Variational calculation of the effective action

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An indication of spontaneous symmetry breaking is found in the two-dimensional $\lambda\phi^4$ model, where attention is paid to the functional form of an effective action. An effective energy, which is an effective action for a static field, is obtained as a functional of the classical field from the ground state of the Hamiltonian $H[J]$ interacting with a constant external field. The energy and wave function of the ground state are calculated in terms of DLCQ (discretized light-cone quantization) under antiperiodic boundary conditions. A field configuration that is physically meaningful is found as a solution of the quantum mechanical Euler-Lagrange equation in the $J\to 0$ limit. It is shown that there exists a nonzero field configuration in the broken phase of Z_2 symmetry because of a boundary effect. [S0556-2821(98)04812-7]

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I. INTRODUCTION

In order to explain the properties of hadrons, it is strongly hoped that QCD particle spectra can be calculated with a reasonable approximation. Light-front field theory is one of the candidates to investigate QCD in the infrared region, since a certain nonperturbative approximation (Tamm-Dancoff truncation) becomes effective [1–4]. Since this method is based on a Hamiltonian formalism, one can obtain mass spectra and wave functions of hadronic states that are important for the calculation of nonperturbative physical quantities such as structure functions [5].

It has been said that spontaneous chiral symmetry breaking is responsible for the finite masses of mesons in the chiral limit. We need to know how the chiral condensate $\langle\bar{\psi}\psi\rangle$ behaves in the chiral region to understand QCD mesons. However, one cannot extract information of spontaneous symmetry breaking from the vacuum, since the light-front vacuum is always trivial. So the question is: How can we understand spontaneous symmetry breaking in light-front field theory? It is standard practice to apply the method of the effective potential to such a problem. In order to define the effective potential on the light front, let us consider a Legendre transform (which we call the effective energy) of the ground state energy of the following Hamiltonian [6,7]:

$$H[J]=H-\int d^{n-1}x\, J(\boldsymbol{x})\phi(\boldsymbol{x}), \tag{1.1}$$

where the external field is independent of time and $\boldsymbol{x}$ indicates the spatial coordinate. The advantage of this consideration is that it is possible to obtain the effective energy if we could know only the ground state of the Hamiltonian $H[J]$. If the external field does not depend on the spatial coordinate, the effective potential is given as the effective energy divided by the total spatial volume of the system. Then, it would be natural to define the system in a finite box $-L\leqslant\boldsymbol{x}<L$ and take the thermodynamic limit $L\to\infty$ after all calculations. This is known as DLCQ (discretized light-cone quantization) [8,9]. There are two possibilities for the consistent boundary condition on the field $\phi(\boldsymbol{x})$: periodic and antiperiodic boundary conditions [10]. If we take a periodic boundary condition and assume a uniform external field $J(\boldsymbol{x})=J$, it would be possible in principle to obtain the effective potential from the effective energy. To do that, we have to know the light-front longitudinal zero mode [8,11,10],

$$\phi_0=\frac{1}{2L}\int_{-L}^{L}dx^-\phi(\boldsymbol{x}), \tag{1.2}$$

which appears in the second term of Eq. (1.1). If we impose a periodic boundary condition on the field, a constraint equation for the zero mode emerges. The light-front zero mode is a dependent variable and then should be represented with other oscillator modes. It has been numerically confirmed, with an approximation, that the zero mode gives rise to a nonzero vacuum expectation value and the entire effect of spontaneous symmetry breaking comes from only one mode [12]. It seems that this scenario holds also in the calculation of the light-front effective potential. The zero mode should have a singular dependence on the external field J to produce a correct convex shape for the effective potential in the broken phase. However, it does not work in practice solving the constraint and calculating the vacuum energy including the zero-mode effect, since the constraint equation is highly complicated and it is difficult to find a reasonable technique to solve it accurately. It is worthwhile to discuss the problem without the zero mode.

The vacuum expectation value that minimizes the effective potential is a particular solution of the following quantum mechanical Euler-Lagrange equation:

$$\frac{\delta\Gamma[\varphi]}{\delta\varphi(x)}=J(x),\quad J(x)\to 0, \tag{1.3}$$

where $\Gamma[\varphi]$ is an effective action. Of course, there should exist also a space-dependent solution and it would be possible to see indications of symmetry breaking in it. Our aim is to extract convexity of the effective potential from the Hamiltonian searching for a nonuniform solution of Eq.

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(1.3). This will be done by assuming an antiperiodic boundary condition and using a trick on the external field. Since the translational invariance is broken by the external field, we can deal with the problem by avoiding the zero mode. The translational invariance of the system is restored by taking the $J\to 0$ limit after all the calculations.

In this paper, we look for an indication of spontaneous symmetry breaking in the two-dimensional $\lambda\phi^4$ model by paying attention to a functional form of the effective energy, where the system is defined and solved using DLCQ [8,9], which is essentially a nonperturbative method and useful also as a conceptual tool [13]. The effective energy is obtained as a functional of the classical field (the expectation value of the field operator) and a space-dependent nonuniform solution of the quantum mechanical Euler-Lagrange equation is found.

The essential point of this consideration is the imposition of an external field on the system. In order to describe the broken phase properly, we have to break the symmetry explicitly by imposing an external field on the system [11,10]. The vanishing limit of the external field has to be taken after the thermodynamic limit. It is impossible to figure out the properties of the effective energy if the order of the limits is changed. In Refs. [14,15], it has been shown that the second derivative of the effective potential is always positive and in particular the potential does not exist for small expectation values if spontaneous symmetry breaking occurs. If the symmetry breaks, the effective potential should have a flat bottom and the finite expectation value of the field survives in the $J\to 0$ limit. The effective energy would also have a convex shape as a functional of the classical field $\varphi(\boldsymbol{x})$, since the energy is a more general quantity than the potential and should contain information of the potential.

This paper is organized as follows. In Sec. II, the effective energy is defined in terms of a Hamiltonian that interacts with an external field $J(\boldsymbol{x})$. It is explained how to obtain a physically meaningful field configuration $\varphi(\boldsymbol{x})$. In Sec. III, the DLCQ method is introduced under an antiperiodic condition to solve the eigenvalue problem given by the Hamiltonian. An approximate value of the critical coupling constant is calculated in a nonperturbative manner. Mass spectra for periodic and antiperiodic boundary conditions are compared with each other for reference. In Sec. IV the effective energy is calculated using DLCQ introduced in Sec. III. We will see that there seems to remain a nonzero configuration in the broken phase as a solution of the Euler-Lagrange equation even if the external field is switched off. Section V is devoted to summary and discussions.

