Is there chaos in low-energy string cosmology?

John D. Barrow*
Astronomy Centre, University of Sussex, Falmer, Brighton BN1 9QJ, United Kingdom

Mariusz P. Dąbrowski†
Astronomy Centre, University of Sussex, Falmer, Brighton BN1 9QJ, United Kingdom,
and Institute of Physics, University of Szczecin, Wielkopolska 15, 70-451 Szczecin, Poland

(Received 10 November 1997; published 29 May 1998)

Bianchi type IX, ''mixmaster'' universes are investigated in low-energy-effective-action string cosmology. We show that, unlike in general relativity, there is no chaos in these string cosmologies for the case of the tree-level action. The characteristic mixmaster evolution through a series of Kasner epochs is studied in detail. In the Einstein frame an infinite sequence of chaotic oscillations of the scale factors on approach to the initial singularity is impossible, as it was in general relativistic mixmaster universes in the presence of a massless scalar field. A finite sequence of oscillations of the scale factors described by approximate Kasner metrics is possible, but it always ceases when all expansion rates become positive. In the string frame the evolution through Kasner epochs changes to a new form which reflects the duality symmetry of the theory. Again, we show that chaotic oscillations must end after a finite time. The need for duality symmetry appears to be incompatible with the presence of chaotic behavior as $t \rightarrow 0$. We also obtain our results using the Hamiltonian qualitative cosmological picture for mixmaster models. We also prove that a time-independent pseudoscalar axion field h is not admitted by the Bianchi type IX geometry. [S0556-2821(98)04312-4]

PACS number(s): 98.80.Hw, 04.50.+h, 11.25.Mj, 98.80.Cq

I. INTRODUCTION

String cosmology has attracted a lot of interest recently, especially in the context of duality symmetry, which is a striking feature of the underlying string theory. The motion of the simplest bosonic string in background fields is governed by the nonlinear sigma-model action [1]. Cosmological solutions (to lowest order in α'—the inverse string tension) have been considered in detail in many papers with special interest in the possibility of inflation, the behavior of inhomogeneities, and the relation between the so called pre- and post-big-bang phases of evolution. [2]. The bosonic string spectrum of particles contains the graviton, dilaton and axion (antisymmetric tensor field). The dilaton can always be accommodated within homogeneous geometries, but this is not the case for the axion. It has been shown that the antisymmetric tensor-field potential does not necessarily have to be time-independent and this can prevent the isotropisation of homogeneous models at late times [3]. Moreover, the time-dependent antisymmetric tensor potential cannot be accommodated in all homogeneous anisotropic universes of Bianchi or Kantowski-Sachs types [4]. In this paper we concentrate on Bianchi type IX (BIX) axion-dilaton cosmology with a homogeneous ansatz as given originally in [5]. This is equivalent to the inclusion of a time-dependent pseudoscalar axion field, $h=h(t)$. The dilaton field is always homogeneous, $\phi=\phi(t)$.

It is well known that in the vacuum BIX homogeneous cosmology one approaches the initial singularity chaotically [6]. An infinite number of oscillations of the orthogonal scale factors occurs in general on any finite interval of proper time including the singularity at $t=0$. These oscillations are created by the 3-curvature anisotropy of the spacetime and are intrinsically general relativistic in origin. Physically, the propagation of homogeneous gravitational waves alters the curvature of spacetime along the direction of propagation, so that their non-linear back-reaction on the curvature reverses the direction of propagation.

In the presence of a massless scalar field (or, alternatively, ''stiff matter,'' with pressure, p, equal to density, ρ) the situation changes. Only a finite number of spacetime oscillations can occur before the evolution is changed into a state in which all directions shrink monotonically to zero as the curvature singularity is reached and the oscillatory behavior ceases [7]. In this paper we want to investigate the oscillatory approach to singularity in BIX low-energy-effective-action homogeneous string cosmologies with both axion and dilaton fields.

This investigation complements other recent studies of the behavior of string cosmologies at early times. The investigation in [8] shows that there is an open neighborhood of initial data space for string cosmologies which display velocity-dominated behavior. That is, on approach to the singularity the spatial gradients become negligible compared to time derivatives, velocities are non-relativistic, and curvature anisotropies are negligible. These solutions resemble inhomogeneously-varying Kasner metrics, but the approximation only hold on scales larger than the particle horizon. Barrow and Kunze [9] found a large family of exact inhomogeneous string cosmologies which are velocity-dominated. They describe the evolution of the dilaton and axion fields on all scales, showing how these fields oscillate once inhomogeneities in their distribution enter the horizon. These studies show that the general behavior of the low-

*Email address: J.D.Barrow@sussex.ac.uk
†Email address: mpd@star.cpes.susx.ac.uk

energy-string cosmology equations are considerably simpler than those of general relativity. The string gravity sector behaves in a quasi-Newtonian fashion in the vicinity of cosmological singularities. In general relativity the velocity-dominated, quasi-Kasner, behavior is disrupted by the chaotic oscillations of the three-curvature anisotropy on approach to the singularity whenever the matter content is a massless scalar field or perfect fluid satisfying $p<\rho$. It is therefore important to determine what happens in the presence of three-curvature anisotropies in string cosmology. It is possible that there exists another stable neighborhood of evolution on approach to the initial singularity which is chaotic rather than of quasi-Kasner form. We will investigate this situation by studying the Bianchi type IX string cosmologies in order to discover whether chaotic behavior is possible or whether it is destroyed by the presence of the dilaton and axion fields.

The paper is organized as follows. In Sec. II we derive the string low-energy-effective-action equations for BIX geometry. We present the equations in both the string and the Einstein frames and we use orthonormal frames in the main body of the paper. In Sec. III we discuss the possible routes to chaos near a curvature singularity in both the Einstein and the string frames with reference to the calculations given in Ref. [6]. In Sec. IV we discuss the relation between the oscillating mixmaster behavior of the scale factors and the duality symmetry of the string equations. In Sec. V we use the Hamiltonian approach to represent the evolution of the BIX string cosmology as the motion of a ‘‘universe point’’ inside a curvature potential. This picture provides a simple picture of the conditions needed for chaos to occur, and for it to be absent, in the Einstein, string and so-called axion frames. In Appendix A we give the relation between Ricci tensors for BIX model in orthonormal and holonomic (coordinate) frames and show why it is impossible to include a time-independent pseudoscalar axion field h in the non-axisymmetric and axisymmetric BIX models.

