Radiation from a charged particle and radiation reaction reexamined

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We study the electromagnetic fields of an arbitrarily moving charged particle and the radiation reaction on the charged particle using a novel approach. We first show that the fields of an *arbitrarily* moving charged particle in an inertial frame can be related in a simple manner to the fields of a *uniformly accelerated* charged particle in its rest frame. Since the latter field is static and easily obtainable, it is possible to derive the fields of an arbitrarily moving charged particle by a coordinate transformation. More importantly, this formalism allows us to calculate the self-force on a charged particle in a remarkably simple manner. We show that the original expression for this force, obtained by Dirac, can be rederived with much less computation and in an intuitively simple manner using our formalism. [S0556-2821(98)05710-5]

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I. MOTIVATION

The field of a charged particle at rest in an inertial frame is a static Coulomb field which falls as $(1/r^2)$ in the standard spherical coordinate system. The field of a charge, moving with uniform velocity, can be obtained by Lorentz transforming the Coulomb field; this field also falls as the inverse square of the distance. The situation changes dramatically for a charged particle which is moving with nonzero acceleration. The field now has a piece which falls only as $(1/r)$, usually called the radiation field. For a field which decreases as $(1/r)$, the energy flux varies as $(1/r^2)$ implying that the same amount of energy flows through spheres of different radii at sufficiently large distances from the charge. For this reason, the radiation fields acquire a life of their own and the entire phenomena of electromagnetic radiation hinges on this feature. Because of the continuous transfer of energy from the charged particle to large distances, there will be a damping force acting on the charged particle which is usually called the radiation reaction force. The derivation of the radiation reaction force is conceptually and operationally quite complicated and the expression — obtained originally by Dirac (see Ref. [4])— has no simple intuitive description.

We analyze these issues from a novel point of view in this paper which throws light on the conceptual and mathematical issues involved in this problem. The analysis is motivated by the following issue: Maxwell's equations are not only Lorentz invariant but can also be written in a generally covariant manner. Given a charged particle moving in some arbitrary trajectory, it is always possible to construct a proper coordinate system for such a charged particle. In such a coordinate system, the charge will be at rest for all times but the background metric will be non-Minkowskian and — in general — time dependent. Maxwell's equations in this coordinate system will correspond to that of a *stationary* charge located in a nontrivial (and in general time-dependent) metric. The solution to Maxwell's equation in this frame receives time-dependent contributions *not* because of the motion of charged particles but because of the nontrivial nature of the *background* metric. But we know that, for internal consistency, these solutions should transform to the standard solutions describing the field of an arbitrarily moving charged particle when we go over to the inertial frame. This is remarkable since the time dependence and nontriviality of the *background metric* have to translate to the correct spatial and time dependence of the *radiation field*. Further, the charged particle has to feel the radiation reaction force in the noninertial frame, even though it is at rest, due to the nontriviality of the background metric. It is not intuitively obvious how these features come about and it is important to understand how the physics in the noninertial frame of the charged particle operates.

We shall explore in this paper both the issues raised in the above paragraph. The key feature which emerges from our analysis is the following. The structure of Maxwell's equations dictate that the static field of a *uniformly accelerated* charged particle in its rest frame can be related to the field of an *arbitrarily moving* charged particle in the inertial frame. This connection also carries over to the computation of the self-force. It turns out that the radiation reaction force has a simple geometrical origin in the uniformly accelerating frame in which the charged particle is instantaneously at rest. The force arises due to the deviation of the trajectory of the charged particle from that of uniform acceleration and hence is proportional to the derivative of the acceleration. We shall now spell out the details of the approach we plan to follow in this paper.

II. THE FORMALISM

Consider the electromagnetic field of a charge moving with a uniform velocity in an inertial frame S. Since Maxwell's equations are Lorentz covariant, the most natural way to calculate the field in S is to find the field in the charge's

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rest frame S' and transform back to S. Let us next consider the problem of calculating the electromagnetic field of a charge which is moving *arbitrarily*. The conventional method (see, e.g., Ref. [1]) is to calculate the Leinard-Weichert potential and to differentiate it to obtain the field. However, we will show that it is possible to approach the problem differently along the following lines.

Consider a charge moving with an arbitrary velocity and acceleration in an inertial frame S. In the Lorentz gauge, Maxwell's equations can be written in terms of the vector potential A^{μ} and the current j^{μ} as

$$\Box A^{\mu}=4\pi j^{\mu}, \tag{1}$$

where $\Box=\partial_{\mu}\partial^{\mu}$. It follows that

$$\Box F^{\mu\nu}=4\pi(\partial^{\mu}j^{\nu}-\partial^{\nu}j^{\mu}). \tag{2}$$

Because of the characteristics of the $\Box$ operator, the fields at an event P can only depend on the trajectory of the charge at the retarded event O, which is the point of intersection of the backward light cone drawn from P, and the world line of the charge $z^{\mu}=z^{\mu}(\tau)$. Since j^{μ} is linear in four velocity, the quantity $\partial_{\mu}j_{\nu}$, in the the right hand side of Eq. (2) can at most depend on $\ddot{z}^{\mu}(\tau)$. Therefore, the fields at P can at most depend on the second derivatives at the retarded position of the charge at O — i.e., at most on the retarded acceleration of the charge. Suppose we now change the trajectory of the charge to that of a uniformly accelerated one without changing the values of the velocity and acceleration at the retarded event O. The field at P, since it depends only on the velocity and acceleration at O will still remain the same. It follows that, if we know the field at P due to a uniformly accelerated charge with a given acceleration and velocity at O, then we can obtain the field due to a general trajectory.

Thus the problem reduces to that of calculating the field of a uniformly accelerated charged particle. This is most easily done by using the fact that Maxwell's equations can be written in a generally covariant manner. Solving the Maxwell's equations in the noninertial, rest frame of charge and transforming the field to the inertial frame, we can obtain the field of a uniformly accelerated particle. Using the argument outlined above, we can then find the field of a charged particle moving in an arbitrary trajectory. To illustrate the power of this technique, we shall directly calculate the field for arbitrary, *rectilinear* motion. (The general case is a straightforward extension, and is treated in Appendix C).

The real power of this formalism, however, lies in calculating the field in the *infinitesimal neighborhood* of the accelerating charge. The general expression for the field in the neighborhood of an accelerating charge, found by Dirac, is a fairly involved expression, and good deal of labor is required to compute it. Our formalism involves computing it in the instantaneous coaccelerating frame of the charge, in which the first and second derivatives of the position of the charge vanish. The only dynamical contribution to the near field comes from the third derivative, which, as we shall see, leads to the radiation reaction term. This, along with the static terms, neatly transforms into expression obtained by Dirac in the inertial frame. In addition, a novel interpretation for radiation reaction emerges in the accelerated frame.