II. EFFECTIVE ACTION AND HAMILTONIAN

Let us consider a generating functional $Z[J]$ of the Green's function to define the effective energy in terms of the Hamiltonian [6,7]

$$Z[J]=e^{iW[J]}=\langle 0|T\exp\left[\int d^n x\, J(x)\phi(x)\right]|0\rangle. \quad (2.1)$$

If the external field $J(x)$ is independent of time, the partition function can be written with a Hamiltonian $H[J]$ that interacts with the external field in the following way:

$$Z[J]=e^{-iw[J]T}=\langle 0|e^{-iH[J]T}|0\rangle, \quad (2.2)$$

$$H[J]=H-\int d^{n-1}x\, J(\boldsymbol{x})\phi(\boldsymbol{x}), \quad (2.3)$$

where $\boldsymbol{x}$ means spatial coordinate and $\phi(\boldsymbol{x})$ is a field operator in the Schrödinger picture. The proof of Eq. (2.2) is given in Appendix A. In the relation (2.2), it is understood that the $-i\epsilon$ prescription is taken, that is, the time coordinate is rotated with a replacement $H[J]\to e^{-i\epsilon}H[J]$ ($\epsilon\ll 1$). By substituting the decomposition of unity into Eq. (2.2) and taking the $T\to\infty$ limit, we can see that the ground state $|0_J\rangle$ of the Hamiltonian $H[J]$ dominates in $Z[J]$,

$$H[J]|0_J\rangle=w[J]|0_J\rangle, \quad (2.4)$$

where the state is normalized as $\langle 0_J|0_J\rangle=1$. The connected generating functional $w[J]$ can be regarded as the ground state energy of the Hamiltonian $H[J]$. By multiplying Eq. (2.4) by the ground state $\langle 0_J|$, we have

$$\langle 0_J|H|0_J\rangle=w[J]+\int d^{n-1}x\, J(\boldsymbol{x})\varphi(\boldsymbol{x}), \quad (2.5)$$

where

$$\varphi(\boldsymbol{x})=\langle 0_J|\phi(\boldsymbol{x})|0_J\rangle. \quad (2.6)$$

Since Eq. (2.5) is a Legendre transform of $w[J]$, this quantity is just an effective action divided by the total time T in the case when the field $\varphi(x)$ is static,

$$\Gamma[\varphi(x)=\varphi(\boldsymbol{x})]=-T\mathcal{E}[\varphi(\boldsymbol{x})]. \quad (2.7)$$

We call this quantity Eq. (2.5) the effective energy $\mathcal{E}[\varphi]$:

$$\mathcal{E}[\varphi]\equiv\langle 0_J|H|0_J\rangle. \quad (2.8)$$

An actual expectation value $\varphi(\boldsymbol{x})$ of the field operator $\phi(\boldsymbol{x})$ should be given as a solution of the following generalized Euler-Lagrange equation with vanishing external field:

$$\frac{\delta\mathcal{E}[\varphi]}{\delta\varphi(\boldsymbol{x})}=J(\boldsymbol{x}), \quad J(\boldsymbol{x})\to 0. \quad (2.9)$$

In order to obtain the solution $\varphi(\boldsymbol{x})$, we have to take three steps: (1) solve the eigenvalue problem (2.4), (2) evaluate the energy $\mathcal{E}[\varphi]$, (3) find the stationary point of $\mathcal{E}[\varphi]$. It is difficult to clear the first step if the field is quantized in the ordinary equal-time coordinate, because vacuum fluctuations dominate and a higher Fock state seems to be needed to represent the ground state of the Hamiltonian. In order to solve Eq. (2.4) in a nonperturbative manner with a reasonable approximation, DLCQ (discretized light-cone quantization) will be used in Sec. IV. The effective energy $\mathcal{E}[\varphi]$ will be obtained as a functional of the classical field $\varphi(\boldsymbol{x})$ by diagonalizing the Hamiltonian $H[J]$.

III. CRITICAL COUPLING CONSTANT WITH DLCQ

A. DLCQ in the $\lambda\phi^4_{1+1}$ model

In this section, we will consider DLCQ [8,9] in order to apply the method introduced in the previous section to the

two-dimensional real scalar model [16–19]. An approximate value of the critical coupling constant λ_c will be calculated.

The Lagrangian density of the model is given by

$$\mathcal{L}=\frac{1}{2}(\partial_\mu\phi\partial^\mu\phi-\mu^2\phi^2)-\frac{\lambda}{4!}\phi^4. \tag{3.1}$$

The light-front coordinate $x^\pm=(x^0\pm x^1)/\sqrt{2}$ is defined and x^+ and x^- are regarded as time and space, respectively. The metric is $g^{+-}=g^{-+}=1$ and $g^{++}=g^{--}=0$. The system is put in a finite size box ($-L\leqslant x^-<L$) and the field is quantized with an antiperiodic boundary condition $\phi(L)=-\phi(-L)$ [10],

$$[\phi(x),\phi(y)]_{x^+=y^+}=-\frac{i}{4}\epsilon(x^--y^-). \tag{3.2}$$

The field is expanded with oscillators at $x^+=0$,

$$\phi(x)|_{x^+=0}=\frac{1}{\sqrt{4\pi}}\sum_{n=1}^{\infty}\frac{1}{\sqrt{\tilde{n}}}[a_n e^{-ik_n^+x^-}+a_n^\dagger e^{ik_n^+x^-}], \tag{3.3}$$

where

$$k_n^+=\frac{\pi\tilde{n}}{L},\quad \tilde{n}=n-\frac{1}{2}, \tag{3.4}$$

and

$$[a_m,a_n^\dagger]=\delta_{m,n},\quad [a_m,a_n]=0,\quad [a_m^\dagger,a_n^\dagger]=0. \tag{3.5}$$

The Hamiltonian and momentum operators are

$$H=\int_{-L}^{L}dx^-:T^{+-}:,\quad P=\int_{-L}^{L}dx^-:T^{++}:, \tag{3.6}$$

where

$$T^{\mu\nu}=\partial^\mu\phi\partial^\nu\phi-g^{\mu\nu}\mathcal{L}. \tag{3.7}$$

Divergent tadpole diagrams are removed by normal ordering. The size of the box L can be extracted from H and P

$$H=\frac{L}{2\pi}\mathcal{H},\quad P=\frac{\pi}{L}\mathcal{K}. \tag{3.8}$$

Explicit forms of $\mathcal{H}$ and $\mathcal{K}$ are written in Appendix B. The invariant mass M of a state is

$$M^2=\mathcal{K}H. \tag{3.9}$$

Note that the invariant mass does not depend on L. The harmonic resolution $\mathcal{K}$ has to be taken to infinity after all calculations so as to give a finite fixed momentum P in the thermodynamic limit $L\to\infty$. Then, we can say that M depends on L implicitly.