II. LOW-ENERGY-EFFECTIVE-ACTION EQUATIONS FOR BIANCHI TYPE IX UNIVERSES

The low-energy-effective-action field equations in the string frame are given by[1] [3,5]

$$R^{\nu}_{\mu}+\nabla_{\mu}\nabla^{\nu}\phi-\frac{1}{4}H_{\mu\alpha\beta}H^{\nu\alpha\beta}=0, \tag{2.1}$$

$$R-\nabla_{\mu}\phi\nabla^{\mu}\phi+2\nabla_{\mu}\nabla^{\mu}\phi-\frac{1}{12}H_{\mu\nu\beta}H^{\mu\nu\beta}=0, \tag{2.2}$$

$$\nabla_{\mu}(e^{-\phi}H^{\mu\nu\alpha})=\partial_{\mu}(e^{-\phi}\sqrt{-g}H^{\mu\nu\alpha})=0, \tag{2.3}$$

where ϕ is the dilaton field, $H_{\mu\nu\beta}=6\partial_{[\mu}B_{\nu\beta]}$ is the field strength of the antisymmetric tensor $B_{\mu\nu}=-B_{\nu\mu}$. In the Einstein frame we have [3]

$$\tilde{R}^{\nu}_{\mu}-\frac{1}{2}\tilde{g}^{\nu}_{\mu}\tilde{R}=\kappa^{2}(\tilde{T}^{\nu(\phi)}_{\mu}+\tilde{T}^{\nu(H)}_{\mu}), \tag{2.4}$$

$$\tilde{\nabla}_{\mu}\tilde{\nabla}^{\mu}\phi+\frac{1}{6}e^{-2\phi}\tilde{H}^{2}=0, \tag{2.5}$$

$$\tilde{\nabla}_{\mu}(e^{-2\phi}\tilde{H}^{\mu\nu\alpha})=\tilde{\partial}_{\mu}(e^{-2\phi}\sqrt{-\tilde{g}}\tilde{H}^{\mu\nu\alpha})=0, \tag{2.6}$$

and ($\kappa^{2}=8\pi G,\ c=1$)

$$\kappa^{2}\tilde{T}^{\nu(\phi)}_{\mu}=\frac{1}{2}\left(\tilde{g}^{\lambda}_{\mu}\tilde{g}^{\nu\sigma}-\frac{1}{2}\tilde{g}^{\nu}_{\mu}\tilde{g}^{\lambda\sigma}\right)\tilde{\nabla}_{\lambda}\phi\tilde{\nabla}_{\sigma}\phi, \tag{2.7}$$

$$\kappa^{2}\tilde{T}^{\nu(H)}_{\mu}=\frac{1}{12}e^{-2\phi}\left(3\tilde{H}_{\mu\alpha\beta}\tilde{H}^{\nu\alpha\beta}-\frac{1}{2}\tilde{g}^{\nu}_{\mu}\tilde{H}^{2}\right). \tag{2.8}$$

Covariant derivatives are formed with respect to the Bianchi type IX metric which, in the string frame, reads as [13]

$$ds^{2}=dt^{2}-a^{2}(t)(\sigma^{1})^{2}-b^{2}(t)(\sigma^{2})^{2}-c^{2}(t)(\sigma^{3})^{2}, \tag{2.9}$$

where the orthonormal forms $\sigma^1,\sigma^2,\sigma^3$ are given by ($i,j,k,l=0,1,2,3$ are orthonormal basis indices)

$$\sigma^{1}=\cos\psi d\theta+\sin\psi\sin\theta d\varphi, \tag{2.10}$$

$$\sigma^{2}=\sin\psi d\theta-\cos\psi\sin\theta d\varphi, \tag{2.11}$$

$$\sigma^{3}=d\psi+\cos\theta d\varphi, \tag{2.12}$$

and the angular coordinates ψ,θ,φ span the following ranges:

$$0\leqslant\psi\leqslant4\pi,\quad 0\leqslant\theta\leqslant\pi,\quad 0\leqslant\varphi\leqslant2\pi. \tag{2.13}$$

In the Einstein frame the metric is

$$d\tilde{s}^{\,2}=d\tilde{t}^{\,2}-\tilde{a}^{\,2}(t)(\sigma^{1})^{2}-\tilde{b}^{\,2}(t)(\sigma^{2})^{2}-\tilde{c}^{2}(t)(\sigma^{3})^{2}, \tag{2.14}$$

and

$$d\tilde{t}=e^{-\phi/2}dt, \tag{2.15}$$

$$\tilde{a}_{i}=e^{-\phi/2}a_{i}, \tag{2.16}$$

where $\tilde{a}_i=\{\tilde{a},\tilde{b},\tilde{c},\}$ and $a_i=\{a,b,c\}$.

The nonzero Ricci tensor components in the orthonormal frame σ^i are (an overdot means a derivative with respect to the synchronous coordinate time t) [6]

$$-R^{0}_{0}=\frac{\ddot{a}}{a}+\frac{\ddot{b}}{b}+\frac{\ddot{c}}{c}, \tag{2.17}$$

$$-R^{1}_{1}=\frac{\ddot{a}}{a}+\frac{\dot{a}}{a}\frac{\dot{b}}{b}+\frac{\dot{a}}{a}\frac{\dot{c}}{c}-\frac{1}{2a^{2}b^{2}c^{2}}[(b^{2}-c^{2})^{2}-a^{4}], \tag{2.18}$$

$$-R^{2}_{2}=\frac{\ddot{b}}{b}+\frac{\dot{a}}{a}\frac{\dot{b}}{b}+\frac{\dot{b}}{b}\frac{\dot{c}}{c}-\frac{1}{2a^{2}b^{2}c^{2}}[(a^{2}-c^{2})^{2}-b^{4}], \tag{2.19}$$

[1]Following standard notation we use Greek indices here to write down the field equations (2.1)–(2.8). However, beginning with the formula (2.9) we replace these Greek indices by Latin ones $i,j,k,l=0,1,2,3$ because we are using an orthonormal basis. Greek indices $\alpha,\beta,\mu,\nu=\bar{0},\bar{1},\bar{2},\bar{3}$ are coordinate basis indices and will be used in Appendix A.

$$-R_3^3=\frac{\ddot{c}}{c}+\frac{\dot{a}}{a}\frac{\dot{c}}{c}+\frac{\dot{b}}{b}\frac{\dot{c}}{c}-\frac{1}{2a^2b^2c^2}[(a^2-b^2)^2-c^4], \qquad (2.20)$$

and the Ricci scalar reads

$$-R=2\frac{\ddot{a}}{a}+2\frac{\ddot{b}}{b}+2\frac{\ddot{c}}{c}+2\frac{\dot{a}}{a}\frac{\dot{b}}{b}+2\frac{\dot{a}}{a}\frac{\dot{c}}{c}+2\frac{\dot{b}}{b}\frac{\dot{c}}{c}-\frac{1}{2a^2b^2c^2}\times[a^4+b^4+c^4-2a^2b^2-2b^2c^2-2a^2c^2]. \qquad (2.21)$$

