Abelian-projected effective gauge theory of QCD with asymptotic freedom and quark confinement

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Starting from SU(2) Yang-Mills theory in 3+1 dimensions, we prove that the Abelian-projected effective gauge theories are written in terms of the maximal Abelian gauge field and the dual Abelian gauge field interacting with magnetic monopole current. This is performed by integrating out all the remaining non-Abelian gauge field belonging to SU(2)/U(1). We show that the resulting Abelian gauge theory recovers exactly the same one-loop beta function as the original Yang-Mills theory. Moreover, the dual Abelian gauge field becomes massive if the monopole condensation occurs. This result supports the dual superconductor scenario for quark confinement in QCD. We give a criterion of dual superconductivity and point out that the magnetic monopole condensation may be estimated from the classical instanton configuration. Therefore there can exist an effective Abelian gauge theory which shows both asymptotic freedom and quark confinement based on the dual Meissner mechanism. The inclusion of an arbitrary number of fermion flavors is straightforward in this approach. Some implications to the lower dimensional case will also be discussed. [S0556-2821(98)05110-8]

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I. INTRODUCTION

It is one of the most important problems in particle physics to clarify the physical mechanism which realizes quark and gluon confinement. An important question is, what are the most relevant degrees of freedom to describe the confinement? In the mid-1970s, the idea of the dual Meissner vacuum of quantum chromodynamics (QCD) was proposed by Nambu [1], 't Hooft [2], and Mandelstam [3]. In this scenario, the monopole degrees of freedom play the most important role in the confinement. This aspect can be seen explicitly through a procedure called *Abelian projection* by 't Hooft [2]. Under Abelian projection the non-Abelian gauge theory can be regarded as an Abelian gauge theory with a magnetic monopole [4]. For the confinement mechanism, there are other proposals [5] which we do not discuss in this paper.

Abelian projection [2] is to fix the gauge in such a way that the maximal torus group of the gauge group G remains unbroken. It goes on as follows for the gauge group SU(N).

(1) One chooses a gauge-dependent local quantity $X(x)=X^A(x)T^A$ which transforms adjointly under the gauge transformation: i.e.,

$$X(x)\rightarrow X'(x):=U(x)X(x)U^{\dagger}(x). \tag{1.1}$$

(2) One performs the gauge rotation so that X becomes diagonal:

$$X'(x)=\mathrm{diag}(\lambda_1(x),\ \ldots\ ,\lambda_N(x)), \tag{1.2}$$

where $\lambda_i(x)$ $(i=1,\ \ldots\ ,N)$ are eigenvalues.

(3) At the space-time point where the eigenvalues are degenerate $\lambda_i(x)=\lambda_j(x)$ $(i\neq j,i,j=1,\ldots,N)$, a monopolelike (hedgehog) singularity appears. The singularity appears in the Abelian gauge field $a_\mu(x)$ extracted from the non-Abelian gauge field $\mathcal{A}'_\mu(x)=U(x)[\mathcal{A}_\mu(x)+(i/g)\,\partial_\mu]U^{\dagger}(x)$. The monopole singularity is characterized as a topological quantity.

(4) At the generic point where the eigenvalues do not coincide, the gauge is not determined completely, since any diagonal gauge rotation U [an element of the largest Abelian subgroup $\mathrm{U}(1)^{N-1}$, the maximal torus group],

$$U(x)=\mathrm{diag}(e^{i\theta_1(x)},\ \ldots\ ,e^{i\theta_N(x)}),\quad \sum_{i=1}^{N}\theta_i(x)=0, \tag{1.3}$$

leaves X invariant. Therefore, within this gauge, the theory reduces to an $(N-1)$ fold Abelian gauge-invariant theory.

Monte Carlo studies of the Abelian projection were initiated by Ref. [6] and the maximal Abelian gauge (MAG) was adopted in the simulation on the lattice [7]. Recent extensive studies of Abelian projection (see [8] for a review) have confirmed the *Abelian dominance* proposed in Ref. [9]. This states that the non-Abelian gauge field A_μ^a in $\mathrm{SU}(N)/\mathrm{U}(1)^{N-1}$, behaving as a charged field under residual $\mathrm{U}(1)^{N-1}$ gauge rotation, is not important in the low energy physics and the maximal Abelian part $\mathrm{U}(1)^{N-1}$ plays the dominant role in quark and gluon confinement. In analytical studies, Abelian dominance was assumed from the beginning to construct the effective low energy theory of QCD [9,10]. Assuming Abelian dominance, one can show that, if monopole condensation occurs, charged quarks and gluons are confined due to the dual Meissner effect. Monopole condensation is expected to bring about mass for the dual gauge field. An effective theory of monopole currents was investigated also on the lattice [11]. In fact, recent Monte Carlo simulations [12] support Abelian dominance and furthermore *monopole dominance*. However, there seems to be no analytical proof of Abelian dominance.

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A deficit of Abelian projection is the gauge dependence of the procedure of Abelian projection. The quantity X is a gauge-dependent quantity and the field variable in which the monopole appears is not a gauge-invariant quantity. Therefore the result seems to depend crucially on the gauge selected in Abelian projection. However, this would not be a real problem, since it is possible to put Abelian projection in a gauge-invariant form, if we desire to do so; see [13,14].

The real problem is another in our view. In the Abelian-projected theory, the magnetic monopole degrees of freedom appear as the singularity in the Abelian gauge field. The magnetic current k_μ is obtained as the divergence of the dual Abelian field strength $\tilde{f}_{\mu\nu}$,

$$\partial^\nu \tilde{f}_{\mu\nu} = k_\mu, \quad \tilde{f}_{\mu\nu} := \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} f^{\rho\sigma}, \tag{1.4}$$

in a similar way that the equation of motion relates the field strength $f_{\mu\nu}$ to the electric current j_μ,

$$\partial^\nu f_{\mu\nu} = j_\mu. \tag{1.5}$$

If the U(1) potential a_μ is nonsingular, the Abelian field strength $f_{\mu\nu} := \partial_\mu a_\nu - \partial_\nu a_\mu$ leads to vanishing magnetic current, $k_\mu = 0$, which is nothing but the Bianchi identity for the U(1) field, $\partial^\nu \tilde{f}_{\mu\nu} \equiv 0$. So if one needs the nonzero magnetic current, the Abelian field must include a singularity. However, we do not think that it is sound as a quantum field theory to treat the singularity of the field variable as the essential ingredient from the very outset. In the lattice gauge theory, such a singularity does not appear due to lattice regularization [15] and the monopole contribution is extracted from the gauge-invariant magnetic flux, although monopole dominance is supported in the Monte Carlo simulation on the lattice. Moreover, it should be noted that the magnetic monopole does not exist in the original non-Abelian gauge theory. The magnetic monopole appears only after Abelian projection (see Appendix C).

The purpose of this paper is to *derive* the Abelian-projected effective gauge theory (APEGT) of QCD *as a quantum field theory*, from which we should start the analysis. For simplicity, we restrict the following argument to the $G = SU(2)$ case. The SU(3) case is more involved and will be presented in a subsequent paper. In this paper, without using various assumptions (actually with no assumptions), we derive the APEGT of Yang-Mills (YM) theory and QCD. This is done by integrating out off-diagonal fields belonging to SU(2)/U(1) based on the functional integral formalism. We use the word ''effective'' in the sense of the Wilson renormalization group [16], since the Abelian-projected theory is obtained after integrating out the degrees of freedom corresponding to the non-Abelian gauge fields $A_\mu^\pm := (A_\mu^1 \pm i A_\mu^2)/\sqrt{2}$ which behave as *massive* charged matter fields and do not play an important role in the low energy physics of confinement. Such a strategy can be exactly performed in the $N = 2$ supersymmetric YM theory and QCD [17].

We show that the off-diagonal field gives rise to a nontrivial magnetic monopole current for the Abelian part,

$$K^\mu = \frac{1}{2}\partial_\nu(\epsilon^{\mu\nu\rho\sigma}\epsilon^{ab3}A_\rho^a A_\sigma^b), \quad a,b = 1,2. \tag{1.6}$$

In other words, the charged off-diagonal gluon field plays the role of the source of the monopole. Although the definition (1.6) of monopole current seems to be different from the usual definition based on the singularity of the Abelian field, we show that both are equivalent to each other (apart from the Dirac string singularity). In the APEGT, the singularity does not appear apparently, although we can always include the singularity if necessary.

The effective dual Ginzburg-Landau (GL) theory derived assuming Abelian dominance does not have sufficient predictive power, since it contains undetermined free parameters. On the contrary, all the quantities in the APEGT are calculable and all the effects of the non-Abelian gauge field are included in the APEGT. In fact, we show that the APEGT recovers exactly the same one-loop beta function as that of the original non-Abelian gauge theory. The dual Abelian gauge field follows naturally in the course of the derivation of the theory and has a coupling with the monopole current. This interaction leads to the dual Meissner effect due to monopole condensation. The resulting nonzero mass of the dual gauge field gives the nonzero string tension, i.e., linear potential for static quarks. Thus the string tension is determined by the monopole loop condensate, $\langle K_\mu(x) K^\mu(x)\rangle/\delta^{(4)}(0)$ (see Sec. IV for a precise definition). The monopole condensate plays the role of the order parameter for confinement.

Moreover, we discuss the possibility that the nonzero monopole condensation is derived from the instanton configuration. Hence the instanton may lead to confinement against conventional wisdom [18].

In our approach, the inclusion of fermions is straightforward. Hence the APEGT is also a starting point to study the relationship between confinement and chiral symmetry breaking (or restoration) [19,20].

This paper is organized as follows. In Sec. II, we derive the APEGT for the maximal Abelian part by integrating out the remaining non-Abelian gauge field. In this step, we introduce an auxiliary tensor field which is converted to the dual gauge field. The dual gauge field is essential for discussing the dual Meissner effect in Sec. IV. The APEGT is first obtained in a form including a logarithmic determinant. The logarithmic determinant is explicitly calculated. It generates the gauge-invariant form due to U(1) gauge invariance. An effect of this term is the renormalization of the Abelian gauge field. In Sec. III, we calculate the one-loop beta function without using the Feynman diagram. It is shown to agree with the original non-Abelian gauge theory. In this sense, the effective theory recovers asymptotic freedom. In Sec. IV, we discuss the dual Meissner effect. If monopole loop condensation occurs, the dual vector field becomes massive. In Sec. V, we include the fermion in the APEGT. In Sec. VI, we discuss the lower dimensional case. In the final section we give our conclusions and a discussion.

II. ABELIAN-PROJECTED EFFECTIVE GAUGE THEORY

A. Separation of the Abelian part and introduction of the dual field

First, we decompose the field $\mathcal{A}_\mu$ into the diagonal [maximal Abelian U(1)] and the off-diagonal parts SU(2)/U(1),

$$\mathcal{A}_\mu(x) = \sum_{A=1}^{3} \mathcal{A}_\mu^A(x) T^A := a_\mu(x) T^3 + \sum_{a=1}^{2} A_\mu^a(x) T^a. \tag{2.1}$$

We adopt the following convention. The generators of the Lie algebra $T^A(A=1,\ldots,N^2-1)$ for the gauge group $G = SU(N)$ are taken to be Hermitian satisfying $[T^A,T^B] = i f^{ABC} T^C$ and normalized as $\mathrm{tr}(T^A T^B) = \frac{1}{2}\delta^{AB}$. The generators in the adjoint representation are given by $[T^A]_{BC} = -i f_{ABC}$. We define the quadratic Casimir operator by $C_2(G)\delta^{AB} = f^{ACD} f^{BCD}$. For SU(2), $T^A = (1/2)\sigma^A$ $(A=1,2,3)$ with Pauli matrices σ^A and the structure constant $f^{ABC} = \epsilon^{ABC}$. The indices $a,b,\ldots$ denote the off-diagonal parts.

Then the field strength

$$\mathcal{F}_{\mu\nu}(x) := \sum_{A=1}^{3} \mathcal{F}_{\mu\nu}^A(x) T^A := \partial_\mu \mathcal{A}_\nu(x) - \partial_\nu \mathcal{A}_\mu(x) - i[\mathcal{A}_\mu(x), \mathcal{A}_\nu(x)] \tag{2.2}$$

is decomposed as

$$\begin{aligned} \mathcal{F}_{\mu\nu}(x) &= [f_{\mu\nu}(x) + \mathcal{C}_{\mu\nu}(x)] T^3 + \mathcal{S}_{\mu\nu}^a(x) T^a, \\ f_{\mu\nu}(x) &:= \partial_\mu a_\nu(x) - \partial_\nu a_\mu(x), \\ \mathcal{S}_{\mu\nu}^a(x) &:= D_\mu[a]^{ab} A_\nu^b - D_\nu[a]^{ab} A_\mu^b, \\ \mathcal{C}_{\mu\nu}(x) T^3 &:= -i[A_\mu(x), A_\nu(x)], \end{aligned} \tag{2.3}$$

where the derivative $D_\mu[a]$ is defined by

$$D_\mu[a] = \partial_\mu + i[a_\mu T^3, \cdot], \quad D_\mu[a]^{ab} := \partial_\mu \delta^{ab} - \epsilon^{ab3} a_\mu. \tag{2.4}$$

Hence the diagonal part $\mathcal{F}_{\mu\nu}^3$ of the field strength is given by

$$\mathcal{F}_{\mu\nu}^3 = f_{\mu\nu} + \mathcal{C}_{\mu\nu}, \quad \mathcal{C}_{\mu\nu} := \epsilon^{ab3} A_\mu^a A_\nu^b. \tag{2.5}$$

Next, we rewrite the Yang-Mills action

$$S_{YM}[\mathcal{A}] = -\frac{1}{2g^2} \int d^4x \, \mathrm{tr}(\mathcal{F}_{\mu\nu} \mathcal{F}^{\mu\nu}). \tag{2.6}$$

By using

$$\mathrm{tr}(f_{\mu\nu} \mathcal{S}^{\mu\nu}) = 0 = \mathrm{tr}(\mathcal{C}_{\mu\nu} \mathcal{S}^{\mu\nu}), \tag{2.7}$$

the YM action is rewritten as

$$S_{YM}[\mathcal{A}] = -\frac{1}{4g^2} \int d^4x [(f_{\mu\nu} + \mathcal{C}_{\mu\nu})^2 + (\mathcal{S}_{\mu\nu}^a)^2]. \tag{2.8}$$

Here we introduce an antisymmetric auxiliary tensor field $B_{\mu\nu}$ in order to linearize the $(\mathcal{C}_{\mu\nu})^2$ term. This procedure enables us to perform a Gaussian integration over the off-diagonal gluon fields A_μ^a $(a=1,2)$.[1] It turns out that the tensor field $B_{\mu\nu}$ plays the role of the "dual" field to the Abelian gluon field a_μ. We find that there are two ways to introduce the "dual" tensor field.