Since H and P commute with each other, we can diagonalize these operators simultaneously. It is convenient to expand a general state as an eigenstate of $\mathcal{K}$:

$$\mathcal{K}|K\rangle=K|K\rangle, \tag{3.10}$$

where

$$|K\rangle=\lim_{N_{\rm TD}\to\infty}\sum_{N=1}^{N_{\rm TD}}\sum_{n_1,n_2,\ldots,n_N}^{K}\times\delta^{\sum_{i=1}^{N}n_i,K}c_{n_1,n_2,\ldots,n_N}|n_1,n_2,\ldots,n_N\rangle, \tag{3.11}$$

and

$$|n_1,n_2,\ldots,n_N\rangle\equiv\prod_{i=1}^{N}a_{n_i}^\dagger|0\rangle. \tag{3.12}$$

The Fock space is truncated by the number of particles $N_{\rm TD}$ and the harmonic resolution K in actual calculations, since the number of states $\mathcal{N}_s$ goes to infinity in the limits $N_{\rm TD}\to\infty$ and $K\to\infty$ and it is impossible to manipulate infinite dimensional matrices. Invariant mass M and wave function c_i are obtained by diagonalizing the finite dimensional matrix $\mathcal{H}_{ij}$ [20]

$$M^2c_i=K\sum_{j=1}^{\mathcal{N}_s}\mathcal{H}_{ij}c_j, \tag{3.13}$$

where

$$\mathcal{H}_{ij}=\langle i|\mathcal{H}|j\rangle,\quad |K\rangle=\sum_{i=1}^{\mathcal{N}_s}c_i|i\rangle. \tag{3.14}$$

After this, all quantities which have mass dimension will be expressed in units of μ^2 due to the absence of transverse component in this model.

The harmonic resolution K is a total sum of N-particle momenta, each of which carries a half-integer piece, $\tilde{n}_i=n_i-1/2$,

$$K=\sum_{i=1}^{N}\tilde{n}_i, \tag{3.15}$$

then we have

$$\sum_{i=1}^{N}n_i=K+\frac{N}{2}. \tag{3.16}$$

Since the left-hand side of Eq. (3.16) is always integer, the number of particles N should be odd or even according to whether K is half-integer or integer, respectively. Then, odd and even sectors decouple from each other. The resolution K is set to be half-integer, because our purpose in this section is to obtain a mass spectrum of the lightest particle state, which belongs to the odd sector and can be seen as a one-bosonic state.

B. Critical coupling constant

The Tamm-Dancoff dependence of the lightest mass is shown in Fig. 1. The mass squared M^2 of the lightest state is plotted as a function of harmonic resolution K in a case $\lambda=25\mu^2$, which is comparatively large and near to the critical

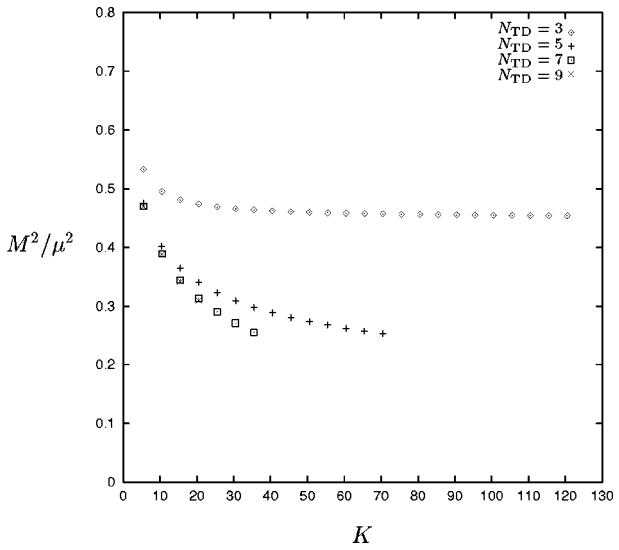

FIG. 1. Mass squared M^2 of the lightest state is plotted as a function of the harmonic resolution K for various Tamm-Dancoff truncations $N_{TD}=3,5,7,9$ under antiperiodic boundary conditions. The coupling constant is taken as $\lambda=25\mu^2$, which is relatively large and near the critical point $\lambda_c\sim 30\mu^2$. The definite value of the critical coupling constant λ_c will be calculated later. The spectrum almost converges at $N_{TD}=7$, which can be confirmed only in the small K region. Mass spectrum calculations will be executed in Fock space truncated with $K=141/2$ and $N_{TD}=5$.

coupling constant λ_c. This state can be seen as a one-bosonic state, since one body component of the wave function is dominant. A definite value of λ_c will be calculated later. Diamonds, pluses, squares, and crosses correspond to $N_{TD}=3,5,7,9$, respectively, where N_{TD} is defined in Eq. (3.11). Since the coupling constant is large, convergence with respect to K and N_{TD} is very slow. The spectrum seems almost to converge at $N_{TD}=7$ in the small K region. The harmonic resolution K cannot be taken to be large values when N_{TD} is large because of the upper bounds of computational resources, especially a shortage of memory size. We will continue our calculations with $N_{TD}=5$ and obtain the critical coupling constant λ_c by extrapolating the results, because the purpose of this section is to prepare a nonperturbative technique for the effective energy calculation and to find an approximate value of the critical coupling constant λ_c that is needed to draw a phase diagram. The effective energy for both phases will be evaluated in the next section using DLCQ.

The K dependence of the one-bosonic masses is shown in Table I for various coupling constants, where the number of particles is truncated with $N_{TD}=5$. Since the mass spectra for large coupling do not converge to the extent of this calculation, let us estimate where the spectra settle in the $K\to\infty$ limit by expanding the mass squared with $1/K$ up to the second order,

$$M^2(\lambda)=m_0+\frac{m_1}{K}+\frac{m_2}{K^2}. \tag{3.17}$$

The coefficients $m_i(\lambda)$ are obtained with least squares fitting using the Marquardt-Levenberg algorithm. The result is shown in Table II. m_0 is regarded as an extrapolated value in the $K\to\infty$ limit. Table III is the same as Table I but for the periodic case, where the zero mode has been removed from the Hamiltonian.

TABLE I. The K dependence of the lightest one-bosonic mass M^2 is shown for various coupling constants, $\lambda/\mu^2=5,10,15,20,25$. The mass spectra are calculated in Fock space truncated with $N_{TD}=5$ under antiperiodic boundary conditions.

	λ/μ^2				
$2K$	5	10	15	20	25
11	0.95800	0.86614	0.74880	0.61655	0.47481
21	0.95471	0.85293	0.71987	0.56736	0.40190
31	0.95341	0.84712	0.70629	0.54317	0.36476
41	0.95268	0.84368	0.69787	0.52769	0.34045
51	0.95221	0.84134	0.69196	0.51657	0.32267
61	0.95188	0.83962	0.68750	0.50804	0.30883
71	0.95163	0.83829	0.68398	0.50120	0.29763
81	0.95144	0.83722	0.68110	0.49555	0.28829
91	0.95129	0.83634	0.67870	0.49079	0.28034
101	0.95116	0.83560	0.67666	0.48669	0.27346
111	0.95105	0.83497	0.67489	0.48312	0.26744
121	0.95096	0.83442	0.67334	0.47997	0.26209
131	0.95088	0.83394	0.67196	0.47716	0.25731
141	0.95081	0.83351	0.67074	0.47465	0.25301