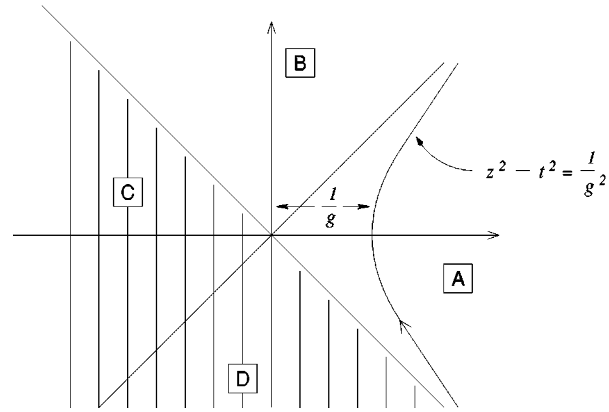

FIG. 1. The four domains of Minkowski spacetime; see text for more discussion.

The rest of the paper is organised as follows. In Sec. III, we obtain the electromagnetic field of a uniformly accelerated charge. This is done by solving Maxwell's equations in the rest frame of the charged particle (which is a noninertial frame) and transforming to the inertial frame. In Sec. IV, we use this result to obtain the field of an arbitrarily moving charged particle. This result is obtained by the procedure outlined above. Section V uses the same formalism to obtain the field in the neighborhood of the charged particle, thereby obtaining the radiation reaction term. The last section summarizes the results of the paper.

III. FIELDS DUE TO A CHARGE AT REST IN A UNIFORMLY ACCELERATED FRAME

A. The coordinate transformation

Since the key idea involves working with a uniformly accelerated frame, we shall review the coordinate transformation connecting the Minkowski frame to the Rindler frame and collect the necessary formulas. Consider a charge moving with uniform acceleration along the z axis of an inertial frame S with the coordinate system (t,x,y,z). The trajectory of the charge is given by

$$t=\frac{1}{g}\sinh(g\tau);\quad z=\frac{1}{g}\cosh(g\tau), \tag{3}$$

where g is the proper acceleration of the charge, and τ is its proper time. The world line

$$z^2-t^2=\left(\frac{1}{g}\right)^2 \tag{4}$$

is a hyperbola. Referring to Fig. 1, one can see that this charge can influence regions A and B of spacetime, which lie along the forward light cone of the charge's trajectory but not the regions C and D. Let us now fix a proper, Fermi-Walker transported coordinate system (τ,ζ,x,y) to the accelerating charge and call it frame U. Separate transformations are defined from S to U for regions A and B. In region A, we take

$$t=\frac{\sqrt{2g\zeta}}{g}\sinh(g\tau),\quad z=\frac{\sqrt{2g\zeta}}{g}\cosh(g\tau),\quad \zeta>0 \tag{5}$$

and in region B we take

$$t=\frac{\sqrt{-2g\zeta}}{g}\cosh(g\tau),\quad z=\frac{\sqrt{-2g\zeta}}{g}\sinh(g\tau),\quad \zeta<0, \tag{6}$$

x and y are mapped to themselves. The spacetime interval, both in regions A and B is

$$ds^2=2g\zeta d\tau^2-\frac{d\zeta^2}{2g\zeta}-d\rho^2-\rho^2 d\phi^2, \tag{7}$$

where $\rho=\sqrt{x^2+y^2}$ and $\phi=\tan^{-1}(y/x)$. The range $[\zeta>0;\ -\infty<\tau<+\infty]$ covers region A, and $[\zeta<0;\ -\infty<\tau<+\infty]$ covers region B. In these coordinates, the charge is at rest, at $\zeta_0=(1/2g)$. Since the metric is same for the transformations defined by Eqs. (5) and (6), we can solve Maxwell's equations in the background metric of Eq. (7) and transform separately in regions A and B to get the fields in the frame S.

B. The fields in the accelerated frame

Let us next obtain the solutions to Maxwell's equations in the noninertial Rindler frame. The generally covariant form of Maxwell's equations are

$$\frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}F^{\mu\nu})=4\pi j^\nu \tag{8}$$

with

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu . \tag{9}$$

The current is

$$j^\mu=\frac{e}{\sqrt{-g}}\delta^3(\mathbf{x}-\mathbf{x}_0)\frac{dx^\mu}{dx^0}, \tag{10}$$

where $\mathbf{x}_0=(\zeta_0,0,0)$ is coordinate of the charged particle in the accelerated frame and the Dirac delta function is defined to be

$$\delta^3(\mathbf{x}-\mathbf{x}_0)=\delta(\zeta-\zeta_0)\delta(\rho)\delta(\phi),\quad \int \delta^3(\vec{\mathbf{x}})d\zeta\ \ d\rho\ \ d\phi=1 \tag{11}$$

for a point charge at $\zeta=\zeta_0$. Since the charge is at rest, $j^i=0$ for $i=1,2,3$ and $j^0\neq 0$. Correspondingly, we can take $A_i=0$ with all time derivatives vanishing. Hence the only relevant components of the field tensor are

$$F^{\zeta 0}=-F_{\zeta 0}=-\partial_\zeta A_0,\quad F^{\rho 0}=-\frac{1}{2g\zeta}(\partial_\rho A_0). \tag{12}$$

Expressing the field tensor in terms of the potential, we get the equation satisfied by A_0:

$$\rho\frac{\partial^2 A_0}{\partial\zeta^2}+\frac{1}{2g\zeta}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial A_0}{\partial\rho}\right)=-4\pi e\,\delta^3(\mathbf{x}-\mathbf{x}_0). \tag{13}$$

This equation has a simple, closed form solution which can be obtained by direct integration of Eq. (13) for $\mathbf{x}\neq\mathbf{x}_0$ and matching the boundary condition at $\mathbf{x}=\mathbf{x}_0$:

$$A_0=ge\frac{\zeta+\zeta_0+(1/2)g\rho^2}{\sqrt{[\zeta-\zeta_0+(1/2)g\rho^2]^2+(2g\rho^2\zeta_0)}}$$
$$=ge\frac{\zeta+\zeta_0+(1/2)g\rho^2}{\sqrt{[\zeta+\zeta_0+(1/2)g\rho^2]^2-4\zeta_0\zeta}}. \tag{14}$$

(An alternative derivation of this solution is given in Appendix A) Also, as mentioned earlier, $A_i=0$ implying that there are no magnetic fields.