One way is to introduce the tensor field $B_{\mu\nu}$ such that the tensor $B_{\mu\nu}$ is the dual of the diagonal field strength $\mathcal{F}_{\rho\sigma}^3$,

$$B_{\mu\nu} \leftrightarrow \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} \mathcal{F}_{\rho\sigma}^3 = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}(f_{\rho\sigma} + \mathcal{C}_{\rho\sigma}). \tag{2.9}$$

This is achieved in the tree level by the following action:

$$S_{apBF\text{-}YM}[\mathcal{A},B] = \int d^4x \left[\frac{1}{4}\epsilon^{\mu\nu\rho\sigma} B_{\rho\sigma}(f_{\mu\nu} + \mathcal{C}_{\mu\nu}) - \frac{1}{4} g^2 B_{\mu\nu} B^{\mu\nu} - \frac{1}{4g^2}(\mathcal{S}_{\mu\nu}^a)^2\right]. \tag{2.10}$$

This theory is equivalent to the BF-YM theory,

$$S_{BF\text{-}YM}[\mathcal{A},\mathcal{B}] = \int d^4x \left[\frac{1}{4}\epsilon^{\mu\nu\rho\sigma} \mathcal{B}_{\rho\sigma}^A \mathcal{F}_{\mu\nu}^A - \frac{1}{4} g^2 \mathcal{B}_{\mu\nu}^A \mathcal{B}^{\mu\nu A}\right]. \tag{2.11}$$

Actually, by identifying $B_{\mu\nu} = \mathcal{B}_{\mu\nu}^3$, the action (2.10) is obtained from Eq. (2.11) by separating the diagonal part from the off-diagonal part and integrating out the off-diagonal auxiliary tensor field $B_{\mu\nu}^a$ $(a=1,2)$. Quite recently, the equivalence of the BF-YM theory with the YM theory has been proved at the quantum level; see [22]. This theory is interesting from the topological point of view.

Another way is to introduce the tensor field as a dual to $\mathcal{C}_{\rho\sigma}$ at the tree level,

$$B_{\mu\nu} \leftrightarrow \frac{1}{2}\epsilon^{\mu\nu\rho\sigma} \mathcal{C}_{\rho\sigma}. \tag{2.12}$$

Thus we are lead to the action,

$$\begin{aligned} S_{apYM}[\mathcal{A},B] = \int d^4x \Bigg[& -\frac{1}{4g^2}(f_{\mu\nu} f_{\mu\nu} + 2 f_{\mu\nu} \mathcal{C}_{\mu\nu}) \\ & + \frac{1}{4}\epsilon^{\mu\nu\rho\sigma} B_{\rho\sigma} \mathcal{C}_{\mu\nu} - \frac{1}{4} g^2 B_{\mu\nu} B^{\mu\nu} \\ & - \frac{1}{4g^2}(\mathcal{S}_{\mu\nu}^a)^2 \Bigg]. \end{aligned} \tag{2.13}$$

In this case, $\frac{1}{2}\epsilon^{\mu\nu\rho\sigma} f_{\rho\sigma}$ is generated through the radiative correction as shown in Sec. II D. In either case, Gaussian integration over $B_{\mu\nu}$ recovers the action (2.8) and hence the original YM action. This model (2.13) is simpler than the model (2.10) in the actual treatment, since the topological theory needs some delicate treatment [22]. (The equivalence of the two formulations is shown in Appendix A.) In what

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[1] This procedure is similar to the field strength approach for non-Abelian gauge theory [21].

follows, we focus on the action (2.13) which is essentially equivalent to that derived by Quandt and Reinhardt [23].

B. Gauge fixing

We discuss the gauge-fixing term. This is independent of the choice of the action. The gauge-fixing term is constructed based on the Becchi-Rouet-Stora-Tyutin (BRST) formalism. We consider a gauge given by

$$F^{\pm}[A,a]:=(\partial^{\mu}\pm i\xi a^{\mu})A_{\mu}^{\pm}=0, \tag{2.14}$$

$$F^{3}[a]:=\partial^{\mu}a_{\mu}=0, \tag{2.15}$$

where we have used the $(\pm,3)$ basis[2]

$$\mathcal{O}^{\pm}:=(\mathcal{O}^{1}\pm i\mathcal{O}^{2})/\sqrt{2}. \tag{2.16}$$

The gauge fixing with $\xi=0$ is the Lorentz gauge, $\partial_{\mu}\mathcal{A}^{\mu}=0$. In particular, $\xi=1$ corresponds to the differential form of the maximal Abelian gauge which is expressed as the minimization of the functional

$$\mathcal{R}[A]:=\frac{1}{2}\int d^4x\{[A_{\mu}^{1}(x)]^2+[A_{\mu}^{2}(x)]^2\}$$
$$=\int d^4x A_{\mu}^{+}(x)A_{\mu}^{-}(x). \tag{2.17}$$

The differential MAG condition (2.14) corresponds to a local minimum of the gauge-fixing functional $R[A]$, while the MAG condition (2.17) requires the global (absolute) minimum. The differential MAG condition (2.14) fixes the gauge degrees of freedom in SU(2)/U(1) and is invariant under the residual U(1) gauge transformation. An additional condition (2.15) fixes the residual U(1) invariance. Both conditions (2.14) and (2.15) then completely fix the gauge except possibly for the Gribov problem. It is known that the differential MAG (2.14) does not spoil renormalizability of YM theory [24]. An implication of this fact is shown in Appendix B.

From physical point of view, we expect that the MAG introduces the nonzero mass m_A for the off-diagonal gluons, A_{μ}^{1},A_{μ}^{2}. This is suggested from the form (2.17) which is equal to the mass term for A_{μ}^{1},A_{μ}^{2}, although we need an independent proof of this statement. This motivates us to integrate out the off-diagonal gluons in the sense of the Wilsonian renormalization group (RG) and allows us to regard the resulting theory as the low energy effective gauge theory written in terms of massless fields alone which describes the physics in the length scale $R>m_A^{-1}$. Abelian dominance will be realized in the physical phenomena occurring in the scale $R>m_A^{-1}$. In this sense the choice of MAG is not unique in realizing Abelian dominance. We can equally take the gauge so that the off-diagonal gluon fields acquire nonzero masses. Then the Abelian-projected effective gauge theory obtained by integrating out the massive off-diagonal gluons will be valid in the low energy region below the energy scale given by the off-diagonal gluon mass.

We introduce the Lagrange multiplier field $\phi^{\pm}$ and ϕ^{3} for the gauge-fixing functions $F^{\pm}[A]$ and $F^{3}[A]$, respectively. It is well known that the gauge fixing term in the BRST quantization is given by [25]

$$\mathcal{L}_{GF}=-i\delta_B G_{gf}, \tag{2.18}$$

where G_{gf} carries the ghost number -1 and is a Hermitian function of the Lagrange multiplier fields $\phi^{\pm},\phi^{3}$, ghost field c^{A}, antighost field $\overline{c^{A}}$, and the remaining field variables of the original Lagrangian. In this paper we consider a simple gauge given by

$$G_{gf}=\sum_{\pm}\overline{c^{\mp}}\left(F^{\pm}[A,a]+\frac{\alpha}{2}\phi^{\pm}\right)+\overline{c^{3}}\left(F^{3}[a]+\frac{\beta}{2}\phi^{3}\right). \tag{2.19}$$

For the most general gauge fixing, see [26].

The BRST transformation in the usual basis is

$$\delta_B\mathcal{A}_{\mu}=\mathcal{D}_{\mu}c:=\partial_{\mu}c-i[\mathcal{A}_{\mu},c],$$
$$\delta_B c=i\frac{1}{2}[c,c],$$
$$\delta_B\overline{c}=i\phi,$$
$$\delta_B\phi=0,$$
$$\delta_B\mathcal{B}_{\mu\nu}=-i[c,\mathcal{B}_{\mu\nu}]. \tag{2.20}$$

Then the BRST transformation in the $(\pm,3)$ basis is given by

$$\delta_B A_{\mu}^{\pm}=(\partial_{\mu}\pm ia_{\mu})c^{\pm}\mp iA_{\mu}^{\pm}c^{3},$$
$$\delta_B a_{\mu}=\partial_{\mu}c^{3}+i(A_{\mu}^{+}c^{-}-A_{\mu}^{-}c^{+}),$$
$$\delta_B c^{\pm}=\mp ic^{3}c^{\pm},$$
$$\delta_B c^{3}=-ic^{+}c^{-},$$
$$\delta_B\overline{c^{\pm,3}}=i\phi^{\pm,3},$$
$$\delta_B\phi^{\pm,3}=0,$$
$$\delta_B B_{\mu\nu}^{\pm}=\mp ic^{\pm}B_{\mu\nu}^{3}\pm ic^{3}B_{\mu\nu}^{\pm},$$
$$\delta_B B_{\mu\nu}^{3}=i(c^{+}B_{\mu\nu}^{-}-c^{-}B_{\mu\nu}^{+}). \tag{2.21}$$

Under a local U(1) gauge transformation,

$$a_{\mu}\to a_{\mu}+\partial_{\mu}\omega,\quad \mathcal{O}^{\pm}\to e^{\mp i\omega}\mathcal{O}^{\pm}\quad \mathcal{O}^{3}\to\mathcal{O}^{3}. \tag{2.22}$$

Hence a_{μ} transforms as a U(1) gauge field, while $A_{\mu}^{\pm}$ and $B_{\mu\nu}^{\pm}$ behave as charged matter fields under the U(1) gauge transformation. It turns out that $B_{\mu\nu}^{3}$ and

$$\mathcal{C}_{\mu\nu}=i\sum_{\pm}(\pm)A_{\mu}^{\pm}A_{\nu}^{\mp} \tag{2.23}$$

are U(1) gauge invariant as expected.

In the usual basis, we can write

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[2]In this basis, $\sum_{\pm}P^{\pm}Q^{\mp}=P^{+}Q^{-}+P^{-}Q^{+}=P^{a}Q^{a}$, $\sum_{\pm}(\pm)P^{\mp}Q^{\pm}=-P^{+}Q^{-}+P^{-}Q^{+}=i\epsilon^{ab3}P^{a}Q^{b}(a,b=1,2)$.

$$G_{gf}=\sum_{a=1,2}\bar{c}^a\left(F^a[A,a]+\frac{\alpha}{2}\phi^a\right)+\bar{c}^3\left(F^3[a]+\frac{\beta}{2}\phi^3\right), \tag{2.24}$$

where

$$F^a[A,a]:=(\partial^\mu\delta^{ab}-\xi\epsilon^{ab3}a^\mu)A^b_\mu:=D^{\mu ab}[a]^\xi A^b_\mu. \tag{2.25}$$

For the gauge-fixing function (2.19) with the BRST transformation (2.21), or (2.24) with (2.20), straightforward calculation leads to the gauge-fixing Lagrangian (2.18),

$$\begin{aligned}\mathcal{L}_{GF}=&\phi^aF^a[A,a]+\frac{\alpha}{2}(\phi^a)^2+i\bar{c}^aD^{\mu ab}[a]^\xi D^{bc}_\mu[a]c^c\\&-i\xi\bar{c}^a[A^a_\mu A^{\mu b}-A^c_\mu A^{\mu c}\delta^{ab}]c^b+\phi^3F^3[a]+\frac{\beta}{2}(\phi^3)^2\\&+i\bar{c}^3\partial^\mu\partial_\mu c^3-i\bar{c}^3\partial^\mu(\epsilon^{ab3}A^a_\mu c^b)\\&+i\bar{c}^a\epsilon^{ab3}[(1-\xi)A^b_\mu\partial^\mu+F^b[A,a]]c^3.\end{aligned} \tag{2.26}$$

This reduces to the usual form in the Lorentz gauge, $\xi=0$.

Finally we introduce the source term

$$\mathcal{L}_J=A^a_\mu J^{\mu a}+\phi^aJ^a_\phi, \tag{2.27}$$

which will be necessary to calculate the correlation functions.

C. Integration over SU(2)/U(1)

Our strategy is to integrate out the off-diagonal fields ϕ^a, A^a_μ, c^a, $\bar{c}^a$ (and $B^a_{\mu\nu}$ for BF-YM case) belonging to the Lie algebra of SU(2)/U(1) and to obtain the effective Abelian gauge theory written in terms of the diagonal fields $a_\mu, B_{\mu\nu}$ [and ghost fields $c^3,\bar{c}^3$ if we need a completely gauge-fixed theory also for the residual U(1) gauge invariance].

First of all, when $\alpha\neq 0$,[3] the Lagrange multiplier field ϕ^a can be easily integrated out. The result is

$$\phi^aF^a[A,a]+\frac{\alpha}{2}(\phi^a)^2+\phi^aJ^a_\phi\to-\frac{1}{2\alpha}(F^a[A,a])^2-\frac{1}{\alpha}F^a[A,a]J^a_\phi. \tag{2.28}$$

Next, as a preliminary procedure to integrate out A^a_μ, we rewrite the last term in the action (2.13) as

$$(\mathcal{S}^a_{\mu\nu})^2=-2A^a_\mu W^{ab}_{\mu\nu}A^b_\nu+2\partial_\mu(A^a_\nu\mathcal{S}^a_{\mu\nu}),$$

$$W^{ab}_{\mu\nu}:=(D^\rho[a]D_\rho[a])^{ab}\delta_{\mu\nu}-\epsilon^{ab3}f_{\mu\nu}-D_\mu[a]^{ac}D_\nu[a]^{cb}, \tag{2.29}$$

where we have used

$$[D_\mu[a]^{ac},D_\nu[a]^{cb}]=-\epsilon^{ab3}f_{\mu\nu}. \tag{2.30}$$

Discarding the surface term,[4] we arrive at

$$\begin{aligned}S_{YM}=S_{YM}[a,A,B,c,\bar{c};J]=&S_1[a,B]+S_2[a,c,\bar{c}]\\&+S_3[a,A,B,c,\bar{c};J],\end{aligned} \tag{2.31}$$

$$S_1=\int d^4x\left[-\frac{1}{4g^2}f_{\mu\nu}f^{\mu\nu}-\frac{1}{4}g^2B_{\mu\nu}B^{\mu\nu}\right], \tag{2.32}$$

$$\begin{aligned}S_2=\int d^4x\Big[&i\bar{c}^aD^{\mu ac}[a]^\xi D^{cb}_\mu[a]c^b+i\bar{c}^3\partial^\mu\partial_\mu c^3\\&+\phi^3(\partial^\mu a_\mu)+\frac{\beta}{2}(\phi^3)^2\Big],\end{aligned} \tag{2.33}$$

$$\begin{aligned}S_3=\int d^4x\Big[&\frac{1}{2g^2}A^a_\mu Q^{ab}_{\mu\nu}A^b_\nu+A^a_\mu\Big(G^a_\mu+\frac{1}{\alpha}D^{\mu ab}[a]^\xi J^b_\phi\\&+J^{\mu a}\Big)\Big],\end{aligned} \tag{2.34}$$

$$\begin{aligned}Q^{ab}_{\mu\nu}:=&(D_\rho[a]D_\rho[a])^{ab}\delta_{\mu\nu}-2\epsilon^{ab3}f_{\mu\nu}\\&+\frac{1}{2}g^2\epsilon^{ab3}\epsilon_{\mu\nu\rho\sigma}B^{\rho\sigma}-2ig^2\xi(\bar{c}^ac^b-\bar{c}^cc^c\delta^{ab})\delta_{\mu\nu}\\&-D_\mu[a]^{ac}D_\nu[a]^{cb}+\frac{1}{\alpha}D_\mu[a]^{ac}_\xi D_\nu[a]^{cb}_\xi,\end{aligned} \tag{2.35}$$

$$\begin{aligned}G^c_\mu:=&i(\partial_\mu\bar{c}^3)\epsilon^{cb3}c^b+i\bar{c}^a\epsilon^{ab3}[(1-\xi)(\partial_\mu c^3)\delta^{bc}\\&-\xi\epsilon^{bc3}a_\mu c^3]-i\partial_\mu(\bar{c}^a\epsilon^{ac3}c^3),\end{aligned} \tag{2.36}$$

where we have rescaled the parameter α to absorb the g dependence.