In Fig. 2, the lightest masses M^2 of two cases (antiperiodic and periodic boundary conditions) are compared with each other. Extrapolated values m_0 of the lightest mass squared are plotted as functions of the coupling constant. In the periodic case, the calculation has been executed tentatively excluding the zero mode from the Hamiltonian. By fitting the points with curves and extrapolating the curves in Fig. 2, the approximate value of the critical coupling constant λ_c is found as

$$\lambda_c=\begin{cases}28.6329\mu^2 & \text{(antiperiodic)},\\ 30.8431\mu^2 & \text{(periodic)}.\end{cases} \tag{3.18}$$

λ_c is defined as the point that gives a massless eigenvalue $M^2(\lambda_c)=0$. These values are nothing but upper bounds of the true value, because DLCQ is a variational method and the size of the variational space cannot be taken to infinity. The results Eq. (3.18) are consistent with the values $22\mu^2$

TABLE II. The coefficients m_i $(i=0,1,2)$ in the $1/K$ expansion (3.17) are obtained from a least squares fitting with the Marquardt-Levenberg algorithm by using a result shown in Table I. m_0 is regarded as the extrapolated value of the lightest state in the thermodynamic limit $K\to\infty$. This is the result under antiperiodic boundary conditions.

λ/μ^2	m_0	m_1	m_2
5	0.95007 ± 0.00002	0.05599 ± 0.00082	-0.06878 ± 0.00432
10	0.82974 ± 0.00022	0.30798 ± 0.00824	-0.59729 ± 0.04307
15	0.66113 ± 0.00077	0.82242 ± 0.02820	-1.88593 ± 0.14742
20	0.45652 ± 0.00175	1.60482 ± 0.06351	-4.01797 ± 0.33191
25	0.22399 ± 0.00317	2.63785 ± 0.11523	-6.97948 ± 0.60219

TABLE III. The K dependence of the lightest one-bosonic mass M^2 is shown for various coupling constants, $\lambda/\mu^2=5,10,15,20,25$. The mass spectra are calculated in Fock space truncated with $N_{\mathrm{TD}}=5$ under periodic boundary conditions, where the zero mode has been removed from the Hamiltonian.

	λ/μ^2				
K	5	10	15	20	25
10	0.97090	0.90555	0.82058	0.72361	0.61871
15	0.96607	0.88968	0.78996	0.67574	0.55186
20	0.96326	0.88025	0.77142	0.64636	0.51036
25	0.96139	0.87385	0.75862	0.62581	0.48105
30	0.96004	0.86915	0.74909	0.61031	0.45873
35	0.95902	0.86551	0.74160	0.59802	0.44089
40	0.95821	0.86259	0.73552	0.58794	0.42613
45	0.95755	0.86018	0.73045	0.57945	0.41361
50	0.95700	0.85814	0.72612	0.57215	0.40279
55	0.95653	0.85640	0.72237	0.56578	0.39329
60	0.95613	0.85488	0.71908	0.56015	0.38485
65	0.95578	0.85355	0.71616	0.55513	0.37727
70	0.95548	0.85236	0.71355	0.55060	0.37041

$<\lambda_c<55\mu^2$ obtained in the conventional equal-time theory [16], but much smaller than $\lambda_c=4\pi(3+\sqrt{3})\mu^2\sim 59.5\mu^2$ [12], $\lambda_c=43.9\mu^2$ [17], and $\lambda_c=40\mu^2$ [19]. A convergence of the spectrum in the antiperiodic case is slightly faster than the periodic one. The spectra are similar but we cannot conclude clearly whether the two results coincide or not, since the spectra are extrapolated to the large K region in this calculation. It is interesting to see how the zero-mode effect dominates in the mass spectrum calculation to confirm the equivalence between both boundary conditions. It seems that a certain renormalization technique needs to be found to get a convergent result with small K [21].

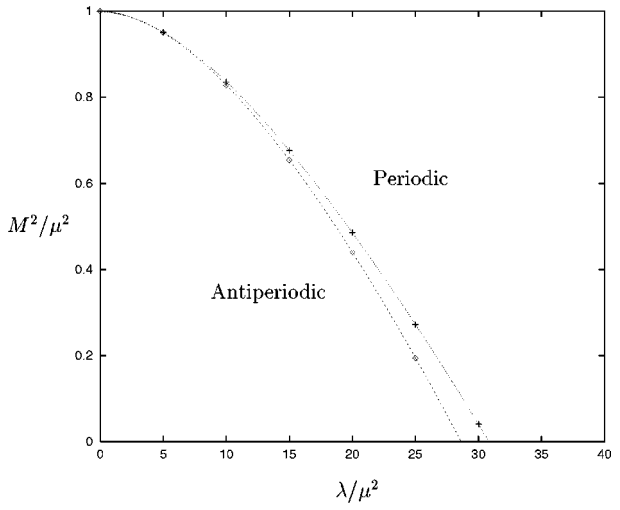

FIG. 2. The extrapolated value m_0 of the lightest state is plotted as a function of the coupling constant λ for antiperiodic and periodic boundary conditions, each of which is represented with diamonds and pluses, respectively. The lines are intended to guide the eye and used to calculated the critical coupling constant λ_c. In the periodic case, the zero mode has been omitted and the extrapolated values of the lightest state are obtained from calculations in Fock space truncated with $K=70$ and $N_{\mathrm{TD}}=5$ in the same manner as the antiperiodic case. The critical coupling constants are calculated as $\lambda_c=28.6329\mu^2$ (antiperiodic) and $\lambda_c=30.8431\mu^2$ (periodic) searching for the massless point $M^2(\lambda_c)=0$.

IV. EFFECTIVE ENERGY

A. Kink solution

In this section, we will calculate the effective energy of the two-dimensional real scalar model with DLCQ and obtain an expectation value $\varphi(x^-)$ as a solution of Eq. (2.9). It is possible to understand all the static physics of the system once a solution of the Euler-Lagrange equation (2.9) is obtained. The configuration $\varphi(x^-)$ must contain information of spontaneous symmetry breaking independent of whether $\varphi(x^-)$ is uniform or not. If we impose a periodic boundary condition on the field $\phi(L)=\phi(-L)$ and assume a uniform external field $J(x^-)=J$, the effective potential $\mathcal{V}(\varphi)$ is obtained as the effective energy divided by the total spatial volume $2L$,

$$\mathcal{V}(\varphi)\equiv\frac{1}{2L}\mathcal{E}[\varphi(x^-)=\varphi]$$

$$=\frac{1}{2L}w(j)+J\langle 0_J|\phi_0|0_J\rangle, \tag{4.1}$$

where ϕ_0 is a zero-mode part of the field operator $\phi(x^-)$

$$\phi_0\equiv\frac{1}{2L}\int_{-L}^{L}dx^-\phi(x^-). \tag{4.2}$$

A vacuum expectation value of the field operator $\varphi=\langle 0_J|\phi(x^-)|0_J\rangle$ is given as a solution of the following equation:

$$\frac{d\mathcal{V}(\varphi)}{d\varphi}=J,\quad J\to 0, \tag{4.3}$$

where φ is independent of space because of translational invariance of the system. In order to evaluate the potential, we have to consider a constrained zero mode ϕ_0, which is given as a solution of the following constraint equation [8],

$$\int_{-L}^{L}dx^-\left(\mu^2\phi+\frac{\lambda}{6}\phi^3-J\right)=0. \tag{4.4}$$

It is expected that the zero mode ϕ_0 has a singular behavior with respect to the external field J in the broken phase due to the convexity of the effective potential [14,15]. However, it is difficult to solve Eq. (4.4) and represent the zero mode as a superposition of other modes, since it is an operator valued nonlinear equation. It would be better if we could understand a mechanism of spontaneous symmetry breaking without such a complicated zero-mode problem.

