BRST–anti-BRST symmetry and observables for topological gravity

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We consider topological four-dimensional (4D) gravity with an independent spin connection by using a superspace formalism. This gives rise to the basic fields of the quantized theory as well as to two pairs of extra fields, which are needed to close the BRST–anti-BRST algebra off shell. Therefore we build a gauge-fixing action written in BRST–anti-BRST exact form leading to an effective one, which allows us to fix all of the symmetries at once. In particular, the topological symmetries are fixed as in the model of topological 4D self-dual gravity. We construct the observables related to both BRST symmetry and anti-BRST symmetry. We find that the anti-BRST invariant observables are not fundamentally different from the BRST invariant ones, since there is a complete mirror symmetry between them. The obtained observables extend those constructed within the equivariant method in the context of topological 4D self-dual gravity. [S0556-2821(98)00712-7]

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Topological field theories provide quantum field theoretic models which incorporate a way of constructing topological invariants, for a review see Ref. [1]. An important example is given by Witten's topological Yang-Mills theory (TYMT) [2–5]. This suggests that one can also describe the global structure of 4D manifolds by constructing topological 4D gravity as a gravitational counterpart of TYMT. In this context, various models were proposed. For example, topological 4D conformal gravity was first discussed in Ref. [6]. This was developed further in Ref. [7], where the authors start with the topological action constructed from the Weyl tensor. Other examples of topological 4D gravity were considered in Refs. [8–10]. There, the theory is constructed from topological combinations of the curvature tensor. In particular, in Ref. [10] the topological symmetries are fixed by anti–self-duality on both the curvature and the torsion; and the observables are derived studying the equivariant cohomology (see also Ref. [9]) as in the case of TYMT [11]. These observables are constructed from operators, which are only 4D forms, contrary to what happens in TYMT where the operators are forms of any degree. Let us note that it is in Ref. [12] where one has shown how to build observables from operators, which are forms of any degree, in the context of the metric approach to topological gravity.

The purpose of this paper is to present topological 4D gravity with an independent spin connection in terms of a superspace formalism. The basic fields of the theory are introduced through a superconnection and its associated supercurvature. The use of the superspace formalism naturally yields the off-shell nilpotent Becchi-Rouet-Stora-Tyutin (BRST) and anti-BRST transformations. Hence it should be possible to consider the BRST and anti-BRST invariant quantum action for topological 4D gravity. At this point, we remark that the coexistence of both BRST symmetry and anti-BRST symmetry was already realized in TYMT [13] and in topological antisymmetric tensor gauge theory [14], so-called BF theory (see Ref. [1]), as well as in topological 2D gravity without torsion [15]. In the present work, we consider topological 4D gravity with torsion and show how the model of topological 4D self-dual gravity developed in Ref. [10] possesses the anti-BRST symmetry as well in addition to the BRST symmetry. However, following the procedure discussed in Ref. [12], we consider the observables based on the BRST symmetry as well as on the anti-BRST symmetry. These observables are defined from operators including forms of any degree constructed out of a generalized curvature. The method used allows us to also see that the anti-BRST invariant observables can be derived from the BRST invariant ones by an obvious mirror symmetry of the ghost numbers.

Now, let ϕ be an ISO(4)-superconnection on the (4,2)-dimensional superspace obtained, as usual, by extending the 4D spacetime manifold with local coordinates (x^μ) with two ordinary anticommuting coordinates (θ^α). The superconnection ϕ can be written as

$$\phi = dZ^M\left(\tfrac{1}{2}\phi_M{}^{ab}M_{ab} + \phi_M{}^a P_a\right), \tag{1}$$

where $\{M_{ab}, P_a\}$ are the generators of the gauge group ISO(4) and $Z^M = (x^\mu, \theta^\alpha)$. The Grassmann degrees of the superfield components $\phi_M{}^{ab}$ and $\phi_M{}^a$ are given by m, where m is the Grassmann degree of Z^M. We assign however to the anticommuting coordinates θ^1 and θ^2 the ghost numbers (-1) and $(+1)$, respectively, and to an even quantity, which could be a coordinate, a generator, or a superform, the ghost number zero. These rules determine the ghost numbers of the superfields $(\phi_M{}^{ab}, \phi_M{}^a)$ which are zero (for $M=\mu$), $(+1)$ (for $M=\alpha=1$), and (-1) (for $M=\alpha=2$).

We note that $\phi_\mu{}^{ab}$ and $\phi_\mu{}^a$ represent the gauge superfields, whereas $\phi_1{}^{ab}$ and $\phi_2{}^{ab}$ ($\phi_1{}^a$ and $\phi_2{}^a$) represent the Lorentz (translation) ghost and antighost superfields, respectively. Therefore, we introduce the coordinate ghost and antighost superfields $\eta_\alpha{}^\mu$ by the following replacement: $\phi_\alpha{}^a \rightarrow \phi_\mu{}^a \eta_\alpha{}^\mu$ and the inverse supervierbein $\phi^\mu{}_a$ by the relations $\phi_\mu{}^a\phi^\mu{}_b = \delta^a_b$ and $\phi_\nu{}^a\phi^\mu{}_a = \delta^\mu_\nu$. Now, it is convenient to use a new basis (b^M), instead of the natural one (dZ^M), so that any superfield component of ϕ could not be a product of the superfields. Indeed, if we also realize the following replacement: $\phi_\alpha{}^{ab} \rightarrow \phi_\alpha{}^{ab} + \phi_\mu{}^{ab}\eta_\alpha{}^\mu$, then the superconnection ϕ can be put in the form