Let us next compute the electric field corresponding to this potential. In an inertial frame, $F^i{}_0, F^0{}_i, F^{i0}$ can all be interpreted as defining the electric field (apart from difference in signs). However, in the metric defined by Eq. (7), these components have different spatial dependence due to raising and lowering by $g_{\mu\nu}$, which is not constant. So, in order to define the *physical* electric field, we go back to the basic definition of electric field as the ''electromagnetic force per unit charge, experienced by a charge at rest.'' The contravariant electromagnetic force vector is

$$f^\mu=eF^\mu{}_\nu\frac{dx^\nu}{ds}$$

which for a charge at rest gives the electric field

$$E^i\equiv F^i{}_0\frac{dx^0}{ds}=\frac{F^i{}_0}{\sqrt{g_{00}}}. \tag{15}$$

Using this in Eqs. (12), we get the electric field components

$$E^\zeta=\frac{(2ge\zeta_0)\sqrt{2g\zeta}\{[\zeta-\zeta_0-(1/2)g\rho^2]\}}{\xi^3},$$

$$E^\rho=\frac{(2ge\zeta_0)\rho\sqrt{2g\zeta}}{\xi^3},\quad E^\phi=0, \tag{16}$$

where

$$\xi\equiv\sqrt{[\zeta-\zeta_0+(1/2)g\rho^2]^2+2g\rho^2\zeta_0}. \tag{17}$$

There are some interesting features which are worth noting about this field. To simplify the analysis let us transform from the coordinates $(\tau,\zeta,\rho,\varphi)$ to (τ,Z,ρ,φ), where $\zeta=(gZ^2/2)$. The metric in region A is now

$$ds^2=g^2Z^2d\tau^2-(dZ^2+d\rho^2+\rho^2d\varphi^2). \tag{18}$$

The Z component of the electric field in this coordinate system is

$$E^Z=\frac{4e}{g^2}\frac{[Z^2-\rho^2-g^{-2}]}{[(Z^2+\rho^2-g^{-2})^2+4g^{-2}\rho^2]^{3/2}}$$
$$=\frac{4e}{g^2}\frac{[Z^2-\rho^2-g^{-2}]}{[(Z^2+\rho^2+g^{-2})^2-4g^{-2}Z^2]^{3/2}}. \tag{19}$$

In this coordinate system, the (apparent) event horizon is at $Z=0$. On this surface, the electric field is along Z axis and has the value

$$E^Z \text{ (at } Z=0) = -\frac{4e}{g^2}\frac{1}{(\rho^2+g^{-2})^2}. \tag{20}$$

This is finite and is equivalent to having a charge density

$$\sigma \text{ (at } Z=0) = +\frac{E^z}{4\pi} = -\frac{e}{\pi g^2}\frac{1}{(\rho^2+g^{-2})^2} \tag{21}$$

at the apparent horizon (this point was earlier noted in Ref. [2]). *Note that this result is coordinate dependent*. The field E^ζ, in the coordinates $(\tau,\zeta,\rho,\varphi)$, *vanishes* at the horizon. In these coordinates, there is no charge density on the horizon.

If we shift the origin of the Z axis by introducing the coordinate $\overline{Z}=Z-g^{-1}$, then the metric becomes

$$ds^2=(1+g\overline{Z})^2 d\tau^2-(dZ^2+d\rho^2+\rho^2 d\varphi^2) \tag{22}$$

and the electric field becomes

$$E^{\bar{z}}=\frac{e\overline{Z}}{r^3}\left(1+\frac{1}{2}\frac{gr^2}{\overline{Z}}\right)\left(1+g\overline{Z}+\frac{1}{4}g^2r^2\right)^{-3/2}, \tag{23}$$

$$E^\rho=-\frac{e\rho}{r^3}\left(1+\frac{1}{2}\frac{gr^2}{\overline{Z}}\right)(1+g\overline{Z})\left(1+g\overline{Z}+\frac{1}{4}g^2r^2\right)^{-3/2} \times\left(1+g\overline{Z}-\frac{g}{2}\frac{r^2}{\overline{Z}}\right)^{-1}, \tag{24}$$

with $E^\varphi=0$. In this form, it is clear that field is the usual Coulomb field for $g\overline{Z}\ll 1$, $gr\ll 1$. The behavior of the field near the charge, compared to its form near the apparent horizon clearly shows the distorting effect of the background line element. We shall now use this result to obtain the fields of an arbitrarily moving change.

IV. FIELD OF A CHARGE MOVING RECTILINEARLY, WITH ARBITRARY VELOCITY AND ACCELERATION

A. The coordinate transformation

We shall calculate the field due to a rectilinearly moving charge using the approach described in Sec. II. Let this charge move along the z axis of the inertial frame S. We are interested in the field at event P with coordinates (t,z,ρ,ϕ). The retarded event is O with coordinates $(t_0,z_0,0,0)$. At O, let $v_{\rm ret}$ be the velocity of the charge and $a_{\rm ret}$ be its acceleration. Then, the proper acceleration is

$$g=\sqrt{-a^\mu a_\mu}=a_{\rm ret}\gamma^3, \tag{25}$$

where $\gamma=(1-v_{\rm ret}^2)^{-1/2}$. We construct a comoving, uniformly accelerating observer with an attached coordinate frame M with coordinates (τ,ζ,ρ,ϕ) such that the origin of M coincides with the world line of the charge up to v^μ and a^μ at the event O. So, at O, in the frame M, the charge is instantaneously at rest without acceleration. With this construction, the constant, proper, acceleration of M is g, as defined by Eq. (25).