Let us look for a nonuniform solution to avoid the zero-mode problem. If we impose an antiperiodic boundary condition on the field and assume a constant external field $J(x^-)=J$, a nonuniform solution $\varphi(x^-)$ that has a kink will be obtained because the external field has a discontinuity at the boundary $x^-=\pm L$. There cannot exist a translationally

invariant solution, since the system has been connected at the boundary with a twist. The purpose of this section is to see whether there exists a nonzero solution $\varphi(x^-)$ of Eq. (2.9) in the limit $J\to 0$ after all of the calculations.

Since the system interacting with the constant external field is not translationally invariant under antiperiodic boundary condition, we cannot diagonalize the Hamiltonian and momentum operators at the same time $[H[J],P]\neq 0$. A general state of Eq. (2.4) should be expanded as a superposition of various momentum states

$$|\Psi\rangle = c_0|0\rangle + \lim_{K_{\rm cut}\to\infty}\sum_{n=1}^{2K_{\rm cut}}|K=n/2\rangle, \tag{4.5}$$

where the resolution takes both half-integer and integer $K=1/2,1,3/2,2,\ldots,K_{\rm cut}$ and $K_{\rm cut}$ is set to be some finite value that leads to convergence in the spectra. The odd and even sectors interact with each other due to the existence of a translationally noninvariant interaction, which can be observed in Eq. (B7).

It is possible to confirm that the state (4.5) is expressed with a complete set of momenta by taking the continuum limit $L\to\infty$. Then

$$\hat{P}|\Psi\rangle = \sum_{K=0}^{\infty}\left(\frac{\pi}{L}K\right)|K\rangle, \tag{4.6}$$

becomes

$$\hat{P}|\Psi\rangle = \int_0^\infty dP\, P|P\rangle, \tag{4.7}$$

where

$$|P\rangle \equiv \sum_{N=0}^{\infty}\int\left[\prod_{i=1}^{N}dp_i\right]\delta\left(\sum_{i=1}^{N}p_i - P\right)\psi_N(p_1,p_2,\ldots,p_N)\times\left[\prod_{i=1}^{N}a^\dagger(p_i)\right]|0\rangle, \tag{4.8}$$

and

$$\psi_N(p_1,p_2,\ldots,p_N)\equiv\frac{L^{N/2}}{2^{(N+2)/2}\pi^{N+1}}c_{n_1,n_2,\ldots,n_N}. \tag{4.9}$$

In order to calculate the effective energy, an eigenvalue problem

$$H[J]|\Psi\rangle = w[J]|\Psi\rangle, \tag{4.10}$$

is solved by numerical diagonalization of the Hamiltonian $\mathcal{H}[J]=(2\pi/L)H[J]$. By substituting the energy $w[J]$ and wave function $|0_J\rangle$ of the ground state into the relation

$$\langle 0_J|\mathcal{H}[J]|0_J\rangle = \frac{2\pi}{L}w[J], \tag{4.11}$$

the effective energy is given as

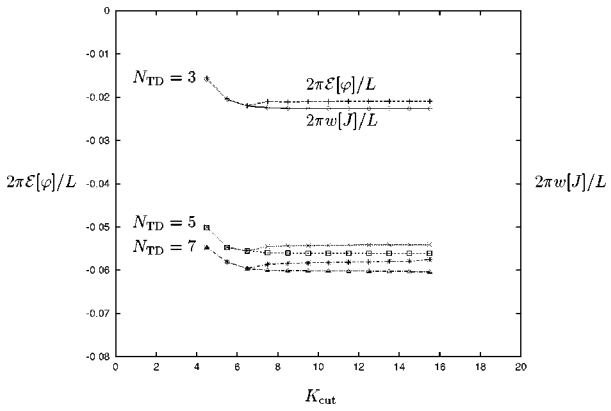

FIG. 3. The effective energy $\mathcal{E}[\varphi]$ and the generating functional $w[J]$ are plotted as functions of $K_{\rm cut}$ for various Tamm-Dancoff truncations $N_{\rm TD}=3,5,7$ in the broken phase, where $\lambda=50\mu^2$ and $J=0.01$ are taken. The effective energy $\mathcal{E}[\varphi]$ always takes a larger value than $w[J]$. A truncation with $K_{\rm cut}=31/2$ and $N_{\rm TD}=7$ seems to give a spectrum that is near the convergent point. This parameter set will be used in the subsequent calculations.

$$\frac{2\pi}{L}\mathcal{E}[\varphi] = \frac{2\pi}{L}w[J] + \sum_{n=1}^{\infty}\langle 0_J|f_n(-L)a_n^\dagger + f_n^*(-L)a_n|0_J\rangle, \tag{4.12}$$

where coefficients f_n are defined in Eq. (B8). The spatial integration has been performed before evaluating contractions of the operators in the second term of the right-hand side. We can obtain the left-hand side as a functional of the classical field $\varphi(x^-)=\langle 0_J|\phi(x^-)|0_J\rangle$, which can be also calculated by using the wave function of the ground state $|0_J\rangle$.

B. Numerical result

In Fig. 3, the generating functional $w[J]$ and the effective energy $\mathcal{E}[\varphi]$ are plotted for various $N_{\rm TD}$'s as functions of the harmonic resolution $K_{\rm cut}$ to check the convergence with respect to $K_{\rm cut}$, where $\lambda=50\mu^2$ and $J=0.01$ are taken. The effective energy has been obtained by evaluating the right-hand side of Eq. (4.12). That is, the Legendre transform is numerically performed in terms of the eigenvalue $w[J]$ and the wave function $|0_J\rangle$ of the ground state of the Hamiltonian $H[J]$. After this, we will take a parameter set $K_{\rm cut}=31/2$ and $N_{\rm TD}=7$ because this parameter set seems to give almost convergent spectra. The convergence of the spectra is slightly faster than mass spectrum calculation, because an expansion of a state $|\Psi\rangle$ starts from the zero-body state $|0\rangle$.

In Fig. 4, the classical field $\varphi(x^-)$ is plotted as a function of the spatial coordinate both in (a) symmetric ($\lambda=0.1\mu^2<\lambda_c$) and (b) broken ($\lambda=50\mu^2>\lambda_c$) phases, where the external field is changed at a regular interval of $\Delta J=0.05$. We can see that the field configuration $\varphi(x^-)$ is nearly uniform except at the boundary and has a twist at $x^-=\pm L$ because of the antiperiodicity of the field operator $\phi(x^-)$.

