even if we maintain all the symmetries of H.[10] Thus we can break one of these symmetries, restricting the dynamics to the superspace ϕ_-, which then would be considered as the superspace of regular states. As we will see this restriction produces the desired time asymmetry. Frequently physicists make an analytic continuation in the complex energy plane supposing that some functions are analytic in one half plane only. In these cases they are implicitly using the kind of mathematical structure we have explicitly introduced here, and so the idea is, by no means, new.

Thus our mathematical structure will essentially be Eq. (13), ϕ_- will be our superspace of regular states, where we must find the states that satisfy the WDW equation (1), and $\phi_-^\times$ will be our generalized state superspace. From Eq. (12) we see that we have restricted our regular state superspace, and so nothing unphysical can happen. We are just adding a new requirement to regular states, in order to assure their asymmetry.

C. Axiomatic structure

We do not pretend to give a completely rigorous axiomatic structure in this paper (but just an approximation of it). Furthermore, we do not know if the proposed axiomatic structure is unique. We are just proposing a first draft of a complete axiomatic structure, and so we will call our axioms just hypotheses.

As we would like the equilibrium state to be contained in our theory we must also consider mixed states ρ and, therefore, the spaces[11]

$$\Phi_- = \phi_- \otimes \phi_-, \ldots, \quad \mathcal{L}_- = \mathcal{H}_- \otimes \mathcal{H}_-, \ldots, \quad \Phi_-^\times = \phi_-^\times \otimes \phi_-^\times, \tag{18}$$

[7]Of course, we will have such kind of continuous spectrum only if the spatial geometry of the universe is open. In closed models we can only suppose that the discrete eigenvalues are so closed that they can be considered as a continuous spectrum in some approximation. This fact and the problems that we will face with an expanding-contracting universe show that our formalism is much better adapted to open geometries.

[8]We do not know which are the necessary and sufficient conditions in order to be sure that this structure would exist. Anyhow there is an obvious necessary condition: The superspace must be time oriented; namely, two subsuperspaces Φ_- and Φ_+ must be found such that $K:\Phi_- \to \Phi_+ \neq \Phi_-$, K being the Wigner operator (namely, the complex conjugation).

[9]A complex function $G(E)$ is a Hardy class function from above (below) if (1) $G(E)$ is the boundary of a function $G(z)$ of the complex plane where $z=E+i\eta$, which is analytic in the half plane $\eta>0$ ($\eta<0$), and (2) $\int_{-\infty}^{\infty}|G(E+i\eta)|^2 dE<k<\infty$ for all η with $0<\eta<\infty$ ($-\infty<\eta<0$). Usually the ω of Eq. (10) is $\omega>0$, so really the functions of ϕ_- satisfy the condition $\langle\omega,\text{out}|\psi\rangle \in \theta(H_2^- \cap \mathcal{S})$.

[10]As $(H^2_-)^* = H^2_+$, then, if $\Psi(\omega)=\langle\omega,\text{out}|\Psi\rangle \in H^2_-$, we have $\Psi^*(\omega)=\langle\omega,\text{out}|\Psi\rangle^* \in H^2_+$. Then we can foresee that the substitution of Φ_- by Φ_+ will become the time inversion when time will be defined. Namely, K, the Wigner operator, will become this inversion.

[11]It is also interesting to study other definitions of these product spaces, as those that can be obtained using the quantum numbers λ and ν. See [16]. Also, if we would like to explicate the singular part of the continuous spectrum, we must choose $\Phi_- = \mathcal{S} \oplus (\phi_- \otimes \phi_-)$. See [17].

and work in the Liouville triplet

$$\Phi_{-}\subset\mathcal{L}_{-}\subset\Phi_{-}^{\times}, \qquad (19)$$

where $\mathcal{L}_{-}\subset\mathcal{L}$ and $\mathcal{L}$ is the usual Liouville space of ordinary mixed states, which actually we will never use, since our regular state superspace is Φ_{-}. So let ρ be a self-adjoint density matrix. Then our main hypotheses are the following.

H_1. *The state* ρ *of the universe satisfies the equations*[12]

$$H\rho=0. \qquad (20)$$

H_2. *The state* ρ *of the universe belongs to the superspace* Φ_{-}, i.e.,

$$\rho\in\Phi_{-}. \qquad (21)$$

H_3. $\rho(q,q')$ *is proportional to the correlation between the configurations* q, *and* q' *and* $\rho(q,q)$ *is proportional to the probability of finding the configuration* q *in the universe.*

These three axioms correspond to the three elements necessary to go into QC mentioned in the Introduction of [3]: *dynamics, ''initial condition'' (precisely the definition of the physically admissible states of the universe in a timeless formalism), and interpretation.* Of course H_2 alone does not fix the actual state of the universe, but, if we want that time asymmetry would appear natural, any state of the universe we choose to build our theory must be contained in Φ_{-}. In this paper we do not address the problem of finding the real and unique state of the universe, but only to define a superspace of admissible states such that the universe would turn out to be time asymmetric.

We will see how far we can go with this axiomatic structure.

The paper is organized as follows. In Sec. II we introduce our model and its semiclassical approximation, and we obtain a new spectral decomposition, using the regular superspace of hypothesis H_2. In Sec. III we obtain the evolution equation of the states. In Sec. IV up to Sec. IX we find the physical characteristics of the model. In Sec. X we draw our main conclusions.

II. MODEL

Let us see how we can implement all we have said in a simple model.

Let us consider the model of Sec. 3, of Ref. [18], or better the one of Ref. [19], where a Robertson-Walker metric

$$ds^2=a^2(\eta)(d\eta^2-dx^2-dy^2-dz^2) \qquad (22)$$

is studied (we will mostly consider the flat space geometry case). The total action is $S=S_g+S_f+S_i$, S_g being the gravitational action, S_f the usual action of a spinless massive field φ, conformally ($\xi=\frac{1}{6}$) coupled, and S_i the interaction given by a mass term in Robertson-Walker geometry. The gravitational action is given by

$$S_g=M^2\int d\eta\left[-\frac{1}{2}\dot{a}^2-V(a)\right], \qquad (23)$$

where M is Planck's mass, η is the conformal time, a is the Robertson-Walker scale, $\dot{a}=da/d\eta$, and $V(a)$ is a potential that arises from the spatial curvature, a possible cosmological constant, and, eventually, a classical matter field. $V(a)$ is a potential with a bounded support contained in $0\leqslant a\leqslant a_1$, with $a_1\gg 0$ [in many examples $V(a)$ is a function of a^2 and $V(a)$ strongly vanishes, for $a^2\to\infty$ [19], and so our potential can be considered as a good approximation of these examples]. This case is the simplest of all, but we believe that the main features, which we will find, will also be present in more general cases.

The WDW equation (1) for our model is

$$H\Psi(a,\varphi)=(h_g+h_f+h_i)\Psi(a,\varphi)=0, \qquad (24)$$

where (in our the flat space geometry case)

$$h_g=\frac{1}{2M^2}\partial_a^2+M^2V(a), \qquad (25)$$

$$h_f=-\frac{1}{2}\int_k(\partial_{\varphi_k}^2-k^2\varphi_k^2)d\mathbf{k}, \qquad (26)$$

$$h_i=\frac{m^2a^2}{2}\int_k\varphi_k^2 d\mathbf{k}, \qquad (27)$$

where m is the mass of the scalar field, and $k^2=|\mathbf{k}|^2$, where $\mathbf{k}/a$ is the linear momentum of the field, in the flat case we are working in. In the two other cases, namely, open and closed space geometry, the integrals of Eqs. (26) and (27) are

(i) integrations on adapted coordinates, in the open case, and (ii) sums, in the closed case, where $\mathbf{k}$ is substituted by a discrete variable.

See the corresponding equations in [18].

Now, let us go to the semiclassical case using the WKB method [3,5].[13] So let

$$\Psi(a,\varphi)=\exp[iM^2S(a)]\chi(a,\varphi) \qquad (28)$$

and let us expand S and χ as

$$S=S_0+M^{-1}S_1+\cdots,$$

$$\chi=\chi_0+M^{-1}\chi_1+\cdots. \qquad (29)$$

Then to satisfy the WDW equation (1), at order M^2, the principal Jacobi function $S(a)$ must satisfy the Hamilton-Jacobi equation

[12] If $\rho=|\Psi\rangle\langle\Psi|$ is a pure state, these equations coincide with the WDW equation (1).