The coordinate transformations from S to M are different in the regions A and B. In region A $(\zeta>0)$

$$t=t_0-\frac{\gamma v_{\rm ret}}{g}+\frac{\sqrt{2g\zeta}}{g}\sinh(g\tau),$$

$$z=z_0-\frac{\gamma}{g}+\frac{\sqrt{2g\zeta}}{g}\cosh(g\tau), \tag{26}$$

while in region B $(\zeta<0)$,

$$t=t_0-\frac{\gamma v_{\rm ret}}{g}+\frac{\sqrt{-2g\zeta}}{g}\cosh(g\tau),$$

$$z=z_0-\frac{\gamma}{g}+\frac{\sqrt{-2g\zeta}}{g}\sinh(g\tau). \tag{27}$$

The constants $(t_0-\gamma v_{\rm ret}/g)$ and $(z_0-\gamma/g)$ ensure the condition that the charge is at rest and with zero acceleration at $\zeta_0=1/(2g)$ in frame M at the event O. The event O has coordinates

$$\zeta_0=\frac{1}{2g},\quad \tau_0=\frac{1}{g}\sinh^{-1}(\gamma v_{\rm ret}),\quad \rho=0,\quad \phi=0 \tag{28}$$

in frame M, as can be verified from Eq. (26). It is convenient to shift the origin and define

$$t'=t-t_0+\frac{\gamma v_{\rm ret}}{g},\quad z'=z-z_0+\frac{\gamma}{g}. \tag{29}$$

In these coordinates, the event O occurs at

$$t'=\frac{\gamma v_{\rm ret}}{g},\quad z_0'=\frac{\gamma}{g}. \tag{30}$$

Given these transformations and the form of the field in the instantaneous Rindler frame, it is straightforward to obtain the field in the inertial frame. Conventionally, the latter fields are expressed in terms of the separation vector between the field point and the retarded position of the particle. To make the comparison we will introduce the null vector R^μ with the components

$$R^\mu=x'^\mu-x'_0{}^\mu=(t'-t_0',z'-z_0',\rho,\phi). \tag{31}$$

Using the condition $R^\mu R_\mu=0$ in region A, we can easily show that

$$\cosh g(\tau-\tau_0)=\frac{\zeta+\zeta_0+(1/2)g\rho^2}{2\sqrt{\zeta}\sqrt{\zeta_0}},\quad \zeta>0,\ \ \zeta_0=\frac{1}{2g}. \tag{32}$$

Further, since the components of $v_{\rm ret}^\mu$ are

$$v_{\rm ret}{}^0=\sqrt{2g\zeta_0}\cosh(g\tau_0),\quad v_{\rm ret}{}^z=\sqrt{2g\zeta_0}\sinh(g\tau_0), \tag{33}$$

we get

$$R_\mu v_{\rm ret}{}^\mu = 2\sqrt{\zeta\zeta_0}\sinh g(\tau-\tau_0) = \sqrt{[\zeta-\zeta_0+(1/2)g\rho^2]^2+2g\rho^2\zeta_0}. \tag{34}$$

Similarly for region B,

$$\sinh g(\tau-\tau_0) = \frac{\zeta+\zeta_0+(1/2)g\rho^2}{2\sqrt{-\zeta}\sqrt{\zeta_0}}, \quad \zeta<0,\ \zeta_0=\frac{1}{2g} \tag{35}$$

and

$$R_\mu v_{\rm ret}{}^\mu = \sqrt{[\zeta-\zeta_0+(1/2)g\rho^2]^2+2g\rho^2\zeta_0} \tag{36}$$

which is the same as that for region A.

B. The field

Given the field in the coaccelerating frame [Eqs. (14) and (16)] and the transformation between the inertial frame and coaccelerating frame [Eqs. (26), (27), and (29)], we can find the field in the inertial frame. We refer to the field tensor in inertial coordinates and the electric and magnetic fields as $F^{\mu\nu}{}_{\rm min}$, $E^i{}_{\rm min}$, and $B^i{}_{\rm min}$, respectively. The electric field, for example, is obtained by

$$E^{z'}{}_{\rm min} = F_{\rm min}{}^{z'}{}_0 = \left(\frac{\partial z'}{\partial\tau}\right)\left(\frac{\partial\zeta}{\partial t'}\right)F^0{}_\zeta + \left(\frac{\partial z'}{\partial\zeta}\right)\left(\frac{\partial\tau}{\partial t'}\right)F^\zeta{}_0 = \frac{g}{2\zeta}(z'^2-t'^2)(-\partial_\zeta A_0). \tag{37}$$

Therefore,

$$E^{z'}{}_{\rm min} = \frac{2ge\zeta_0[\zeta-\zeta_0-(1/2)g\rho^2]}{\xi^3} = \frac{4e}{g^2}\,\frac{[z'^2-t'^2-\rho^2-(1/g)^2]}{\{[z'^2-t'^2+\rho^2-(1/g)^2]^2+4(\rho^2/g^2)\}^{3/2}}. \tag{38}$$

Similarly, we obtain

$$E^\rho{}_{\rm min} = F_{\rm min}{}^\rho{}_0 = \frac{8ez'\rho}{g^2\{[z'^2-t'^2+\rho^2-(1/g)^2]^2+4(\rho^2/g^2)\}^{3/2}}, \tag{39}$$

$$B_{\rm min}{}^\phi = F_{\rm min}{}^{\rho z} = \frac{8et'\rho}{g^2\{[z'^2-t'^2+\rho^2-(1/g)^2]^2+4(\rho^2/g^2)\}^{3/2}}, \tag{40}$$

$$E^\phi{}_{\rm min} = B^z{}_{\rm min} = B^\rho{}_{\rm min} = 0. \tag{41}$$

This can be recast in a more familiar form by using Eqs. (34) and (36):

$$E^{z'}{}_{\rm min} = \frac{ge[z'^2-t'^2-\rho^2-(1/g)^2]}{2(R_\mu v^\mu{}_{\rm ret})^3} = \frac{ge[z'^2-t'^2-\rho^2-(1/g)^2]}{2\gamma^3(R-R_{z'}v_{\rm ret})^3}, \tag{42}$$

where $R=R^0=t'-t'_0$, $R_{z'}=z'-z'_0$. Note that

$$z'^2-t'^2-\rho^2-(1/g)^2 = \frac{2\gamma}{g}(R_{z'}-Rv_{\rm ret})-2\rho^2 \tag{43}$$

$$= \frac{2}{a_{\rm ret}}[(1-v_{\rm ret}{}^2)(R_z-Rv_{\rm ret})-a_{\rm ret}\rho^2]. \tag{44}$$

Therefore, we can write our answer as

$$E^{z'}{}_{\rm min} = \frac{e[(1-v_{\rm ret}{}^2)(R_{z'}-Rv_{\rm ret})-a_{\rm ret}\rho^2]}{(R-R_{z'}v_{\rm ret})^3}. \tag{45}$$

Similarly,

$$E^\rho{}_{\rm min} = \frac{e[(1-v_{\rm ret}{}^2)\rho+a_{\rm ret}R_{z'}\rho]}{(R-R_{z'}v_{\rm ret})^3} \tag{46}$$

$$B^\phi{}_{\rm min} = \frac{e[a_{\rm ret}R\rho+v_{\rm ret}(1-v_{\rm ret}{}^2)\rho]}{(R-R_{z'}v_{\rm ret})^3}. \tag{47}$$

These components can be expressed in a more familiar vector notation as

$$\mathbf{E} = \frac{e(1-v_{\rm ret}{}^2)(\mathbf{R}-\mathbf{v}_{\rm ret}R)}{(R-\mathbf{R}\cdot\mathbf{v}_{\rm ret})^3} + \frac{e\mathbf{R}\times[(\mathbf{R}-\mathbf{v}_{\rm ret})\times\mathbf{a}_{\rm ret}]}{(R-\mathbf{R}\cdot\mathbf{v}_{\rm ret})^3}, \tag{48}$$

$$\mathbf{B} = \frac{\mathbf{R}\times\mathbf{E}}{R}. \tag{49}$$

This is the standard result for the electromagnetic field of an arbitrary moving charged particle (see Ref. [1]).