In order to see how the magnitude of the classical field behaves with changing J, the J dependence of the classical field $\varphi(x^-)$ is shown in Fig. 5. The maximum value $\varphi_{\rm max}\equiv\max\{\varphi(x^-)\}$ of the classical field is plotted as a function of J both in symmetric ($\lambda=0.1\mu^2$) and broken ($\lambda=50\mu^2$)

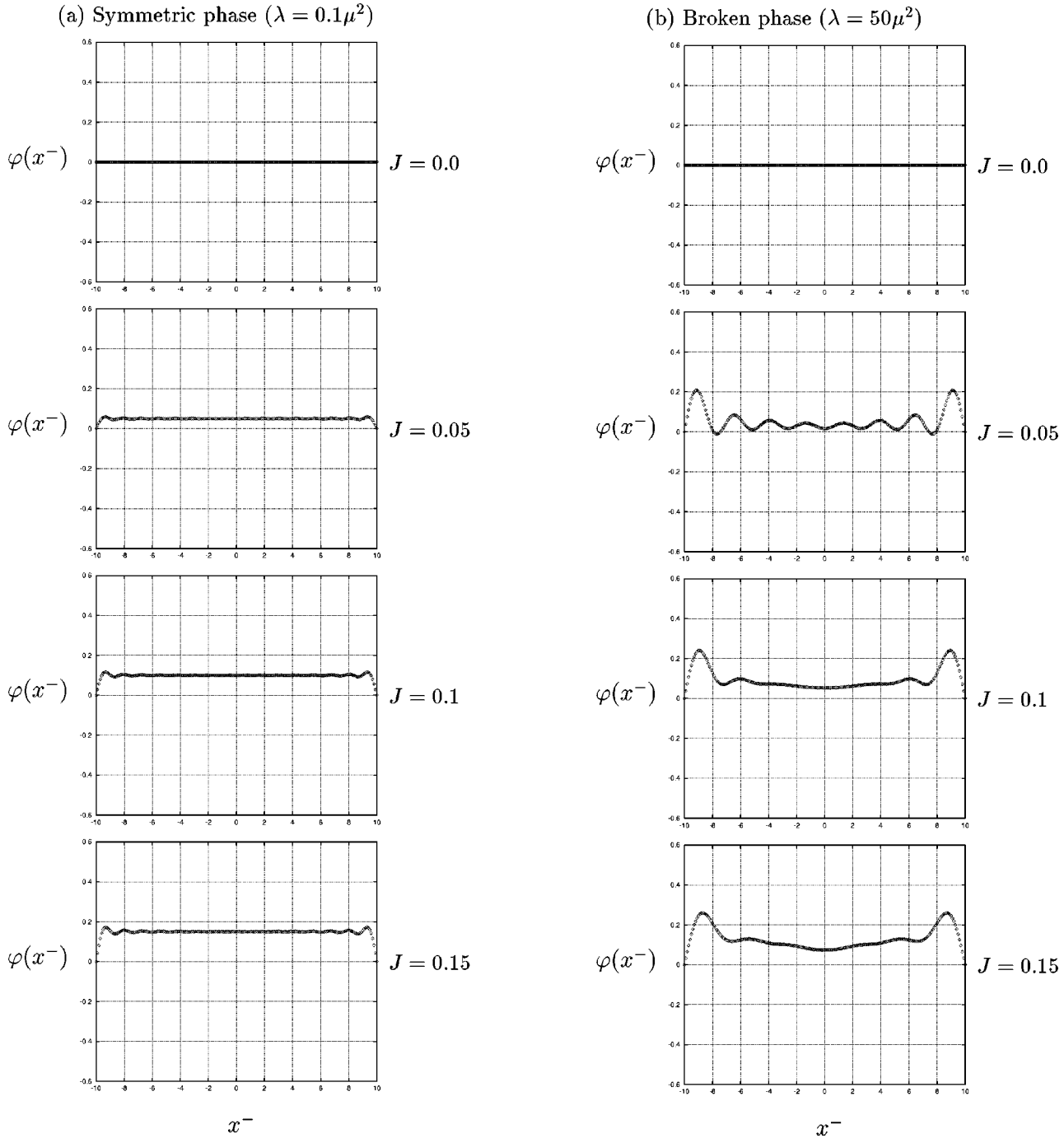

FIG. 4. The classical field $\varphi(x^-)=\langle 0_J|\phi(x^-)|0_J\rangle$ is plotted as a function of x^- both in (a) symmetric ($\lambda=0.1\mu^2$) and (b) broken ($\lambda=50\mu^2$) phases for $J=0,0.05,0.1$, and 0.15, where Fock space is truncated with $K=31/2$ and $N_{\mathrm{TD}}=7$. The vertical and horizontal axes stand for $\varphi(x^-)$ and x^-, respectively. We can observe that the classical field $\varphi(x^-)$ has a twist at the boundary $x^-=\pm L$ due to the antiperiodicity of the field operator $\phi(x)$.

phases, which are represented with diamonds and pluses, respectively. J is increased at regular intervals. There is a one-to-one correspondence between the field configuration $\varphi(x^-)$ and the external field J. In the symmetric phase, the expectation value vanishes in the $J\to 0$ limit. In the broken phase, the curve tends to be closer to the $\varphi_{\max}$ axis with increasing harmonic resolution K_{cut}. The classical field $\varphi(x^-)$ approaches the origin quickly as decreasing J. This fact suggests the presence of a nonzero field configuration $\varphi(x^-)$ of the $\lambda\phi^4_{1+1}$ model in the thermodynamic limit $L\to\infty$ ($K_{\mathrm{cut}}\to\infty$).

In Fig. 6, the effective energy $\mathcal{E}[\varphi]$ is plotted as a function of $\varphi_{\max}$ instead of as a functional of $\varphi(x^-)$. In the symmetric phase ($\lambda=0.1\mu^2$), we can see that the energy $\mathcal{E}[\varphi]$ has a minimum at the origin, where the state is composed only of a zero-momentum state $|0\rangle$ (Fock vacuum). The ground state of the Hamiltonian $H[J]$ goes to $|0\rangle$ and gives zero energy $\mathcal{E}[\varphi]=0$ in the $J\to 0$ limit. In the broken phase ($\lambda=50\mu^2$), however, a situation is completely different from the symmetric one. The effective energy has a flat bottom. A state on the flat region has a wave function where finite K components are dominant because of a twist at the