[13] Following D. Bohm we can also say that WDW equation is exactly equivalent to the system $[S'(a)]^2=2V(a)+(i/2M^2)S''(a)$, $iS'(a)(\partial/\partial a)\chi=h\chi-(1/2M^2)(\partial^2/\partial a^2)\chi$, where the M^{-2} order term can be considered as a gravitational correction to the Hamilton-Jacobi equation and as a quantum potential that must be added to the classical potential of Hamiltonian h. Theoretically this system can be solved exactly. See [20].

$$\left(\frac{dS}{da}\right)^2 = 2V(a). \tag{30}$$

Now we can define the time, in our up to now, timeless theory. It is the (semi)classical time parameter $\eta = \eta(a)$ given by

$$\frac{d}{d\eta} = \frac{dS}{da}\frac{d}{da} = \pm\sqrt{2V(a)}\frac{d}{da}. \tag{31}$$

From Eqs. (30) and (31) we can find the set of classical solutions,

$$a = \pm f(\eta, C), \tag{32}$$

where C is an arbitrary integration constant. Using different values for this constant and different choices of the $\pm$ sign we obtain different classical geometries (in more general cases many constants would be necessary). For $a > a_1$, it is $\sqrt{2V(a)} = 0$ [since $V(a)$ has a bounded support, contained in $[0, a_1]$], and we cannot define the time η using Eq. (31); thus we must choose another hand for our clock to define the time there. To avoid this problem let us consider that when $a > a_1$ it is $\sqrt{2V(a)} = \varepsilon = \text{const} \cong 0$. We can always make $\varepsilon = 0$ to reobtain the primitive case. Then a will be

$$a = \pm \varepsilon \eta + C. \tag{33}$$

So we can see that the potential can also be considered as a function with bounded support in the variable η. We will always consider that $\varepsilon > 0$. The role of C is just to fix the origin of time, and so we can take any C we want. As the coupling is conformal we will have well-defined vacua [21,22]. In particular we can consider two scales a_{in} and a_{out} such that $0 < a_{\text{in}} \ll a_1 \ll a_{\text{out}}$ and define the $|0,\text{in}\rangle, |0,\text{out}\rangle$ vacua there. (We can as well transform all the equations to the nonrescaled case, consider the proper time $t = \int a\, d\eta$, and the physical momentum $\mathbf{k}/a$, and define the $|0,\text{out}\rangle$ in the $\eta \to \infty$ limit, as in Appendix A of [19], but here we will use the first simpler formalism.)

For our model we obtain

$$h(a) = h_f(\varphi_k) + h_i(a, \varphi_k), \tag{34}$$

where we have omitted the φ_k in $h(a)$. Then

$$h(a) = \frac{1}{2}\int_{\mathbf{k}}\left[-\frac{\partial^2}{\partial\varphi_{\mathbf{k}}^2} + \Omega_k^2(a)\varphi_{\mathbf{k}}^2\right]d\mathbf{k}, \tag{35}$$

where [cf. Eqs. (26) and (27)]

$$\Omega_{\varpi}^2(a) = m^2a^2 + k^2 = m^2a^2 + \varpi, \tag{36}$$

where $\varpi = k^2, k = |\mathbf{k}|$. So $h(a)$ is a time-dependent Hamiltonian, where all its time dependence comes from a scale variable mass m^2a^2. It is well known [21,22] that we can diagonalize this time-dependent Hamiltonian at a_{in} and at a_{out} and define the corresponding vacua, the corresponding creation and annihilation operators, and the corresponding Fock spaces. For the out geometry the vacuum will be the adiabatic vacuum, since $a_{\text{out}} \gg a_1$; therefore all the out elements will coincide with those defined in the Introduction. In fact, the out geometry is almost constant during the final time evolution (which goes up to $\eta \to \infty$) and therefore they correspond to the geometry with the maximum probability. $h(a_{\text{out}})$ reads

$$h(a_{\text{out}}) = \int_{\mathbf{k}} \Omega_{\varpi}(a_{\text{out}}) a_{\text{out},\mathbf{k}}^{\dagger} a_{\text{out},\mathbf{k}}\, d\mathbf{k}, \tag{37}$$

where $a_{\text{out},\mathbf{k}}^{\dagger}$ and $a_{\text{out},\mathbf{k}}$ are the creation and annihilation operators corresponding to the out vacuum. With these objects we can construct the Fock space with a basis

$$|\mathbf{k}_1 \cdot \mathbf{k}_2, \ldots, \mathbf{k}_{n,\text{out}}\rangle = |\{k\},\text{out}\rangle \sim a_{\text{out},\mathbf{k}_1}^{\dagger} a_{\text{out},\mathbf{k}_2}^{\dagger} \cdots a_{\text{out},\mathbf{k}_n}^{\dagger}|0,\text{out}\rangle, \tag{38}$$

where we have called $\{k\}$ the set $\mathbf{k}_1, \mathbf{k}_2, \ldots, \mathbf{k}_n$. These states are eigenvectors of $h(a_{\text{out}})$, precisely

$$h(a_{\text{out}})|\{k\},\text{out}\rangle = \Omega(a_{\text{out}})|\{k\},\text{out}\rangle \tag{39}$$

[of course, this equation corresponds to Eq. (10)], where

$$\Omega(a_{\text{out}}) = \sum_{k\in\{k\}} \Omega_{\varpi}(\text{out}) = \sum_{k\in\{k\}} (m^2a^2 + \varpi)^{1/2}. \tag{40}$$

We can use this energy to label the eigenvectors as

$$|\{k\},\text{out}\rangle = |\varpi,[k],\text{out}\rangle, \tag{41}$$

where $[k]$ is the remaining set of labels necessary to define the vector unambiguously. $\{|\varpi,[k],\text{out}\rangle\}$ is an orthonormal basis: namely,

$$\langle\varpi,[k],\text{out}|\varpi'[k'],\text{out}\rangle = \delta(\varpi - \varpi')\,\delta([k] - [k']), \tag{42}$$

$$1 = \int_0^{\infty} d\varpi \int_{[k]} |\varpi,[k],\text{out}\rangle\langle\varpi,[k],\text{out}|\, d[k], \tag{43}$$

where the meaning of the symbols related to $[k]$ is evident. In the closed space geometry case the indices would be discrete and the integral a sum.

The same can also be done at a_{in}. We can now define the S matrix between the in and out states:

$$S_{\varpi,[k];\varpi',[k']} = \langle\varpi,[k],\text{in}|\varpi',[k'],\text{out}\rangle = S_{\varpi,[k],[k']}\,\delta(\varpi - \varpi'). \tag{44}$$

According to [23,24], this matrix has an infinite set of complex poles as we will demonstrate in Sec. IV (also an example is given in [19] and using this paper and [22] and [25] other examples can be obtained).

If we forget the indices $[k]$, as we will always do below, and consider again Eq. (10), we see that $|\varpi,[k],\text{out}\rangle$ is the $|\omega,\text{out}\rangle$ of this equation. In the Introduction we have defined the triplets (13) and (16) only using the Hamiltonian $h(\text{out}) = h(a_{\text{out}})$. These triplets correspond to a Fock space defined for a_{out}. But there will also be two similar triplets defined in the Fock space at a_{in}. We make the following choice (motivated by reasons that will be evident in a moment): For the in Fock space we will use functions $|\psi\rangle \in \phi_{+\text{in},}$ namely, such that $\langle\varpi,\text{in}|\psi\rangle\in\mathcal{S}$ and $\langle\varpi,\text{in}|\psi\rangle \in H^2_+$, and for the out Fock space we will use functions such that $\langle\varpi,\text{out}|\varphi\rangle\in\phi_{-,\text{out}}$, namely, such that $|\varphi\rangle\in\mathcal{S}$ and $\langle\varpi,\text{out}|\varphi\rangle\in H^2_-$. So the ϕ_- of the Introduction is now $\phi_{-,\text{out}}$ and our regular states belong to this space.[14] The role of $\phi_{+,\text{in}}$ is to allow us to define the corresponding functional space $\phi^{\times}_{+,\text{in}}$ so we can use the functionals of this space in some spectral decompositions.[15] As both vacua, at a_{in}, and a_{out}, are well defined and the particle production between these vacua is finite the theory is implementable [26]. We can then multiply the state of both Fock spaces.

So let us again write Eq. (43) with no $[k]$:

$$1=\int_0^\infty d\varpi|\varpi,\text{out}\rangle\langle\varpi,\text{out}|. \tag{45}$$

Of course there is an analogous equation for the ''in'' case. Now using this equation and Eq. (44) we have

$$\begin{aligned}|\varpi,\text{out}\rangle&=\int_0^\infty d\varpi'|\varpi',\text{in}\rangle\langle\varpi',\text{in}|\varpi,\text{out}\rangle\\&=\int_0^\infty d\varpi'|\varpi',\text{in}\rangle S_{\varpi',\varpi}.\end{aligned} \tag{46}$$

Then

$$1=\int_0^\infty d\varpi\int_0^\infty d\varpi'|\varpi',\text{in}\rangle S_{\varpi',\varpi}\langle\varpi,\text{out}| \tag{47}$$

or

$$\begin{aligned}\langle\psi|\varphi\rangle&=\int_0^\infty d\varpi\int_0^\infty d\varpi'\langle\psi|\varpi',\text{in}\rangle S_{\varpi',\varpi}\langle\varpi,\text{out}|\varphi\rangle\\&=\int_0^\infty d\varpi\int_0^\infty d\varpi'\langle\psi|\varpi',\text{in}\rangle S_{\varpi}\delta(\varpi-\varpi')\langle\varpi,\text{out}|\varphi\rangle\\&=\int_0^\infty d\varpi\langle\psi|\varpi,\text{in}\rangle S_{\varpi}\langle\varpi,\text{out}|\varphi\rangle.\end{aligned} \tag{48}$$

Now let $|\varphi\rangle\in\phi_{-,\text{out}}$ and $|\psi\rangle\in\phi_{+,\text{in}}$ and let us promote ϖ to a complex variable z. Then $\langle z,\text{out}|\varphi\rangle\in H^2_-$, $\langle z,\text{in}|\psi\rangle\in H^2_+$ and therefore $\langle\psi|z,\text{in}\rangle\in H^2_-$. So in the integrand of the last equation all the factors are analytic in the lower half plane, with the exception of $S_{\omega,\omega'}$, which has an infinite number of poles z_n as we have already said. Then we can choose any curve Γ, beginning at the origin, and going below all the poles of the lower half plane up to the infinity of the positive real axis.[16] We can now change the integration contour of Eq. (48) from $[0,\infty)$ to the curve Γ. If we add the pole contribution, as in [14,27,28], we obtain

$$\begin{aligned}\langle\psi|\varphi\rangle&=\sum_n\langle\psi|\overline{n\rangle}\langle\tilde{n}|\varphi\rangle+\int_\Gamma dz\langle\psi|z,\text{in}\rangle S_z\langle z,\text{out}|\varphi\rangle\\&=\sum_n\langle\psi|\overline{n\rangle}\langle\tilde{n}|\varphi\rangle+\int_\Gamma dz\langle\psi|\overline{z\rangle}\langle\tilde{z}|\varphi\rangle,\end{aligned} \tag{49}$$

where the sum comes from the residues of the poles (each pole z_n is labeled by a discrete index n, and of course[17] $\text{Im}z_n\leqslant 0$). Then, in a weak sense, we have found a new spectral decomposition of 1:

$$1=\sum_n|\overline{n\rangle}\langle\tilde{n}|+\int_\Gamma dz|\overline{z\rangle}\langle\tilde{z}|. \tag{50}$$