Our results in Eqs. (45), (46), and (47) have been derived for the special case of a charge in rectilinear motion. This was done to show clearly the use of our formalism. In fact, one can obtain the general result quite easily. Consider the general case, in which the motion is not restricted to a straight line. Then, one can always transform to an *inertial* frame of reference S'' in which the charge was at rest at the retarded event O. This requires us to make the usual transformation to the accelerated frame

$$t = t_0 + \frac{\sqrt{2g\zeta}}{g}\sinh(g\tau), \quad z = z_0 - \frac{1}{g} + \frac{\sqrt{2g\zeta}}{g}\cosh(g\tau), \tag{50}$$

followed by a Lorentz transformation to bring $\mathbf{v}_{\rm ret}$ to zero. Working in a similar fashion, we will land up with simpler expressions for the fields:

$$\mathbf{E} = \frac{e\mathbf{R}}{R^3} + \frac{\mathbf{R}\times(\mathbf{R}\times\mathbf{a}_{\rm ret})}{R^3}, \quad \mathbf{B} = \frac{\mathbf{R}\times\mathbf{E}}{R}. \tag{51}$$

This gives the field in the Lorentz frame in which the charge has zero velocity at the retarded event. By making a Lorentz transformation with an arbitrary velocity **v**, we can get fields in Eqs. (48), (49). More formally, one can show that the fields in Eq. (51) can be obtained from the following Lorentz invariant expression:

$$F_{\mu\nu}=\frac{e}{(R^{\sigma}v_{\sigma})^{3}}[(R_{\mu}v_{\nu}-R_{\nu}v_{\mu})+R_{\nu}R^{\sigma}(v_{\mu}a_{\sigma}-v_{\sigma}a_{\mu})-R_{\mu}R^{\sigma}(v_{\nu}a_{\sigma}-v_{\sigma}a_{\nu})]. \tag{52}$$

Then, since this is a tensor equation under Lorentz transformations, it will give the fields in a frame S in which the retarded velocity is arbitrary. A simple calculation shows that Eq. (52) reduces to Eqs. (48) and (49) in this case. [This form is derived in Ref. [3] in a very complicated manner. A simple derivation of Eq. (52) is given in Appendix C.]

V. ELECTROMAGNETIC FIELDS IN THE INFINITESIMAL NEIGHBORHOOD OF AN ACCELERATING CHARGE

As before, let us consider a rectilinearly moving charge with arbitrary velocity and acceleration along the z axis of frame S. The instantaneous, uniformly accelerating frame is M. At the event O, the charge is at rest in M and with zero acceleration at the point $\zeta=\zeta_0$. We are interested in calculating the finite part of the force exerted on the charge by it's own field at the event O. It is now convenient to use the coordinate Z introduced earlier with the definition

$$Z=\frac{\sqrt{2g\zeta}}{g}. \tag{53}$$

In the frame with coordinates (τ,Z,ρ,ϕ), the charge, is at rest in M at $Z=Z_0=(1/g)$ at event O. In these coordinates, the fields corresponding to the retarded event O are

$$E^{Z}=\frac{4e(Z^2-Z_0{}^2-\rho^2)}{g^2[(Z^2-Z_0{}^2+\rho^2)^2+4\rho^2Z_0{}^2]^{3/2}},$$

$$E^{\rho}=\frac{8e\rho Z}{g^2[(Z^2-Z_0{}^2+\rho^2)^2+4\rho^2Z_0{}^2]^{3/2}}. \tag{54}$$

These are same as those given by Eqs. (38) and (39), but expressed in the new coordinate Z.

We will analyze the situation described in Fig. 2. The event O, at which the charge has zero velocity and acceleration, corresponds to $\tau=0$. Consider now an event O' along the world line of the charge. The forward light ray traveling from this event is seen to intersect the Z axis at the point P, which is an event simultaneous with event O. We want to study the fields at P in the limit $P\rightarrow O$. In this limit, $O'\rightarrow O$ and the fields at P are those due to the charge in its own infinitesimal neighborhood. (Since the metric is noninertial, the curve $O'P$ will not be a straight line. But we are only interested in the limit of $O'\rightarrow O$ when the actual form of the curve is irrelevant.)

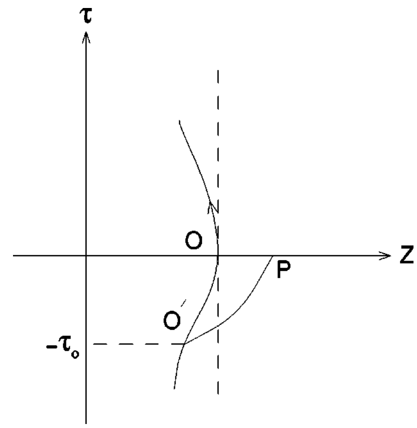

FIG. 2. The geometry of the word lines indicating how radiation reaction force arises; see text for more discussion.