$$\phi = b^M\phi_M, \tag{2}$$

with

$$b^{\mu}=dx^{\mu}+d\theta^{\alpha}\eta_{\alpha}{}^{\mu}, \quad b^{\alpha}=d\theta^{\alpha}, \tag{3}$$

$$\phi_{\mu}=\tfrac{1}{2}\phi_{\mu}{}^{ab}M_{ab}+\phi_{\mu}{}^{a}P_{a}, \quad \phi_{\alpha}=\tfrac{1}{2}\phi_{\alpha}{}^{ab}M_{ab}. \tag{4}$$

Using the exterior covariant superdifferential on ϕ we define then its associated supercurvature Ω given by the structure equation, $\Omega=d\phi+\frac{1}{2}[\phi,\phi]$, and satisfying the Bianchi identity, $d\Omega+[\phi,\Omega]=0$. In the new basis (b^{M}), we express Ω as

$$\Omega=\tfrac{1}{2}b^{N}b^{M}\Omega_{MN}, \tag{5}$$

where

$$\Omega_{MN}=\tfrac{1}{2}\Omega_{MN}{}^{ab}M_{ab}+\Omega_{MN}{}^{a}P_{a}. \tag{6}$$

The Grassmann degrees and the ghost numbers of the superfield components of Ω are determined as in the case of ϕ. For example, the Grassmann degree and the ghost number of the superfield $\Omega_{11}{}^{ab}$ are given by zero and $(+2)$, respectively.

Inserting Eqs. (2) and (5) into the structure equation, we obtain

$$\Omega_{\mu\nu}=\partial_{\mu}\phi_{\nu}-\partial_{\nu}\phi_{\mu}+[\phi_{\mu},\phi_{\nu}], \tag{7a}$$

$$\Omega_{\mu\alpha}=\partial_{\mu}\phi_{\alpha}-\partial_{\alpha}\phi_{\mu}+\eta_{\alpha}{}^{\nu}\partial_{\nu}\phi_{\mu}+\partial_{\mu}\eta_{\alpha}{}^{\nu}\phi_{\nu}+[\phi_{\mu},\phi_{\alpha}], \tag{7b}$$

$$\begin{aligned}\Omega_{\alpha\beta}=&(\partial_{\alpha}\eta_{\beta}{}^{\mu}-\eta_{\alpha}{}^{\nu}\partial_{\nu}\eta_{\beta}{}^{\mu})\phi_{\mu}+(\partial_{\alpha}\phi_{\beta}-\eta_{\beta}{}^{\mu}\partial_{\mu}\phi_{\alpha})\\&+\tfrac{1}{2}[\phi_{\alpha},\phi_{\beta}]+(\alpha\leftrightarrow\beta).\end{aligned} \tag{7c}$$

Similarly, from the Bianchi identity, we find

$$\sum_{(\mu\nu\rho)}(\tilde{D}_{\mu}\Omega_{\nu\rho})=0, \tag{8a}$$

$$\tfrac{1}{2}\tilde{D}_{\alpha}\Omega_{\mu\nu}+\tilde{D}_{\mu}\Omega_{\nu\alpha}-\tfrac{1}{2}\eta_{\alpha}{}^{\rho}\partial_{\rho}\Omega_{\mu\nu}-\partial_{\mu}\eta_{\alpha}{}^{\rho}\Omega_{\rho\nu}-(\mu\leftrightarrow\nu)=0, \tag{8b}$$

$$\tilde{D}_{\alpha}\Omega_{\mu\beta}-\tfrac{1}{2}\tilde{D}_{\mu}\Omega_{\alpha\beta}+(\partial_{\alpha}\eta_{\beta}{}^{\nu}-\eta_{\alpha}{}^{\rho}\partial_{\rho}\eta_{\beta}{}^{\nu})\Omega_{\mu\nu}-\eta_{\alpha}{}^{\nu}\partial_{\nu}\Omega_{\mu\beta}-\partial_{\mu}\eta_{\alpha}{}^{\nu}\Omega_{\nu\beta}+(\alpha\leftrightarrow\beta)=0, \tag{8c}$$

$$\sum_{(\alpha\beta\gamma)}\{\tilde{D}_{\alpha}\Omega_{\beta\gamma}+[\partial_{\alpha}\eta_{\beta}{}^{\mu}-\eta_{\alpha}{}^{\nu}\partial_{\nu}\eta_{\beta}{}^{\mu}+(\alpha\leftrightarrow\beta)]\Omega_{\mu\gamma}-\eta_{\alpha}{}^{\mu}\partial_{\mu}\Omega_{\beta\gamma}\}=0, \tag{8d}$$

where $\tilde{D}_{M}=\partial_{M}+[\phi_{M},.]$ and $\Sigma_{(MNR)}$ means a cyclic sum over M, N, and R.