Following the same procedure with $\langle\psi|h(\text{out})|\varphi\rangle$ we can obtain the spectral decomposition of $h(\text{out})$ (always in a weak sense):

$$h(\text{out})=\sum_n\Omega_n|\overline{n\rangle}\langle\tilde{n}|+\int_\Gamma\Omega_z|\overline{z\rangle}\langle\tilde{z}|dz. \tag{51}$$

We have three possibilities to choose the curve Γ:

(i) to use all possible curves Γ as in [28], (ii) to take the curve $(-\infty,0]$, in the second sheet, as in Ref. [27], provided we have a good behavior at infinity in the lower half plane, or (iii) to use the Nakanishi trick [29], as in [31], namely, to define tilded functionals such that

$$\int_\Gamma\langle\psi|\overline{z\rangle}\langle\tilde{z}|\varphi\rangle dz=\int_0^\infty\langle\psi|\overline{\varpi\rangle}\langle\tilde{\varpi}|\varphi\rangle d\varpi \tag{52}$$

for all $|\psi\rangle\in\phi_{+,\text{in}},|\varphi\rangle\in\phi_{-,\text{out}}$. We will use this last notation. Then we have

$$1=\sum_n|\overline{n\rangle}\langle\tilde{n}|+\int_0^\infty d\varpi|\overline{\varpi\rangle}\langle\tilde{\varpi}|, \tag{53}$$

$$h(\text{out})=\sum_n\Omega_n|\overline{n\rangle}\langle\tilde{n}|+\int_0^\infty\Omega_\varpi|\overline{\varpi\rangle}\langle\tilde{\varpi}|d\varpi \tag{54}$$

[14] Precisely and repeating what we have anticipated in footnote 10, $|\chi\rangle\in\phi^{\text{out}}_-$ will be a function of η and a functional of field φ. But $a=a(\eta)$, and so $\chi=\chi(a,\varphi)$ is the function of Eq. (28), which multiplied by the prefactor $\exp[iMS(a)]$ is $\Psi(a,\varphi)$, a solution of the WDW equation. So it is $\Psi(a,\varphi)$, the function that really belongs to ϕ_-. This fact proves that the asymmetry we use exists either in the spaces of χ or Ψ functions and therefore in the full quantum superspace.

[15] The main difference of this curved space-time formalism, with the flat space usual one, is that in the former case we have two nonequivalent vacua $|0,\text{in}\rangle$ and $|0,\text{out}\rangle$, while in the second one we have just one, $|0\rangle$, such that $K|0\rangle=|0\rangle$, where K is the Wigner time-reversal operator. In the curved space time case $K|0,\text{in}\rangle\neq|0,\text{out}\rangle$ and therefore we have two equations $K:\phi^{\text{out}}_-\to\phi^{\text{out}}_+\neq\phi^{\text{out}}_-$ and $K:\phi^{\text{in}}_+\to\phi^{\text{in}}_-\neq\phi^{\text{in}}_+$. In the usual case the ''in'' and ''out'' superscripts would be absent and, therefore, there would only be one equation. These differences, with the usual case, must be taken into account but they are not very important. Taken into account that $\overline{\omega}>0$ the functions of ϕ_- must satisfy the condition $\langle\overline{\omega},\text{in}|\psi\rangle\in\theta(\mathcal{S}\cap H^2_+)$ and those of ϕ_+ the condition $\langle\overline{\omega},\text{out}|\psi\rangle\in\theta(\mathcal{S}\cap H^2_-)$.

[16] Eventually the location of the poles can be such that it turns out to be impossible to find the curve Γ. Then we can use a set of curves $\{\Gamma_n\}$, such that Γ_n goes below the poles $z_1,z_2,\ldots,z_n$, and take $n\to\infty$ at the end of calculations.

[17] The first term of the right-hand side (rhs) of Eq. (49) can be also obtained if we eliminate the short time effect (Zeno effect) and the long time effect (Khalfin effect). This will also originate Eq. (100).

(see also [30] and [14], where the Nakanishi trick is explained). From its own definition it is evident that $|\overline{n}\rangle,|\overline{z}\rangle,|\overline{\varpi}\rangle\in\phi^{\times}_{+,\mathrm{out}}$ since these vectors are functionals over $|\psi\rangle\in\phi_{+,\mathrm{out}}$ and that $|\widetilde{n}\rangle,|\widetilde{z}\rangle,|\widetilde{\varpi}\rangle\in\phi^{\times}_{+,\mathrm{in}}$ since these vectors are functionals over $|\phi\rangle\in\phi_{-,\mathrm{in}}$. Restoring the $[k]$ the last equation reads

$$h(\mathrm{out})=\sum_n \Omega_n|\overline{n}\rangle\langle\widetilde{n}|+\int_0^\infty d\varpi\int_k\Omega_k|\overline{\varpi,[k]}\rangle\langle\widetilde{\varpi,[k]}|d[k], \tag{55}$$

but we will continue with the previous shorthand notation and we will not write the $[k]$ anymore. It can be proved that the bases $\{|\overline{n}\rangle,|\overline{\varpi}\rangle\},\{|\widetilde{n}\rangle,|\widetilde{\varpi}\rangle\}$ are a biorthonormal system [27,28,31]: namely,

$$\langle\overline{n}|\widetilde{n}'\rangle=\delta_{nn'},\quad\langle\overline{n}|\widetilde{\varpi}\rangle=0,$$

$$\langle\overline{\varpi}|\widetilde{n}\rangle=0,\quad\langle\overline{\varpi}|\widetilde{\varpi}'\rangle=\delta(\varpi-\varpi'). \tag{56}$$

From all these equations we have that

$$h(\mathrm{out})|\overline{n}\rangle=\Omega_n|\overline{n}\rangle,$$

$$\langle\widetilde{n}|h(\mathrm{out})=\Omega_n\langle\widetilde{n}|, \tag{57}$$

where Ω_n is a complex eigenvalue, and $|\overline{n}\rangle$ are right eigenvectors and $\langle\widetilde{n}|$ left eigenvectors of $h(\mathrm{out})$. Even if $h(\mathrm{out})$ is Hermitian, it has complex eigenvalues because we are using a new spectral decomposition, which is only possible because we are working in a convenient Gel'fand triplet. This fact will be the main tool that we will use below. The eigenvalues and their squared will be written as

$$\Omega_n=\omega_n-\frac{i}{2}\gamma_n,\qquad\gamma_n\geqslant 0, \tag{58}$$

since, from Eq. (36),

$$\Omega_n^2=m^2a(\mathrm{out})^2+z_n,\qquad z_n\in\mathbb{C} \tag{59}$$

and, by its own construction, the poles z_n are in the fourth quadrant of their complex plane, and therefore also the Ω_n^2 and thus the Ω_n are in the lower half plane of the corresponding unphysical sheet and so $\gamma_n\geqslant 0$.

III. TIME EVOLUTION

Coming back to the WKB expansion (28), if we now consider the next order and the time defined in Eq. (31), the function $\chi(a,\varphi)$ must satisfy the Schrödinger equation,

$$i\frac{d\chi}{d\eta}=h(\eta)\chi, \tag{60}$$

where $h(\eta)$ is Hamiltonian h written as a function of η. Even if this Hamiltonian is time dependent we can consider that for scales $a>a_{\mathrm{out}}$ there is no particle creation and therefore we have an invariant adiabatic vacuum $|0,\mathrm{out}\rangle$ and a definitive pole structure for the S matrix [19]; so for $a(\eta)\mathrm{a}(\eta)>a_{\mathrm{out}}$ expansion (55) will always have the same structure. Thus the time evolution of χ will be

$$\chi(\eta)=\exp\left[-i\int_0^\eta h(\eta')d\eta'\right]\chi(0). \tag{61}$$

From this equation we can obtain some conclusions:

(i) In particular the time evolution of the right eigenvector $|\overline{n}\rangle$ reads

$$|\overline{n(\eta)}\rangle=\exp\left[-i\int_0^\eta\Omega_n(\eta')d\eta'\right]|\overline{n(0)}\rangle, \tag{62}$$

because even if the pole structure remains fixed, the poles move as can be seen in the example of Ref. [19], Eq. (3.3). So, from Eq. (58), we can see that, if there are some $\gamma_n>0$, the corresponding eigenvectors have a dumped evolution. Therefore, these eigenvectors correspond to decaying states. Thus our formalism naturally yields decaying states that vanish towards the direction of time that we can call the *future*.