The field due to the *retarded point* O is a static field, given by Eq. (54). However, we are interested in calculating the field at event P due to the *event* O'. At O', the charge is not at rest in frame M. So, the field given by Eq. (54) will not work. But, in the limit $O\rightarrow O'$, we can make a correction to the field given by Eq. (54), which can account for its motion at event O'. To illustrate this, let us approach the infinitesimal neighborhood of the charge along the Z axis. If we ignore for the moment the motion of the charge then the field along the Z axis at event P (in the limit $O\rightarrow O'$) is

$$E^{Z}=\frac{4e}{g^2}\frac{1}{(Z^2-Z_0{}^2)^2}, \quad E^{\rho}=0. \tag{55}$$

Let the event O' occur at $\tau=-\tau_0$. In the approximation that the charge was at rest at O', it can be verified that

$$R_{\mu}v^{\mu}{}_{\rm ret}=\frac{g}{2}(Z^2-Z_0{}^2), \tag{56}$$

which can be obtained from Eqs. (34) and (53) after setting $\rho=0$. So the field can be expressed as

$$E^{Z}=\frac{e}{(R_{\mu}v_{\rm ret}{}^{\mu})^2}. \tag{57}$$

Let us next account for the charge's motion at O'. We begin by noting that, in arriving at Eq. (34), we used the fact that the charge was at rest at the retarded event. When we take into account the motion of the charge at O', the expression for $(R_{\mu}v^{\mu}_{\rm ret})$ will be modified. Since the Coulomb part of the field dominates as $O'\rightarrow O$, the leading term in the field is still given by Eq. (57) with the corrected expression for $R_{\mu}v_{\rm ret}{}^{\mu}$. If velocity of the charge at O' is $u^{Z}(-\tau_0)$, then, in the limit we are interested in, the updated $(R_{\mu}v_{\rm ret}{}^{\mu})$ is

$$R_{\mu}v_{\rm ret}{}^{\mu}=\frac{g}{2}[Z^2-Z_0{}^2(-\tau_0)]-[Z-Z_0(-\tau_0)]u^{Z}(-\tau_0). \tag{58}$$

So the expression for the field which accounts for the motion of the charge at O' will be

$$E^Z(P) = \frac{4e}{g^2\{Z^2 - Z_0{}^2(-\tau_0) - (2/g)[Z - Z_0(-\tau_0)]u^Z(-\tau_0)\}^2}. \tag{59}$$

At event O, the velocity and acceleration of the charge are zero in M. However, the rate of change of acceleration is nonzero, and we denote this quantity by $\dot{\alpha}$. Then, all the relevant quantities at time $\tau = -\tau_0$ can be expressed in terms of $\dot{\alpha}$ alone, in the limit $O \rightarrow O'$ (that is when $\tau_0 \rightarrow 0$). Making a Taylor expansion about $\tau = 0$, we get

$$Z_0(-\tau_0) \cong Z_0 - \frac{\tau_0{}^3}{6}\dot{\alpha} \tag{60}$$

and

$$u^Z(-\tau_0) \cong \frac{\tau_0{}^2}{2}\dot{\alpha}, \tag{61}$$

where $Z_0 = Z_0(0) = g^{-1}$. (Here, only terms up to order $\tau_0{}^3$ need to be retained, in the limit of $\tau_0 \rightarrow 0$.) In this limit, writing $Z = Z_0 + \delta$, we find that

$$E^Z(P) \cong \frac{4e}{g^2[\delta^2 + 2Z_0\delta + (Z_0\tau_0{}^3\dot{\alpha}/3) - Z_0\delta\tau_0{}^2\dot{\alpha}]^2}. \tag{62}$$

From Eq. (32), we get, after setting $\tau = 0$, $\rho = 0$ and replacing τ_0 by $-\tau_0$

$$\sinh(g\tau_0) = \frac{\zeta - \zeta_0}{2\sqrt{\zeta\zeta_0}} \cong \frac{\delta}{Z_0} \tag{63}$$

in the limit $\delta \rightarrow 0$. Hence, in the limit $\tau_0 \rightarrow 0$,

$$\tau_0 \cong \delta. \tag{64}$$

We are interested in the force exerted on the charge by its own field, which is equal to $eE^Z(P)$ in the limit $\delta \rightarrow 0$. This force is given by

$$F^Z = eE^Z(P) \cong \frac{e^2}{\delta^2} - \frac{e^2 g}{\delta} + \frac{2e^2}{3}\dot{\alpha} + \frac{3e^2 g^2}{4}, \tag{65}$$

where we have expanded the expression for E^Z in the binomial series in the infinitesimal parameter δ. It is understood that we should evaluate this expression in the limit of $\delta \rightarrow 0$.

The first two terms diverge as $\delta \rightarrow 0$. This point is extensively discussed in literature, and these terms arise from the self-energy of a charged particle due to interaction with its own electromagnetic field and are expected to be absorbed by mass renormalization. There is also the constant (last) term, which is uninteresting, since the first two terms are already divergent. We would have landed up with these three terms, even if we had not accounted for the motion of the charge at event O'. It is the third term, which has the derivative of the acceleration, which is the most interesting term. We have been able to obtain it because we accounted for the motion of the charge at event O'. *It is this term which accounts for the effect of radiation reaction on the charge.*

All these terms were first found by Dirac for arbitrary motion of the charge in an inertial frame. The general expression in the inertial frame obtained by Dirac is (see Ref. [4]; also see pp. 141–144 of Ref. [3])

$$f^\mu \cong \frac{e^2}{\sqrt{1-(\delta)a^\lambda u_\lambda}}\left[\frac{u^\mu}{\delta^2} - \frac{a^\mu}{2\delta} - \frac{a^\mu\ (a^\lambda u_\lambda)}{2} + \frac{g a^\mu}{8} - \frac{u^\mu\ (\dot{a}_\lambda v^\lambda)}{2} + \frac{2}{3}[\dot{a}^\mu - v^\mu(\dot{a}_\lambda v^\lambda)]\right]. \tag{66}$$

Here, a^μ is the four-acceleration, v^μ the 4-velocity, $\delta = R^\lambda v_\lambda$ and $u^\mu \equiv [(R^\mu - \delta v^\mu)/(\delta)]$. These are the leading terms in an expansion in δ in the limit $\delta \rightarrow 0$.

Computing the above expression for f^μ in the inertial frame is a laborious task. Our formalism makes the corresponding computation very simple. In fact, we calculated it the same way we calculated the fields: We transform to an inertial frame S'', in which the charge is at rest at the retarded event and find the expression for the force in the comoving accelerating frame, which is given by Eq. (65). By transforming back to S'' and making an arbitrary Lorentz transformation to S gives the force (66) in the general inertial frame starting from Eq. (65). (We give the explicit procedure in Appendix B.) Here we shall transform only the radiation reaction term, and show that it gives the correct radiation reaction in the inertial frame, for the case of rectilinear motion.