All the terms appearing in the resulting YM action are at most quadratic in A^a_μ. Therefore the field $A^a_\mu(a=1,2)$ in S_3 can be eliminated using the Gaussian integration and we obtain

[3]The case of $\alpha=0$ should be treated separately. Since $F^a[A,a]=DA$ is linear in A^a_μ, the ϕ^a integration can be performed finally after integrating out the A^a_μ field. However, it generates the additional complicated logarithmic determinant lndet$[DQ^{-1}D]$. Such a case was treated in [23]. The choice of gauge-fixing parameter should not change the physics, since it appears due to the gauge choice. Therefore we do not treat this case in this paper.

[4]This will be justified, since the off-diagonal gluons become massive due to the MAG.

$$
\begin{aligned}
iS_0 &= \ln \int [dA_\mu^a] \exp\left\{ i \int d^4x \left[\frac{1}{2g^2} A_\mu^a Q_{\mu\nu}^{ab} A_\nu^b + A_\mu^a \left(G_\mu^a + \frac{1}{\alpha} D^{\mu ab}[a]^\xi J_\phi^b + J^{\mu a} \right) \right] \right\} \\
&= -\frac{1}{2} \ln\det(Q_{\mu\nu}^{ab}) + \frac{g^2}{2} G_\mu^a (Q^{-1})_{\mu\nu}^{ab} G_\nu^b + g^2 \left(\frac{1}{\alpha} D^{\mu ac}[a]^\xi J_\phi^c + J^{\mu a} \right) (Q^{-1})_{\mu\nu}^{ab} G_\nu^b \\
&\quad - \frac{g^2}{2\alpha} J_\phi^a D^{ab}[a]^\xi (Q^{-1})_{\mu\nu}^{ab} D^{\nu cd}[a]^\xi J_\phi^d + \frac{g^2}{\alpha} J^{\mu a} (Q^{-1})_{\mu\nu}^{ab} D^{\nu bc}[a]^\xi J_\phi^c + \frac{g^2}{2} J^{\mu a} (Q^{-1})_{\mu\nu}^{ab} J^{\nu b}.
\end{aligned} \tag{2.37}
$$

Thus we obtain the effective Abelian gauge theory

$$
\begin{aligned}
S_E &= S_0[a,B,c,\bar{c};J] + S_1[a,B] + S_2[a,c,\bar{c}], \\
S_0 &= -\frac{1}{2} \ln\det(Q_{\mu\nu}^{ab}) + \frac{g^2}{2} G_\mu^a (Q^{-1})_{\mu\nu}^{ab} G_\nu^b + g^2 \left(\frac{1}{\alpha} D^{\mu ac}[a]^\xi J_\phi^c + J^{\mu a} \right) (Q^{-1})_{\mu\nu}^{ab} G_\nu^b - \frac{g^2}{2\alpha} J_\phi^a D^{ab}[a]^\xi (Q^{-1})_{\mu\nu}^{ab} D^{\nu cd}[a]^\xi J_\phi^d \\
&\quad + \frac{g^2}{\alpha} J^{\mu a} (Q^{-1})_{\mu\nu}^{ab} D^{\nu bc}[a]^\xi J_\phi^c + \frac{g^2}{2} J^{\mu a} (Q^{-1})_{\mu\nu}^{ab} J^{\nu b}.
\end{aligned} \tag{2.38}
$$

As will be shown in the next subsection, $\ln\det Q$ gives the renormalization of the fields a_μ, $B_{\mu\nu}$, and c^a. The residual U(1)-invariant theory is obtained by putting $\phi^3=0$ and $\bar{c}^3 = c^3 = 0$ (hence $G_\mu^a=0$). Therefore, the resulting APEGT is greatly simplified.

On the other hand, the effective Abelian BF-YM theory is obtained if S_1 and $Q_{\mu\nu}^{ab}$ in S_3 are replaced by

$$
S_1 = \int d^4x \left[\frac{1}{4} \epsilon^{\mu\nu\rho\sigma} B_{\rho\sigma} f_{\mu\nu} - \frac{1}{4} g^2 B_{\mu\nu} B^{\mu\nu} \right],
$$

$$
\begin{aligned}
Q_{\mu\nu}^{ab} &:= (D_\rho[a] D_\rho[a])^{ab} \delta_{\mu\nu} - \epsilon^{ab3} f_{\mu\nu} + \frac{1}{2} g^2 \epsilon^{ab3} \epsilon_{\mu\nu\rho\sigma} B^{\rho\sigma} \\
&\quad - 2ig^2 \xi (\bar{c}^a c^b - \bar{c}^c c^c \delta^{ab}) \delta_{\mu\nu} - D_\mu[a]^{ac} D_\nu[a]^{cb} \\
&\quad + \frac{1}{\alpha} D_\mu[a]_\xi^{ac} D_\nu[a]_\xi^{cb},
\end{aligned} \tag{2.39}
$$

where the G is the same as Eq. (2.36). This case is discussed in Appendix A.

D. Calculation of logarithmic determinant

In the MAG ($\xi=1$), the last two terms in Q cancel by taking $\alpha=1$ (they disappear also for $\alpha=0$ [23]),

$$
\begin{aligned}
Q_{\mu\nu}^{ab} &:= (D_\rho[a] D_\rho[a])^{ab} \delta_{\mu\nu} - 2\epsilon^{ab3} f_{\mu\nu} \\
&\quad + \frac{1}{2} g^2 \epsilon^{ab3} \epsilon_{\mu\nu\rho\sigma} B^{\rho\sigma} \\
&\quad - 2ig^2 (\bar{c}^a c^b - \bar{c}^c c^c \delta^{ab}) \delta_{\mu\nu}.
\end{aligned} \tag{2.40}
$$

In order to calculate $\ln\det Q$, we use the ζ function regularization or heat kernel method (see, e.g., [27]),

$$
\ln\det Q = -\lim_{s\to 0} \frac{d}{ds} \frac{\mu^{2s}}{\Gamma(s)} \int_0^\infty dt\, t^{s-1} \mathrm{Tr}(e^{-tQ}), \tag{2.41}
$$

where Tr is understood in the functional sense. In this subsection the calculations are performed in a Euclidean formulation.

First, we calculate the trace of e^{-tQ}. To estimate this quantity, we use the plane wave basis,

$$
\begin{aligned}
\mathrm{Tr}(e^{-tQ}) &= \int d^4x\ \mathrm{tr}\langle x | e^{-tQ} | x \rangle \\
&= \int d^4x\ \mathrm{tr} \int \frac{d^4k}{(2\pi)^4} e^{-ikx} e^{-tQ} e^{ikx}.
\end{aligned} \tag{2.42}
$$

By making use of the relation,

$$
[D_\mu^{ab}, e^{\pm ikx}] = \pm ik_\mu e^{\pm ikx} \delta^{ab}, \tag{2.43}
$$

we find

$$
\begin{aligned}
e^{-ikx} e^{-t(D_\rho[a]^2)^{ab}\delta_{\mu\nu}} e^{ikx} &= \exp\{-t(D_\rho[a]^{ac} + ik_\rho \delta^{ac}) \\
&\quad \times (D_\rho[a]^{cb} + ik_\rho \delta^{cb}) \delta_{\mu\nu}\}.
\end{aligned} \tag{2.44}
$$

Furthermore, the rescaling of k_μ, $k_\mu \to k_\mu/\sqrt{t}$, leads to

$$
\begin{aligned}
\mathrm{Tr}(e^{-tQ}) &= \int d^4x \frac{1}{t^2}\ \mathrm{tr} \int \frac{d^4k}{(2\pi)^4} e^{k_\mu k^\mu} \\
&\quad \times \exp[-(2i\sqrt{t} k^\mu D_\mu + tQ)] \\
&= \int d^4x \frac{1}{t^2} \sum_{n=0}^\infty \frac{(-1)^n}{n!} \\
&\quad \times \mathrm{tr} \int \frac{d^4k}{(2\pi)^4} e^{k_\mu k^\mu} (2i\sqrt{t} k^\mu D_\mu + tQ)^n,
\end{aligned} \tag{2.45}
$$

where we have omitted the unit operator, $\delta_{ab}\delta_{\mu\nu}$. It is obvious that all terms odd with respect to k_μ in the expansion go to zero in the integration. Thus we obtain[5]

$$\mathrm{Tr}(e^{-tQ})-\mathrm{Tr}(e^{-tQ_0})=\int\frac{d^4x}{16\pi^2}\,\mathrm{tr}\Big[\frac{1}{2}Q^2-D^2Q+\frac{1}{6}(2D^2D^2+D_\mu D_\nu D_\mu D_\nu)\Big]+O(t), \tag{2.47}$$

where we have used the cyclicity of trace and the replacement

$$k_\mu k_\nu\to\frac{1}{4}k^2\delta_{\mu\nu},$$
$$k_\mu k_\nu k_\alpha k_\beta\to\frac{1}{24}(k^2)^2(g_{\mu\nu}g_{\alpha\beta}+g_{\mu\alpha}g_{\nu\beta}+g_{\mu\beta}g_{\nu\alpha}), \tag{2.48}$$

which is applied in the integrand of the integration formula

$$\int\frac{d^4k}{(2\pi)^4}e^{k^2}(k^2)^m=\frac{(-1)^m}{16\pi^2}(m+1)!\quad(m=0,1,2,\ \dots\ ). \tag{2.49}$$

Separating the first term from the other terms in Q,

$$Q^{ab}_{\mu\nu}:=(D_\rho[a]D_\rho[a])^{ab}\delta_{\mu\nu}+\tilde{Q}^{ab}_{\mu\nu}, \tag{2.50}$$

we see that

$$\begin{aligned}\mathrm{Tr}(e^{-tQ})-\mathrm{Tr}(e^{-tQ_0})&=\frac{1}{16\pi^2}\int d^4x\ \mathrm{tr}\Big[\frac{1}{2}\tilde{Q}^2+\frac{1}{6}D_\mu D_\nu(D_\mu D_\nu-D_\nu D_\mu)\Big]+O(t)\\&=\frac{1}{16\pi^2}\int d^4x\ \mathrm{tr}\Big(\frac{1}{2}\tilde{Q}^2+\frac{1}{12}[D_\mu,D_\nu][D_\mu,D_\nu]\Big)+O(t),\end{aligned} \tag{2.51}$$

where any cross term between D and $\tilde{Q}$ does not appear.

The first term is obtained as

$$\mathrm{tr}\Big(\frac{1}{2}\tilde{Q}^2\Big)=2\kappa f_{\mu\nu}f^{\mu\nu}-\frac{1}{2}g^4\kappa B_{\mu\nu}B^{\mu\nu}-\kappa g^2\epsilon^{\mu\nu\rho\sigma}B_{\rho\sigma}f_{\mu\nu}-8g^4(\overline{c^a}c^b-\overline{c^c}c^c\delta^{ab})(\overline{c^b}c^a-\overline{c^d}c^d\delta^{ba}), \tag{2.52}$$

and the second term is

$$\mathrm{tr}\Big(\frac{1}{12}[D_\mu,D_\nu][D_\mu,D_\nu]\Big)=\frac{-1}{3}\kappa f_{\mu\nu}f^{\mu\nu}, \tag{2.53}$$

where

$$\kappa:=C_2(G):=f^{3cd}f^{3cd}=2. \tag{2.54}$$

Thus we obtain (apart from the four-ghost interaction terms; see Appendix B) the U(1)-invariant result

$$\frac{1}{2}\ \ln\det Q^{ab}_{\mu\nu}=\int d^4x\Big[\frac{1}{4g^2}z_af_{\mu\nu}f^{\mu\nu}+\frac{1}{4}z_bg^2B_{\mu\nu}B^{\mu\nu}+\frac{1}{2}z_cB_{\mu\nu}\tilde{f}_{\mu\nu}+\cdots\Big], \tag{2.55}$$

where

$$z_a=-\frac{20}{3}\kappa\frac{g^2}{16\pi^2}\ \ln\mu,\quad z_b=+2\kappa\frac{g^2}{16\pi^2}\ \ln\mu,$$
$$z_c=+4\kappa\frac{g^2}{16\pi^2}\ \ln\mu. \tag{2.56}$$

Therefore, in the absence of the source $J^a_\mu=0=J^a_\phi$,

$$S_0+S_1=\int d^4x\Big[-\frac{1+z_a}{4g^2}f_{\mu\nu}f^{\mu\nu}-\frac{1+z_b}{4}g^2B_{\mu\nu}B^{\mu\nu}+\frac{1}{2}z_cB_{\mu\nu}\tilde{f}_{\mu\nu}+\cdots\Big]. \tag{2.57}$$

Integrating out the $B_{\mu\nu}$ field, we will obtain an additional contribution

$$-\frac{1}{4g^2}z_c^2(1+z_b)^{-1}f_{\mu\nu}f^{\mu\nu}. \tag{2.58}$$

However, at the one-loop level, this term is irrelevant. Therefore, the cross term does not contribute at the one-loop level.

For later convenience, we calculate another determinant coming from the integration over ghost fields. For the action

$$S_F=\int d^4xi\overline{c^a}D^{ac}_\mu[a]D^{cb}_\mu[a]c^b, \tag{2.59}$$

we obtain, up to one loop,

$$\begin{aligned}S_c&=\ln\int[d\bar{c}][dc]\exp\Big\{-\int d^4x\overline{c^a}D^{ac}_\mu[a]D^{cb}_\mu[a]c^b\Big\}\\&=\ln\det(D^{ac}_\mu[a]D^{cb}_\mu[a])\\&=\int d^4x\frac{1}{4g^2}z'_af_{\mu\nu}f^{\mu\nu}+\cdots,\end{aligned}$$

$$z'_a:=\frac{2}{3}\kappa\frac{g^2}{16\pi^2}\ \ln\mu. \tag{2.60}$$

---

[5]The zero-order term of the expansion with respect to t is equal to the free term

$$\mathrm{Tr}(\exp[-tQ_0]):=\mathrm{Tr}(\exp[-t\partial^2\delta^{ab}\delta_{\mu\nu}])=\frac{4N(N-1)\int d^4x}{16\pi^2t^2}. \tag{2.46}$$

For the Abelian-projected effective BF-YM theory, see Appendix A.