The evaluation of the Eqs. (7a) and (8a) at $\theta^{\alpha}=0$ allows the determination of the curvature $R_{\mu\nu}{}^{ab}=\Omega_{\mu\nu}{}^{ab}|$, the torsion $T_{\mu\nu}{}^{a}=\Omega_{\mu\nu}{}^{a}|$, and their associated Bianchi identities, where $e_{\mu}{}^{a}=\phi_{\mu}{}^{a}|$ is the vierbein and $\omega_{\mu}{}^{ab}=\phi_{\mu}{}^{ab}|$, the spin connection. On the other hand, to translate the remaining equations into the equations determining the BRST and anti-BRST transformations of topological 4D gravity, we should interpret all the fields occurring in such a theory geometrically. Besides the vierbein and the spin connection, the superconnection permits us to introduce the following fields: $c^{\mu}=\eta_{1}{}^{\mu}|$ is the coordinate ghost, $\overline{c^{\mu}}=\eta_{2}{}^{\mu}|$ is the antighost of c^{μ}, $h^{\mu}=\partial_{1}\eta_{2}{}^{\mu}|$ is the associated auxiliary field, $c^{ab}=\phi_{1}{}^{ab}|$ is the Lorentz ghost, $\overline{c^{ab}}=\phi_{2}{}^{ab}|$ is the antighost of c^{ab}, $h^{ab}=\partial_{1}\phi_{2}{}^{ab}|$ is the associated auxiliary field. In addition, besides the curvature and the torsion, the supercurvature also permits us to consider the following now: $\psi_{\mu}{}^{a}=-\Omega_{\mu 1}{}^{a}|$ is the superpartner of $e_{\mu}{}^{a}$, $\bar{\psi}_{\mu}{}^{a}=-\Omega_{\mu 2}{}^{a}|$ is the antighost of $\psi_{\mu}{}^{a}$, $h_{\mu}{}^{a}=-\partial_{1}\Omega_{\mu 2}{}^{a}|$ is the associated auxiliary field, $\psi_{\mu}{}^{ab}=-\Omega_{\mu 1}{}^{ab}|$ is the superpartner of $\omega_{\mu}{}^{ab}$, $\bar{\psi}_{\mu}{}^{ab}=-\Omega_{\mu 2}{}^{ab}|$ is the antighost of $\psi_{\mu}{}^{ab}$, $h_{\mu}{}^{ab}=-\partial_{1}\Omega_{\mu 2}{}^{ab}|$ is the associated auxiliary field, $\varphi^{\mu}=\frac{1}{2}\Omega_{11}{}^{\mu}|$ is the ghost for the ghost of c^{μ}, $\overline{\varphi^{\mu}}=\frac{1}{2}\Omega_{22}{}^{\mu}|$ is the antighost of φ^{μ}, $k^{\mu}=\frac{1}{2}\partial_{1}\Omega_{22}{}^{\mu}|$ is the associated auxiliary field, $\varphi^{ab}=\frac{1}{2}\Omega_{11}{}^{ab}|$ is the ghost for the ghost of c^{ab}, $\overline{\varphi^{ab}}=\frac{1}{2}\Omega_{22}{}^{ab}|$ is the antighost of φ^{ab}, $k^{ab}=\frac{1}{2}\partial_{1}\Omega_{22}{}^{ab}|$ is the associated auxiliary field, $(\Phi^{\mu}=\Omega_{12}{}^{\mu}|, K^{\mu}=\partial_{1}\Omega_{12}{}^{\mu}|)$ and $(\Phi^{ab}=\Omega_{12}{}^{ab}|, K^{ab}=\partial_{1}\Omega_{12}{}^{ab}|)$ are two pairs of extra fields which take into account further degeneracies associated with the fact that we wish to incorporate both BRST symmetry and anti-BRST symmetry into the quantization of topological gravity. In the above identifications, we have used, instead of the supercurvature components $\Omega_{\alpha\beta}{}^{a}$, the superfields $\Omega_{\alpha\beta}{}^{\mu}$ defined by $\Omega_{\alpha\beta}{}^{\mu}=\phi^{\mu}{}_{a}\Omega_{\alpha\beta}{}^{a}$. We also realize the usual identifications: $\partial_{\alpha}S|=Q_{\alpha}(S|)$, where S is any superfield and $Q=Q_{1}(Q=Q_{2})$ is the BRST (anti-BRST) operator.