Since the model under consideration is homogeneous the dilaton field can only depend on time and so we have (cf. Appendix A)

$$\nabla_0\nabla^0\phi=\ddot{\phi}, \qquad (2.22)$$

$$\nabla_i\nabla^i\phi=\frac{\dot{a}_i}{a_i}\dot{\phi}, \qquad (2.23)$$

$$\nabla_0\phi\nabla^0\phi=\dot{\phi}^2. \qquad (2.24)$$

As for the axion field, we can use the following homogeneous ansatz [3,5]

$$H=\frac{A}{abc}e^1\wedge e^2\wedge e^3=A\sigma^1\wedge\sigma^2\wedge\sigma^3, \qquad (2.25)$$

where A is constant and

$$e^i=a_i\sigma^i. \qquad (2.26)$$

Since

$$(H_i^j)^2\equiv H_{ikl}H^{jkl}, \qquad (2.27)$$

and

$$H^2\equiv H_{ikl}H^{ikl}, \qquad (2.28)$$

we have

$$(H_0^0)^2=0, \qquad (2.29)$$

$$(H_1^1)^2=(H_2^2)^2=(H_3^3)^2=-2\frac{A^2}{a^2b^2c^2}, \qquad (2.30)$$

$$H^2=-6\frac{A^2}{a^2b^2c^2}. \qquad (2.31)$$

The special case of axial symmetry $a=b$ gives the orthonormal forms (2.10)–(2.12) as below [11]

$$\sigma^1=d\theta, \qquad (2.32)$$

$$\sigma^2=\sin\theta d\varphi, \qquad (2.33)$$

$$\sigma^3=d\psi+\cos\theta d\varphi. \qquad (2.34)$$

One can easily check (cf. Appendix A) that the homogeneous ansatz (2.25) is also appropriate for the axisymmetric case.

III. TIME-DEPENDENT PSEUDOSCALAR AXION FIELD AND THE OSCILLATORY APPROACH TO A SINGULARITY

The ansatz (2.25) is, in fact, equivalent to the inclusion of a time-dependent pseudoscalar axion field $h=h(t)$ [3] (cf. Appendix A). We are going to investigate the possible emergence of chaos in such models. Using Eqs. (2.17)–(2.24) and Eqs. (2.29)–(2.31) the field equations (2.1) in the string frame read ($\dot{}=d/dt$)

$$\frac{\ddot{a}}{a}+\frac{\ddot{b}}{b}+\frac{\ddot{c}}{c}-\ddot{\phi}=0, \qquad (3.1)$$

$$\frac{\ddot{a}}{a}+\frac{\dot{a}}{a}\frac{\dot{b}}{b}+\frac{\dot{a}}{a}\frac{\dot{c}}{c}-\frac{\dot{a}}{a}\dot{\phi}-\frac{1}{2a^2b^2c^2}[(b^2-c^2)^2-a^4]-\frac{1}{2}\frac{A^2}{a^2b^2c^2}=0, \qquad (3.2)$$

$$\frac{\ddot{b}}{b}+\frac{\dot{a}}{a}\frac{\dot{b}}{b}+\frac{\dot{b}}{b}\frac{\dot{c}}{c}-\frac{\dot{b}}{b}\dot{\phi}-\frac{1}{2a^2b^2c^2}[(a^2-c^2)^2-b^4]-\frac{1}{2}\frac{A^2}{a^2b^2c^2}=0, \qquad (3.3)$$

$$\frac{\ddot{c}}{c}+\frac{\dot{a}}{a}\frac{\dot{c}}{c}+\frac{\dot{b}}{b}\frac{\dot{c}}{c}-\frac{\dot{c}}{c}\dot{\phi}-\frac{1}{2a^2b^2c^2}[(a^2-b^2)^2-c^4]-\frac{1}{2}\frac{A^2}{a^2b^2c^2}=0. \qquad (3.4)$$

Equation (2.2), with Eq. (3.1), reads [4,5]

$$-2\left(\frac{\dot{a}}{a}\frac{\dot{b}}{b}+\frac{\dot{a}}{a}\frac{\dot{c}}{c}+\frac{\dot{b}}{b}\frac{\dot{c}}{c}\right)+2\dot{\phi}\left(\frac{\dot{a}}{a}+\frac{\dot{b}}{b}+\frac{\dot{c}}{c}\right)-\dot{\phi}^2+\frac{1}{2a^2b^2c^2}[a^4+b^4+c^4-2a^2b^2-2a^2c^2-2b^2c^2]+\frac{1}{2}\frac{A^2}{a^2b^2c^2}=0. \qquad (3.5)$$

Using Eq. (3.5), together with the sum of Eqs. (3.2)–(3.4), we have the dilaton equation of motion

$$\ddot{\phi}-(\dot{\phi})^2+\left(\frac{\dot{a}}{a}+\frac{\dot{b}}{b}+\frac{\dot{c}}{c}\right)\dot{\phi}-\frac{A^2}{a^2b^2c^2}=0. \qquad (3.6)$$

In the Einstein frame, the field equations (2.4) become [3] ($'=d/d\tilde{t}$)

$$-\frac{\tilde{a}''}{\tilde{a}}-\frac{\tilde{b}''}{\tilde{b}}-\frac{\tilde{c}''}{\tilde{c}}=\frac{1}{2}\phi'^2+\frac{1}{2}A^2\frac{e^{-2\phi}}{\tilde{a}^2\tilde{b}^2\tilde{c}^2}, \tag{3.7}$$

$$\frac{\tilde{a}''}{\tilde{a}}+\frac{\tilde{a}'}{\tilde{a}}\frac{\tilde{b}'}{\tilde{b}}+\frac{\tilde{a}'}{\tilde{a}}\frac{\tilde{c}'}{\tilde{c}}=\frac{1}{2\tilde{a}^2\tilde{b}^2\tilde{c}^2}[(\tilde{b}^2-\tilde{c}^2)^2-\tilde{a}^4], \tag{3.8}$$

$$\frac{\tilde{b}''}{\tilde{b}}+\frac{\tilde{a}'}{\tilde{a}}\frac{\tilde{b}'}{\tilde{b}}+\frac{\tilde{b}'}{\tilde{c}}\frac{\tilde{c}'}{\tilde{c}}=\frac{1}{2\tilde{a}^2\tilde{b}^2\tilde{c}^2}[(\tilde{a}^2-\tilde{c}^2)^2-\tilde{b}^4], \tag{3.9}$$

$$\frac{\tilde{c}''}{\tilde{c}}+\frac{\tilde{c}'}{\tilde{c}}\frac{\tilde{b}'}{\tilde{b}}+\frac{\tilde{a}'}{\tilde{a}}\frac{\tilde{c}'}{\tilde{c}}=\frac{1}{2\tilde{a}^2\tilde{b}^2\tilde{c}^2}[(\tilde{a}^2-\tilde{b}^2)^2-\tilde{c}^4], \tag{3.10}$$

and Eq. (2.5) now becomes

$$\phi''+\phi'\left(\frac{\tilde{a}'}{\tilde{a}}+\frac{\tilde{b}'}{\tilde{b}}+\frac{\tilde{c}'}{\tilde{c}}\right)=A^2\frac{e^{-2\phi}}{\tilde{a}^2\tilde{b}^2\tilde{c}^2}. \tag{3.11}$$