(ii) Using Eq. (53) we can expand any function $|\varphi\rangle\in\phi_-$ as

$$|\varphi\rangle=\sum_n|\overline{n}\rangle\langle\widetilde{n}|\varphi\rangle+\int_0^\infty|\overline{\varpi}\rangle\langle\widetilde{\varpi}|\varphi\rangle d\omega; \tag{63}$$

then its time evolution will be

$$|\varphi(\eta)\rangle=\sum_n\exp\left[-i\int_0^\eta\Omega_n(\eta')d\eta'\right]|\overline{n}\rangle\langle\widetilde{n}|\varphi\rangle+\int_0^\infty\exp\left[-i\int_0^\eta\Omega_\varpi(\eta')d\eta'\right]|\overline{\varpi}\rangle\langle\widetilde{\varpi}|\varphi\rangle d\varpi, \tag{64}$$

where all the terms in the sum, such that $\gamma_n\neq 0$, have a decaying evolution, while the rest of the terms and the integral have an oscillatory behavior. So in the time evolution of (almost) any state we have a decaying term that vanishes towards the future.

(iii) In this way the asymmetry introduced in hypothesis H_2 produces an effective *time asymmetry*, because it allows us to define a future time direction, the one pointed by the dumping process. Moreover, it can be proved that, if in Eq. (61) the evolution operator is considered as an operator from space ϕ_- to space ϕ_-, namely, if we restrict the dynamics to space ϕ_-, Eq. (61) is only defined for $\eta\geqslant 0$. Therefore the evolution operator cannot be inverted and so it is really an *irreversible operator* (see [32,27,14]).

(iv) Let us consider the case of mixed states. For a mixed state $\rho\in\Phi_-$ we can generalize the spectral decomposition (63) to obtain

$$\rho=\sum_{n,m}\rho_{nm}|\overline{n}\rangle\langle\overline{n}|+\sum_{n}\int_{0}^{\infty}\rho_{n\varpi}|\overline{n}\rangle\langle\overline{\varpi}|d\varpi+\int_{0}^{\infty}\sum_{n}\rho_{\varpi n}|\overline{\varpi}\rangle\langle\overline{n}|d\varpi+\int_{0}^{\infty}\int_{0}^{\infty}\rho_{\varpi\varpi'}|\overline{\varpi}\rangle\langle\overline{\varpi}|d\varpi d\varpi'. \tag{65}$$

Repeating the computation of the pure state case, we can compute the time evolution of state $\rho(\eta)$. Since either $\gamma_n=0$ or $\gamma_n>0$, there will be oscillating terms and dumped ones. Then we obtain

$$\rho(\eta)=\rho_*(\eta)+\exp\left[-\frac{1}{2}\int_0^\eta\gamma(\eta')d\eta'\right]\rho_1(\eta), \tag{66}$$

where the first term of the rhs is an oscillatory term and the second a decaying term, where we have written a first factor corresponding to the slowest dumping factor; namely, γ is the smallest of the nonzero γ_n. When $\eta\to\infty$ we have

$$\rho(\eta)\to\rho_*(\eta). \tag{67}$$

$\rho_*(\eta)$ is a thermodynamical equilibrium state. In fact, since in its evolution there are no dumping factors, it behaves like an ordinary stable quantum state and its entropy is time constant (as we will see), namely, the one that corresponds to thermodynamical equilibrium (below we will normalize this constant to zero). It is logically a nonstationary oscillatory equilibrium state, because, even if it is in thermic equilibrium, the field cannot go to dynamical equilibrium since, in our simple model, there are no interaction terms among the field components. If these terms were present, new dumping factors would also be present and the final equilibrium would be a stationary state.[18]

This is the essence of our formalism. Below we will see the results that we can obtain if we follow this road.

[18]In a practical complete case microscopic motions would always remain. These microscopic motions are, in our model, those of the field, namely, those that correspond to the integral in the spectral decomposition. In practice they have a maximum (very small) length, since the integral in the spectral decomposition really does not begin with zero, but with a finite value. If we were to introduce a coarse graining, these microscopic motions would be hidden and the equilibrium would be the usual stationary equilibrium state $\rho_*=\text{const}$. But in our formalism there is no need of a coarse graining to hide the microscopic motions. Really we have only a *mathematical graining* (the choice of the right mathematical structure) to produce time asymmetry. The macroscopic energy we had at the beginning goes into microscopic oscillations in order to fulfill the law of energy conservation. Also if we multiply Eq. (66) by a smooth distribution σ, the microscopic oscillation would be smeared, $[\rho_*(\eta)|\sigma]$ becomes a constant, and we have a typical equilibrium weak limit [34]: $\lim_{\eta\to\infty}[\rho(\eta)|\sigma]=[\rho_*(\eta)|\sigma]$.

IV. DECOHERENCE AND CORRELATIONS

In Ref. [19], using our formalism, it is proved that, if the S matrix has an infinite number of complex poles, we have decoherence and that, in unstable states, configuration and momentum are correlated, in such a way that the universe ends in a classical phase. In this demonstration hypothesis H_3 plays an essential role. In [19] it was not proved that, in general, the S matrix, relevant for our problem, has an infinite set of complex poles, but that set was computed in one example, while other examples were proposed.

Here we will review the demonstration of Ref. [19], in a condensed but more general way, and we will complete this paper observing that using a potential, with a bounded support, as in the present paper, the existence of an infinite set of poles is a consequence of Refs. [23] and [24].

In fact, a massive scalar field, conformally coupled, in metric (22) satisfies Klein-Gordon equation

$$\left(\nabla_\mu\nabla^\mu+m^2+\frac{1}{6}R\right)\psi=0,\quad R=6a^{-3}\frac{\partial^2 a}{\partial\eta^2}. \tag{68}$$

This equation corresponds to Hamiltonian (35) and it leads, by variable separation, to

$$\psi_{\mathbf{k}}=\frac{1}{(2\pi)^{3/2}a(\eta)}f_{\mathbf{k}}(\eta)\exp(\pm i\mathbf{k}\cdot\mathbf{x}), \tag{69}$$

and $f_{\mathbf{k}}(\eta)$ satisfies a generalized oscillator equation with time-dependent frequency:

$$f''_{\mathbf{k}}+\omega^2(\eta)f_{\mathbf{k}}=0,\quad \omega=[a^2(\eta)m^2+k^2]^{1/2}. \tag{70}$$

On the other hand, in ordinary quantum mechanics, the Hamiltonian of a massive particle in a potential $W(r)$ is

$$H=-\frac{1}{2m}\Delta\psi+W(r)\psi \tag{71}$$

and a stationary solution

$$\psi_{klm}=\frac{u_k(r)}{r}Y_l^m(\theta,\varphi) \tag{72}$$

satisfies the equation

$$u''_k(r)+\omega^2(r)u_k(r)=0,$$
$$\omega(r)=\left[k^2-\frac{l(l+1)}{r^2}-2mW(x)\right]^{1/2}. \tag{73}$$

So both phenomena can be mathematically related according to the analogy

$$\eta\leftrightarrow r;\quad \mathbf{k}\leftrightarrow k,l,m;\quad a^2(\eta)m^2\leftrightarrow k^2-\frac{l(l+1)}{r^2}-2mW(x);$$
$$k^2\leftrightarrow 2mE. \tag{74}$$

More details about this analogy can be obtained from [23].

Now from Ref. [24], p. 218, we know that the S matrix of a cutoff potential $W(r)$, namely, a potential with a bounded support, has an infinite number of complex poles. Our potential is $\sim a^2(\eta)$ which is practically a constant for $a \gg a_1$, since $\varepsilon \cong 0$, and exactly a constant if we consider the value $\varepsilon = 0$, and so, subtracting this constant final value, we can say that it is a cutoff potential, with an S matrix endowed with an infinite set of poles.

Almost all potentials used in the literature of quantum field theory in curved space-time [22] are very well behaved in the infinities and can be approximated by this, bounded support, kind of potential (this is not the case for some QC potentials, which we will discuss in the Conclusions). So the existence of an infinite set of poles seems quite a general feature of the theory. Thus, using the equation of [19], it can be proved that our formalism leads to decoherence, to correlations, as will be reviewed below, and to the outcome of a classical universe.

Finally, in particular subsystems of the universe the S matrix has poles if unstable quantum states exist in the subsystem [27]. Of course, these poles will also appear in any complete S matrix of the universe.

A. Decoherence

Decoherence naturally appears in systems where the S matrix has complex poles [14], and therefore in the system we are studying. The classical geometries are defined by a choice of the sign $\pm$ and the constant C in Eq. (32); we will call these labels $\alpha, \beta, \ldots$. We will call φ_N the field φ of Eq. (28), where N will label the possible modes; precisely, we will use n for the discrete unstable states coming from the poles, and k for the continuous stable states coming from the continuous spectrum. When we will be referring to both kinds of modes we will use the index N. Then Eq. (28) reads

$$\Psi(a,[\varphi_N]) = \exp[iM^2 S(a)]\chi(a,[\varphi_N]), \tag{75}$$

where $\chi(a,[\varphi_N])$ can be written as

$$\chi(a,[\varphi_N]) = \prod_N \chi_N(\eta,\varphi_N). \tag{76}$$

We can obtain $\chi_N(\eta,\varphi_N)$ via a Gaussian ansatz:

$$\chi_N(\eta,\varphi_N) = A_N(\eta)\exp[i\alpha_N(\eta) - B_N(\eta)\varphi_N^2]. \tag{77}$$