Let f^z denote the radiation reaction force in the inertial frame S. Then, if $F_{\rm rad}{}^Z$ denotes the radiation reaction force in frame M, then using Eqs. (26), (28), and (53), we have

$$f^z = \left(\frac{\partial z}{\partial Z}\right)_{\rm ret} F_{\rm rad}{}^Z = \frac{2e^2}{3}\dot{\alpha}\cosh(g\tau_0). \tag{67}$$

The derivative of the retarded acceleration $\dot{a}^\mu_{\rm ret} = da^\mu/(d\tau)$, as measured in the frame S is related to $\dot{\alpha}$ by

$$\dot{a}^0 = g^2\cosh(g\tau_0) + \dot{\alpha}\sinh(g\tau_0), \tag{68}$$

$$\dot{a}^z = g^2\sinh(g\tau_0) + \dot{\alpha}\cosh(g\tau_0). \tag{69}$$

Then, using Eq. (67), we get

$$f^z = \frac{2e^2}{3}\dot{\alpha}\cosh(g\tau_0) = \frac{2e^2}{3}[\dot{a}^z - v^z(\dot{a}^\mu v_\mu)], \tag{70}$$

which is indeed the radiation reaction force in the inertial frame S. For an arbitrarily moving charge, one can first transform to the instantaneous Lorentz frame, in which (putting $v^z = 0$)

$$f^z = \frac{2e^2}{3}\dot{a}^z \tag{71}$$

and make a Lorentz transformation to an arbitrary inertial frame, to get

$$f^{\mu}=\frac{2e^{2}}{3}[\dot{a}^{\mu}-v^{\mu}(\dot{a}^{\nu}v_{\nu})] \tag{72}$$

which is the correct expression for radiation reaction.

An attempt is made in Ref. [2] to relate the *radiated power* to the force acting between the charge and the fictitious charge density at the horizon. Our result is more general and gives the actual *radiation reaction force* itself. Further we did not have to use the *ficticious*, coordinate dependent, charge density to interpret a *real* effect.

VI. CONCLUSIONS

The radiation of electromagnetic waves by a charged particle and the consequent radiation reaction force are issues of considerable theoretical significance and have attracted the attention of researchers over the decades. We believe that the approach outlined in this paper throws light on these processes and clarifies the conceptual issues involved in the problem. To begin with, we have been able to derive the radiation field as arising out of a static Coulomb field in a noninertial frame through a general coordinate transformation. This is of some conceptual importance since one believes that the physics should be independent of the coordinate system. Secondly, we have been able to show that the key contribution to radiation reaction arises because of the deviation of the trajectory from that of a uniformly accelerated one. This deviation, which essentially modifies the expression $R^{\mu}v_{\mu}$, has a purely geometrical origin in the locally coaccelerating frame. Since the lowest order deviation will be proportional to the rate of change of acceleration, it is clear that the radiation reaction force should be proportional to the same; that is, it should be proportional to the third derivative of the trajectory.

It will be of interest to see whether these results allow us to tackle the question of self-force in curved space time and to generalize the various expressions to an arbitrary curved background. We hope to address these questions in a future publication.

APPENDIX A: SOLUTION TO MAXWELL'S EQUATIONS FOR A CHARGE AT REST IN A UNIFORMLY ACCELERATING FRAME

The scalar potential A_0 due to a charge at rest in a uniformly accelerating frame satisfies the following equation [see Eq. (13)]:

$$\rho\partial_{\zeta}{}^{2}A_{0}+\frac{1}{2g\zeta}\partial_{\rho}(\rho\partial_{\rho}A_{0})=-4\pi e\,\delta(\zeta-\zeta_{0})\quad\delta(\rho)\quad\delta(\phi). \tag{A1}$$

We shall find a solution to this equation by studying a different, but related problem.

Consider the problem of a charge placed at rest outside the horizon of a Schwarzschild black hole. In the spherical polar coordinates (r,θ,ϕ), the charge is placed at $r=r',\ \theta=0$. The metric is given by

$$ds^{2}=\left(1-\frac{2M}{r}\right)dt^{2}-\frac{dr^{2}}{(1-2M/r)}-r^{2}(d\theta^{2}+\sin^{2}\theta d\phi^{2}). \tag{A2}$$

Maxwell's equations are separable in these coordinates. The differential equation satisfied by A_0 in this metric is

$$\frac{1}{r^{2}}\partial_{r}(r^{2}\partial_{r}A_{0})+\frac{1}{(1-2M/r)}\frac{1}{r^{2}\sin\theta}\partial_{\theta}(\sin\theta\partial_{\theta}A_{0})$$
$$=-4\pi e\,\delta(r-r')\,\delta(\cos\theta-1). \tag{A3}$$

The solution to this equation can be expressed in a closed form (see Ref. [5]):

$$A_{0}=\frac{e[(r-M)(r'-M)-M^{2}\cos\theta]}{rr'\sqrt{(r-M)^{2}+(r'-M)^{2}-2(r-M)(r'-M)\cos\theta-M^{2}\sin^{2}\theta}}. \tag{A4}$$

Now, if the charge is placed infinitesimally close to the horizon and all measurements made in an arbitrarily small region around the charge, then the horizon would appear "flat." Mathematically, if we introduce a coordinate ζ by

$$r=2M+\zeta;\quad 2M\gg\zeta, \tag{A5}$$

then we can write

$$\left(1-\frac{2M}{r}\right)\cong\frac{\zeta}{2M}=2g\zeta, \tag{A6}$$

where $g=(1/4M)$ is the effective surface gravity of the horizon. If we fix our origin at $\theta=0,\ r\cong 2M$ and restrict all observations perpendicular to the z axis to a very small region, then $\rho\cong 2M\sin\theta\cong 2M\theta$. In this limit,

$$r^{2}(d\theta^{2}+\sin^{2}\theta\ d\phi^{2})\cong(2M)^{2}(d\theta^{2}+\theta^{2}\ d\phi^{2})$$
$$\cong d\rho^{2}+\rho^{2}\ d\phi^{2} \tag{A7}$$

giving

$$ds^{2}=2g\zeta\ dt^{2}-\frac{d\zeta^{2}}{2g\zeta}-d\rho^{2}-\rho^{2}\ d\phi^{2} \tag{A8}$$

which is identical to the metric given by Eq. (7).

In this approximation, we are neglecting curvature of spacetime such that the horizon appears as a plane ($\zeta=0$) as in a uniformly accelerating frame. More importantly, it gives us an ansatz to find a solution to Eq. (13), by using this approximation in the expression for A_0 given by Eq. (B3). Straightforward algebra gives

$$A_0 \cong \frac{ge[\zeta+\zeta_0+(1/2)g\rho^2]}{\sqrt{[\zeta-\zeta_0+(1/2)g\rho^2]^2+2g\rho^2\zeta_0}} \tag{A9}$$

which is an exact solution to equation (A1).