A new time coordinate is introduced to simplify the field equations by defining [3,5]

$$d\eta=\frac{e^{\phi}}{abc}dt=\frac{1}{\tilde{a}\tilde{b}\tilde{c}}d\tilde{t}. \tag{3.12}$$

First, notice that the string-frame equation (3.6) then simplifies to

$$\phi_{,\eta\eta}-A^2e^{-2\phi}=0, \tag{3.13}$$

where $(,_\eta=d/d\eta)$. The Einstein-frame Eq. (3.11), with the time coordinate $\tilde{t}$, gives the same result, Eq. (3.13). The solution of Eq. (3.13) is [5]

$$e^{\phi}=\cosh\Lambda M\eta+\sqrt{1-\frac{A^2}{\Lambda^2M^2}}\sinh\Lambda M\eta, \tag{3.14}$$

with M,Λ constant $(\Lambda^2M^2>A^2)$. A useful relation, implied by Eq. (3.14) is

$$\phi_{,\eta\eta}+\phi_{,\eta}^2=\Lambda^2M^2. \tag{3.15}$$

Let us introduce new forms for the scale factors;

$$a=e^{\alpha}\qquad b=e^{\beta}\qquad c=e^{\gamma}, \tag{3.16}$$

and

$$\tilde{a}=e^{\tilde{\alpha}}\qquad \tilde{b}=e^{\tilde{\beta}}\qquad \tilde{c}=e^{\tilde{\gamma}}, \tag{3.17}$$

so, from Eq. (2.16),

$$\tilde{\alpha}=\alpha-\phi/2\qquad \tilde{\beta}=\beta-\phi/2\qquad \tilde{\gamma}=\gamma-\phi/2, \tag{3.18}$$

and $dt=\exp(\phi/2)d\tilde{t}$ in Eq. (3.12). The field equations (3.1)–(3.4) in the string frame take the form

$$(\alpha+\beta+\gamma)_{,\eta\eta}-M^2=2(\alpha_{,\eta}\beta_{,\eta}+\alpha_{,\eta}\gamma_{,\eta}+\beta_{,\eta}\gamma_{,\eta})-2(\alpha_{,\eta}+\beta_{,\eta}+\gamma_{,\eta})\phi_{,\eta}, \tag{3.19}$$

$$2e^{2\phi}\alpha_{,\eta\eta}=(b^2-c^2)^2-a^4+A^2, \tag{3.20}$$

$$2e^{2\phi}\beta_{,\eta\eta}=(a^2-c^2)^2-b^4+A^2, \tag{3.21}$$

$$2e^{2\phi}\gamma_{,\eta\eta}=(a^2-b^2)^2-c^4+A^2. \tag{3.22}$$

Equations (3.20)–(3.22), using Eq. (3.13), can be rewritten to give

$$(-\phi+2\alpha)_{,\eta\eta}=[(b^2-c^2)^2-a^4]e^{-2\phi}, \tag{3.23}$$

$$(-\phi+2\beta)_{,\eta\eta}=[(a^2-c^2)^2-b^4]e^{-2\phi}, \tag{3.24}$$

$$(-\phi+2\gamma)_{,\eta\eta}=[(a^2-b^2)^2-c^4]e^{-2\phi}. \tag{3.25}$$

Notice that there is no explicit dependence of the axion, A, in these equations (3.23)–(3.25).

On the other hand, using Eq. (3.12), Eqs. (3.7)–(3.10) in the Einstein frame reduce to

$$2(\tilde{\alpha}_{,\eta}\tilde{\beta}_{,\eta}+\tilde{\alpha}_{,\eta}\tilde{\gamma}_{,\eta}+\tilde{\beta}_{,\eta}\tilde{\gamma}_{,\eta})=(\tilde{\alpha}+\tilde{\beta}+\tilde{\gamma})_{,\eta\eta}+\frac{1}{2}M^2, \tag{3.26}$$

$$2\tilde{\alpha}_{,\eta\eta}=(\tilde{b}^2-\tilde{c}^2)^2-\tilde{a}^4, \tag{3.27}$$

$$2\tilde{\beta}_{,\eta\eta}=(\tilde{a}^2-\tilde{c}^2)^2-\tilde{b}^4, \tag{3.28}$$

$$2\tilde{\gamma}_{,\eta\eta}=(\tilde{a}^2-\tilde{b}^2)^2-\tilde{c}^4. \tag{3.29}$$

These equations [except for the constant M which appears in Eq. (3.26)] have exactly the same form as the standard BIX equations of general relativity [6]. Equations (3.26)–(3.29) can be explicitly transformed into Eqs. (3.23)–(3.25) by using Eqs. (3.16)–(3.18).

In order to discuss possible emergence of chaos we consider suitable initial conditions expressed in terms of the Kasner parameters.

A. Einstein frame

The Kasner solutions are obtained as approximate solutions of the equations (3.7)–(3.11) when the right-hand sides (describing the curvature anisotropies) are neglected. In the Einstein frame, in terms of $\tilde{t}$-time, they are

$$\begin{aligned}\tilde{a}&=\tilde{a}_0\tilde{t}^{\tilde{p}_1},\\ \tilde{b}&=\tilde{b}_0\tilde{t}^{\tilde{p}_2},\\ \tilde{c}&=\tilde{c}_0\tilde{t}^{\tilde{p}_3},\end{aligned} \tag{3.30}$$

while $(A^2\leqslant\tilde{\Lambda}^2M^2)$

$$\phi=\ln\tilde{d}_0+\frac{M}{\tilde{\Lambda}}\ln|\tilde{t}|, \qquad (3.31)$$

for $A=0$, and

$$\phi=\ln\tilde{d}_0+\ln\left[\frac{1}{2}\left(1+\sqrt{1-\frac{A^2}{\tilde{\Lambda}M^2}}\right)|\tilde{t}|^{M/\tilde{\Lambda}}+\frac{1}{2}\left(1-\sqrt{1-\frac{A^2}{\tilde{\Lambda}M^2}}\right)|\tilde{t}|^{-M/\tilde{\Lambda}}\right], \qquad (3.32)$$

for $A\neq 0$, where

$$\tilde{\Lambda}=\tilde{a}_0\tilde{b}_0\tilde{c}_0, \quad \tilde{d}_0=\text{const.} \qquad (3.33)$$

From Eqs. (3.8)–(3.10), irrespective to the presence of the axion term A, we have the following algebraic conditions on the Kasner indices, $\tilde{p}_i$:

$$\sum_{i=1}^{3}\tilde{p}_i=1, \qquad (3.34)$$

and, from Eq. (3.7),

$$\sum_{i=1}^{3}\tilde{p}_i^{\,2}=1-\frac{M^2}{2\tilde{\Lambda}^2}, \qquad (3.35)$$

which is exactly the case considered first by Belinskii and Khalatnikov [7].[2] So, in the Einstein frame their argument follows.