The functions $A_N(\eta)$ and $\alpha_N(\eta)$ are real while $B_N(\eta) = B_{NR}(\eta) + iB_{NI}(\eta)$ is complex. They can be obtained by solving the system

$$A_N(\eta) = \pi^{-1/4}[2B_{NR}(\eta)]^{1/2},$$

$$\dot{\alpha}_N(\eta) = -B_{NR}(\eta),$$

$$\dot{B}_N = -2iB_N^2(\eta) + \frac{i}{2}\Omega_N^2(\eta), \tag{78}$$

where the overdot denotes derivatives with respect to η. From these equations, just working with the real continuous spectrum, decoherence can be proved under some restricted conditions [18,19]. But we will show that if we use both the complex discrete and the real continuous spectra, decoherence can be proved for almost all initial conditions. From Eq. (75), after the integration of the modes of the scalar field (considered as the environment) we obtain the following reduced density matrix:

$$\begin{aligned}\rho_r(a,a') = {} & \exp[-iM^2S_\alpha(a) + iM^2S_\alpha(a')]\rho^{\alpha\alpha}(a,a') \\ & + \exp[-iM^2S_\alpha(a) + iM^2S_\beta(a')]\rho^{\alpha\beta}(a,a') \\ & + \exp[-iM^2S_\beta(a) + iM^2S_\alpha(a')]\rho^{\beta\alpha}(a,a') \\ & + \exp[-iM^2S_\beta(a) + iM^2S_\beta(a')]\rho^{\beta\beta}(a,a'),\end{aligned} \tag{79}$$

where α and β symbolize two classical solutions and

$$\begin{aligned}\rho^{\alpha\beta}(a,a') &= \prod_N \rho_N^{\alpha\beta}(a,a') \\ &= \prod_N \int d\varphi_N \chi_N^{\alpha *}(\eta,\varphi_N)\chi_N^{\beta}(\eta',\varphi_N).\end{aligned} \tag{80}$$

On the other hand, from Refs. [18,19], it is

$$B_N = -\frac{i}{2}\frac{\dot{g}_N}{g_N}, \tag{81}$$

where g_N is a solution of

$$\ddot{g}_N + \Omega_N^2 g_N = 0, \tag{82}$$

where Ω_N can be complex, as in Eq. (58), where we know that there are an infinite number of modes n.

Let us now consider the asymptotic (or adiabatic) expansion of the function g_N, when $\eta \to \infty$, in the basis of the out modes. As this g_N corresponds to an arbitrary initial state its expansion reads

$$g_N = \frac{P_N}{\sqrt{2\Omega_N}}\exp\left(-i\int_0^\eta \Omega_N d\eta\right) + \frac{Q_N}{\sqrt{2\Omega_N}}\exp\left(i\int_0^\eta \Omega_N d\eta\right), \tag{83}$$

where P_N and Q_N are arbitrary coefficients. It is obvious that if all the Ω_N are real, as in the case of the Ω_k, the last equation will have an oscillatory nature, as well as its derivatives. This will also be the behavior of B_k in Eq. (81). Therefore the limit $\eta \to +\infty$ will be not well defined, even if B_k could be bounded. But if Ω_N is complex, the first term of Eq. (83) will have a dumping factor and the second one a growing one. In this case $N = n$ and when $\eta \to +\infty$ we have

$$B_n \approx -\frac{i}{2}\frac{\dot{g}_n}{g_n} = \frac{1}{2}\Omega_n. \tag{84}$$

Then we have two cases.

(i) $\Omega_N = \Omega_k \in \mathbb{R}^+$ for the real factor corresponding to the stable states. Then we see that when $\eta \to +\infty$ the RHS of Eq. (80) is an oscillatory function with no limit in general. We will only have good limits for some particular cases listed in Refs. [18] and [19].

(ii) $\Omega_N = \Omega_n = \omega_n - (i/2)\gamma_n \in \mathbb{C}$ for the complex factor corresponding to decaying states. Then for $\eta \to +\infty$ we will have a definite limit of Eq. (84). Therefore, in this case we can calculate the $\rho_{rn}^{\alpha\beta}$ corresponding to the complex factors, and we obtain

$$\rho_{rn}^{\alpha\beta}(a,a')=\left[\frac{4B_{nR}(\eta,\alpha)B_{nR}(\eta',\beta)}{[B_n^*(\eta,\alpha)+B_n(\eta',\beta)]^2}\right]^{1/4} \times\exp[-i\alpha_n(\eta,\alpha)+i\alpha_n(\eta',\beta)]. \quad (85)$$

Now when $\mathrm{Im}B_n\approx\frac{1}{2}\mathrm{Im}\Omega_n\neq 0$ and $B_n^*(\eta,\alpha)\neq B_n(\eta',\beta)$ it can be proved that

$$|\rho_{rn}^{\alpha\beta}(a,a')|<1, \quad (86)$$

and as there is an infinite number of these complex factors the product (80) vanishes when $\eta\to+\infty$. Then we have decoherence when $B_n^*(\eta,\alpha)\neq B_n(\eta',\beta)$, namely, if $\Omega_n^*(\eta,\alpha)\neq\Omega_n(\eta',\beta)$ or

$$a(\eta,C_\alpha,(\pm)_\alpha)\neq a(\eta',C_\beta,(\pm)_\beta). \quad (87)$$

So we have decoherence (i) for different classical solutions, i.e., $C_\alpha\neq C_\beta$ or $(\pm)_\alpha\neq(\pm)_\beta$, even if the time is the same $\eta=\eta'$, or (ii) for the same classical solution, i.e., $C_\alpha=C_\beta$ and $(\pm)_\alpha=(\pm)_\beta$, if the times are different, $\eta\neq\eta'$.

B. Correlations

From Ref. [18] we know that the existence of correlations can be proved using only the real continuous spectrum, and nothing new can be added in this case. We must only study the correlation for the unstable states of the discrete spectrum. Correlations take place inside each classical solution and, therefore, they can be computed using the Wigner function associated with $\rho_{rn}^{\alpha\alpha}(a,a')$ [18,19], namely,

$$F_W^{\alpha\alpha}(n)(a,P_a)=\int_{-\infty}^{+\infty}d\Delta\exp(-2iP_a\Delta)\rho_{rn}^{\alpha\alpha}\left(a-\frac{\Delta}{M},a+\frac{\Delta}{M}\right), \quad (88)$$

where $a,a'=a\pm\Delta/M$, and P_a is the canonical momentum conjugated to a. Then we can repeat the reasonings of [18], from Eqs. (2.24) to (2.28), and we will arrive at

$$F_W^{\alpha\alpha}(n)(a,P_a)\approx C^2(a)\sqrt{\frac{\pi}{\sigma^2}}\exp\left[-\left(P_a-MS'+\alpha'-\frac{B'_{nI}}{4B_{nR}}\right)^2\bigg/\sigma^2\right], \quad (89)$$

where the prime symbolizes derivatives with respect to a and $\sigma^2=|B_n'|^2/4B_{nR}^2$. When $\eta\to+\infty$ we know that $B_n\approx\frac{1}{2}\Omega_n$, and so

$$\sigma^2=\frac{1}{2}\times\frac{m^4a^2}{\sqrt{(m^2a^2+x_n)^2+y_n^2}[m^2a^2+x_n+\sqrt{(m^2a^2+x_n)^2+y_n^2}]}, \quad (90)$$

where $z_n=x_n+iy_n$ is the corresponding complex pole. Thus, if $a_{\mathrm{out}}\gg 1$, we have that $\sigma\approx 1/a$ and we have correlation in the solutions corresponding to unstable states.

For more details about these two subsections, see [19].

V. ENTROPY

Let $\rho(\eta)$ be the density matrix, of the universe or one of its subsystems, for a physical admissible state $(\rho\in\Phi_-)$ and let $\rho_*(\eta)$ be the corresponding thermodynamical equilibrium matrix. In the universe these matrices are related by Eq. (66). The Hamiltonian of the subsystem is necessarily a term of the general Hamiltonian of Eq. (24) and its S matrix must have poles if the subsystem is not trivial (see [14] and [27]). So we can repeat all that we have said for the universe for the case of the subsystem and we can also choose a t-asymmetric regular space state for the subsystem. We must take care that the dumping or future direction of the subsystem coincides with the dumping or future direction of the universe for consistency. I.e., the local and global arrows of time must coincide. Then we will also find Eq. (66) for the subsystem. The only difference would be that, if the subsystem Hamiltonian h is not time dependent (with respect, e.g., to the proper time t), the integral of Eq. (66) must be subtituted by the usual product ht.

Then we can define the conditional entropy $S=S[\rho(\eta)|\rho_*(\eta)]$ of state $\rho(\eta)$ with respect to state $\rho_*(\eta)$ [34] both for the universe or the subsystem,

$$S=S[\rho(\eta)|\rho_*(\eta)]=-\mathrm{tr}\{\rho(\eta)\ln[\rho_*^{-1}(\eta)\rho(\eta)]\}, \quad (91)$$

such that $S[\rho_*(\eta)|\rho_*(\eta)]=0$; namely, the entropy vanishes at equilibrium (in this definition we consider that the vanishing trace ghosts have been eliminated by that procedure explained in [35], and also other technicalities, explained in this paper, are taken into account). We can, as well, use the corresponding classical definition, since we are really interested in the classical phase of the universe, but in order to use one notation only, we will use the quantum formulas.[19]

From Eq. (66) we see that

$$\mathrm{tr}\rho(\eta)=\mathrm{tr}\rho_*(\eta)=1\Rightarrow\mathrm{tr}\rho_1(\eta)=0; \quad (92)$$

i.e., if the states $\rho(\eta)$ and $\rho_*(\eta)$ are normalized as it should be, $\rho_1(\eta)$ has a vanishing trace, and thus $\rho_1(\eta)$ is not a state

[19] There is a close relation between the quantum and classical cases, and so Eq. (66) can also be obtained in the second one [14]. This relation can be obtained using [33], as is shown in Ref. [50]. The Wigner function is not positive definite, and this problem is studied in Ref. [35].

but the coefficient of a correction to the equilibrium state to obtain state $\rho(\eta)$. The vanishing of the trace of ρ_1 is directly proved in [14,32,36], using our formalism. Now if we expand the logarithm in Eq. (91), and use Eq. (92) we obtain

$$S=S[\rho(\eta)|\rho_*(\eta)]=-\exp\left[-\frac{1}{2}\int \gamma d\eta\right]\mathrm{tr}[\rho_*^{-1}(\eta)\rho_1^2(\eta)] + \cdots, \tag{93}$$

where the ellipsis symbolizes higher order terms.[20]

This entropy has the property:

$$\lim_{\eta\to\infty} S[\rho(\eta)|\rho_*(\eta)]=0; \tag{94}$$

namely, the entropy evolves towards its null equilibrium value. This is so because the prefactor in Eq. (93) dominates any other time variation, since $\rho_*(\eta)$ is usually oscillatory [namely, it will be oscillatory in case (a), but not in case (b); see below] and $\rho_1(\eta)$ has oscillatory terms and dumping factors that vanish faster than the dominant decaying factor.