APPENDIX B: THE LORENTZ-DIRAC FORMULA

Consider a charge moving in an arbitrary trajectory in an inertial frame S. As before, we construct a comoving, uniformly accelerating frame M, in which the expression for the self-force is given by Eq. (65). Consider now the expression (66) in a frame in which

$$v^\mu=(1,0,0,0), \quad \dot{a}^\mu=(g^2,\dot{\alpha},0,0), \quad a^\mu=(0,g,0,0). \tag{B1}$$

The expression reduces to

$$f^\mu \cong e^2 \; a^\mu\left(\frac{1}{g\delta^2}-\frac{1}{\delta}+\frac{3g}{4}\right)+\frac{2e^2}{3}[\dot{a}^\mu-v^\mu(\dot{a}_\lambda v^\lambda)] \tag{B2}$$

so that

$$f^z \cong e^2\left(\frac{1}{\delta^2}-\frac{g}{\delta}+\frac{3g^2}{4}+\frac{2e^2}{3}\dot{a}^z\right). \tag{B3}$$

(It is assumed that $\delta \rightarrow 0$ limit is considered; also note that $u^\mu \rightarrow a^\mu$ in this limit.) This is identical to Eq. (65) when we use Eq. (69) with $\tau_0=0$. If one wants the expression for f^μ in the frame S, all one has to do is find the expression in M (which is a simpler task compared to directly calculating it in the frame S, as is normally done), given by Eq. (65), transform it to the inertial frame [to get expression (B3)], and finally to transform it to S by making a Lorentz transformation. This leads to the Lorentz-Dirac expression, given by Eq. (66).

APPENDIX C: COVARIANT FORM OF THE FIELD OF AN ARBITRARILY MOVING CHARGE

It is possible to express the field tensor $F^{\mu\nu}$ produced by an arbitrarily moving charge, in a manifestedly Lorentz invariant form. Though the result is obtained in Ref. [3], the derivation is quite cumbersome. We give here a clearer and simpler derivation of this formula. Maxwell's equations in inertial coordinates in flat spacetime are

$$\partial_\mu F^{\mu\nu}=4\pi j^\nu, \quad F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu. \tag{C1}$$

Combined together in the Lorentz gauge $(\partial^\mu A_\mu)=0$, we have

$$\Box A_\mu=4\pi j_\mu \tag{C2}$$

which has the solution

$$A_\mu(x)=4\pi\int d^4x G_{\rm ret}(x-y)j_\mu(y), \tag{C3}$$

where $G_{\rm ret}$ is the retarded Green's function, satisfying

$$\Box G_{\rm ret}(x-y)=\delta^4(x-y), \quad G_{\rm ret}(x-y)=0 \quad \text{for } x^0<y^0. \tag{C4}$$

The current $j^\mu(x)$ for a point charge moving along a world line $z^\mu=z^\mu(\tau)$ with a four-velocity $u^\mu(\tau)$ is given by

$$j^\mu(x)=e\int d\tau\delta^4[x-z(\tau)]u^\mu(\tau) \tag{C5}$$

so that

$$A_\mu(x)=4\pi e\int d\tau G_{\rm ret}[x-z(\tau)]u_\mu(\tau). \tag{C6}$$

Now, let $R^\mu=x^\mu-z^\mu(\tau)$. Then,

$$G_{\rm ret}[x-z(\tau)]=\frac{1}{2\pi}\delta(s^2) \quad \theta(x^0-z^0), \quad s^2\equiv R^\mu R_\mu \tag{C7}$$

giving

$$A_\mu(x)=2e\int d\tau\delta(s^2)u_\mu(\tau) \tag{C8}$$

and

$$\partial_\nu A_\mu(x)=2e\int d\tau\partial_\nu\delta(s^2)u_\mu(\tau)$$
$$=2e\int d\tau \; \frac{d\delta(s^2)}{ds^2}\frac{\partial s^2}{\partial x^\nu} \; u_\mu(\tau). \tag{C9}$$

Now,

$$\frac{\partial s^2}{\partial x^\nu}=2R_\nu. \tag{C10}$$

Therefore,

$$\partial_\nu A_\mu(x)=4e\int d\tau\frac{d\delta(s^2)}{d\tau}\left(\frac{ds^2}{d\tau}\right)^{-1} R_\nu u_\mu(\tau). \tag{C11}$$

Also,

$$\frac{ds^2}{d\tau}=-2\rho, \quad \rho\equiv R^\mu u_\mu \tag{C12}$$

leading to

$$\partial_\nu A_\mu(x)=-2e\int d\tau\frac{d\delta(s^2)}{d\tau}\left(\frac{R_\nu u_\mu}{\rho}\right)$$
$$=2e\int d\tau\delta(s^2)\frac{d}{d\tau}\left(\frac{R_\nu u_\mu}{\rho}\right)=\frac{e}{\rho}\frac{d}{d\tau}\left(\frac{R_\nu u_\mu}{\rho}\right)\bigg|_{\rm ret}. \tag{C13}$$

It follows that

$$F_{\mu\nu}=\partial_\mu A_\nu-\partial\nu A_\mu=\frac{e}{\rho}\frac{d}{d\tau}\left(\frac{R_\mu u_\nu-R_\nu u_\mu}{\rho}\right)\bigg|_{\rm ret}. \tag{C14}$$

Differentiating the expression, we get

$$F_{\mu\nu}=\frac{e}{\rho^3}[(R^\sigma u_\sigma)(R_\mu a_\nu - R_\nu a_\mu) + (1-R^\sigma a_\sigma)(R_\mu u_\nu - R_\nu u_\mu)]. \tag{C15}$$

Using $R^\sigma R_\sigma = 0$, $R^\mu = (R,\mathbf{R})$, $u^\mu = \gamma(1,\mathbf{u})$, $a^\mu = \gamma(\dot{\gamma},\mathbf{u}\dot{\gamma} + \gamma\mathbf{a})$, we get the electric and magnetic fields as

$$\mathbf{E}=\frac{e(1-u^2)(\mathbf{R}-\mathbf{u}R)}{(R-\mathbf{R}\cdot\mathbf{u})^3}+\frac{e\mathbf{R}\times[(\mathbf{R}-\mathbf{u}R)\times\mathbf{a}]}{(R-\mathbf{R}\cdot\mathbf{u})^3}, \tag{C16}$$

$$\mathbf{B}=\frac{\mathbf{R}\times\mathbf{E}}{R}. \tag{C17}$$

These are the standard textbook expressions for the fields.

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