Substituting Eqs. (4) and (6) into the structure equations (7b) and (7c), evaluating these at $\theta^{\alpha}=0$ and using the above identifications as well as the structure constants of ISO(4), we obtain the following BRST transformations:

$$\begin{gathered}Qe_{\mu}{}^{a}=\psi_{\mu}{}^{a}+L_{c}e_{\mu}{}^{a}-c^{ab}e_{\mu b},\\ Q\omega_{\mu}{}^{ab}=\psi_{\mu}{}^{ab}+L_{c}\omega_{\mu}{}^{ab}+D_{\mu}c^{ab},\\ Qc^{\mu}=\varphi^{\mu}+\tfrac{1}{2}L_{c}c^{\mu}, \quad Q\overline{c^{\mu}}=h^{\mu}, \quad Qh^{\mu}=0,\\ Qc^{ab}=\varphi^{ab}-\varphi^{\mu}\omega_{\mu}{}^{ab}+L_{c}c^{ab}-c^{ad}c_{d}{}^{b},\\ Q\overline{c^{ab}}=h^{ab}, \quad Qh^{ab}=0,\end{gathered} \tag{9}$$

where L_{v} represents the Lie derivative along a vector field $v=(v^{\mu})$ and D_{μ}, the covariant derivative with respect to the spin connection. We also obtain the anti-BRST transformations, which can be derived from Eq. (9) by the following mirror symmetry of the ghost numbers: $X\rightarrow X$ (for $X=e_\mu{}^a,\omega_\mu{}^{ab}$), $X\rightarrow\bar{X}$ (for $X=Q,c^\mu,h^\mu,c^{ab},h^{ab}$), and $\bar{\bar{X}}=X$, where

$$\bar{h}^\mu=-h^\mu+\Phi^\mu+\tfrac{1}{2}L_c\bar{c}^\mu+\tfrac{1}{2}L_{\bar{c}}c^\mu,$$

$$\begin{aligned}\bar{h}^{ab}=&-h^{ab}+\Phi^{ab}-\Phi^\mu\omega_\mu{}^{ab}+L_c\bar{c}^{ab}+L_{\bar{c}}c^{ab}-c^{ad}\bar{c}_d{}^b\\&-\bar{c}^{ad}\bar{c}_d{}^b.\end{aligned}\tag{10}$$

However, after a similar straightforward calculation, from the Bianchi identities (8b)–(8d) we get the following BRST transformations:

$$QT_{\mu\nu}{}^a=D_\mu\psi_\nu{}^a-D_\nu\psi_\mu{}^a+L_cT_{\mu\nu}{}^a-c^a{}_bT_{\mu\nu}{}^b,$$

$$\begin{aligned}QR_{\mu\nu}{}^{ab}=&D_\mu\psi_\nu{}^{ab}-D_\nu\psi_\mu{}^{ab}+L_cR_{\mu\nu}{}^{ab}-c^a{}_dR_{\mu\nu}{}^{db}\\&+c^b{}_dR_{\mu\nu}{}^{da},\end{aligned}$$

$$Q\psi_\mu{}^a=\varphi^{ab}e_{\mu b}-\varphi^\nu\omega_\nu{}^{ab}e_{\mu b}-L_\varphi e_\mu{}^a+L_c\psi_\mu{}^a-c^{ab}\psi_{\mu b},$$

$$Q\bar{\psi}_\mu{}^a=h_\mu{}^a,\quad Qh_\mu{}^a=0,$$

$$\begin{aligned}Q\psi_\mu{}^{ab}=&-D_\mu\varphi^{ab}+\varphi^\nu R_{\mu\nu}{}^{ab}+L_c\psi_\mu{}^{ab}-c^a{}_d\psi_\mu{}^{db}\\&+c^b{}_d\psi_\mu{}^{da},\end{aligned}$$

$$Q\bar{\psi}_\mu{}^{ab}=h_\mu{}^{ab},\quad Qh_\mu{}^{ab}=0,$$

$$Q\varphi^\mu=L_c\varphi^\mu,\quad Q\bar{\varphi}^\mu=k^\mu,\quad Qk^\mu=0,$$

$$Q\varphi^{ab}=\varphi^\mu\psi_\mu{}^{ab}+L_c\varphi^{ab}-c^a{}_d\varphi^{db}+c^b{}_d\varphi^{da},$$

$$Q\bar{\varphi}^{ab}=k^{ab},\quad Qk^{ab}=0,$$

$$Q\Phi^\mu=K^\mu,\quad QK^\mu=0,\quad Q\Phi^{ab}=K^{ab},\quad QK^{ab}=0,\tag{11}$$

as well as the anti-BRST transformations, which can be derived from Eq. (11) by the mirror symmetry of the ghost numbers given by $X\rightarrow X$ (for $X=e_\mu{}^a,\omega_\mu{}^{ab},T_{\mu\nu}{}^a,R_{\mu\nu}{}^{ab},\Phi^\mu,\Phi^{ab}$), $\bar{\bar{X}}=X$, and $X\rightarrow\bar{X}$ (otherwise), where

$$\bar{h}_\mu{}^a=-h_\mu{}^a+\Phi^{ab}e_{\mu b}-\Phi^\nu\omega_\nu{}^{ab}e_{\mu b}-L_\Phi e_\mu{}^a+L_c\bar{\psi}_\mu{}^a+L_{\bar{c}}\psi_\mu{}^a-c^a{}_b\bar{\psi}_\mu{}^b-\bar{c}^a{}_b\psi_\mu{}^b,$$