In Eq. (91) we have two matrices $\rho(\eta)$ and $\rho_*(\eta)$; then we also have two possibilities. Either both matrixes have the same kind of evolution or they have different ones. The first is the case of a closed system, e.g., the universe, but the second case appears when $\rho_*(\eta)$ follows a different evolution due to, e.g., an external agency; this would be the case of an open system within the universe. Let us consider the two cases.

(a) The closed case. Both matrices follow the same evolution law. Namely, if we have a time variable Hamiltonian $h(\eta)$, as in the case of the universe, the evolution will be

$$\rho(\eta)=\exp\left[-i\int_{\eta'}^{\eta} l(\eta)d\eta\right]\rho(\eta'),$$

$$\rho_*(\eta)=\exp\left[-i\int_{\eta'}^{\eta} l(\eta)d\eta\right]\rho_*(\eta'), \tag{95}$$

where $l(\eta)$ is the corresponding Liouville operator, i.e., $l(\eta)\rho=h(\eta)\rho-\rho h(\eta)$.

In the case of a closed subsystem of the universe, where the Hamiltonian h is not a proper-time variable (e.g., in a subsystem which does not expand or contract due to an external agency), we would have

$$\rho(t)=\exp[-il(t-t')]\rho(t'),$$

$$\rho_*(t)=\exp[-il(t-t')]\rho_*(t'), \tag{96}$$

where l is the Liouville operator corresponding to the Hamiltonian h and t is the proper time. In this case the equilibrium matrix $\rho_*(t)$ also evolves in the same way as the matrix $\rho(t)$. Now, from what we have said in Sec. III, point (iii), Eq. (95) is only valid if $\eta>\eta'$. Analogously, if the subsystem S matrix has poles and, for the subsystem, we have also chosen an admissible function space with the same criteria as those used to choose space Φ_-, Eq. (96) is only valid if $t-t'\geqslant 0$ or $t\geqslant t'$. Therefore the last two evolutions are irreversible. Now, from Ref. [34] (using the classical-quantum analogy [33,35], since we are in the classical phase, the ''exp'' operator of the last two equations will be a Frobenius-Perron operator, and we can use the classical definition of conditional entropy) we know that

$$S[\rho(\eta)|\rho_*(\eta)]\geqslant S[\rho(\eta')|\rho_*(\eta')],$$

$$S[\rho(t)|\rho_*(t)]\geqslant S[\rho(t')|\rho_*(t')], \tag{97}$$

respectively. It would be ''='' if the evolution operator would be reversible, *but it is not* [consider also Eq. (93)]. Then these entropies are really monotonically growing. Therefore we have proved the second law of thermodynamics for the whole universe or for any nontrivial closed subsystem. So our formalism yields this fundamental law naturally (compare with the much more complicated coarse-graining method of [37]).

The demonstration is based on the fact that (in both cases) ρ is an admissible state (like the ink drop spreading in the glass of water), and so $\rho\in\Phi_-$ and $\gamma_n>0$. If we would have taken $\rho\in\Phi_+$, it would be $\gamma_n<0$ and the entropy would decrease showing the following:

(i) *In the case of the closed subsystem within the universe*: Φ_+ is a space of a clearly nonadmissible solution (the ink drop contracting spontaneously). In fact, in this case the arrow of time is the one of the universe and not the one of the subsystem, and in the subsystem we will see a decay of the entropy, showing that these states are not physically admissible.

(ii) *In the case of the universe*: Going from Φ_- to Φ_+ we have simply changed our convention (since all possible arrows of time are embodied in the universe evolution), conventionally calling the ''future'' the direction of decreasing entropy.

(b) The open case. Let us now consider the important case of an open subsystem of the universe, e.g., the matter and radiation within an expanding universe (since in this case we do not take into account the entropy of the gravitational field, and we will consider that this field as an external agency that expands the space, where the matter and the radiation are located). Then the conditional entropy is not necessarily monotonically increasing, at least for short times. In fact, we cannot use Eq. (97) since $\rho_*(\eta)$ does not satisfy an equation such as the second equation of Eq. (95), because its evolution could be fixed by an agency external to the system, e.g., a thermostat or the universe expansion as we will see. In this last case this fact is completely logical since $S[\rho(\eta)|\rho_*(\eta)]$ is just the matter-radiation entropy (with no gravitational field entropy contribution) in an expanding (or contracting) universe with an equilibrium state $\rho_*(\eta)$, which varies independently. Therefore it is not the total entropy. It is well

[20]As in this equation distributions are multiplied, some care must be taken in order to convince ourselves that what we are doing is mathematically correct. E.g., the distributions can be transformed in ordinary density matrices by a Λ transformation [14]. This transformation maintains the dumping factors, and so the results obtained remain valid, but the distributions become ordinary matrices, which can be multiplied. In this way the rhs of Eq. (93) becomes well defined. The multiplication can be also done using matrix M of Ref. [51]. There are also more refined mathematical ways to reach to the desired result, as the one of Refs. [32,35].

known that matter (let us say a gas) can have decreasing entropy into a variable geometry (let us say a box with moving walls). A phenomenological study of the problem can be found in Refs. [38,39]. In this case what we have called, up to now, S is just the entropy gap ΔS with respect to a variable maximal possible entropy $S_{\max}$. The actual entropy, which grows monotonically, is $S=S_{\rm act}=S_{\max}+\Delta S$. But ΔS does not have this property. Furthermore, the diminishing of ΔS for short times is welcome, as we will see in the next two sections.

VI. ENTROPY GAP

In this section we study the universe entropy gap $\Delta S=S_{\rm act}-S_{\max}$, following a qualitative idea of Davies [38]. Actually we will complete this idea computing the entropy gap after decoupling time. Therefore we will change our model; it still will be homogeneous and isotropic, with metric (22), but obviously the particle production will be finished, and so we will consider that we are simply in a flat geometry, matter-dominated, universe.

It is well known that the isotropic and homogeneous expansion of the universe is a reversible process with constant entropy [40]. In this case the matter and the radiation of the universe are in a thermic equilibrium state $\rho_*(t)$ at any time t. As the radiation is the only important component, from the thermodynamical point of view, we can choose $\rho_*(t)$ as a blackbody radiation state [41]; i.e., $\rho_*(t)$ will be a diagonal matrix with a main diagonal:

$$\rho_*(\omega)=ZT^{-3}\frac{1}{e^{\omega/T}-1}, \tag{98}$$

where T is the temperature, ω the energy, and Z a normalization constant [[42], Eqs. (60.4) and (60.10)]. The total entropy is

$$S=\frac{16}{3}\sigma VT^3 \tag{99}$$

[[42], Eq. (60.13)], where σ is the Stefan-Boltzmann constant and V a comoving volume.

Let us consider our isotropic and homogeneous model of universe with scale a. Any comoving volume evolves as $V\sim a^3$, and, since from the conservation of the energy-momentum tensor and radiation state equation, we know that $T\sim a^{-1}$, we can verify that $S=$const. Thus the irreversible nature of the universe evolution is not produced by the universe expansion, even if $\rho_*(t)$ has a slow time variation.

Therefore, after decoupling time, the main process that has an irreversible nature is the burning of unstable H in the stars (which produces He and, after a chain of nuclear reactions, Fe). This unstable state produces poles in the corresponding S matrix and a nuclear reaction process, with mean lifetime $t_{\rm NR}=2\gamma^{-1}$. Therefore, using Eq. (66), and considering that γ is constant (under proper-time variation), since it corresponds to a local process considered in case (a) of the last section (or simply on phenomenological grounds), we can then say that the state of the universe, at time t, is

$$\rho(t)=\rho_*(t)+\rho_1e^{-\gamma t}+o[e^{-\gamma t}], \tag{100}$$

where ρ_1 is a certain phenomenological coefficient, which is constant in time since all the time variation of nuclear reactions is embodied in the exponential law $e^{-\gamma t}$. Also, on phenomenological grounds, we can foresee that ρ_1 must peak strongly around ω_1, the characteristic energy of the nuclear process. All these reasonable phenomenological facts can also be theoretically explained in different ways; e.g., Eq. (100) can be computed with the theory of [28] or [35]. In Ref. [32] it is explicitly proved that ρ_1 peaks strongly at the energy ω_1. So using Eq. (91) we can compute the entropy gap

$$\Delta S=-\operatorname{tr}[\rho\ln(\rho_*^{-1}\rho)]. \tag{101}$$

Now using Eq. (100) and considering only times $t\gg t_{\rm NR}$ γ^{-1} we can expand the logarithm, as in Eq. (93), to obtain

$$\Delta S\approx -e^{-\frac{1}{2}\gamma t\operatorname{tr}(\rho_*^{-1}\rho_1^2)}, \tag{102}$$

where we have used Eq. (92). We now introduce the equilibrium state (98) for $\omega\gg T$. Then

$$\Delta S\approx -Z^{-1}T^3e^{-\frac{1}{2}\gamma t}\operatorname{tr}(e^{\omega/T}\rho_1^2), \tag{103}$$

where $e^{\omega/T}$ is a diagonal matrix with this function as the diagonal. But as ρ_1 is peaked around ω_1 we arrive to a final formula for the entropy gap:

$$\Delta S\approx -CT^3e^{-\frac{1}{2}\gamma t}e^{\omega_1/T}, \tag{104}$$

where C is a positive constant.