In terms of η-time from Eqs. (3.27)–(3.29)

$$\tilde{\alpha}=\tilde{\Lambda}\tilde{p}_1\eta+\tilde{r}_1,$$
$$\tilde{\beta}=\tilde{\Lambda}\tilde{p}_2\eta+\tilde{r}_2, \qquad (3.36)$$
$$\tilde{\gamma}=\tilde{\Lambda}\tilde{p}_3\eta+\tilde{r}_3,$$

and the constraint equation (3.26) becomes[3]

$$\tilde{p}_1\tilde{p}_2+\tilde{p}_1\tilde{p}_3+\tilde{p}_2\tilde{p}_3=\frac{1}{4}\frac{M^2}{\tilde{\Lambda}^2}. \qquad (3.37)$$

This constraint is equivalent to the constraints (3.34)–(3.35). In order to get Eq. (3.30) from Eq. (3.36) we need to use Eq. (3.12) to relate $\tilde{t}$ -time with η-time, i.e.,

$$\eta=\tilde{\Lambda}^{-1}\ln\tilde{t}+\text{const.} \qquad (3.38)$$

[2]In their notation this occurs if we put $M^2/2\tilde{\Lambda}^2=q^2$. Also, they define the Lagrangian of the scalar field without a factor of 1/2 [i.e., $L_\phi=(\nabla\phi)^2$] while we follow standard notation.

[3]In [5] there is A^2 instead of M^2 while in all our calculations we obtained M^2 both in the string frame and in the Einstein frame. They claim they use the so-called ‘‘sigma frame,’’ which, however, seems to be just the Einstein frame [10].

When we approach singularity $\eta\to-\infty$ $(\tilde{t}\to 0)$ we cannot neglect all the terms on the right-hand side of Eqs. (3.27)–(3.29) since one of the Kasner indices in Eq. (3.37) can be negative. We assume $\tilde{p}_1\equiv-|\tilde{p}_1|<0$, which means that $\tilde{a}(\eta)\gg\tilde{b}(\eta)\gg\tilde{c}(\eta)$, so Eqs. (3.27)–(3.29) are approximated by

$$\tilde{\alpha}_{,\eta\eta}=-\frac{1}{2}e^{4\tilde{\alpha}},$$
$$\tilde{\beta}_{,\eta\eta}=\frac{1}{2}e^{4\tilde{\alpha}}, \qquad (3.39)$$
$$\tilde{\gamma}_{,\eta\eta}=\frac{1}{2}e^{4\tilde{\alpha}}.$$

Far away from singularity the approximate axisymmetric Kasner regime is fulfilled, but it is broken by the $\tilde{a}^2$ term when we approach singularity, so our Kasner solutions (3.36) will be fulfilled for $\eta\to\infty$ $(\tilde{t}\to\infty)$; suitable solutions of (3.39) which satisty (3.36) are[4]

$$\tilde{\alpha}(\eta)=-\frac{1}{2}\ln\left[\frac{1}{2|\tilde{p}_1|\tilde{\Lambda}}\cosh(-2|\tilde{p}_1|\tilde{\Lambda}\eta)\right],$$
$$\tilde{\beta}(\eta)=\frac{1}{2}\ln\left[\frac{1}{2|\tilde{p}_1|\tilde{\Lambda}}\cosh(-2|\tilde{p}_1|\tilde{\Lambda}\eta)\right]+\tilde{\Lambda}(-|\tilde{p}_1|+\tilde{p}_2)\eta, \qquad (3.40)$$
$$\tilde{\gamma}(\eta)=\frac{1}{2}\ln\left[\frac{1}{2|\tilde{p}_1|\tilde{\Lambda}}\cosh(-2|\tilde{p}_1|\tilde{\Lambda}\eta)\right]+\tilde{\Lambda}(-|\tilde{p}_1|+\tilde{p}_3)\eta.$$

In the limit $\eta\to-\infty$ $(\tilde{t}\to 0)$ from Eq. (3.40) we have

$$\tilde{\alpha}(\eta)=\tilde{\Lambda}|\tilde{p}_1|\eta=-\tilde{\Lambda}\tilde{p}_1\eta,$$
$$\tilde{\beta}(\eta)=\tilde{\Lambda}(\tilde{p}_2-2|\tilde{p}_1|)\eta=\tilde{\Lambda}(\tilde{p}_2+2\tilde{p}_1)\eta, \qquad (3.41)$$
$$\tilde{\gamma}(\eta)=\tilde{\Lambda}(\tilde{p}_2-2|\tilde{p}_1|)\eta=\tilde{\Lambda}(\tilde{p}_3+2\tilde{p}_1)\eta,$$

which means that one Kasner epoch Eq. (3.36) with indices $\tilde{p}_i$ is replaced by another Kasner epoch with indices given by Eq. (3.41). By virtue of Eq. (3.12)

$$\eta=\tilde{\Lambda}'^{-1}\ln\tilde{t}+\text{const}, \qquad (3.42)$$

and

$$\tilde{a}=\tilde{a}_0'\tilde{t}^{\tilde{p}_1'},$$

[4]This necessarily requires $\tilde{p}_1<0$. Of course, one has the same solutions for $\tilde{p}_1>0$, but in such a case our initial conditions would have to be taken at $\eta\to\infty$ rather than at $\eta\to-\infty$. This does not change the physics of the problem and was assumed in [19,21], but we prefer to follow the spirit of [6].

$$\tilde{b}=\tilde{b}_0'\tilde{t}^{\tilde{p}_2'}, \qquad (3.43)$$

$$\tilde{c}=\tilde{c}_0'\tilde{t}^{\tilde{p}_3'},$$

where

$$\tilde{p}_1'=-\frac{|\tilde{p}_1|}{1-2|\tilde{p}_1|}=\frac{\tilde{p}_1}{1+2\tilde{p}_1},$$

$$\tilde{p}_2'=\frac{\tilde{p}_2-2|\tilde{p}_1|}{1-2|\tilde{p}_1|}=\frac{\tilde{p}_2+2\tilde{p}_1}{1+2\tilde{p}_1}, \qquad (3.44)$$

$$\tilde{p}_3'=\frac{\tilde{p}_3-2|\tilde{p}_1|}{1+2\tilde{p}_1}=\frac{\tilde{p}_3+2\tilde{p}_1}{1+2\tilde{p}_1},$$

and

$$\tilde{\Lambda}'\tilde{t}=\tilde{a}\tilde{b}\tilde{c}\tilde{d}, \qquad \frac{\tilde{\Lambda}'}{\tilde{\Lambda}}=1+2\tilde{p}_1,$$

$$\sum_{i=1}^{3}\tilde{p}_i'=1, \qquad \sum_{i=1}^{3}\tilde{p}_i'^2=1-\frac{1}{2}\frac{M^2}{\tilde{\Lambda}'^2}. \qquad (3.45)$$

As we can see from Eq. (3.32), the axion does not influence these asymptotic solutions.