$$\bar{h}_\mu{}^{ab}=-h_\mu{}^{ab}-D_\mu\Phi^{ab}+\Phi^\nu R_{\mu\nu}{}^{ab}+L_c\bar{\psi}_\mu{}^{ab}+L_{\bar{c}}\psi_\mu{}^{ab}-c^a{}_d\bar{\psi}_\mu{}^{db}+c^b{}_d\bar{\psi}_\mu{}^{da}-\bar{c}^a{}_d\psi_\mu{}^{db}+\bar{c}^b{}_b\psi_\mu{}^{da},$$

$$\bar{k}^\mu=-K^\mu+L_c\Phi^\mu+L_{\bar{c}}\varphi^\mu,$$

$$\bar{k}^{ab}=-K^{ab}+\varphi^\mu\bar{\psi}_\mu{}^{ab}+\Phi^\mu\psi_\mu{}^{ab}+L_c\Phi^{ab}+L_{\bar{c}}\varphi^{ab}-c^a{}_d\Phi^{db}+c^b{}_d\Phi^{da}-\bar{c}^a{}_d\varphi^{db}+\bar{c}^b{}_d\varphi^{da},$$

$$\bar{K}^\mu=-k^\mu+L_c\bar{\varphi}^\mu+L_{\bar{c}}\Phi^\mu,$$

$$\bar{K}^{ab}=-k^{ab}+\bar{\varphi}^\mu\psi_\mu{}^{ab}+\Phi^\mu\bar{\psi}_\mu{}^{ab}+L_c\bar{\varphi}^{ab}+L_{\bar{c}}\Phi^{ab}-c^a{}_d\bar{\varphi}^{db}+c^b{}_d\bar{\varphi}^{da}-\bar{c}^a{}_d\Phi^{db}+\bar{c}^b{}_d\Phi^{da}.\tag{12}$$

We note that the BRST–anti-BRST algebra is a nilpotent off shell, i.e., $Q^2=\bar{Q}^2=[Q,\bar{Q}]=0$ on all the fields. In particular, the two pairs of extra fields (Φ^μ,K^μ) and (Φ^{ab},K^{ab}) are required to achieve the anticommutation of the BRST and anti-BRST operators.

Let us now turn to the construction of the complete quantum action S_q of topological 4D gravity which is defined at the classical level by the topological invariant S_0 given as a linear combination of the Euler number and the Pontryagin number [8–10]. This is similar to the case of TYMT, where the topological classical action can be taken as the second Chern class. In order to build the quantum action S_q we shall add to S_0 the gauge-fixing action S_{gf} which includes all the relevant gauge-fixing terms associated with all the invariances of S_0. We note that S_0 is invariant under local Lorentz transformations and diffeomorphisms as well as under topological (shift) symmetries, since S_0 is a topological invariant. These transformations are redundant so that topological 4D gravity is a first stage reducible gauge theory. To realize the quantization of such a theory, one may apply the formalism of Batalin-Vilkovisky [16]. In this framework, the necessary fields to describe the quantized theory have been introduced and the action of the off-shell nilpotent BRST operator on these fields has been determined [8–10]. There, the antighosts and their corresponding auxiliary fields are introduced in relation to the choice of the gauge-fixing conditions.

It is worth noting that the superspace formalism permits us to recast all the fields in topological 4D gravity in a geometric way. It also permits us to see that the anti-BRST symmetry necessarily coexists with the BRST symmetry. To this end, we will establish that the gauge-fixing action S_{gf} can be written in BRST and anti-BRST exact form. The resulting full quantum action $S_q=S_0+S_{gf}$ will then be a BRST and anti-BRST invariant, thanks to the off-shell nilpotency of the BRST–anti-BRST algebra. Let us start with S_{gf} given as

$$S_{gf}=Q\bar{Q}\int d^4xe(\tfrac{1}{2}\tilde{R}_{\mu\nu}{}^{ab}\tilde{R}^{\mu\nu}{}_{ab}+\tfrac{1}{2}\tilde{T}_{\mu\nu}{}^{a}\tilde{T}^{\mu\nu}{}_{a}-\psi_\mu{}^{ab}\bar{\psi}^\mu{}_{ab}$$
$$-\psi_\mu{}^a\bar{\psi}^\mu{}_a-\tfrac{1}{2}\omega_\mu{}^{ab}\omega^\mu{}_{ab}-\tfrac{1}{2}e), \tag{13}$$

where $e=\det e_\mu{}^a$, $\tilde{R}_{\mu\nu}{}^{ab}$ and $\tilde{T}_{\mu\nu}{}^{ab}$ represent the self-dual components of the curvature and the torsion with respect to the coordinate indices,

$$\tilde{R}_{\mu\nu}{}^{ab}=R_{\mu\nu}{}^{ab}+\tfrac{1}{2}\varepsilon_{\mu\nu}{}^{\rho\sigma}R_{\rho\sigma}{}^{ab}, \tag{14}$$

$$\tilde{T}_{\mu\nu}{}^{a}=T_{\mu\nu}{}^{a}+\tfrac{1}{2}\varepsilon_{\mu\nu}{}^{\rho\sigma}T_{\rho\sigma}{}^{a}. \tag{15}$$