Let us now compute the time evolution of the entropy gap. We have computed ΔS for times larger than decoupling time and therefore, as $a\sim t^{2/3}$ and $T\sim a^{-1}$, we have

$$T=T_0\left(\frac{t_0}{t}\right)^{2/3}, \tag{105}$$

where t_0 is the age of the universe and T_0 the present temperature. Then

$$\Delta S\approx -C_1e^{-\frac{1}{2}\gamma t}t^{-2}\exp\left[\frac{\omega_1}{T_0}\left(\frac{t_0}{t}\right)^{2/3}\right], \tag{106}$$

where C_1 is a positive constant. Drawing the corresponding curve [39] it can be seen that ΔS has a maximum at $t=t_{{\rm cr}_1}$ and a minimum at $t=t_{{\rm cr}_2}$. Let us compute these critical times. The time derivative of the entropy reads

$$\Delta\dot{S}\approx\left[-\frac{1}{2}\gamma-2t^{-1}+\frac{2}{3}\frac{\omega_1}{t_0T_0}\left(\frac{t_0}{t}\right)^{1/3}\right]\Delta S. \tag{107}$$

This equation shows two antagonistic effects. The universe expansion effect is embodied in the second and third terms in the square brackets, being an external agency to the matter-radiation system such that, if we neglect the second term, it tries to increase the entropy gap and, therefore, to take the system away from equilibrium (as we will see the second term is practically negligible). On the other hand, the nuclear reactions embodied in the γ term try to convey the matter-radiation system towards equilibrium. These effects become equal at the critical times $t_{\rm cr}$ such that

$$\frac{1}{2}\gamma t_0 + 2\frac{t_0}{t_{cr}} = \frac{2}{3}\frac{\omega_1}{T_0}\left(\frac{t_0}{t_{cr}}\right)^{1/3}. \tag{108}$$

For almost any reasonable numerical values this equation has two positive roots: $t_{cr_1} \ll t_0 \ll t_{cr_2}$.

Precisely, (i) for the first root we can neglect the γt_0 term and we obtain

$$t_{cr_1} \approx t_0 \left(3\frac{T_0}{\omega_1}\right)^{3/2} \tag{109}$$

(this quantity, with a minus sign, gives the third nonphysical root) and (ii) for the second root we can neglect the $2(t_0/t_{cr})$ term, and we find

$$t_{cr_2} \approx t_0 \left(\frac{2}{3}\frac{\omega_1}{T_0}\frac{t_{NR}}{t_0}\right)^3. \tag{110}$$

Let us make now some numerical estimates. We must choose numerical values for four parameters: $\omega_1 = T_{NR}$, $t_{NR} = \gamma^{-1}$, t_0, and T_0.

T_{NR} and t_{NR} can be chosen between the following values [43]:

$$T_{NR} = 10^6 - 10^8 \ \ \text{K}, \tag{111}$$

$$t_{NR} = 10^6 - 10^9 \ \ \text{yr},$$

while for t_0 and T_0 we can take

$$t_0 = 1.5 \times 10^{10} \ \ \text{yr}, \tag{112}$$

$$T_0 = 3 \ \ \text{K}.$$

In order to obtain a reasonable result we choose the lower bounds for T_{NR} and t_{NR} and for t_{cr_1} we obtain

$$t_{cr_1} \approx 1.5 \times 10^3 \ \ \text{yr}. \tag{113}$$

So t_{cr_1} is smaller than the decoupling time and it should not be considered since the physical processes before this time are different than those we have used in our model. Also, we must only consider times $t > t_{NR} = \gamma^{-1}$, in order to use Eq. (102).

For t_{cr_2} we obtain

$$t_{cr_2} \lesssim 10^4 t_0. \tag{114}$$

From Eqs. (113) and (114) we can see that really $t_{cr_1} \ll t_0 \ll t_{cr_2}$.

Thus (i) from t_{NR} to t_{cr_2} the expansion of the universe produces a decreasing of the entropy gap, according to a prediction of Davies [38] . Also, it probably produces a growing order, and therefore the creation of structures like clusters, galaxies, and stars [44].

(ii) After t_{cr_2} we have a growing of entropy, a decreasing order, and a spreading of the structures: Star energy is spread in the universe, which ends in a thermic equilibrium [45]. In fact, when $t \to \infty$ the entropy gap vanishes [see Eq. (106)] and the universe reaches a thermic equilibrium final state.

$t_{cr_2} \lesssim 10^4 t_0$ is the frontier between the two periods. Is the order of magnitude of t_{cr_2} a realistic one? In fact it is, since $10^4 t_0 \approx 1.5 \times 10^{14}$ yr after the big bang all the stars will exhaust their fuel [45], and so the border between the two periods should have this order of magnitude. Furthermore, it should also be smaller than this number. This is precisely the result of our calculations contained in Eq. (114) (see also [39]).

So we are at the edge of a correct physical prediction, even if our model is extremely naive and simplified, a homogeneous universe, and besides we have neglected the higher order terms in Eq. (100) which perhaps may be important for finite times. Besides in the real universe nuclear reactions take place within the stars, which can only be properly considered in an inhomogeneous geometry. Nevertheless, this rough numerical estimate shows that the theory can be used for practical purposes. Furthermore the decreasing of the entropy gap, in the period $t_{NR} < t < t_{cr_2}$, will be crucial in the next section.

VII. BRANCH SYSTEM

The set of *irreversible* processes within the universe, each one beginning in an unstable nonequilibrium state, can be considered a *branch system* [38,46]. Namely, every one of these processes begins in a nonequilibrium state, such that this state was produced by a previous process of the set. E.g., a Gibbs ink drop (initial unstable state) spreading in a glass of water (irreversible process) is only probable (since the probability to create an ink drop by fluctuations is extremely small) if there was first an ink factory, which extracted the necessary energy from an oven, where coal (initial unstable state) was burnt (branched irreversible process); in turn coal was created with energy coming from the Sun, where H (initial unstable state) is burnt into He (branched irreversible process); finally H was created using energy obtained from the unstable initial state of the universe (the absolute initial state of the branch system). Therefore, using this hierarchical chain, all the irreversible processes are related to the cosmological initial condition, the only one that must be explained. Let us observe the following:

(i) The branch system defines its own arrow of time, the *branch arrow of time (BAT),* as the direction that goes from the unstable initial state of every member of the system towards equilibrium. Probably the BAT is the most useful of all arrows of time, since it is present in any irreversible local process.

(ii) Once we have the branch system the irreversible evolution of each system is easy to explain, since once we have understood the origin of the initial unstable state of each irreversible process within the universe (even if we have not yet discussed the origin of the initial state of the whole universe) it is not difficult to obtain Lyapunov variables (or irreversible evolution equations), if we consider, e.g., that the subsystems where these processes take place are not isolated. If it is so, forces of a stochastic nature penetrate from the exterior of each subsystem and, it is well known, that if we add stochastic terms to the time-symmetric evolution equation, we obtain time-asymmetric ones, yielding Lyapunov variables, e.g., a nondecreasing entropy [34]. We can also consider that each subsystem has an enormous amount of information and we are able to measure, compute, and control a part of this information, which we will call *relevant.* If we neglect the rest of the information, the *irrelevant* one, we can obtain also irreversible evolution equations and Lyapunov variables [34,47]. These two procedures can be considered within the coarse-graining usual formalism.

(iii) But of course, if we follow the ideas of this paper, we will use more refined mathematical tools and, in each of the subsystems, we will introduce a model similar to the one we have used for the whole universe, as we have done in Sec. V, introducing the hypothesis H_2 in each subsystem. (It has already been done in [14,27,32], and the same results are obtained, i.e., irreversible evolution equation, Lyapunov variables, etc.) Then we see that entropy grows in each subsystem provided the state of the subsystem would be chosen among the physically admissible states of space Φ_-. Then each subsystem of the branch system begins in an unstable, low entropy state and evolves towards thermal equilibrium. The physical nonadmissible growing states (those of space Φ_+) correspond to theoretical evolutions that would only exist before the instant of creation of the subsystem (the instant when we put the ink in the glass of water), evolving with decreasing entropy, towards that instant (namely, the ink drop contracting spontaneously). These evolutions simply do not exist in nature because, before the instant of its creation, the subsystem really does not exist as such. In fact, before that instant a different subsystem existed with different evolution laws (e.g., the ink factory that creates the ink drop). Therefore all the scenarios we are using turn out to be realistic and satisfactory.