One can easily show that

$$-\sqrt{\frac{2}{3}}\leqslant\frac{M}{\tilde{\Lambda}\sqrt{2}}\leqslant\sqrt{\frac{2}{3}} \qquad (3.46)$$

and, after ordering the Kasner indices by $\tilde{p}_1\leqslant\tilde{p}_2\leqslant\tilde{p}_3$, we require

$$-1/3\leqslant\tilde{p}_1\leqslant1/3,$$

$$0\leqslant\tilde{p}_2\leqslant2/3, \qquad (3.47)$$

$$1/3\leqslant\tilde{p}_3\leqslant1.$$

This means that, unlike the vacuum case, all the Kasner indices can be positive and, as in the analysis of [7], the final situation is that the universe inevitably reaches a monotonic stage of evolution in which all three Kasner indices are positive (see also the discussion of Sec. V).[5] By means of Eqs. (3.30) and (3.32), which lead to the same relations between the Kasner indices (3.34), (3.35), one can extend this conclusion into the axion-dilaton cosmology. The presence of the axion field cannot change the fate of the universe near to a singularity and the universe finally reaches a monotonic stage of evolution with all three scale factors tending to zero as $t\rightarrow0$. Thus, there is no chaos in BIX homogeneous string cosmology in the Einstein frame. Finally, we note that there is an isotropic limit here, provided all the Kasner indices are equal with $\tilde{p}_1=\tilde{p}_2=\tilde{p}_3=1/3$.

B. String frame

For the string frame we can follow the discussion given in [19] (see also [20,21]).

In terms of t-time, Kasner solutions of the system (3.1)–(3.4), for $A=0$, are given by

$$a=a_0t^{p_1},$$

$$b=b_0t^{p_2}, \qquad (3.48)$$

$$c=c_0t^{p_3},$$

$$e^{-\phi}=d_0t^{p_4},$$

or, alternatively, the last of these conditions (3.48) reads as

$$\phi(t)=-\ln d_0-p_4\ln t, \quad \text{and} \quad p_4=-M. \qquad (3.49)$$

After putting Eq. (3.48) into Eq. (3.6) we have

$$p_4=1-\sum_{i=1}^{3}p_i, \qquad (3.50)$$

and from Eq. (3.1) we have

$$\sum_{i=1}^{3}p_i^2=1, \qquad (3.51)$$

so Eq. (3.49) can be rewritten to give

$$\phi(t)=-\ln d_0+\left(\sum_{i=1}^{3}p_i-1\right)\ln t. \qquad (3.52)$$

The last condition means, for instance, that a choice $p_i=(-1/3,2/3,2/3)$ gives $\Sigma p_i=1$ and leads to a vanishing or constant dilaton ($M=0$), i.e., to general relativity. (Other permutations of the three p_i with the signs changed are allowed). Thus, the difference between general relativity and string theory depends on the ''fourth Kasner index'' $p_4=-M$.

Now, from Eqs. (3.23)–(3.25) (one can see that $\Lambda p_i=2\tilde{\Lambda}\tilde{p}_i$ here)

$$\alpha=\phi/2+\Lambda(p_1/2)\eta+r_1/2=\phi/2+\tilde{\Lambda}\tilde{p}_1\eta+\tilde{r}_1,$$

$$\beta=\phi/2+\Lambda(p_2/2)\eta+r_2/2=\phi/2+\tilde{\Lambda}\tilde{p}_2\eta+\tilde{r}_2, \qquad (3.53)$$

$$\gamma=\phi/2+\Lambda(p_3/2)\eta+r_3/2=\phi/2+\tilde{\Lambda}\tilde{p}_3\eta+\tilde{r}_3,$$

and, from Eq. (3.19),

[5]Then, the terms of the type $e^{4\tilde{\alpha}},e^{4\tilde{\beta}},e^{4\tilde{\gamma}}$ decrease if $\eta\rightarrow-\infty$ ($t\rightarrow0$), and they do not allow a transition into another Kasner epoch to take place. This is just monotonic evolution such as in the isotropic Friedman case.

$$p_1p_2+p_1p_3+p_2p_3=\frac{M^2}{\Lambda^2}. \qquad (3.54)$$

The condition (3.54), in fact, can be obtained from Eqs. (3.51),(3.52). In order to get Eq. (3.48) we need to use Eq. (3.12), i.e.,

$$\eta=\Lambda^{-1}\ln t+\text{const}, \quad \Lambda=a_0b_0c_0d_0. \qquad (3.55)$$

From the conditions (3.50),(3.51) on the p_i, we have necessarily that [7]

$$-1-\sqrt{3}\leqslant M\leqslant -1+\sqrt{3}. \qquad (3.56)$$

However, the domain of M given by Eq. (3.56) covers the whole duality-related region since in the case of a Kasner regime the duality symmetry (cf. Sec. IV) simply means that we change

$$p_1\to -p_1, \quad p_2\to -p_2, \quad p_3\to -p_3,$$
$$M\to M-p_1-p_2-p_3=-(M+2). \qquad (3.57)$$

Having given Eq. (3.57), we can delineate the two duality-related domains of Kasner indices as follows:

$$-1-\sqrt{3}\leqslant M\leqslant -1, \quad -\frac{1}{\sqrt{3}}\leqslant p_1\leqslant \sqrt{\frac{2}{3}},$$
$$-\frac{2}{3}\leqslant p_2\leqslant \frac{1}{2}\sqrt{\frac{2}{3}}, \quad -1\leqslant p_3\leqslant -\frac{1}{2}\sqrt{\frac{2}{3}}; \qquad (3.58)$$

and

$$-1\leqslant M\leqslant -1+\sqrt{3}, \quad -\sqrt{\frac{2}{3}}\leqslant p_1\leqslant \frac{1}{\sqrt{3}},$$
$$-\frac{1}{2}\sqrt{\frac{2}{3}}\leqslant p_2\leqslant \frac{2}{3}, \quad \frac{1}{2}\sqrt{\frac{2}{3}}\leqslant p_3\leqslant 1. \qquad (3.59)$$

Of course, for $-1-\sqrt{3}\leqslant M\leqslant -1$, the indices are ordered so that $p_3\leqslant p_2\leqslant p_1$; while, for $-1\leqslant M\leqslant -1+\sqrt{3}$, they are ordered as $p_1\leqslant p_2\leqslant p_3$.