In order to show that S_{gf} effectively represents a gauge-fixing action, we shall verify that it leads to an appropriate set of symmetry constraints, which allow us to fix all the symmetries present in topological 4D gravity. To this purpose, we use the fact that we are in the presence of a topological field theory of the Witten type. However, one can introduce a coupling constant by rescaling the various fields. In this case all the interaction terms are proportional to the higher power of the coupling [2,4]. Using the weak coupling limit which is in fact an exact limit, we can then write Eq. (13) up to negligible higher-order terms, since only its quadratic terms contribute to the partition function. We recall that in topological fields theories of the Witten type the semiclassical approximation is exact [2]. Let us remark that similar arguments have been used in TYMT [13] and in BF theory [14] as well as in topological 2D gravity [15]. Thus, from the BRST and anti-BRST transformations together with the relations (10) and (12) and after some computations, it follows that S_{gf} as in Eq. (13) can be put modulo higher-order terms and modulo a total divergence in the form

$$S_{gf}=Q\int d^4xe(\bar{\chi}^{\mu\nu}{}_{ab}\tilde{R}_{\mu\nu}{}^{ab}+\bar{\lambda}^{\mu\nu}{}_a\tilde{T}_{\mu\nu}{}^a+\bar{\varphi}^{ab}\nabla^\mu\psi_{\mu ab}$$
$$+\bar{\varphi}^a\nabla^\mu\psi_{\mu a}+\bar{c}^{ab}\nabla^\mu\omega_{\mu ab}+\bar{c}^\mu ee^\nu{}_a\partial_\mu e_\nu{}^a)+S^0_{gf}, \tag{16}$$

with

$$\bar{\chi}^{\mu\nu}{}_{ab}=(\nabla^\mu\bar{\psi}^\nu{}_{ab}-\nabla^\nu\bar{\psi}^\mu{}_{ab})+\tfrac{1}{2}\varepsilon^{\mu\nu}{}_{\rho\sigma}(\nabla^\rho\bar{\psi}^\sigma{}_{ab}-\nabla^\sigma\bar{\psi}^\rho{}_{ab}), \tag{17}$$

$$\bar{\lambda}^{\mu\nu}{}_{a}=(\nabla^\mu\bar{\psi}^\nu{}_{a}-\nabla^\nu\bar{\psi}^\mu{}_{a})+\tfrac{1}{2}\varepsilon^{\mu\nu}{}_{\rho\sigma}(\nabla^\rho\bar{\psi}^\sigma{}_{a}-\nabla^\sigma\bar{\psi}^\rho{}_{a}), \tag{18}$$

$$S^0_{gf}=Q\int d^4xe\{\bar{\psi}^\mu{}_{ab}(h_\mu{}^{ab}+\nabla_\mu\Phi^{ab}-\omega_\mu{}^{ab})$$
$$+\bar{\psi}^\mu{}_a(h_\mu{}^a+\nabla_\mu\Phi^a-ee_\mu{}^a)\}, \tag{19}$$

where ∇_μ denotes the covariant derivative with both spin connection $\omega_\mu{}^{ab}$ and Christoffel symbol $\Gamma^\rho_{\mu\nu}$.

Note that in deriving Eq. (16), we have used that $Qe=ee^\mu{}_aQe_\mu{}^a=ee^\mu{}_a\psi_\mu{}^a+\partial_\mu(e\bar{c}^\mu)$, and $\Gamma^\nu_{\nu\mu}=\partial_\mu e/e$. We have also used relations such as $D_\mu\psi_\nu{}^a-D_\nu\psi_\mu{}^a=\nabla_\mu\psi_\nu{}^a-\nabla_\nu\psi_\mu{}^a$, since $\Gamma^\rho_{\mu\nu}=\Gamma^\rho_{\nu\mu}$.

Furthermore, making use of the replacements: $\nabla_\mu\Phi^{ab}\to\nabla_\mu\Phi^{ab}-h_\mu{}^{ab}+\omega_\mu{}^{ab}$ and $\nabla_\mu\Phi^a\to\nabla_\mu\Phi^a-h_\mu{}^a+ee_\mu{}^a$, we can write S^0_{gf} given by Eq. (19) modulo higher-order terms as follows:

$$S^0_{gf}=-\int d^4xe(K_{ab}\nabla^\mu\bar{\psi}_\mu{}^{ab}+\Phi_{ab}\nabla^\mu B_\mu{}^{ab}+K_a\nabla^\mu\bar{\psi}_\mu{}^a$$
$$+\Phi_a\nabla^\mu B_\mu{}^a). \tag{20}$$