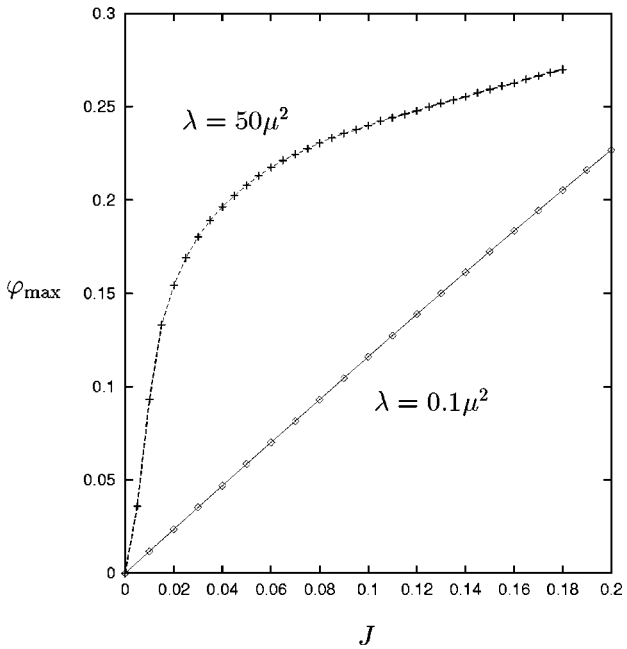

FIG. 5. The maximum value $\varphi_{\max} \equiv \max\{\varphi(x^-)\}$ of the nonuniform classical field $\varphi(x^-)$ is plotted as a function of J, where Fock space is truncated with $K_{\rm cut}=31/2$ and $N_{\rm TD}=7$. Diamonds and pluses correspond to symmetric ($\lambda=0.1\mu^2$) and broken ($\lambda=50\mu^2$) phases, respectively. The external field J is changed with an interval $\Delta J=0.005$. In the broken phase, the magnitude of the classical field $\varphi(x^-)$ rapidly approaches the origin as J decreases. This suggests the presence of a nonzero field configuration $\varphi(x^-)$ in the thermodynamic limit $L\to\infty$ ($K_{\rm cut}\to\infty$).

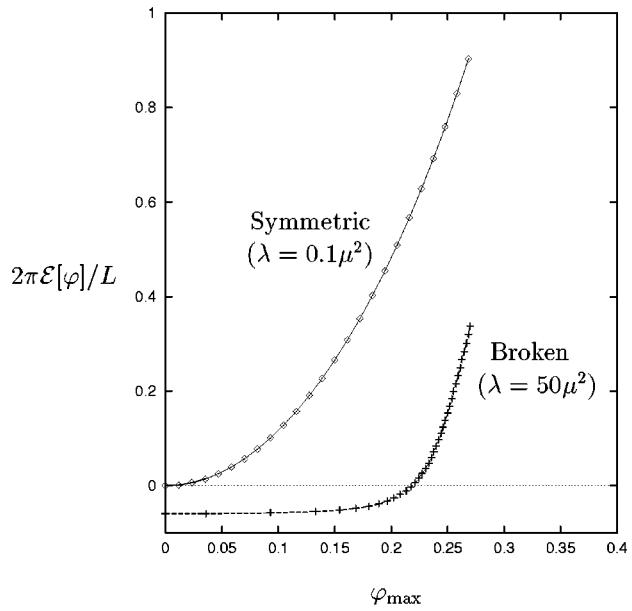

FIG. 6. The effective energy $2\pi\mathcal{E}[\varphi]/L$ is plotted as a function of $\varphi_{\max}$ instead of as a functional of $\varphi(x^-)$, where Fock space is truncated with $K_{\rm cut}=31/2$ and $N_{\rm TD}=7$. Diamonds and pluses correspond to symmetric ($\lambda=0.1\mu^2$) and broken ($\lambda=50\mu^2$) phases, respectively. In the symmetric phase, a physically meaningful configuration is at the origin, where the state is composed only of the Fock vacuum $|0\rangle$. In the broken phase, there seems to exist a nonzero field configuration as a solution of the extended Euler-Lagrange equation in the $J\to 0$ limit, since the bottom of the effective energy is flat. This demonstrates the existence of an infinite number of configurations $\varphi(x^-)$ which are energetically equivalent.

boundary. The classical field which is placed in the edge of the bottom jumps to zero in the $J\to 0$ limit. The classical field shows a singular behavior if the symmetry breaks spontaneously. This fact supports the existence of an infinite number of configurations $\varphi(x^-)$, which are energetically equivalent, and a nonzero field configuration as a kink solution of the Euler-Lagrange equation (2.9) in the broken phase.

V. SUMMARY AND DISCUSSIONS

We have found an approximate value of the critical coupling constant and obtained the effective energy by using DLCQ. In the symmetric phase, the effective energy has a minimum at the origin, which is composed only of the trivial Fock vacuum. In the broken phase, Z_2 symmetry spontaneously breaks, which has been confirmed by seeing that the bottom of the effective energy is flat. In the vanishing J limit, a nonzero expectation value of the field seems to remain. A field configuration, which has a twist, can be a solution of the quantum mechanically extended Euler-Lagrange equation.

In Sec. III, mass spectrum calculation of particle states has been done with DLCQ. In the critical region, convergence of the spectra is very slow because the coupling constant is large there. This is due to an insufficiency of the harmonic resolution to represent small k^+ components of wave functions. By plotting the three-body wave function, we can see that the wave function increases rapidly at $k^+ \sim 0$. A small finite K value cannot represent such a sharp increase because the resolution is not sufficient. It would be hopeless to extend the DLCQ method by brute force to realistic models such as QCD, which has higher dimensions, since we could not get a convergent result near the critical region even in this two-dimensional model. We need to renormalize the degrees of freedom of the harmonic resolution K [21]. It would be better if we could also construct an effective Hamiltonian by renormalizing higher Fock space truncated by the number of particles $N_{\rm TD}$ [22].

In the latter part of this paper, we have discussed spontaneous symmetry breaking by searching for a state that minimizes the effective energy. We have succeeded in finding an indication of spontaneous symmetry breaking, which is just contained in the Hamiltonian with antiperiodic boundary condition. This suggests that the Hamiltonian knows the existence of symmetry breaking in spite of an absence of a zero mode. We have considered how to extract information about symmetry breaking from the effective energy even though it is natural to use an effective potential for such investigations. It is easier to evaluate the effective energy than the effective potential because there is no vacuum fluctuation and the truncation of Fock space with respect to particle numbers works well in the light-front field theory.

If the order parameter one would like to consider is a vacuum expectation value of a composite field such as $\langle 0|\bar{\psi}(x)\psi(x)|0\rangle$, it is possible to trace spontaneous symmetry breaking by using a Hamiltonian that has an interaction between the composite operator and an external field. There are a couple of possibilities to figure out symmetry breaking

for the composite operator. One way is to define the effective potential with the zero mode of the composite operator $\bar{\psi}\psi$. Another way is to find a nonuniform (kink) solution of a classical field $\langle 0_J|\bar{\psi}\psi|0_J\rangle$ through the effective energy as discussed in this paper. To do that, we use a trick on $J(x)$, because the operator $\bar{\psi}\psi$ is always periodic even if any kinds of boundary conditions are imposed on the fermionic field ψ. Since the expectation value of the periodic operator can only have an even number of kinks, we have to assume an even number of kinks also on the external field.