From Eqs. (3.57)–(3.59), we can draw some interesting conclusions. First, that the vacuum general relativity case $M=0$ (with $-1/3\leqslant p_1\leqslant 0$, $0\leqslant p_2\leqslant 2/3$, $2/3\leqslant p_3\leqslant 1$) is dual to the case $M=-2$ (with $-1\leqslant p_1\leqslant -2/3$, $-2/3\leqslant p_2\leqslant 0$, $0\leqslant p_3\leqslant 1/3$), while the case $M=-1$ is ''self-dual'' in M giving $M=-1$ again, although the p_i's change. Second, the $M=-1$ plane is the dividing plane for the duality-related range of the parameters which are defined by

$$-\sqrt{\frac{2}{3}}\leqslant p_1\leqslant -\frac{1}{2}\sqrt{\frac{2}{3}},$$
$$-\frac{1}{2}\sqrt{\frac{2}{3}}\leqslant p_2\leqslant \frac{1}{2}\sqrt{\frac{2}{3}},$$
$$\frac{1}{2}\sqrt{\frac{2}{3}}\leqslant p_3\leqslant \sqrt{\frac{2}{3}}.$$

The isotropic Friedmann solution is given when we choose $M=-1+\sqrt{3}$, $p_1=p_2=p_3=1/\sqrt{3}$, while its dual with $M=-1-\sqrt{3}$, $p_1=p_2=p_3=-1/\sqrt{3}$ does not appear in the vacuum general relativity case. This shows that one can generally have two different types of change in the Kasner indices, and in M (effectively, a fourth Kasner index). There are oscillations as in general relativistic vacuum or stiff fluid cases [6,7], but there are also duality-related exchanges of indices. We will discuss some points related to duality that refer to the work of Ref. [16].

There are eight possible permutations of the Kasner indices in which some are equal to $-1/\sqrt{3}$ or $1/\sqrt{3}$, with suitable M. These are the first two quadruples $(p_1,p_2,p_3,M)=(1/\sqrt{3},1/\sqrt{3},1/\sqrt{3},-1+\sqrt{3}),(-1/\sqrt{3},-1/\sqrt{3},-1/\sqrt{3},-1-\sqrt{3})$, plus another three pairs with $[-1/\sqrt{3},1/\sqrt{3},1/\sqrt{3},-1+(\sqrt{3}/3)],[1/\sqrt{3},-1/\sqrt{3},-1/\sqrt{3},-1-(\sqrt{3}/3)]$ and the permutations which are dual to each other. One can easily show that these last three pairs of cases are duality related and describe exactly the transitions from one Kasner epoch to another given by Eq. (3.80). Another interesting set of ''self-dual'' $(M=-1)$ combinations which describe LRS (locally rotationally symmetric) Kasner solutions are given by $(\sqrt{2/3},-1/\sqrt{6},-1/\sqrt{6},-1),(-\sqrt{2/3},1/\sqrt{6},1/\sqrt{6},-1)$ and their permutations. Other interesting duality-related combinations are $(-1/3,2/3,2/3,0)$, $(1/3,-2/3,-2/3,-2)$ and $(0,1/\sqrt{2},-1/\sqrt{2},-1),(0,-1/\sqrt{2},1/\sqrt{2},-1)$. The last two give the so-called Taub points (flat Minkowski spacetime) [16].

The question arises whether the changes of the Kasner indices (chaotic and duality-related) have something in common. In order to answer this question one should try to find a suitable parametrization of the Kasner indices analogous to the one given in Ref. [7] for the stiff-fluid case. We have checked that such a parametrization does not cover the right range of the indices given by Eq. (3.58) unless we multiply p_1 by $\sqrt{3}$. After some searching, we have found the following parametrization of the indices which has advantages which we will explain in due course.

The u-parametrization we use is defined as follows:[6]

$$\overline{p_1}=\frac{2u}{1+u^2}, \qquad (3.60)$$

$$\overline{p_2}=\frac{1}{2}\frac{1}{1+u^2}[(1+M)(1+u^2)-2u+P^+(M,u)],$$

$$\overline{p_3}=\frac{1}{2}\frac{1}{1+u^2}[(1+M)(1+u^2)-2u-P^+(M,u)]$$

where u is constant and

[6]In the general relativity case $(M=0)$ this parametrization can be written down in terms of the one suggested in [16] with $p_1=(1/3)(1-2\cos\psi)$, $p_{2,3}=(1/3)(1+\cos\psi\pm\sin\psi)$ with a suitable choice of $\cos\psi=(1/2)(u^2-6u+1)(u^2+1)$ in our case.

$$P^{+}(M,u)=\sqrt{(1-2M-M^{2})(1+u^{2})^{2}+4u(M+1)(1+u^{2})-12u^{2}}, \tag{3.61}$$

and to achieve $p_1<0$ we need $u<0$. Alternatively, we have

$$p_1=-\frac{2u}{1+u^2},$$
$$p_2=\frac{1}{2}\frac{1}{1+u^2}[(1+M)(1+u^2)+2u+P^{-}(M,u)] \tag{3.62}$$
$$p_3=\frac{1}{2}\frac{1}{1+u^2}[(1+M)(1+u^2)+2u-P^{-}(M,u)]$$

where

$$P^{-}(M,u)=\sqrt{(1-2M-M^{2})(1+u^{2})^{2}-4u(M+1)(1+u^{2})-12u^{2}}, \tag{3.63}$$

and for $p_1<0$ we need $u>0$. These are very general transformations that include all duality-related cases (3.58),(3.59) and the vacuum general relativity case when $M=0$. Similarly, as for stiff-fluid models in Ref. [7], we draw a plot the allowed values of the parameter u against M and show the duality-related regions in Figs. 1 and 2.

There are many possible transformations of the general homographic type $u\rightarrow(au+b)/(cu+d)$ or some more general types related to Padé or other rational approximants [18] which may be useful in the discussion of mixmaster oscillations. Although we are mainly interested in T-duality symmetry of the low-energy-effective-action equations (2.1)–(2.3) in this paper (see the more extensive discussion of this in Sec. IV), it is quite interesting to comment on the fact these homographic transformation with $ad-bc=1$ fulfill the requirements for S-duality. This suggests T-duality may be embedded at a certain level in S-duality here, and vice versa. However, we do not consider these general transformations in this paper; we discuss only the simplest cases and mainly address the T-duality.

First, notice that a transformation of the type

$$u\rightarrow\frac{1}{u}, \tag{3.64}$$

gives

$$\overline{p_i}(M,1/u)=\overline{p_i}(M,u),$$
$$p_i(M,1/u)=p_i(M,u), \tag{3.65}$$

where $i=1,2,3$, unless we take negative value of the root of $P^{\pm}$ after applying the transformation. On the other hand the transformation

$$u\rightarrow-\frac{1}{u}, \tag{3.66}$$

changes parametrization (3.60) into (3.62), i.e.,

$$\overline{p_i}(M,-1/u)=p_i(M,u)=-\overline{p_i}(M,u), \tag{3.67}$$

$$p_i(M,-1/u)=\overline{p_i}(M,u)=-p_i(M,u). \tag{3.68}$$

Second, one can consider the transformation

$$u\rightarrow-u, \tag{3.69}$$

which results in the same rules as Eq. (3.65). However, this transformation is not enough to describe duality relations between the considered solutions. As one can see, the duality-reflecting transformations should be of the type (see also Fig. 1)

$$u\rightarrow-u,\quad M\rightarrow-(M+2), \tag{3.70}$$

or, alternatively

$$u\rightarrow-\frac{1}{u},\quad M\rightarrow-(M+2). \tag{3.71}$$

From Eqs. (3.70) and (3.71) we see that

$$P^{\pm}(-M-2,-u)=P^{\pm}(-M-2,-1/u)=P^{\pm}(M,u), \tag{3.72}$$

and

$$\overline{p_1}(-u)=-\overline{p_1}(u),$$
$$\overline{p_2}(-M-2,-u)=-\overline{p_3}(M,u), \tag{3.73}$$
$$\overline{p_3}(-M-2,-u)=-\overline{p_2}(M,u);$$

similarly,

$$p_1(-u)=-p_1(u),$$
$$p_2(-M-2,-u)=-p_3(M,u), \tag{3.74}$$
$$p_3(-M-2,-u)=-p_2(M,u).$$