E. APEGT with a monopole

The antisymmetric (Abelian) tensor $B_{\mu\nu}$ has the Hodge decomposition in 3+1 dimensions (see Sec. VI for other dimensions):

$$B_{\mu\nu}=b_{\mu\nu}+\tilde{\chi}_{\mu\nu}, \quad b_{\mu\nu}:=\partial_\mu b_\nu-\partial_\nu b_\mu .$$

$$\tilde{\chi}_{\mu\nu}=\frac{1}{2}\epsilon_{\mu\nu\alpha\beta}(\partial^\alpha\chi^\beta-\partial^\alpha\chi^\beta). \tag{2.61}$$

The tensor $B_{\mu\nu}$ has six degrees of freedom, while the fields b_μ and χ_μ have eight. This mismatch is not a problem, since two degrees are redundant; the gauge transformations

$$b_\mu(x)\to b'_\mu(x)=b_\mu(x)-\partial_\mu\theta,$$

$$\chi_\mu(x)\to\chi'_\mu(x)=\chi_\mu(x)-\partial_\mu\varphi \tag{2.62}$$

leave $B_{\mu\nu}$ invariant. In the function integral, the integration over $B_{\mu\nu}$ is replaced by an integration over b_μ and χ_μ, provided that the gauge degrees of freedom are fixed in Eq. (2.62). These gauge fixings are not explicitly presented in the following, since they can be easily implemented.

In this case, we obtain

$$S_0+S_1=\int d^4x\left[-\frac{1+z_a}{4g^2}f_{\mu\nu}f^{\mu\nu}-\frac{1+z_b}{4}g^2(b_{\mu\nu}b^{\mu\nu}+\tilde{\chi}_{\mu\nu}\tilde{\chi}^{\mu\nu})+\frac{1}{2}z_c b_{\mu\nu}\tilde{f}_{\mu\nu}+\frac{1}{2}z_c\chi_{\mu\nu}f_{\mu\nu}+\cdots\right]. \tag{2.63}$$

At the one-loop level, integration over χ leads to

$$S_E=\int d^4x\left[-\frac{1+z_a}{4g^2}f_{\mu\nu}f^{\mu\nu}+i\bar{c}^a D_\mu^{ac}[a]D_\mu^{cb}[a]c^b-\frac{1+z_b}{4}g^2 b_{\mu\nu}b^{\mu\nu}-z_c b_\mu k^\mu\right], \tag{2.64}$$

where we have defined the magnetic current

$$k^\mu:=\partial^\nu\tilde{f}_{\mu\nu}, \quad \tilde{f}_{\mu\nu}:=\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}f^{\rho\sigma}. \tag{2.65}$$

Here we have neglected ghost self-interaction terms (see Appendix B) and higher derivative terms coming from the logarithmic determinant of Q. This is the APEGT written in terms of the Abelian gauge field a_μ and the dual gauge field b_μ (the effect of the off-diagonal ghost field is studied in the next section). This theory has $U(1)_e\times U(1)_m$ symmetry where the Abelian gauge field a_μ has $U(1)_e$ symmetry and the dual Abelian gauge field b_μ has $U(1)_m$ symmetry which is guaranteed by the conservation $\partial_\mu k^\mu=0$. If the field a_μ is singular, the magnetic current k_μ is nonzero and couples with the dual field b_μ. This interaction leads to the dual Meissner effect; see Sec. IV. In the absence of magnetic current, the dual field b_μ decouples from the theory. Note that the renormalizations of the fields a_μ, b_μ are different from each other.

The APEGT can be considered as an interpolating theory which reduces to a theory with an action $S[a]$ by integrating out the b_μ field or to another theory with $S[b]$ by integrating out the a_μ field. The theory $S[a]$ is suitable for describing the weak coupling region, while $S[b]$ is more suitable for the strong coupling region. However, both theories give a dual description of the same physics. In the next section, we see an aspect of this picture.

III. ONE-LOOP BETA FUNCTION AND ASYMPTOTIC FREEDOM

Neglecting the contribution from the dual gauge field, the APEGT is reduced to the U(1) gauge theory,

$$S_E[a,c,\bar{c}]=\int d^4x\left[-\frac{1+z_a}{4g^2}f_{\mu\nu}f^{\mu\nu}+i\bar{c}^a D_\mu^{ac}[a]D_\mu^{cb}[a]c^b\right]. \tag{3.1}$$

This APEGT is similar to scalar quantum electrodynamics. But the scalar field is replaced with the ghost field. We can show that the running coupling g exhibits asymptotic freedom; i.e., the beta function has a negative coefficient. The beta function is obtained from the calculation of the logarithmic determinant in the previous section.

We define the wave function renormalization for a_μ and c^a by

$$a_\mu^R=Z_a^{-1/2}a_\mu, \quad c_R=Z_c^{-1/2}c. \tag{3.2}$$

For the three-point $a_\mu c\bar{c}$ vertex, the renormalized coupling constant is defined by

$$g_R=Z_a^{1/2}Z_c Z_g^{-1}g. \tag{3.3}$$

It should be remarked that the effective Abelian gauge theory (3.1) has U(1) gauge invariance and we can derive the Ward-Takahashi (WT) identity for this symmetry. For example, the three-point vertex function and the ghost propagator obey the well-known WT identity which is similar to that in scalar QED. This implies that $Z_g=Z_c$ (independently of the order of the perturbation). Therefore the coupling constant for the $a_\mu c\bar{c}$ vertex is determined by Z_a alone,

$$g_R=Z_a^{1/2}g. \tag{3.4}$$

Note that Z_a is obtained by integrating the ghost field, i.e., $\ln\det D^2$, if we remember Eq. (2.60). Adding this contribution to Eq. (3.1), we obtain

$$Z_a=1-z_a+z'_a=1+\frac{g^2}{16\pi^2}\frac{22C_2(G)}{3}\ln\mu,$$

$$C_2(G):=f^{3cd}f^{3cd}=2. \tag{3.5}$$

Thus the β function is easily calculated:

$$\beta(g):=\mu\frac{dg_R}{d\mu}=-\frac{b_0}{16\pi^2}g_R^3,\quad b_0=\frac{11C_2(G)}{3}>0. \tag{3.6}$$

Thus the APEGT exhibits asymptotic freedom as the original YM theory.[6]

In order to obtain the RG beta function, we could have used the Feynman graph technique. By perturbation expansion in the coupling constant, we can ascertain the Ward relation $Z_g=Z_c$.[7] The origin of asymptotic freedom (z_a') is understood as follows. By the Ward relation, asymptotic freedom is explained by the vacuum polarization of the Abelian gauge field alone. This diagram up to order g^2 is quite similar to those of scalar QED by replacing the complex scalar fields ϕ,ϕ^* with the ghost, antighost fields $c^a,\overline{c^a}$:

$$\overline{c^a}D_\mu^{ab}[a]D_\mu^{bc}[a]c^c\leftrightarrow|(\partial_\mu-iea_\mu)\phi|^2$$
$$=-\phi^*(\partial_\mu-iea_\mu)^2\phi. \tag{3.8}$$

An essential difference is the signature due to a ghost loop. This minus sign changes the nonasymptotic freedom of scalar QED into asymptotic freedom in the effective Abelian gauge theory in question. The additional dominant contribution (z_a) comes from the gluon self-interaction which is already included in the action of the APEGT through the calculation of $-(1/2)\ln\det Q$. A summation of two contributions gives exactly the same beta function as the original YM theory.

In other words, the APEGT is the Abelian gauge theory with a QCD-like running coupling constant $g(\mu)$,

$$S_E[a]=\int d^4x\left[-\frac{1}{4g(\mu)^2}f_{\mu\nu}f^{\mu\nu}\right],$$

$$\frac{1}{g(\mu)^2}=\frac{1}{g(\mu_0)^2}+\frac{b_0}{8\pi^2}\ln\frac{\mu}{\mu_0}. \tag{3.9}$$

IV. MONOPOLE CONDENSATION AND THE DUAL MEISSNER EFFECT

In Sec. II, we have obtained the APEGT with magnetic current (after the ghost integration),

$$S_E[a,b,k]=\int d^4x\left[-\frac{1}{4g^2}f_{\mu\nu}^Rf^{R\mu\nu}-\frac{1}{4}b_{\mu\nu}^Rb^{R\mu\nu}-\frac{1}{g}z_c/Z_b^{1/2}b_\mu^Rk^\mu\right]. \tag{4.1}$$

The interaction term between the dual gauge field b_μ and the magnetic current k_μ is generated by the radiative correction through the gluon self-interaction. The action leads to the field equation for the renormalized field,

$$\partial_\mu f_R^{\mu\nu}=j_R^\nu,\quad \partial_\mu b_R^{\mu\nu}=k_R^\nu, \tag{4.2}$$

where we have defined

$$k_R^\mu:=\frac{1}{g}(z_c/Z_b^{1/2})k^\mu,\quad Z_b^{1/2}=1-z_b/2. \tag{4.3}$$

Integrating out the dual field b_μ, we obtain the effective action for the monopole loop,

$$S_E[a,k]\cong\int d^4x\left[-\frac{Z_a^{-1}}{4g^2}f_{\mu\nu}f^{\mu\nu}+\frac{1}{g^2}k^\mu D_{\mu\nu}k^\nu\right], \tag{4.4}$$

where $D_{\mu\nu}$ is the massless vector propagator. Such a monopole action was predicted on a lattice in [11].

For our purposes, it is more convenient to use the local Lagrangian formalism invented by Zwanziger [28] for a system having both electric and magnetic currents.

Before that, we will give a different treatment which is helpful to discuss the relationship between the monopole condensation and the instanton. We show how the magnetic monopole current is calculated in the original YM theory.

A. Definition of the monopole current

We show that the current K_μ defined by

$$K^\mu=\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\partial_\nu(\epsilon^{ab3}A_\rho^aA_\sigma^b)=\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\partial_\nu\mathcal{C}_{\rho\sigma} \tag{4.5}$$

is interpreted as the magnetic monopole current. This current is topologically conserved, i.e., $\partial_\mu K^\mu=0$. For a while, we use a different normalization of the field $\mathcal{A}\to g\mathcal{A}$. Usually the Abelian gauge field a_μ defined by $a_\mu(x):=\mathrm{tr}[T^3\mathcal{A}_\mu(x)]$ can have singularities if the field $\mathcal{A}$ is gauge transformed by the rotation matrix U(x) as $\mathcal{A}_\mu(x)\to\mathcal{A}_\mu^U(x)$,

$$\mathcal{A}_\mu^U(x):=U(x)\mathcal{A}_\mu(x)U^\dagger(x)+\frac{i}{g}U(x)\partial_\mu U^\dagger(x), \tag{4.6}$$

such that the gauge transformed field $\mathcal{A}_\mu^U(x)$ satisfies the Abelian gauge-fixing condition, e.g., the MAG. It is this singularity that leads to a nonzero magnetic current. Under the gauge transformation (4.6), the field strength is transformed as $\mathcal{F}_{\mu\nu}(x)\to\mathcal{F}_{\mu\nu}^U(x)$,

$$\mathcal{F}_{\mu\nu}^U(x)=U(x)\mathcal{F}_{\mu\nu}(x)U^\dagger(x)$$
$$=\partial_\mu\mathcal{A}_\nu^U(x)-\partial_\nu\mathcal{A}_\mu^U(x)-ig[\mathcal{A}_\mu^U(x),\mathcal{A}_\nu^U(x)], \tag{4.7}$$

---

[6]This fact was first obtained in the gauge $\alpha=0$ based on quite complicated calculations [23].

[7]Explicit calculation based on perturbation theory shows that

$$Z_g=Z_c=1-\frac{g^2}{16\pi^2}2(\beta-3)\ln\mu, \tag{3.7}$$

where β is the gauge-fixing parameter.

see Appendix C. The Abelian gauge field strength is extracted as

$$f_{\mu\nu}:=\partial_\mu a_\nu^U-\partial_\nu a_\mu^U=\mathrm{tr}[T^3(\partial_\mu\mathcal{A}_\nu^U-\partial_\nu\mathcal{A}_\mu^U)] \tag{4.8}$$

$$=\mathrm{tr}[T^3(U\mathcal{F}_{\mu\nu}U^\dagger+ig[\mathcal{A}_\mu^U,\mathcal{A}_\nu^U])]. \tag{4.9}$$

The definition of the magnetic current is

$$k_\mu:=\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\partial^\nu f^{\rho\sigma}. \tag{4.10}$$

The first term in Eq. (4.6) is nonsingular. Hence Eq. (4.8) shows that the first term gives a vanishing contribution in the magnetic current. Only the second term

$$\tilde{\mathcal{A}}_\mu(x):=\frac{i}{g}U(x)\partial_\mu U^\dagger(x) \tag{4.11}$$

gives a nonvanishing magnetic current. If $U(x)$ is not singular, $\tilde{\mathcal{A}}_\mu$ is a pure gauge and hence the field strength constructed from $\tilde{\mathcal{A}}_\mu$ is zero, $\tilde{\mathcal{F}}_{\mu\nu}(x):=\partial_\mu\tilde{\mathcal{A}}_\nu(x)-\partial_\nu\tilde{\mathcal{A}}_\mu(x)-ig[\tilde{\mathcal{A}}_\mu(x),\tilde{\mathcal{A}}_\nu(x)]\equiv 0$. For the singular $U(x)$, this is modified as

$$\tilde{\mathcal{F}}_{\mu\nu}(x):=\partial_\mu\tilde{\mathcal{A}}_\nu(x)-\partial_\nu\tilde{\mathcal{A}}_\mu(x)-ig[\tilde{\mathcal{A}}_\mu(x),\tilde{\mathcal{A}}_\nu(x)]$$

$$=\frac{i}{g}U(x)[\partial_\mu,\partial_\nu]U^\dagger(x). \tag{4.12}$$

Thus we obtain the expression of the magnetic current,

$$k_\mu=\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\partial^\nu\mathrm{tr}[T^3(\partial_\rho\tilde{\mathcal{A}}_\sigma-\partial_\sigma\tilde{\mathcal{A}}_\rho)]$$

$$=\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\partial^\nu\mathrm{tr}(T^3ig[\tilde{\mathcal{A}}_\rho,\tilde{\mathcal{A}}_\sigma])$$

$$+\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\partial^\nu\,\mathrm{tr}\left(T^3\frac{i}{g}U[\partial_\rho,\partial_\sigma]U^\dagger\right). \tag{4.13}$$

The magnetic current is composed of two parts. The second part corresponds to the contribution from the Dirac string. Therefore the first part is the contribution from the magnetic monopole which agrees with Eq. (4.5) in the original normalization of the field $\mathcal{A}$. This can be seen also from Eq. (4.9), since

$$k_\mu=\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\partial^\nu\mathrm{tr}(T^3ig[\tilde{\mathcal{A}}_\rho,\tilde{\mathcal{A}}_\sigma])+\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\partial^\nu\mathrm{tr}(T^3\mathcal{F}_{\mu\nu}^U)$$

$$=\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\partial^\nu\mathrm{tr}(T^3ig[\tilde{\mathcal{A}}_\rho,\tilde{\mathcal{A}}_\sigma])$$

$$+\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\partial^\nu\mathrm{tr}\left(T^3\frac{i}{g}U[\partial_\rho,\partial_\sigma]U^\dagger\right). \tag{4.14}$$

For details, see Appendix C.