(iv) So only one problem is left: Why did the universe begin in an unstable, out of equilibrium, low entropy state? Let us first observe that really we are referring not to the ''whole'' universe (with its gravitational field) but only to the matter-radiation subsystem of the universe.[21]

Then, in the no-time version of the introduction we have postulated H_1 and H_2. Using these hypotheses we have reconstructed time and demonstrated, in the sections above, that the universe expansion creates, in its matter-radiation subsystem, an entropy gap ΔS that takes it out of equilibrium, not only at $t=0$, but in a long period of its history, since the actual entropy is $S_{\mathrm{act}}<S_{\max}$. We have also demonstrated that the matter-radiation subsystem of the universe evolves to a final state of thermic equilibrium since $\Delta S\to 0$, when $t\to\infty$ [cf. Eq. (94)]. *So the answer to the only problem left is hypothesis H_2* since all the facts above are based on this hypothesis. Certainly, someone will think that we have solved a problem by postulating an axiom, and this is not a very exciting result. But if the axiom yields the solutions of many problems, and this is the case of H_2, the axiom must be welcome. After all, this is the role of axioms.

(v) Finally we can ask ourselves if, in the perspective of the branch system idea, H_2 is a natural hypothesis. H_2 says that $\rho\in\Phi_-\subset\mathcal{S}\otimes\mathcal{S}$. So first it is postulated that $\rho\in\mathcal{S}\otimes S$, and therefore ρ is a smooth function, with infinite derivatives, and well behaved in the infinities of the configuration space. This part of H_2 seems quite natural, certainly much more natural than the two other alternative possibilities, namely, (a) $\rho\in\mathcal{L}$, in which case ρ can be, e.g., a square integrable function of $\mathcal{H}\otimes\mathcal{H}$, where in a set of points, the function can have noncontinuous and arbitrary values. What is the physical meaning of this discontinuity? (b) $\rho\in\mathcal{S}^\times\otimes\mathcal{S}^\times$, namely, a distribution, e.g., a δ function, certainly a quite unnatural state.

So the first part of H_2 is natural. The second part is to ask why ρ would be endowed with a natural asymmetry, the one of Φ_-. Is it too much to ask? Let us study this question according to the branch system idea and our formalism. There will be no branch system only if the universe (and now we are referring to the whole universe with the gravitational field included) would begin in an equilibrium state ρ_*, since in this case it will always remain in equilibrium. Now, from Eqs. (65), (66), and (67) we see that, in this case, $\rho_*\in\Phi_-^\times$, and so ρ_* would be a distribution,[22] something like a δ function, and we have just considered this choice as unnatural. On the contrary $\rho\in\Phi_-$ is a much more natural (i.e., regular) state. Any state of Φ_- will produce a branch system, since any state of Φ_- yields Eq. (66). So we can, at least, conclude that H_2 is the requirement that the state of the universe would be a natural and an asymmetric one. H_2 is also intimately related to the branch system idea and in consequence it is also related to the fact that really our universe is a branch system. H_2 is just the transcription of these physical facts.

VIII. COORDINATION OF THE ARROWS OF TIME

In this section we will only consider the coordination of the arrows of time related to our model, namely,

(i) *The branch arrow of time (BAT)*, the arrow that goes from the unstable initial state of every process of the universe branch system to its equilibrium final state. As we have seen in the last section this arrow is a direct consequence of the asymmetry introduced by H_2. (ii) *The thermodynamic arrow of time (TAT)*, which points to the direction of the growing of the universe entropy S. (iii) *The cosmological arrow of time (CAT)*, which points to the direction of the growing of the universe scale a.

Of course, all these arrows are related to time and therefore they must only be considered in the classical (or semiclassical) period where the time η, given by Eq. (31), is well defined. As shown in Sec. VI, η has the BAT direction, since it points away from the initial unstable state. In the timeless quantum period we only have the asymmetry defined by H_2.

Then Eq. (97), for nonexpanding or contracting subsystems within the universe, which is a consequence of H_2,

[21]Let us also observe that for our purposes we can put the ''beginning of the universe'' (precisely the unstable beginning of our branch system or $t=0$) after decoupling time. With this change we avoid the problem we could have if we put $t=0$ in the quantum gravity period and still demonstrate our thesis.

[22]This fact is evident if we consider a Baker's transformation [34]: Let us consider a distribution function with compact support, and such that it belongs to $\mathcal{L}=L^2([0,1]\times[0,1])$. In the far future this support becomes a set of horizontal strips, and in the limit a set of infinite horizontal straight lines, such that it is dense in $[0,1]\times[0,1]$. No function of $\mathcal{L}$ has such a support and therefore the equilibrium state (considered as a ''strong'' limit) does not belong to $\mathcal{L}$. Of course, a weak limit exists and therefore equilibrium is reached in a weak sense, e.g., as a coarse-grained average [34].

shows that BAT=TAT, and that t, or more generally η, grows in the same direction than S.

The relation between BAT (=TAT) and CAT is given by the $\pm$ sign in Eqs. (32) or (33). Then these two arrows of time, *a priori*, are not coordinated in our model. But in the classical period we have just *one* classical universe and therefore the $\pm$ sign and the constant C are fixed; so in the classical period we have just one sign: either + or − . Therefore, once the sign is fixed, a clear relation appears between BAT and CAT.

(i) If the model is an expanding one (and we choose the sign +), we will have BAT=CAT, at least in the final evolution where Eq. (33) is valid (if we make the unusual choice of the − sign, we are just changing the conventional direction of time η, with no physical consequences).

(ii) If the model is an expanding-contracting one (and we choose the sign +), we will have BAT=CAT in the expanding period and BAT≠CAT in the contracting period. BAT =CAT is, in fact, the definition of the expanding period and BAT≠CAT is the definition of the contracting one. But BAT =TAT does not change when we go from the expanding to the contracting period or vice versa, since the choice of Φ_- (or Φ_+) is made once and for all.

So the study of the correlation of the arrows of time is completed, and almost trivial, because we have H_2, which defines BAT.[23]

IX. OTHER RESULTS

The main results related to quantum cosmology are stated in the above sections. But we must comment that using the present formalism all the relevant results of irreversible statistical mechanics can also be obtained, e.g., all the results of Ref. [49], as is proved in Ref. [14], because the main Π projector of the quoted book can be defined using Gel'fand triplets. Also, in some simple cases, we can go from the quantum models to the classical ones [50], where we find the same philosophy, in classical cases. Chaotic models such as Baker's transformation and Renyi's maps are also treated with the same method, with good results [51]. Other interesting results are contained in [13,27,28,31, and 48]. So what we have explained is just the quantum cosmological chapter of a general method to deal with irreversible processes.

X. CONCLUSIONS

Let us summarize our main conclusions.

(i) Our entire scheme is based in the existence of a physically admissible state quantum superspace Φ_- and of a physically forbidden state quantum superspace Φ_+. Thus, the time inversion that goes from Φ_- to Φ_+ is also forbidden. Namely, no Maxwell demon can change the direction of all the velocities of the universe. This is, of course, a practically impossible task. Is it also theoretically impossible? In fact it is, even if the Maxwell demon were to change all these directions (while we are sleeping), we will not notice the change (when we wake up), since *all arrows of time would be changed* and we would not have any extra arrow to verify the change. Thus this global demon task is also conceptually impossible.

(ii) What we have presented is not a mathematical theorem, but a model that can be generalized in many ways. These more general models will have a behavior similar to the present one if two essential features are present: the existence, at the quantum gravity level, of a geometry of maximal relative probability which allows us to construct ''out'' states for the fields, and an S matrix with infinite complex poles. The first requirement seems natural for any realistic model of the universe. On the other hand, we have restricted the class of possible potentials in order to be able to prove that the corresponding S matrix has infinite complex poles. But several QC potentials do not belong to this class, because they have a bad behavior at infinity. Nevertheless, they usually also have an infinite set of poles, as can be proved case by case [50]. So the two basic features seem usual enough to consider that our model is a good sample of the general behavior of the universe. Then we can say the following.

(iii) If we introduce an adequate regular state space, it seems that all the known results of statistical irreversible physics can be reobtained. It must be emphasized that we are *not adding a new object to the theory, since a regular state space (or the corresponding topology) must be defined anyhow*. We are just choosing the most convenient one. Let us repeat the general relativity example: The space-time has a metric, and we can choose a flat space-time metric or a curved one. In the second case we explain gravity in the best possible way. We add nothing; we just choose the best mathematical structure. The same thing happens in the present case. If we choose the usual regular state space $\mathcal{S}$, we are forced to make a coarse graining (and there is nothing experimentally wrong with coarse graining, as there is nothing experimentally wrong with post-Newtonian theories, but both are ''noneconomical'' formalisms). If we choose the new regular state space ϕ_-, we make two steps in one, and so we have a conceptual advantage.

(iv) Precisely, because the new formalism is conceptually clearer, we can see that time asymmetry is just a kind of spontaneous symmetry breaking.

(v) Most probably the old and new formalisms will always yield the same physical results, because they are both based in the same physical base: The limited amount of information, contained in $\mathcal{P}$ (see Sec. I B) must be somehow worked out to obtain a complete theory (to complete $\mathcal{P}$ using $\mathcal{S}$ and then make a coarse graining or to complete $\mathcal{P}$ using ϕ_-; see [32]). Therefore, most likely, they are as experimentally equivalent, as general relativity and post-Newtonian gravity with an infinite number of terms are.

So, even if we have not found any new or spectacular result, we think that the introduced formalism presents a quite coherent picture of the real time-asymmetric universe and shows us how time asymmetry forces us to choose a Gel'fand triplet as the mathematical structure of the theory.

[23] The *quantum arrow of time (QAT)*, which goes from preparation to measurement, and coincides with the collapse arrow of time, can also be considered, as in [48]. It is not difficult to see that this arrow also coincides with BAT=TAT, since the measurement process is an irreversible decoherence process which is also contained in (Reichenbach) branch system and, therefore, the measurement arrow (QAT) must coincide with BAT.

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