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APPENDIX A: PARTITION FUNCTION AND HAMILTONIAN

Let us prove that the following relation holds for the partition function $Z[J]$ when the external field is static $J(x)=J(\boldsymbol{x})$ [6],

$$Z[J]\equiv\langle 0|Te^{i\int d^n x J(x)\phi(x)}|0\rangle=\langle 0|e^{-iH[J]T}|0\rangle, \quad \text{(A1)}$$

where

$$H[J]=H+H_J, \quad H_J\equiv-\int d^{n-1}x\, J(\boldsymbol{x})\boldsymbol{\phi}(\boldsymbol{x}), \quad \text{(A2)}$$

and $\boldsymbol{x}$ indicates $(n-1)$-dimensional spatial coordinates and $\boldsymbol{\phi}$ is a field operator represented in the Schrödinger picture. In order to prove the formula (A1), the interaction picture will be defined regarding H_J as a perturbed part.

The Schrödinger equation for a general state vector is

$$i\frac{\partial}{\partial t}|\Psi_{\mathrm{S}}(t)\rangle=H[J]|\Psi_{\mathrm{S}}(t)\rangle. \quad \text{(A3)}$$

and a formal solution is readily obtained by writing

$$|\Psi_{\mathrm{S}}(t)\rangle=e^{-iH[J](t-t_0)}|\Psi_{\mathrm{S}}(t_0)\rangle. \quad \text{(A4)}$$

Define the interaction state vector in the following way:

$$|\Psi_{\mathrm{I}}(t)\rangle=e^{iHt}|\Psi_{\mathrm{S}}(t)\rangle. \quad \text{(A5)}$$

The equation of motion of this state is easily found by carrying out the time derivative

$$i\frac{\partial}{\partial t}|\Psi_{\mathrm{I}}(t)\rangle=H_{\mathrm{I}}(t)|\Psi_{\mathrm{I}}(t)\rangle, \quad \text{(A6)}$$

$$H_{\mathrm{I}}(t)\equiv e^{iHt}H_J e^{-iHt}=-\int d^{n-1}x\, J(\boldsymbol{x})\phi(t,\boldsymbol{x}), \quad \text{(A7)}$$

where $\phi(t,\boldsymbol{x})$ is the field operator in the interaction picture. The solution of Eq. (A6) is

$$|\Psi_{\mathrm{I}}(t)\rangle=T\exp\left(i\int_{t_0}^{t}ds\, d^{n-1}x\, J(\boldsymbol{x})\phi(s,\boldsymbol{x})\right)|\Psi_{\mathrm{I}}(t_0)\rangle, \quad \text{(A8)}$$

where the time-ordered product is used due to the time dependence of the Hamiltonian $H_{\mathrm{I}}(t)$. By substituting Eqs. (A4) and (A8) into Eq. (A5), we have

$$e^{-iH[J](t_1-t_0)}=e^{-iHt_1}T\exp\left(i\int_{t_0}^{t_1}dt\, d^{n-1}x\, J(\boldsymbol{x})\phi(t,\boldsymbol{x})\right), \quad \text{(A9)}$$

where $|\Psi_{\mathrm{I}}(t_0)\rangle=|\Psi_{\mathrm{S}}(t_0)\rangle$ is used. By sandwiching Eq. (A9) with the ground state $|0\rangle$ of the Hamiltonian H, which satisfies $H|0\rangle=0$, and setting $t_1=-t_0=T/2$ (T is assumed to be large), we have

$$\langle 0|e^{-iH[J]T}|0\rangle = \langle 0|T\exp\left(i\int_{-T/2}^{T/2}dt\, d^{n-1}x\, J(\boldsymbol{x})\phi(t,\boldsymbol{x})\right)|0\rangle. \quad \text{(A10)}$$

APPENDIX B: HAMILTONIAN

The unperturbed $\mathcal{H}$ and the perturbed $\mathcal{H}_J$ parts of the Hamiltonian (2.3)

$$\mathcal{H}[J]=\frac{2\pi}{L}H[J]=\mathcal{H}+\mathcal{H}_J, \quad \text{(B1)}$$

are expressed using creation and annihilation operators in the two-dimensional $\lambda\phi^4$ model. The unperturbed part $\mathcal{H}$ is

$$\mathcal{H}=\sum_{n=1}^{\infty}\frac{\mu^2}{\tilde{n}}a_n^\dagger a_n + \frac{1}{4}\frac{\lambda}{4\pi}\sum_{n_1,n_2,n_3,n_4=1}^{\infty}\frac{\delta_{\tilde{n}_1+\tilde{n}_2,\tilde{n}_3+\tilde{n}_4}}{\sqrt{\tilde{n}_1\tilde{n}_2\tilde{n}_3\tilde{n}_4}}a_{n_1}^\dagger a_{n_2}^\dagger a_{n_3}a_{n_4} \quad \text{(B2)}$$

$$+\frac{1}{6}\frac{\lambda}{4\pi}\sum_{n_1,n_2,n_3,n_4=1}^{\infty}\frac{\delta_{\tilde{n}_1+\tilde{n}_2+\tilde{n}_3,\tilde{n}_4}}{\sqrt{\tilde{n}_1\tilde{n}_2\tilde{n}_3\tilde{n}_4}} \times[a_{n_1}^\dagger a_{n_2}^\dagger a_{n_3}^\dagger a_{n_4}+\text{H.c.}]. \quad \text{(B3)}$$

and the harmonic resolution is

$$\mathcal{K}=\sum_{n=1}^{\infty}\tilde{n}a_n^\dagger a_n, \quad \text{(B4)}$$

where

$$\tilde{n}=n-\frac{1}{2}. \quad \text{(B5)}$$

Even if the external field is assumed to be constant $J(x^-)=J$, $J(x^-)$ is not continuous at the boundary $x^-=\pm L$ because of the antiperiodicity of the field operator $\phi(x)$.

This discontinuity is reflected as a twist in the expectation value of the field operator $\phi(x)$. It is possible to shift the position of the kink by taking the following external field:

$$J(x^-)=\begin{cases} -J(-L\leqslant x^-<a), \\ J(a<x^-<L), \end{cases} \tag{B6}$$

where a indicates the position of the kink, $-L\leqslant a<L$. The perturbed part of $\mathcal{H}[J]$ is given by

$$\mathcal{H}_J=-\sum_{n=1}^{\infty}\left[f_n(a)a_n^\dagger+f_n^*(a)a_n\right] \tag{B7}$$

where

$$f_n(a)\equiv\frac{2J}{\sqrt{\pi}\tilde{n}^{3/2}}\left[-\sin\left(\frac{\pi a}{L}\tilde{n}\right)+i\cos\left(\frac{\pi a}{L}\tilde{n}\right)\right]. \tag{B8}$$

Of course, physics should be independent of the position a. It has been numerically confirmed that the ground state eigenvalue of $H[J]$ is independent of a.

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