B. Dual effective Abelian theory

In the following we present a somewhat different picture of monopole condensation leading to the dual Meissner effect. By extracting the b_μ-dependent pieces from the action (2.31), and inserting the identity

$$1=\int[dK^\mu]\delta\left(K^\mu-\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\partial_\nu(\epsilon^{ab3}A_\rho^aA_\sigma^b)\right), \tag{4.15}$$

the partition function Z_{YM} is written as

$$Z_{YM}[J]:=\int d\mu e^{-S_{YM}}$$

$$=\int d\mu\int[dK^\mu]\delta\left(K^\mu-\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\partial_\nu(\epsilon^{ab3}A_\rho^aA_\sigma^b)\right)$$

$$\times\exp\Bigg\{-S_{YM}[a,A,\chi,c,\bar{c};J]$$

$$-\int d^4x\left[-\frac{1}{4}g^2b_{\mu\nu}b^{\mu\nu}+b^\mu K_\mu\right]\Bigg\}, \tag{4.16}$$

where the measure $d\mu$ denotes the integration over all the fields.

In order to see that the APEGT can exhibit a dual Meissner effect, we consider the effective action $S[b]$ written in terms of b_μ which is obtained by integrating out all the fields except for b_μ,

$$Z_{YM}[J]:=\int[db_\mu]\exp\{-S[b]\}. \tag{4.17}$$

Then $S[b]$ is obtained as

$$S[b]=\frac{-1}{4}g^2\int d^4xb_{\mu\nu}b^{\mu\nu}$$

$$+\ln\left\langle\exp\left[\int d^4xb_\mu(x)K_\mu(x)\right]\right\rangle_0, \tag{4.18}$$

where the expectation value for a function f of the field is defined by

$$\langle f(A)\rangle_0:=\int d\tilde{\mu}\int[dK^\mu]\delta\left(K^\mu-\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\partial_\nu(\epsilon^{ab3}A_\rho^aA_\sigma^b)\right)$$

$$\times\exp\{-S_{YM}[a,A,\chi,c,\bar{c};J]\}f(A), \tag{4.19}$$

where $d\tilde{\mu}$ denotes the normalized measure without $[db_\mu]$ so that $\langle 1\rangle_0=1$. It turns out that

$$S[b]=\frac{-1}{4}g^2\int d^4xb_{\mu\nu}(x)b^{\mu\nu}(x)+\int d^4x\langle K_\mu(x)\rangle_0 b^\mu(x)$$

$$+\frac{1}{2}\int d^4x\int d^4y\langle K_\mu(x)K_\nu(y)\rangle_c b^\mu(x)b^\nu(y)$$

$$+O(b^3), \tag{4.20}$$

where $\langle K_\mu(x)K_\nu(y)\rangle_c$ is the connected correlation function obtained from the normalized expectation value $\langle f(A)\rangle := \langle f(A)\rangle_0/\langle 1\rangle_0$, e.g., $\langle f(A)g(A)\rangle_c = \langle f(A)g(A)\rangle - \langle f(A)\rangle\langle g(A)\rangle$.

We can obtain a similar expression for the APEGT using the action (2.64). Hence the argument in the next subsection can be extended also to the APEGT.

C. Dual Meissner effect due to monopole condensation

The effective dual Abelian theory $S[b]$ has U(1) symmetry, $b_\mu \to b_\mu + \partial_\mu \theta$, which is different from the U(1) symmetry for the Abelian field a_μ and is called the magnetic $U(1)_m$ symmetry hereafter. The magnetic current satisfies the conservation $\partial_\mu K^\mu = 0$, consistently with the $\mathrm{U}(1)_m$ symmetry. This implies that the correlation function of the magnetic monopole current is transverse,

$$\langle K_\mu(x)K_\nu(y)\rangle_c = (\delta_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu)M(x-y). \quad (4.21)$$

As long as the magnetic $\mathrm{U}(1)_m$ symmetry is not broken, the dual gauge field b_μ is always massless as can be seen from Eqs. (4.20) and (4.21). Therefore a nonzero mass for the dual gauge field implies breakdown of the $\mathrm{U}(1)_m$ symmetry.

If $\mathrm{U}(1)_m$ symmetry is broken in such a way that

$$\langle K_\mu(x)K_\nu(y)\rangle_c = g^2\delta_{\mu\nu}\delta^{(4)}(x-y)f(x) + \cdots, \quad (4.22)$$

the mass term is generated,

$$S[b] = \int d^4x \left[\frac{-1}{4} g^2 b_{\mu\nu}(x) b^{\mu\nu}(x) + \frac{1}{2} g^2 m_b^2 b_\mu(x) b_\mu(x) + \cdots \right], \quad (4.23)$$

if we write $f(x) = m_b^2$. This can be called the dual Meissner effect; the dual gauge field acquires a mass given by

$$m_b^2 = \frac{1}{4g^2}\Phi(0), \quad (4.24)$$

if the monopole loop condensation occurs in the sense that

$$\Phi(x) := \lim_{y\to x} \frac{\langle K_\mu(x)K_\mu(y)\rangle_c}{\delta^{(4)}(x-y)} \neq 0. \quad (4.25)$$

This is a criterion of the dual superconductivity of QCD.[8] It is consistent with the picture of a dual superconductor scenario for quark confinement proposed by Nambu [1], 't Hooft [2], and Mandelstam [3]. In the translation-invariant theory, $\Phi(x)$ is an x-independent constant which depends only on the gauge coupling constant g. If we take a specific classical configuration to estimate them, an x dependence may appear; see the effective dual GL theory in the latter half of this subsection.

It should be remarked that Φ is not a local order parameter in the usual sense. In order to find the nonzero value of m_b, we must extract, from the magnetic monopole current correlation function $\langle K_\mu(x)K_\nu(y)\rangle_c$, a piece which is proportional to the Dirac delta function $\delta^{(4)}(x-y)$ diverging as $y\to x$. Therefore, if such a type of strong short-range correlation between two magnetic monopole loops does not exist, Φ is obviously zero. This observation seems to be consistent with the result of lattice simulations. Monopole loops exist both in the confinement and the deconfinement phases. However, in the deconfinement phase the monopole currents are dilute and the vacuum contains only short monopole loops with some nonzero density. In the confinement phase, on the other hand, the monopole trajectories form infinite long loops and the monopole currents form a dense cluster, although there is a number of small mutually disjoint clusters [30].

It should be remarked that the APEGT does not need any scalar field. In this sense, the mechanism in which the dual gauge field acquires a mass is different from the dual Higgs mechanism. Nevertheless, we can always introduce a scalar field into the APEGT so as to recover the spontaneously broken $\mathrm{U}(1)_m$ symmetry,

$$\frac{1}{2}m_b^2 b_\mu(x)b_\mu(x) \to \frac{1}{2}m_b^2[b_\mu(x) - \partial_\mu\theta(x)]^2 = |[\partial_\mu - i b_\mu(x)]\phi(x)|^2, \quad (4.26)$$

where we identify

$$\phi(x) = \frac{m_b}{\sqrt{2}} e^{i\theta(x)}. \quad (4.27)$$

Indeed, the result is invariant under $b_\mu \to b_\mu + \partial_\mu\alpha$ and $\theta \to \theta + \alpha$ $(\phi \to e^{i\alpha}\phi)$. Such a scalar field is called a Stückelberg field or Batalin-Fradkin field [31]. The case (4.27) is obtained as an extreme type II limit (London limit),

$$\lim_{\lambda\to\infty} V(\phi), \quad V(\phi) := \lambda(|\phi(x)|^2 - m_b^2/2)^2, \quad (4.28)$$

or nonlinear σ model with a constraint,

$$\delta(|\phi(x)|^2 - m_b^2/2). \quad (4.29)$$

The value ϕ_0 at which the potential $V(\phi)$ has a minimum is proportional to the mass m_b of a dual gauge field,

$$m_b = \sqrt{2}\phi_0 = \frac{\sqrt{\Phi}}{2g}. \quad (4.30)$$

In the deconfinement phase, the minimum is given by $\phi_0 = 0$ $(m_b = 0)$, while in the confinement phase the minimum is shifted from zero $\phi_0 \neq 0$ $(m_b \neq 0)$ which corresponds to monopole condensation. Thus the dual Abelian gauge theory with an action $S[b]$ is equivalent to (the London limit of) the dual GL theory (or the dual Abelian Higgs model with radial part of the Higgs field being frozen),

$$S_{dGL}[b] = \int d^4x \left[\frac{-1}{4} b_{\mu\nu}b^{\mu\nu} + |(\partial_\mu - i g^{-1} b_\mu)\phi|^2 + \lambda(|\phi|^2 - \phi_0^2)^2 + \cdots \right], \quad (4.31)$$

where we have rescaled the field $b_\mu \to b_\mu/g$. Note that the inverse coupling g^{-1} has appeared as a coupling constant. This implies that the dual theory is suitable for describing the strong coupling region.

[8]For other proposals, see [29] and references therein.

Now we compare our approach with the previous approach [32,33] where a summation over the monopole trajectories is performed. The monopole trajectories are expressed by the four-vector $x^\mu = \overline{x}_l^\mu(\tau_l), l=1,2, \ldots, N$, where τ_l is an arbitrary parameter characterizing the trajectory and N is the total number of loops. Then the monopole current is written as

$$K^\mu(x) = \frac{4\pi}{g}\sum_{l=1}^{N} n_l \int d\tau_l \dot{\overline{x}}_l^\mu(\tau_l)\,\delta^{(4)}(x-\overline{x}_l(\tau_l)),$$

$$\dot{\overline{x}}^\mu := \frac{\partial \overline{x}^\mu}{\partial \tau}, \tag{4.32}$$

with n_l being the winding number. Then the interaction $b_\mu K^\mu$ between the dual field and the monopole current is written as

$$\int d^4x b_\mu(x)K^\mu(x) = \frac{4\pi}{g}\sum_{l=1}^{N} n_l \int d\tau_l b_\mu(\overline{x}_l(\tau_l))\dot{\overline{x}}_l^\mu(\tau_l). \tag{4.33}$$

The summation over all configurations containing an arbitrary number of monopole loops with all possible winding number and trajectories is performed based on the identity [33]

$$\begin{aligned}\sum_{N=0}^{\infty}\frac{1}{N!}\int \prod_{l=1}^{N}[d\overline{x}_l]\exp\Bigg\{ i\sum_{l=1}^{N}\int d\tau_l [M\sqrt{\dot{\overline{x}}_l^{2}(\tau_l)} \\ + Q_\mu(\overline{x}_l(\tau_l))\dot{\overline{x}}_l^\mu(\tau_l)]\Bigg\} \\ = \exp \mathrm{Tr} \ln H = \det(H) \\ = \int [d\phi]\exp\Bigg\{ i\int d^4x\{|[\partial_\mu + iQ_\mu(x)]\phi(x)|^2 \\ - M^2|\phi(x)|^2\}\Bigg\},\end{aligned} \tag{4.34}$$

$$H := \frac{1}{2}(p_\mu - Q_\mu)^2 - \frac{1}{2}M^2,$$

where ϕ is a complex scalar field. Both sides are equal to the vacuum-to-vacuum transition amplitude of the theory consisting of charged scalar particles of mass M in the presence of an external electromagnetic field Q_μ.

If n_l is restricted to $n_l = \pm 1$, a field theoretical quantity is obtained,

$$|(\partial_\mu + ig_m b_\mu(x))\phi(x)|^2, \quad g_m := \frac{4\pi}{g}n,$$

where $Q_\mu = g_m b_\mu$ and ϕ plays the role of the monopole. Assuming a mass term of the monopole field and the repulsive self-interaction among the monopoles, a low energy (infrared) effective theory of the GL type, the effective dual GL theory, was proposed [10],

$$\frac{-1}{4}b_{\mu\nu}(x)b^{\mu\nu}(x) + |[\partial_\mu + ig_m b_\mu(x)]\phi(x)|^2 - \lambda[|\phi(x)|^2 - v^2]^2. \tag{4.35}$$

If monopole condensation occurs in the sense that $|\phi(x)| \equiv v \neq 0$, the mass term of the dual gauge field is generated, and the GL theory reduces to [note that the normalization for the field b_μ is different from Eq. (4.23)]

$$\frac{-1}{4}b_{\mu\nu}(x)b^{\mu\nu}(x) + \frac{1}{2}m_b^2 b_\mu(x)^2, \quad m_b \equiv g_m v.$$

This is the so-called dual Meissner effect. Precisely speaking, the classical solution [34] $\phi(x)$ is not a constant and is a function of x such that $\phi(x)\to v$ as $|x|\to\infty$ and $\phi(x)\to 0$ as $|x|\to 0$. The characteristic length separating both behaviors is the coherence length $\xi := \sqrt{2}/m_\phi$. The ratio

$$\kappa_{GL} := \delta/\xi = m_\phi/(\sqrt{2}m_b) \tag{4.36}$$

is called the GL parameter where $\delta := 1/m_b$ is the penetration depth. The constant $|\phi(x)| \equiv v \neq 0$ corresponds to $m_\phi = \infty$ or $\xi = 0$ and $\kappa_{GL} = \infty$, a special case of type II superconductors $\kappa_{GL} > 1/\sqrt{2}$. In the APEGT, this effect is expressed by the x-dependent mass $m_b(x)$.

D. Monopole action

It is easy to show that monopole condensation actually occurs, if we use the lattice version [11,30] of the monopole action[9] (4.4),

$$S_m = -\frac{1}{4g^2}\sum_x f_{\mu\nu}(x)f^{\mu\nu}(x) + \sum_{x,y}\frac{1}{g^2}k^\mu(x)D_{\mu\nu}(x-y)k^\nu(y). \tag{4.37}$$

The monopole condensate (4.25) is calculated as follows. From Eq. (4.37), we can extract the self-mass term

$$S_{ma} = \frac{D(0)}{g^2}\sum_x k^\mu(x)k^\mu(x), \quad D(0) < \infty. \tag{4.38}$$

The self-mass term with constant $|k_\mu(x)| = 1$ (see [11]) is proportional to the length of monopole loops. The probability that a monopole loop with length L will appear somewhere is

$$P_L = 7^L \exp(-S_{ma}) = \exp\{[C - D(0)/g^2]L\}, \tag{4.39}$$

[9] On the lattice, the monopole action is obtained from the radially fixed Abelian Higgs model (of Villain type) by lattice duality transformation [35].

where $C=\ln 7$ for a nonbacktracking walk on a four-dimensional hypercubic lattice. For sufficiently large g^2 $[g^2>D(0)/C]$, $P_L\uparrow\infty$ as $L\uparrow\infty$ and long loops give a dominant contribution to the functional integral. On the other hand, $P_L\downarrow 0$ as $L\uparrow\infty$, if g^2 is small $[g^2<D(0)/C]$. This indicates that in the infinite volume limit long monopole loops make no finite contribution. This is a simple energy-entropy (action-entropy) argument. Taking into account that the entropy contribution is equivalent to adding an action,

$$S_{en}=-C\sum_x k^\mu(x)k^\mu(x), \quad C<\infty. \tag{4.40}$$

Therefore we obtain

$$\Phi=\left(C-\frac{D(0)}{g^2}\right)^{-1}. \tag{4.41}$$

This shows that, if the coupling g is sufficiently strong, we have a positive Φ and nonzero m_b. In other words, if the entropy of a monopole loop exceeds the energy, monopole condensation occurs. The region exhibiting monopole condensation extends to smaller and smaller values of g for longer loops due to recent studies [30]. The above argument is valid for long loops. For more details, see [30]. The monopole action in the continuum needs more careful treatment as in three-dimensional case [36] which will be treated in a subsequent paper.