Therefore, we remark that the path integrals over the extra fields K^{ab}, Φ^{ab}, K^a, and Φ^a yield delta functions which enforce the conditions that

$$\nabla^\mu\bar{\psi}_\mu{}^{ab}=0,\quad \nabla^\mu h_\mu{}^{ab}=0, \tag{21}$$

$$\nabla^\mu\bar{\psi}_\mu{}^{a}=0,\quad \nabla^\mu h_\mu{}^{a}=0. \tag{22}$$

Thus, we end up with an effective action S^{eff}_{gf} given by Eq. (16) without the extra term S^0_{gf}. It is now easy to see that this effective action represents a true symmetry-fixing action for topological 4D gravity. Indeed, from the different terms present in S^{eff}_{gf}, we learn that the topological symmetries are fixed by $\tilde{R}_{\mu\nu}{}^{ab}=0$ and $\tilde{T}_{\mu\nu}{}^a=0$, i.e., anti–self-duality on both the curvature and the torsion. However, the reduced symmetries are fixed by $\nabla^\mu\psi_\mu{}^{ab}=0$ and $\nabla^\mu\psi_\mu{}^a=0$, while local Lorentz transformations and diffeomorphisms are fixed by $\nabla^\mu\omega_\mu{}^{ab}=0$ and $e^\nu{}_a\partial_\mu e_\nu{}^a=0$. Except the latter, these gauge conditions are the same as in the model of topological 4D self-dual gravity proposed in Ref. [10]. Here, to fix diffeomorphism invariance, the authors adopt the harmonic gauge condition. At this point, we remark that there are the antighosts $\bar{\psi}_\mu{}^{ab}$ and $\bar{\psi}_\mu{}^a$ and their auxiliary fields $h_\mu{}^{ab}$ and $h_\mu{}^a$, which permit us to introduce the antighosts $\bar{\chi}^{\mu\nu}{}_{ab}$ and $\bar{\lambda}^{\mu\nu}{}_a$ and their auxiliary fields $H^{\mu\nu}{}_{ab}=Q\bar{\chi}^{\mu\nu}{}_{ab}$ and $H^{\mu\nu}{}_a=Q\bar{\lambda}^{\mu\nu}{}_a$ [see Eqs. (17) and (18)], and to take off the extra fields K^{ab}, Φ^{ab}, K^a, and Φ^a through the conditions (21) and (22). However, in order to impose other gauge conditions, in particular, for the topological symmetries, we have to change the first two terms in S_{gf} as in Eq. (13).

Let us note that the analysis of Ref. [10] considers the reduced BRST operator, which is used to fix the topological and reducible symmetries, whereas diffeomorphisms and local Lorentz transformations are fixed using a second BRST operator. The reduced BRST operator is defined from a full BRST operator (see also Refs. [8, 9]), which is equivalent to that given in Eqs. (9) and (11). In our approach, we have used a single BRST operator and the obtained BRST exact effective action S^{eff}_{gf} is also anti-BRST exact, since it is, up to negligible higher-order terms, the same as the BRST–anti-BRST exact action S_{gf} given by Eq. (13). We can now ask how to construct the observables for both BRST symmetry and anti-BRST symmetry, and whether the fixing of all the symmetries at once could be used to extend the set of observables constructed in Ref. [10]. For this purpose, we shall determine a set of operators satisfying descent equations, which involve the BRST and anti-BRST operators Q and $\bar{Q}$. Inspired mainly by the results obtained in Ref. [12], we can write down the generalized descent equation

$$(d+Q+\bar{Q})(e^{i(c+\bar{c})}P)=0. \tag{23}$$

$i(v)$ is the inner product along a vector field $v=(v^\mu)$ and $P(\tilde{R},\tilde{R},\ldots,\tilde{R})$ is a characteristic polynomial in the generalized curvature $\tilde{R}$ given by

$$\tilde{R}=R+\psi+\overline{\psi}+\varphi+\overline{\varphi}+\Phi, \tag{24}$$

where we have used differential forms to represent the SO(4)-valued components. For example, we have $R=\frac{1}{4}dx^\nu dx^\mu R_{\mu\nu}{}^{ab}M_{ab}$, $\psi=\frac{1}{2}dx^\mu\psi_\mu{}^{ab}M_{ab}$. Indeed, working in the same spirit as in Ref. [12] (see also Ref. [8]), we define the generalized covariant derivative as $\tilde{D}=e^{-i(c+\overline{c})}(d+Q+\overline{Q}+\omega+c_0+\overline{c}_0)e^{i(c+\overline{c})}$, where $\omega=\frac{1}{2}dx^\mu\omega_\mu{}^{ab}M_{ab}$, $c_0=\frac{1}{2}c^{ab}M_{ab}$, and $\overline{c}_0=\frac{1}{2}\overline{c}^{ab}M_{ab}$. Therefore, we find that $\tilde{D}^2=\tilde{R}$ satisfying the generalized Bianchi identity $\tilde{D}\tilde{R}=0$, so that Eq. (23) holds for any characteristic polynomial. Now, expanding Eq. (23) in terms of form degree, we obtain the relations

$$(Q+\overline{Q})W_n+dW_{n-1}=0,\quad 0\leqslant n\leqslant 4, \tag{25}$$

with $W_n=\Sigma_{m=0}^{4-n}(1/m!)i^m(c+\overline{c})P_{n+m}$ and $W_{-1}=0$, where $P_k(0\leqslant k\leqslant 4)$ denotes the k form of P.