In the usual language of field theory, the term $k^\mu(x)D_{\mu\nu}(x-y)k^\nu(y)$ corresponds to the quartic self-interaction, especially the self-mass term $k^\mu(x)k^\nu(x)$ to the contact quartic self-interaction.[10] Therefore, it is assumed that the self-interaction among monopole loops does not essentially change the above picture. It should be remarked that higher order expansion generates interactions between monopole loops. For example, the self-interaction among the monopoles,

$$\begin{aligned}\langle K_\mu(x)K_\nu(y)K_\rho(z)K_\sigma(w)\rangle=\lambda(g)[&\delta_{\mu\nu}\delta_{\rho\sigma}\delta^{(4)}(x-y)\\ &\times\delta^{(4)}(z-w)\,\delta^{(4)}(x-z)\\ &+\delta_{\mu\rho}\delta_{\nu\sigma}\delta^{(4)}(x-z)\\ &\times\delta^{(4)}(y-w)\,\delta^{(4)}(x-y)\\ &+\delta_{\mu\sigma}\delta_{\nu\rho}\delta^{(4)}(x-w)\\ &\times\delta^{(4)}(y-z)\,\delta^{(4)}(x-y)]+\cdots,\end{aligned} \tag{4.42}$$

induces quartic self-interactions for b_μ,

$$\begin{aligned}&\int d^4x d^4y d^4z d^4w\, b_\mu(x)b_\nu(y)b_\rho(z)b_\sigma(w)\\ &\quad\times\langle K_\mu(x)K_\nu(y)K_\rho(z)K_\sigma(w)\rangle\\ &\quad=3\lambda(g)\int d^4x[b_\mu(x)b_\mu(x)]^2+\cdots.\end{aligned} \tag{4.43}$$

This renormalizes the mass term in Eq. (4.23) through radiative corrections. In this sense the criterion (4.25) is the tree-level criterion. The monopole interaction is expected to be weakly repulsive.

E. Another effective Abelian gauge theory and confinement

The effective Abelian theory $S[a]$ written in terms of a_μ is obtained by integrating out the dual gauge field. This theory with an action $S[a]$ gives a dual description of the same physics as that given by $S[b]$. Following the Zwanziger formalism [28] (we do not repeat the details; see [10] and [19]), if the dual gauge field acquires nonzero mass m_b (due to monopole condensation), we obtain

$$\begin{aligned}\mathcal{S}_{eff}[a]=\int d^4x\Bigg[&\frac{-1}{4g(\mu)^2}f_{\mu\nu}(x)f^{\mu\nu}(x)\\ &+\frac{1}{2}a^\mu(x)\frac{n^2m_b^2(x)}{(n\cdot\partial)^2+n^2m_b^2(x)}X_{\mu\nu}(\partial)a^\nu(x)\Bigg],\end{aligned} \tag{4.44}$$

$$X_{\mu\nu}(\partial):=\frac{1}{n^2}\epsilon^{\lambda\mu\alpha\beta}\epsilon^{\lambda\nu\gamma\delta}n_\alpha n_\gamma\partial_\beta\partial_\delta,$$

where n is an arbitrary fixed four-vector appearing in the Zwanziger formalism. The coupling constant $g(\mu)$ is the running coupling constant obeying the same β function as the YM theory. In the limit $m_b\to 0$, Eq. (4.44) reduces to Eq. (3.9). Note that the local $\mathrm{U}(1)_e$ symmetry is not broken and a_μ is massless, since

$$\partial^\mu X_{\mu\nu}=0=\partial^\nu X_{\mu\nu}. \tag{4.45}$$

The low energy effective theories (4.44) and (4.31) lead to the linear static potential $V(r)$ between static color charges and the string tension σ is given by

$$V(r)=\sigma r, \quad \sigma=\frac{Q^2}{4\pi}m_b^2 f(\kappa_{GL}), \tag{4.46}$$

where $f(x)$ is a function depending on the method of calculation [10,19]. The essential part m_b^2 in the string tension follows simply due to the dimensional analysis, irrespective of the details of the calculation.

Monopole condensation can be estimated based on the classical configuration of $\mathcal{A}(x)$ satisfying the gauge-fixing condition $F^a[A,a]=0$,

[10] We remember that the quartic self-interaction in the scalar $\lambda\varphi^4$ theory can be understood as the intersection probability of two random walks with a repulsive interaction.

$$\langle K_\mu(x)K_\mu(y)\rangle = Z_{YM}^{-1}\int [d\mathcal{A}(x)]e^{-S_{YM}[\mathcal{A}]}\delta(F[A,a]) \times \int [dK_\mu]\delta\left(K_\mu - \frac{1}{2}g^2\epsilon_{\mu\nu\rho\sigma}\partial^\nu \times (\epsilon^{ab3}A_\rho^a A_\sigma^b)\right)K_\mu(x)K_\mu(y). \tag{4.47}$$

Note that the MAG condition $F[A,a]=0$ is satisfied by the classical multi-instanton solution [37,38] of 't Hooft type,

$$A_\mu^a(x) = \overline{\eta}_{\mu\nu}^a\partial_\nu f(x),$$
$$\overline{\eta}_{\mu\nu}^a := \epsilon_{a\mu\nu} + \delta_{a\mu}\delta_{\nu 4} - \delta_{a\nu}\delta_{\mu 4} = -\overline{\eta}_{\nu\mu}^a. \tag{4.48}$$

Therefore the classical instanton configuration may have a possibility to generate monopole condensation. Actually, it has been shown that monopole loop formation and its condensation are intimately correlated with the instanton configuration [13,39–45]. Therefore it is quite interesting to clarify whether the instanton configuration gives sufficient monopole loop condensation for quark confinement. The details of this problem will be given in a subsequent paper [46].

V. INCLUSION OF A FERMION

In order to discuss QCD, we add the fermionic action

$$S_F = \int d^4x\,\overline{\psi}[i\gamma^\mu\mathcal{D}_\mu[\mathcal{A}] - m]\psi, \quad \mathcal{D}_\mu[\mathcal{A}] := \partial_\mu - i\mathcal{A}_\mu. \tag{5.1}$$

The contribution from the fermionic action is evaluated as

$$\int [d\overline{\psi}][d\psi]\exp\left\{-\int d^4x\,\overline{\psi}\{i\gamma^\mu\mathcal{D}_\mu[\mathcal{A}] - m\}\psi\right\}$$
$$= (\det\{i\gamma^\mu\mathcal{D}_\mu[\mathcal{A}] - m\})^{N_f}$$
$$= \exp(N_f \ln\det\{i\gamma^\mu\mathcal{D}_\mu[\mathcal{A}] - m\})$$
$$= \exp\left[\frac{N_f}{2}\ln\det\{i\gamma^\mu\mathcal{D}_\mu[\mathcal{A}] - m\}^2\right]. \tag{5.2}$$

In a similar way as in Sec. II, we can calculate the logarithmic determinant

$$\mathrm{Tr}(\exp\{-t(i\gamma^\mu\mathcal{D}_\mu[\mathcal{A}])^2\}) - \mathrm{Tr}\{\exp[-t(i\gamma^\mu\partial_\mu)^2]\} = \int d^4x\frac{g^2}{16\pi^2}\frac{2}{3}r(F)(\mathcal{F}_{\mu\nu}^a)^2 + O(t) \tag{5.3}$$

and

$$\ln\frac{\det(i\gamma^\mu D_\mu[\mathcal{A}])^2}{\det(i\gamma^\mu\partial_\mu)^2} = \int d^4x\frac{g^2}{16\pi^2}\frac{2}{3}r(F)\ln\mu^2(\mathcal{F}_{\mu\nu}^a)^2, \tag{5.4}$$

where $r(F)$ is the dimension of fermion representation. In this calculation, we have used the commutator

$$[\mathcal{D}_\mu[\mathcal{A}],\mathcal{D}_\nu[\mathcal{A}]] = -i\mathcal{F}_{\mu\nu}. \tag{5.5}$$

At the one-loop level, it is easy to see that we can replace $(\mathcal{F}_{\mu\nu}^a)^2$ in this contribution by $(f_{\mu\nu})^2$. If we add this contribution to the APEGT obtained in Sec. II, the APEGT of QCD is obtained (apart from the gauge-fixing term and the Abelian ghost term),

$$S = \int d^4x\left[-\frac{1+z_a}{4g^2}f_{\mu\nu}f^{\mu\nu} - \frac{1+z_b}{4}g^2b_{\mu\nu}b^{\mu\nu} - z_c b_\mu K^\mu + \overline{\psi}(i\gamma^\mu\mathcal{D}_\mu[a] - m)\psi + i\overline{c}^a D_\mu^{bc}[a]D_\mu^{bc}[a]c^c\right],$$
$$\mathcal{D}_\mu[a] := \partial_\mu - ia_\mu T^3. \tag{5.6}$$

In the region $K_\mu \cong 0$, it is clear that this theory recovers the one-loop beta function of QCD,

$$b_0 = \frac{11}{3}C_2(G) - \frac{4}{3}N_f r(F). \tag{5.7}$$

Monopole condensation and resulting dual Meissner effect can be treated in a similar way as Sec. IV. We can discuss chiral symmetry breaking based on the APEGT of QCD, Eq. (5.6); see, e.g., [19].

VI. LOWER DIMENSIONAL CASE

In the (2+1)-dimensional case, we introduce the auxiliary vector field B_μ [instead of the tensor field $B_{\mu\nu}$ in (3+1)-dimensional case]. Then, corresponding to Eq. (2.10) or (2.13), the action is rewritten as

$$S_{apBF\text{-}YM}[\mathcal{A},B] = \int d^3x\left[\frac{1}{4}\epsilon^{\mu\nu\rho}B_\rho(f_{\mu\nu} + \mathcal{C}_{\mu\nu}) - \frac{1}{4}g^2B_\mu B^\mu - \frac{1}{4g^2}(\mathcal{S}_{\mu\nu}^a)^2\right], \tag{6.1}$$

or

$$S_{apYM}[\mathcal{A},B] = \int d^3x\left[-\frac{1}{4g^2}(f_{\mu\nu}f_{\mu\nu} + 2f_{\mu\nu}\mathcal{C}_{\mu\nu}) + \frac{1}{4}\epsilon^{\mu\nu\rho}B_\rho\mathcal{C}_{\mu\nu} - \frac{1}{4}g^2B_\mu B^\mu - \frac{1}{4g^2}(\mathcal{S}_{\mu\nu}^a)^2\right]. \tag{6.2}$$

At the tree level, the dual vector field has the respective correspondence

$$B_\mu \leftrightarrow \frac{1}{2}\epsilon^{\mu\rho\sigma}(f_{\rho\sigma} + \mathcal{C}_{\rho\sigma}), \quad \frac{1}{2}\epsilon^{\mu\rho\sigma}\mathcal{C}_{\rho\sigma}. \tag{6.3}$$

In order to discuss the monopole contribution, we use the decomposition[11]

$$B_\mu = \partial_\mu \phi + \frac{1}{2}\epsilon_{\mu\alpha\beta}\chi^{\alpha\beta}, \quad \chi_{\mu\nu} := \partial_\mu \chi_\nu - \partial_\nu \chi_\mu . \quad (6.4)$$

Hence the APEGT of the (2+1)-dimensional YM theory is given by

$$S_1[a,\phi,\chi] = \int d^3x \left[-\frac{1}{4g^2} f_{\mu\nu} f_{\mu\nu} - \frac{1}{4} g^2 [(\partial_\mu \phi)^2 + \chi_{\mu\nu}^2] \right] \quad (6.5)$$

and

$$Q^{ab}_{\mu\nu} := (D_\rho[a] D_\rho[a])^{ab} \delta_{\mu\nu} - 2\epsilon^{ab3} f_{\mu\nu} + \frac{1}{2} g^2 \epsilon^{ab3} (\epsilon_{\mu\nu\rho}\partial^\rho \phi + \chi_{\mu\nu}) - 2ig^2 \xi(\overline{c^a} c^b - \overline{c^c} c^c \delta^{ab}) \delta_{\mu\nu} - D_\mu[a]^{ac} D_\nu[a]^{cb} + \frac{1}{\alpha} D_\mu[a]^{ac}_\xi D_\nu[a]^{cb}_\xi . \quad (6.6)$$

In the (2+1)-dimensional case, instead of the interaction $b_\mu K^\mu$ between the dual gauge field and the magnetic current, we obtain the interaction term between the dual scalar ϕ and the monopole density ρ,

$$\rho(x)\phi(x), \quad \rho(x) := \epsilon^{\mu\nu\rho} \partial_\rho \mathcal{C}_{\mu\nu}(x), \quad (6.7)$$

since

$$\int d^3x \epsilon^{\mu\nu\rho} B_\rho \mathcal{C}_{\mu\nu} = \int d^3x [-\phi \epsilon^{\mu\nu\rho} \partial_\rho \mathcal{C}_{\mu\nu} + \chi_{\mu\nu} \mathcal{C}_{\mu\nu}]. \quad (6.8)$$

The effective dual theory is the scalar theory with

$$S[\phi] = \int d^3x [\partial_\mu \phi(x)]^2 + \int d^3x \langle \rho(x) \rangle \phi(x) + \frac{1}{2} \int d^3x \int d^3y \langle \rho(x) \rho(y) \rangle_c \phi(x) \phi(y) + \cdots . \quad (6.9)$$

In the (1+1)-dimensional case, the dual tensor reduces to a one-component scalar B,

$$S_{apBFYM}[\mathcal{A},\phi] = \int d^2x \left[\frac{1}{4} \epsilon^{\mu\nu} (f_{\mu\nu} + \mathcal{C}_{\mu\nu}) \phi - \frac{1}{4} g^2 \phi^2 - \frac{1}{4g^2} (\mathcal{S}^a_{\mu\nu})^2 \right] \quad (6.10)$$

or

$$S_{apYM}[\mathcal{A},\phi] = \int d^2x \left[-\frac{1}{4g^2} (f_{\mu\nu} f_{\mu\nu} + 2 f_{\mu\nu} \mathcal{C}_{\mu\nu}) + \frac{1}{4} \epsilon^{\mu\nu} \phi \mathcal{C}_{\mu\nu} - \frac{1}{4} g^2 \phi^2 - \frac{1}{4g^2} (\mathcal{S}^a_{\mu\nu})^2 \right]. \quad (6.11)$$

The tree-level correspondence is given by

$$\phi \leftrightarrow \frac{1}{2} \epsilon^{\rho\sigma} (f_{\rho\sigma} + \mathcal{C}_{\rho\sigma}), \quad \frac{1}{2} \epsilon^{\rho\sigma} \mathcal{C}_{\rho\sigma} . \quad (6.12)$$

Thus (1+1)-dimensional YM theory is reduced to an effective Abelian gauge theory with

$$S_1[a,\phi] = \int d^2x \left[-\frac{1}{4g^2} f_{\mu\nu} f_{\mu\nu} - \frac{1}{4} g^2 \phi^2 \right], \quad (6.13)$$

and

$$Q^{ab}_{\mu\nu} := (D_\rho[a] D_\rho[a])^{ab} \delta_{\mu\nu} - 2\epsilon^{ab3} f_{\mu\nu} + \frac{1}{2} g^2 \epsilon^{ab3} \epsilon_{\mu\nu} \phi - 2ig^2 \xi(\overline{c^a} c^b - \overline{c^c} c^c \delta^{ab}) \delta_{\mu\nu} - D_\mu[a]^{ac} D_\nu[a]^{cb} + \frac{1}{\alpha} D_\mu[a]^{ac}_\xi D_\nu[a]^{cb}_\xi . \quad (6.14)$$

In this case, the interaction term is induced,

$$\phi(x) \epsilon_{\mu\nu} f_{\mu\nu}(x). \quad (6.15)$$

It is interesting to compare these formulations with the previous approaches [36,47–49]. Detailed analyses of the lower dimensional case will be given in a forthcoming paper.

VII. CONCLUSION AND DISCUSSION

We have derived Abelian-projected effective gauge theories (APEGT) of YM theory and QCD. This has been performed by integrating out all off-diagonal non-Abelian gauge fields belonging to SU(2)/U(1). The obtained APEGT is written in terms of the maximal Abelian gauge field a_μ and the dual Abelian gauge field b_μ which couples to the magnetic monopole current K_μ. First, we have shown that the APEGT has the same one-loop beta function as the original non-Abelian gauge theories. Hence the APEGT exhibits asymptotic freedom (at the one-loop level).

Next, we have shown that the dual vector field introduced to linearize the gluon self-interaction has an interaction with the magnetic current. Because of this interaction, the dual gauge field can become massive if monopole loop condensation occurs. This is interpreted as the dual Meissner effect. We have shown that the mass of the dual gauge field is given by the monopole loop condensation $\langle K_\mu(x) K^\mu(x) \rangle / \delta^{(4)}(0) \neq 0$. This is our criterion of dual superconductivity. A method of showing monopole condensation is to consider the monopole action. The lattice monopole action [11,30] gives a simple proof of monopole condensation.

If we apply the Zwanziger formalism to the APEGT with a magnetic monopole, we can show that the static quark

[11]The vector B_μ has three degrees of freedom, while the real scalar ϕ has one and the vector χ_μ has three. One redundant degree of freedom corresponds to that of the gauge transformation of χ_μ.

potential contains a linear part proportional to the quark separation. The APEGT with a monopole is sufficient to show quark confinement. This supports Abelian dominance. Monopole dominance will be confirmed by evaluating the monopole condensate, since the string tension is determined from the mass m_b of the dual gauge field. We have pointed out that this condensation can be estimated by the classical instanton configuration. The intimate relationship between confinement and instanton will be understood from the viewpoint of a topological field theory of the Schwarz type, BF-YM theory.

This work justifies some aspects of the pioneering works of Ezawa and Iwazaki [9] and Suzuki [10] based on the effective dual GL model. However, the APEGT has no free parameter and is of predictive power in sharp contrast with previous works where Abelian dominance was assumed from the beginning. The APEGT has a complete correspondence to the original YM theory.

We have chosen the gauge group SU(2) for mathematical simplicity. To discuss confinement in the real world, we must discuss the SU(3) case. This case will be treated in a subsequent paper [46].

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APPENDIX A: APEGT OF BF-YM THEORY

In a similar way as in Sec. II, the APEGT of BF-YM theory is obtained as

$$S_0+S_1+S_2=\int d^4x\bigg[-z_a\frac{1}{4g^2}f_{\mu\nu}f^{\mu\nu}-\frac{1}{4}(1+z_b)g^2B_{\mu\nu}B^{\mu\nu}+\frac{1}{2}(1-z_c)B_{\mu\nu}\tilde{f}_{\mu\nu}+i\bar{c}^aD^{\mu ac}[a]^{\xi}D_{\mu}^{cb}[a]c^b\bigg], \tag{A1}$$

where

$$z_a=-\frac{2}{3}\kappa\frac{g^2}{16\pi^2}\ln\mu,\quad z_b=+2\kappa\frac{g^2}{16\pi^2}\ln\mu,$$
$$z_c=+2\kappa\frac{g^2}{16\pi^2}\ln\mu. \tag{A2}$$

Integrating out the tensor field B, we obtain

$$S_E=\int d^4x\bigg[-z_a\frac{1}{4g^2}f_{\mu\nu}f^{\mu\nu}-\frac{1}{4g^2}(1+z_b)^{-1}\times(1-z_c)^2f_{\mu\nu}f^{\mu\nu}+i\bar{c}^aD^{\mu ac}[a]^{\xi}D_{\mu}^{cb}[a]c^b\bigg]. \tag{A3}$$

Hence, at the one-loop level, this reduces to

$$S_E=\int d^4x\bigg[-(1+h)\frac{1}{4g^2}f_{\mu\nu}f^{\mu\nu}+i\bar{c}^aD^{\mu ac}[a]^{\xi}D_{\mu}^{cb}[a]c^b\bigg], \tag{A4}$$

where

$$h=z_a-z_b-2z_c=-\frac{20}{3}\kappa\frac{g^2}{16\pi^2}\ln\mu. \tag{A5}$$

This agrees with the APEGT of YM theory given in Sec. II. Therefore, two types of APEGT are equivalent to each other.

APPENDIX B: GHOST INTERACTION AND GAUGE FIXING

If we adopt a more general gauge-fixing functional

$$G_{gf}=\sum_{\pm}\bar{c}^{\mp}\left(F^{\pm}[A,a]+\frac{\alpha}{2}\phi^{\pm}\right)+\bar{c}^3\left(F^3[a]+\frac{\beta}{2}\phi^3\right)+\alpha\eta\sum_{\pm}(\pm)\bar{c}^3\bar{c}^{\pm}c^{\mp}+\alpha\zeta c^3\bar{c}^{+}\bar{c}^{-}, \tag{B1}$$

the gauge-fixing part $\mathcal{L}_{GF}=-i\delta_BG_{gf}$ has the additional contribution

$$\mathcal{L}'_{GF}=-\sum_{\pm}(\pm)\bar{c}^{\mp}\frac{\alpha}{\beta}\eta F^3[a]c^{\pm}-\sum_{\pm}(\pm)\bar{c}^3\eta F^{\pm}[A,a]c^{\mp}+\sum_{\pm}(\pm)\bar{c}^{\mp}\zeta F^{\pm}[A,a]c^3-\alpha(1+\zeta)\eta\sum_{\pm}\bar{c}^3c^3\bar{c}^{\pm}c^{\mp}-\alpha\left(\zeta+\frac{\alpha}{\beta}\eta^2\right)\bar{c}^{+}\bar{c}^{-}c^{+}c^{-}. \tag{B2}$$

Therefore, the U(1)-invariant four-ghost interaction term $\bar{c}^{+}\bar{c}^{-}c^{+}c^{-}$ coming from the expansion of $\ln\det Q$,

$$(\bar{c}^ac^b-\bar{c}^cc^c\delta^{ab})(\bar{c}^bc^a-\bar{c}^dc^d\delta^{ba})=-2\bar{c}^1c^1\bar{c}^2c^2=-2\bar{c}^{+}\bar{c}^{-}c^{+}c^{-}, \tag{B3}$$

is canceled by adding the BRST exact term $-i\delta_B(c^3\bar{c}^+\bar{c}^-)=-i\{Q_B,c^3\bar{c}^+\bar{c}^-\}$. Such a term does not influence the physical state characterized by $Q_B|\mathrm{phys}\rangle=0$. This is an implication of the renormalizability of YM theory in the MAG.

APPENDIX C: MAGNETIC MONOPOLE AND DIRAC STRING IN SU(2) GAUGE THEORY

In this appendix, we discuss how the Abelian objects, the Dirac magnetic monopole and Dirac string, are produced due to a singular gauge transformation in SU(2) non-Abelian gauge theory.[12]

The non-Abelian field strength $\mathcal{F}_{\mu\nu}$ is defined using the covariant derivative

$$\mathcal{D}_\mu := \partial_\mu - ig\mathcal{A}_\mu \tag{C1}$$

as

$$\mathcal{F}_{\mu\nu} = \frac{i}{g}[\mathcal{D}_\mu,\mathcal{D}_\nu] = \frac{i}{g}[\partial_\mu - ig\mathcal{A}_\mu, \partial_\nu - ig\mathcal{A}_\nu]. \tag{C2}$$

This is rearranged as

$$\begin{aligned}\mathcal{F}_{\mu\nu} &= \frac{i}{g}[\partial_\mu,\partial_\nu] + [\partial_\mu,\mathcal{A}_\nu] - [\partial_\nu,\mathcal{A}_\mu] - ig[\mathcal{A}_\mu,\mathcal{A}_\nu]\\ &= \frac{i}{g}[\partial_\mu,\partial_\nu] + \partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu - ig[\mathcal{A}_\mu,\mathcal{A}_\nu].\end{aligned} \tag{C3}$$

It should be remarked that the first term on the RHS in the final line cannot be neglected when there is a singularity. We consider the local gauge transformation

$$\mathcal{A}_\mu \to \mathcal{A}'_\mu := U\mathcal{A}_\mu U^\dagger + \frac{i}{g}U\partial_\mu U^\dagger. \tag{C4}$$

Straightforward calculation using Eq. (C4) leads to

$$\mathcal{F}'_{\mu\nu} := \partial_\mu\mathcal{A}'_\nu - \partial_\nu\mathcal{A}'_\mu - ig[\mathcal{A}'_\mu,\mathcal{A}'_\nu] \tag{C5}$$

$$= U(\partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu - ig[\mathcal{A}_\mu,\mathcal{A}_\nu])U^\dagger + \frac{i}{g}U[\partial_\mu,\partial_\nu]U^\dagger. \tag{C6}$$

This is consistent with Eq. (C3); that is, the field strength transforms covariantly,

$$\mathcal{F}_{\mu\nu} \to \mathcal{F}'_{\mu\nu} := U\mathcal{F}_{\mu\nu}U^\dagger. \tag{C7}$$

In what follows we assume that $\mathcal{A}_\mu$ is not singular and that the singularity in $\mathcal{A}'_\mu$ comes from the gauge rotation U. In such a case, we call U the singular gauge rotation. Therefore, the gauge-transformed field strength is composed of two parts, the regular and the singular part,

$$\begin{aligned}\mathcal{F}'_{\mu\nu} &= \mathcal{F}^r_{\mu\nu}{}' + \mathcal{F}^s_{\mu\nu}{}',\\ \mathcal{F}^r_{\mu\nu}{}' &:= U(\partial_\mu\mathcal{A}_\nu - \partial_\nu\mathcal{A}_\mu - ig[\mathcal{A}_\mu,\mathcal{A}_\nu])U^\dagger,\\ \mathcal{F}^s_{\mu\nu}{}' &:= \frac{i}{g}U[\partial_\mu,\partial_\nu]U^\dagger.\end{aligned} \tag{C8}$$

First, we show that only the second part of the potential $\mathcal{A}'_\mu(x)$,

$$\mathcal{A}^s_\mu(x) := \frac{i}{g}U(x)\partial_\mu U^\dagger(x), \tag{C9}$$

gives rise to the nonvanishing magnetic current. The diagonal part a^s_μ of the gauge potential $\mathcal{A}^s_\mu$ is singular on the point where the Dirac string exists. The direction of the Dirac string can be changed arbitrarily by the gauge transformation. Hence the Dirac string is not a physical object. Actually, the magnetic charge is shown to obey the Dirac quantization condition. This can be seen as follows.

The local SU(2) matrix $U(x)$ can be written in terms of three Euler's angles α,β,γ,

$$U(x) = e^{i\gamma(x)\sigma_3/2}e^{i\beta(x)\sigma_2/2}e^{i\alpha(x)\sigma_3/2} = \begin{pmatrix} e^{(i/2)[\alpha(x)+\gamma(x)]}\cos\frac{\beta(x)}{2} & e^{(i/2)[\alpha(x)-\gamma(x)]}\sin\frac{\beta(x)}{2} \\ -e^{(-i/2)[\alpha(x)-\gamma(x)]}\sin\frac{\beta(x)}{2} & e^{(-i/2)[\alpha(x)+\gamma(x)]}\cos\frac{\beta(x)}{2}\end{pmatrix}. \tag{C10}$$

Using the residual U(1) invariance after the MAG, we can choose $\gamma(x)=-\alpha(x)$. A convenient choice is to take $\alpha(x)=-\gamma(x)=\varphi(x)$, $\beta(x)=\theta(x)$, and identity the angles θ and φ with the polar and the azimuthal angles in the three-dimensional polar coordinate of SU(2) so that

[12]This appendix is deeply indebted to Suganuma and Ichie [50].

$$U(x)_{\theta,\varphi}=\exp(i\theta\vec{e}_\varphi\cdot\vec{\sigma}/2)$$
$$=\begin{pmatrix} \cos\frac{\theta(x)}{2} & e^{i\varphi(x)}\sin\frac{\theta(x)}{2} \\ -e^{-i\varphi(x)}\sin\frac{\theta(x)}{2} & \cos\frac{\theta(x)}{2}\end{pmatrix}$$
$$=\cos\frac{\theta(x)}{2}+i\vec{\sigma}\cdot\vec{e}_\varphi\sin\frac{\theta(x)}{2}, \tag{C11}$$

$$\vec{e}_\varphi:=-\sin\varphi(x)\vec{e}_X+\cos\varphi(x)\vec{e}_Y, \tag{C12}$$

where (X,Y,Z) is identified with the space coordinates of $x^\mu=(0,\vec{r})=(0,X,Y,Z)$ and

$$0<\theta:=\arctan\frac{\sqrt{X^2+Y^2}}{Z}<\pi,$$

$$0<\varphi:=\arctan\frac{Y}{X}<2\pi. \tag{C13}$$

This choice does not lose generality, since we can always rotate the matrix using the residual U(1) degrees of freedom; see [39] for details.[13]

For the gauge rotation (C12), the three-dimensional part of $\mathcal{A}_\mu^s$ is

$$\vec{\mathcal{A}}^s(x)=\frac{1}{gr}[\cos\varphi(x)\vec{e}_\varphi+\sin\varphi(x)\vec{e}_\theta]T^1$$
$$+\frac{1}{gr}[\sin\varphi(x)\vec{e}_\varphi-\cos\varphi(x)\vec{e}_\theta]T^2$$
$$+\frac{1}{gr}\tan\frac{\theta(x)}{2}\vec{e}_\varphi T^3, \tag{C16}$$

where we have used

$$\nabla:=\vec{e}_r\frac{\partial}{\partial r}+\frac{\vec{e}_\theta}{r}\frac{\partial}{\partial\theta}+\frac{\vec{e}_\varphi}{r\sin\theta}\frac{\partial}{\partial\varphi}. \tag{C17}$$

The diagonal Abelian part is defined by

$$a'_\mu:=2\,\mathrm{tr}(T^3\mathcal{A}'_\mu). \tag{C18}$$

[13]If we take $\gamma(x)=\alpha(x)$ and write $\alpha(x)=\gamma(x)=\varphi(x)$, $\beta(x)=\theta(x)$,

$$U(x)=\begin{pmatrix} e^{i\varphi(x)}\cos\frac{\theta(x)}{2} & \sin\frac{\theta(x)}{2} \\ -\sin\frac{\theta(x)}{2} & e^{-i\varphi(x)}\cos\frac{\theta(x)}{2}\end{pmatrix}. \tag{C14}$$

For this choice of γ, the Dirac string appears on the positive Z axis, since $\beta=0,\pi$ corresponds to

$$U(x)_{0,\varphi}=\begin{pmatrix} e^{i\varphi(x)} & 0 \\ 0 & e^{-i\varphi(x)}\end{pmatrix},\quad U(x)_{\pi,\varphi}=\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}. \tag{C15}$$

In this case,[14]

$$\vec{a}^s(x)=\frac{1}{gr}\tan\frac{\theta}{2}\vec{e}_\varphi=\frac{1}{gr}\frac{1-\cos\theta}{\sin\theta}\vec{e}_\varphi \tag{C21}$$

or

$$a_s^\mu(x)=(a_0^s(x),\vec{a}^s(x))=\frac{1}{gr(r+Z)}(0,-Y,X,0). \tag{C22}$$

The vector potential $\vec{a}^s$ is singular on the negative Z axis and is not defined for $\theta=\pi$. Then the rotation is given by

$$\nabla\times\vec{a}^s(x)=\vec{B}_m+\vec{B}_{DS}=\frac{\vec{r}}{gr^3}+\frac{4\pi}{g}\delta(X)\delta(Y)\theta(-Z)\vec{e}_Z. \tag{C23}$$

This implies that $\nabla\times\vec{a}^s(x)=\vec{r}/gr^3$ except along the negative Z axis. The singularity along the negative Z axis is called the Dirac string. This cannot be avoided as long as one uses a single expression for the gauge potential in the whole space. A method to avoid the singularity is using the Wu-Yang monopole [51]. It is impossible to construct a single singularity-free potential which is defined everywhere. When considering the total space, we need at least two expressions for the vector potential.