Moreover, denoting the number of $\tilde{R}$ in P by N and in view of Eq. (24), we can expand P_k in ghost number as follows: $P_k=\Sigma_g P_k^g$, where $-(2N-k)\leqslant g\leqslant(2N-k)$ and g is even (odd) for k even (odd). This allows the expansion of W_n in ghost number and from Eq. (25) it follows that the integrals

$$W_{2N-n}=\int_{\gamma_n}W_n^{2N-n} \tag{26}$$

over nontrivial homology cycles γ_n of the 4D spacetime of dimension n are observables with ghost number $(2N-n)$ since W_{2N-n} are BRST invariant and metric independent. We also have anti-BRST invariant observables given by

$$W_{-(2N-n)}=\int_{\gamma_n}W_n^{-(2N-n)}. \tag{27}$$

We remark that Eq. (25) also gives rise to the relations $QW_n^g+\overline{Q}W_n^{g+2}=-dW_{n-1}^{g+1}$, $-(2N-n)\leqslant g\leqslant(2N-n)-2$ and g is even (odd) for n even (odd), which will not contribute to the construction of observables.

Let us note that the observables in Ref. [10] are constructed by integrating $W_4^{2N-4}=P_4^{2N-4}$ over the 4D spacetime manifold. These are the only observables provided by the equivariant Bianchi identity in the context of the reduced BRST symmetry (see also Ref. [9]). By using a single BRST symmetry, one has more observables given in Eq. (26), which are constructed from the operators

$$W_n^{2N-n}=\sum_{m=0}^{4-n}\frac{1}{m!}i^m(c)P_{n+m}^{2N-n-m}. \tag{28}$$

Besides W_4^{2N-4}, we have other operators, namely W_n^{2N-n} $(0\leqslant n\leqslant 3)$, which are only present within the single BRST method. However, the incorporation of the anti-BRST symmetry permits us to construct other observables given in Eq. (27). These anti-BRST invariant observables are not fundamentally different from the BRST invariant ones, since they are in complete duality, with respect to the mirror symmetry of the ghost numbers. In fact, such observables are defined from the operators $W_n^{-(2N-n)}$ which can simply be derived from Eq. (28) by replacing the ghosts with their associated anti-ghosts.

Finally, let us also note that the constructed observables can lead to the Euler number and the Pontryagin one used in the construction of the classical action S_0. This is the same result obtained in Ref. [10]. For example, explicitly given $P=\mathrm{tr}(\tilde{R}\wedge\tilde{R})$, we have the operators given by Eq. (28) with $N=2$ and

$$P_4^0=\mathrm{tr}(R\wedge R),\quad P_3^1=2\ \mathrm{tr}(\psi\wedge R),$$
$$P_2^2=\mathrm{tr}(\psi\wedge\psi+2\varphi R),\quad P_1^3=2\ \mathrm{tr}(\varphi\psi),\quad P_0^4=\mathrm{tr}(\varphi\varphi). \tag{29}$$

We remark that the operators as in Eq. (29) are analogous to those characterizing TYMT with SO(4) as the gauge group. Working with the gauge group ISO(4) allows us to introduce the diffeomorphism ghost and to construct, besides the Pontryagin number given by integrating $W_4^0=P_4^0$ over the 4D manifold [10], the observables $W_{4-n}=\int_{\gamma_n}W_n^{4-n}$ $(0\leqslant n\leqslant 3)$ which only appear when we fix all of the symmetries at once. We also have the observables $W_{-(4-n)}$ which are just W_{4-n}, but with negative ghost numbers.

We summarize the results of this paper. We have applied the superspace formalism to topological 4D gravity. In our treatment, the gauge fields (the spin connection and vierbein), the Lorentz, and diffeomorphism ghosts and their antighosts have been introduced via an ISO(4)-superconnection, whereas the curvature, the torsion, and the topological ghosts associated to the gauge fields and their antighosts as well as the second generation ghosts and their antighosts have been introduced through the supercurvature. In addition, the supercurvature allows us to introduce two pairs of extra fields. These are necessary in order to close the BRST–anti-BRST algebra off shell by using the structure equations and the Bianchi identities.

Therefore, we have performed a construction of a gauge-fixing action which is BRST–anti-BRST exact, so that we have the anti-BRST symmetry as well in addition to the BRST symmetry. We have also shown how this action effectively includes all the relevant gauge-fixing terms associated with all the invariances of the classical action given as a linear combination of the Euler number and the Pontryagin one. In particular, the fixing of the topological symmetries has been realized through the anti–self-duality on both the curvature and the torsion, in analogy to what happens in topological 4D self-dual gravity [10]. In our approach, we have considered the single BRST method in order to fix all of the symmetries at once, rather than the equivariant method used in Ref. [10]. We note that the inclusion of the anti-BRST symmetry has been ensured by the presence of the extra fields. These are necessary in order to define the anti-ghosts and their auxiliary fields associated with the topological symmetries from those coming from the supercurvature and corresponding to the topological ghosts.

Furthermore, the recursive relations which are essential to construct the BRST invariant observables as well as the anti-

BRST invariant ones are derived from the characteristic polynomials in the generalized curvature. The latter has been constructed from the Lorentz curvature and the SO(4)-valued fields introduced via the supercurvature. The obtained anti-BRST invariant observables are not fundamentally different from the BRST invariant ones, since there is a complete mirror symmetry between them. We have also seen how the observables constructed in Ref. [10] are supplemented by other observables defined from operators including forms of any degree.

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