The magnetic monopole sits at $\vec{r}=0$,

$$\nabla\cdot\vec{B}_m=k_0^m(x),\quad k_0^m(x)=\frac{4\pi}{g}\delta^{(3)}(x). \tag{C24}$$

The four-dimensional expression of the magnetic current is

$$\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\partial_\sigma f_{\mu\nu}=k_\rho(x),\quad k_\mu(x)=\frac{4\pi}{g}\delta_{\mu0}\delta^{(3)}(x), \tag{C25}$$

where the Abelian-projected field strength is defined,

$$f_{\mu\nu}:=\partial_\mu a_\nu^s-\partial_\nu a_\mu^s:=\mathrm{tr}[T^3(\partial_\mu\mathcal{A}_\nu^s-\partial_\nu\mathcal{A}_\mu^s)]. \tag{C26}$$

The magnetic flux Φ obtained by integrating $\vec{B}_m$ over any closed surface containing the origin is

$$\Phi_m:=\int_S\vec{B}_m\cdot d\vec{S}=\frac{4\pi}{g}. \tag{C27}$$

[14]The four-dimensional expression is given by

$$a_\mu^s(x)=-\frac{1}{g}[\cos\beta(x)\partial_\mu\alpha(x)+\partial_\mu\gamma(x)]. \tag{C19}$$

The angle $\gamma(x)$ does not appear in the U(1)-invariant quantity. Actually, the magnetic current given by

$$k_\mu(x)=\frac{1}{g}\epsilon_{\mu\nu\rho\sigma}\partial_\nu[\partial_\rho\cos\beta(x)\partial_\sigma\alpha(x)] \tag{C20}$$

does not contain the angle γ. For more details, see [13].

On the other hand, the magnetic flux Φ obtained by integrating $\vec{B}_{Ds}$ over any closed surface containing the origin is

$$\Phi_{Ds}:=\int_S \vec{B}_{Ds}\cdot d\vec{S}=-\frac{4\pi}{g}. \tag{C28}$$

We observe that the singular gauge potential $\mathcal{A}_\mu^s$ satisfies the following relation:

$$\begin{aligned}\partial_\mu\mathcal{A}_\nu^s-\partial_\nu\mathcal{A}_\mu^s&=\frac{i}{g}\{\partial_\mu(U\partial_\nu U^\dagger)-\partial_\nu(U\partial_\mu U^\dagger)\}\\ &=\frac{i}{g}\{(\partial_\mu U)(\partial_\nu U^\dagger)-(\partial_\nu U)(\partial_\mu U^\dagger)\}+\frac{i}{g}(U\partial_\mu\partial_\nu U^\dagger-U\partial_\nu\partial_\mu U^\dagger)\\ &=\frac{i}{g}\{(\partial_\mu U)U^\dagger U(\partial_\nu U^\dagger)-(\partial_\nu U)U^\dagger U(\partial_\mu U^\dagger)\}+\frac{i}{g}(U[\partial_\mu,\partial_\nu]U^\dagger)\\ &=\frac{i}{g}\{-(U\partial_\mu U^\dagger)(U\partial_\nu U^\dagger)+(U\partial_\nu U^\dagger)(U\partial_\mu U^\dagger)\}+\frac{i}{g}(U[\partial_\mu,\partial_\nu]U^\dagger)\\ &=\frac{i}{g}[iU\partial_\mu U^\dagger,iU\partial_\nu U^\dagger]+\frac{i}{g}(U[\partial_\mu,\partial_\nu]U^\dagger)\\ &=ig[\mathcal{A}_\mu^s,\mathcal{A}_\nu^s]+\frac{i}{g}(U[\partial_\mu,\partial_\nu]U^\dagger),\end{aligned} \tag{C29}$$

where we have used

$$UU^\dagger=1,\quad \partial_\mu(UU^\dagger)=(\partial_\mu U)U^\dagger+U(\partial_\mu U^\dagger)=0. \tag{C30}$$

Hence the Abelian-projected field strength reads

$$f_{\mu\nu}=\mathrm{tr}(T^3 ig[\mathcal{A}_\mu^s,\mathcal{A}_\nu^s])+\mathrm{tr}\left(T^3\frac{i}{g}U[\partial_\mu,\partial_\nu]U^\dagger\right). \tag{C31}$$

If U is not singular, the last term in Eq. (C29) or (C31) is absent, since $\mathcal{A}_\mu^s$ is a pure gauge which gives a vanishing field strength for nonsingular $U(x)$,

$$\mathcal{F}_{\mu\nu}:=\partial_\mu\mathcal{A}_\nu^s-\partial_\nu\mathcal{A}_\mu^s-ig[\mathcal{A}_\mu^s,\mathcal{A}_\nu^s]\equiv 0. \tag{C32}$$

The last term in Eq. (C29) corresponds to the singularity due to a Dirac string as shown shortly.

Now we clarify the physical meaning of the last term $(i/g)(U[\partial_\mu,\partial_\nu]U^\dagger)^{(3)}$. We show that[15]

$$U(x)[\partial_X,\partial_Y]U^\dagger(x)=-2\pi n i\,\delta(X)\delta(Y)\theta(-Z)\sigma_3. \tag{C33}$$

To prove this, we first show that

$$[\partial_X,\partial_Y]\varphi(x)=2\pi n\,\delta(X)\delta(Y). \tag{C34}$$

This is a result of the Stokes theorem; for the arbitrary two-dimensional region S including $(X,Y)=(0,0)$,

$$\begin{aligned}\int_S dXdY[\partial_X,\partial_Y]\varphi&=\int_S d^2S\ \det\begin{pmatrix}\partial_X & \partial_Y\\ \partial_X\varphi & \partial_Y\varphi\end{pmatrix}\\ &=\int_S d^2S\nabla\times(\nabla\varphi)\\ &=\oint_{C=\partial S}\partial_\mu\varphi dx^\mu=\Delta\varphi=2\pi n\\ &=2\pi n\int_S dXdY\,\delta(X)\delta(Y),\end{aligned} \tag{C35}$$

where the integer n comes from the multivaluedness of φ.

When $\theta=0$ (i.e., on the positive Z axis),

$$U(x)_{0,\varphi}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \tag{C36}$$

which does not give a nontrivial contribution in Eq. (C33). On the other hand, for $\theta=\pi$ (i.e., on the negative Z axis),

$$U(x)_{\pi,\varphi}=\begin{pmatrix}0 & e^{+i\varphi(x)}\\ -e^{-i\varphi(x)} & 0\end{pmatrix}. \tag{C37}$$

Then, using Eq. (C34),

$$\begin{aligned}U(x)_{\pi,\varphi}[\partial_X,\partial_Y]U(x)_{\pi,\varphi}^\dagger&=-i[\partial_X,\partial_Y]\varphi(x)\sigma_3\\ &=-2\pi i n\,\delta(X)\delta(Y)\sigma_3.\end{aligned} \tag{C38}$$

This proves the statement (C33).

[15]This is derived also from homotopy theory, $\Pi_2(\mathrm{SU}(N)/\mathrm{U}(1)^{N-1})=\Pi_1(\mathrm{U}(1)^{N-1})=Z^{N-1}$. In particular, $\Pi_2(\mathrm{SU}(2)/\mathrm{U}(1))=\Pi_1(\mathrm{U}(1))=Z$; see argument in Ref. [6].

The relation (C33) shows that the term $(i/g)\,(U[\partial_\mu,\partial_\nu]U^\dagger)^{(3)}$ produces a magnetic field only along the negative Z axis,

$$B_Z^{Ds}:=\frac{i}{g}(U[\partial_X,\partial_Y]U^\dagger)^{(3)}=\frac{4\pi n}{g}\,\delta(X)\,\delta(Y)\,\theta(-Z). \tag{C39}$$

So this is identified with the Dirac string (not the magnetic monopole) extending from the origin to infinity along the negative Z axis (due to the above choice of U) in three-dimensional space. Hence the divergence of B_Z^{Ds} is nonzero at the origin,

$$k_0^{Ds}=\nabla\cdot B_Z^{Ds}:=\partial_Z\frac{i}{g}(U[\partial_X,\partial_Y]U^\dagger)^{(3)}=-\frac{4\pi n}{g}\,\delta^3(x), \tag{C40}$$

which should be compared with Eq. (C28).

Finally, we give an alternative definition of the Abelian-projected field strength,

$$f_{\mu\nu}=\mathrm{tr}(T^3 ig[\mathcal{A}_\mu^s,\mathcal{A}_\nu^s])+\mathrm{tr}\left(T^3\frac{i}{g}U[\partial_\mu,\partial_\nu]U^\dagger\right). \tag{C41}$$

This is the Abelian field strength obtained from the singular gauge potential and consists of the magnetic monopole part and the Dirac string part as shown above. In the RHS, the second term $\mathrm{tr}(T^3\,(i/g)U[\partial_\mu,\partial_\nu]U^\dagger)$ expresses a magnetic field on the Dirac string and vanishes elsewhere. Therefore, the remaining part $\mathrm{tr}(T^3 ig[\mathcal{A}_\mu^s,\mathcal{A}_\nu^s])$ denotes the field strength of the magnetic monopole defined everywhere. Hence, the magnetic monopole part of the magnetic current defined by

$$k_\rho:=\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\partial^\sigma f_{\mu\nu} \tag{C42}$$

is equivalent to

$$K_\rho=\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\partial^\sigma(g\epsilon^{ab3}A_\mu^a A_\nu^b). \tag{C43}$$

In the three-dimensional slices, this describes a magnetic monopole with magnetic charge

$$g_m:=\int K_0(x)d^3x=\frac{4\pi}{g}n,\quad \frac{gg_m}{4\pi}=n, \tag{C44}$$

where n is an integer. This is nothing but the Dirac quantization condition.

In the original YM theory, as a result of the Jacobi identity

$$\epsilon_{\mu\nu\rho\sigma}[\mathcal{D}_\nu,[\mathcal{D}_\rho,\mathcal{D}_\sigma]]=0, \tag{C45}$$

the Bianchi identity always holds,

$$0=\epsilon_{\mu\nu\rho\sigma}\mathcal{D}_\nu\mathcal{F}_{\rho\sigma},$$

$$\epsilon_{\mu\nu\rho\sigma}\partial_\nu\mathcal{F}_{\rho\sigma}=ig\,\epsilon_{\mu\nu\rho\sigma}\mathcal{A}_\nu\mathcal{F}_{\rho\sigma}=2ig\mathcal{A}_\nu\tilde{\mathcal{F}}_{\mu\nu}. \tag{C46}$$

After gauge fixing, the Bianchi identity for the residual U(1) is violated,

$$\epsilon_{\mu\nu\rho\sigma}\partial^\rho f'_{\mu\nu}\neq 0, \tag{C47}$$

which leads to the magnetic monopole. In the original YM theory, the magnetic monopole does not exist. However, note that

$$\epsilon_{\mu\nu\rho\sigma}\partial^\rho\mathcal{F}'_{\mu\nu}{}^{(3)}\neq 0,\quad \mathcal{F}'_{\mu\nu}{}^{(3)}:=2\,\mathrm{tr}(T^3\mathcal{F}'_{\mu\nu}), \tag{C48}$$

since

$$\begin{aligned}\epsilon_{\mu\nu\rho\sigma}\partial^\rho\mathcal{F}'_{\mu\nu}{}^{(3)}&=\epsilon_{\mu\nu\rho\sigma}\partial^\rho(\partial_\mu a'_\nu-\partial_\nu a'_\mu)\\&\quad-\epsilon_{\mu\nu\rho\sigma}\partial^\rho ig([\mathcal{A}'_\mu,\mathcal{A}'_\nu])^{(3)}\\&=\epsilon_{\mu\nu\rho\sigma}\partial^\rho\frac{i}{g}(U[\partial_\mu,\partial_\nu]U^\dagger)^{(3)}.\end{aligned} \tag{C49}$$

The RHS is equal to the Dirac string contribution [45,39]

Incidentally, the four-vector

$$K_\rho^{Ds}=\frac{1}{2}\epsilon_{\mu\nu\rho\sigma}\partial^\rho\frac{i}{g}(U[\partial_\mu,\partial_\nu]U^\dagger)^{(3)} \tag{C50}$$

denotes the trajectory

$$K_\mu(x)=\int d\tau\frac{\partial y_\mu(\tau)}{\partial\tau}\,\delta^{(4)}(x-y(\tau,0)), \tag{C51}$$

as the boundary $x_\mu=y_\mu(\tau,0)$ of the Dirac sheet described by $y_\mu(\tau,\sigma)$ (world sheet of the Dirac string, i.e., two-dimensional surface swept by the Dirac string in four-dimensional space),

$$\begin{aligned}\omega_{\mu\nu}(x)&:=\frac{i}{g}(U(x)[\partial_\mu,\partial_\nu]U^\dagger(x))^{(3)}\\&=\int d\tau d\sigma\frac{\partial(y^\mu,y^\nu)}{\partial(\tau,\sigma)}\,\delta^{(4)}(x-y(\tau,\sigma)).\end{aligned} \tag{C52}$$

The Yang-Mills theory is further discussed from the topological point of view in a subsequent paper [52].

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