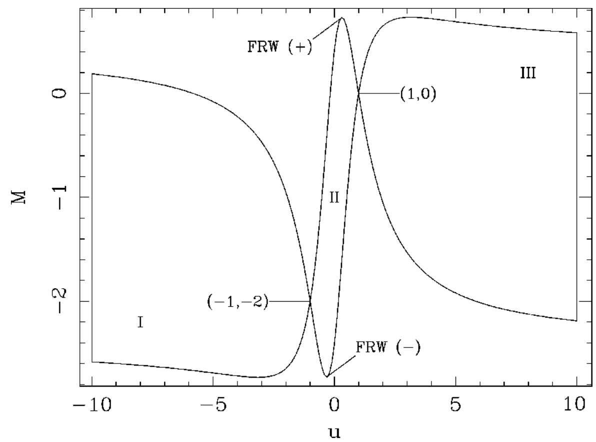

FIG. 1. The regions I, II, III of permissible values of the parameters u and M for the parametrization (3.60) of the Kasner indices p_1, p_2 and p_3. We restricted ourselves to $-10 \leqslant u \leqslant 10$ although regions I and III extend to infinity. Special isotropic FRW (+) and FRW (−) points are given for $(\sqrt{3}-\sqrt{2}, -1+\sqrt{3})$ and $(-\sqrt{3}+\sqrt{2}, -1-\sqrt{3})$ respectively.

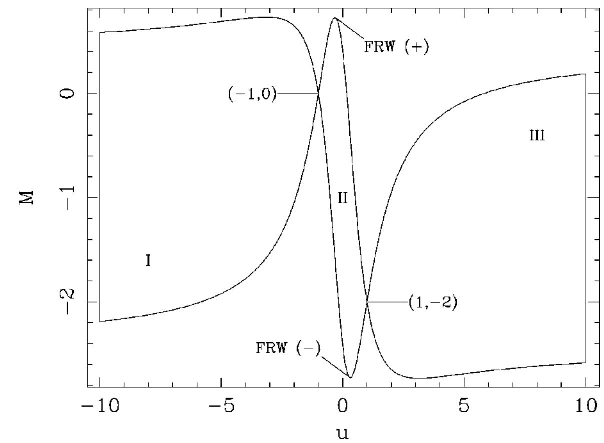

FIG. 2. The regions I, II, III of permissible values of the parameters u and M for the parametrization (3.62) of the Kasner indices p_1, p_2 and p_3. We restricted ourselves to $-10 \leqslant u \leqslant 10$ although regions I and III extend to infinity. Special isotropic FRW (+) and FRW (−) points are given for $(-\sqrt{3}+\sqrt{2}, -1+\sqrt{3})$ and $(\sqrt{3}-\sqrt{2}, -1-\sqrt{3})$ respectively.

We notice that the duality in Eqs. (3.73),(3.74) would entail the exchange of the Kasner indices p_2 and p_3.

If we assume that $a \gg b \gg c$ in Eqs. (3.23), (3.24) then we obtain the following set of approximating equations (for $A=0$):

$$\alpha_{,\eta\eta} = -\frac{1}{2} e^{4\alpha} e^{-2\Lambda M \eta},$$

$$\beta_{,\eta\eta} = \frac{1}{2} e^{4\alpha} e^{-2\Lambda M \eta}, \tag{3.75}$$

$$\gamma_{,\eta\eta} = \frac{1}{2} e^{4\alpha} e^{-2\Lambda M \eta},$$

$$\phi_{,\eta\eta} = 0,$$

together with the constraint (3.19). The Kasner solutions in terms of η-time are given by

$$\alpha(\eta) = \Lambda p_1 \eta + \text{const},$$

$$\beta(\eta) = \Lambda p_2 \eta + \text{const}, \tag{3.76}$$

$$\gamma(\eta) = \Lambda p_3 \eta + \text{const},$$

$$\phi(\eta) = \Lambda M \eta + \text{const}.$$

However, these solutions are not directly obtained by using the relations (3.18) between the Einstein frame and string-frame scale factors. This is very important in the case when the axion field is taken into account and will be discussed below.

The solutions of equations (3.75) which satisfy the above conditions, (3.76), in the limit $\eta \to \infty$ ($t \to \infty$, i.e. far from the singularity), provided $p_1 = -|p_1| < 0$ and $2p_1 - M = -2|p_1| - M < 0$, can be chosen to be[7]

$$\alpha(\eta) = -\frac{1}{2} \ln\left(\frac{1}{\Lambda(2p_1 - M)} \cosh \Lambda(2p_1 - M)\eta \right) + \frac{1}{2} \Lambda M \eta,$$

$$\beta(\eta) = \frac{1}{2} \ln\left(\frac{1}{\Lambda(2p_1 - M)} \cosh \Lambda(2p_1 - M)\eta \right) + (p_1 + p_2 - M)\Lambda \eta + \frac{1}{2} \Lambda M \eta, \tag{3.77}$$

$$\gamma(\eta) = \frac{1}{2} \ln\left(\frac{1}{\Lambda(2p_1 - M)} \cosh \Lambda(2p_1 - M)\eta \right) + (p_1 + p_3 - M)\Lambda \eta + \frac{1}{2} \Lambda M \eta.$$

In the limit $\eta \to -\infty$ ($t \to 0$, i.e. on the approach to the singularity) they approach the following asymptotic forms:[8]

$$\alpha(\eta) \sim -\Lambda(p_1 - M)\eta,$$

$$\beta(\eta) \sim \Lambda(p_2 + 2p_1 - M)\eta, \tag{3.78}$$

$$\gamma(\eta) \sim \Lambda(p_3 + 2p_1 - M)\eta,$$

$$\phi(\eta) \sim \Lambda M \eta.$$

[7]Note, we can derive them in a similar way to the $A \neq 0$ case (see the rest of this section), but here it is more convenient to follow the results given in [21].

[8]If we assume $p_1 < 0$ and $2p_1 - M < 0$, then $p_1 - M < 0$, provided $M > 0$, which means we have changed expansion of $a(\eta)$ into contraction.

One can check the solutions (3.77) by putting them into the constraint (3.19) in order to recover the condition (3.54), as expected.

Now, we can express the scale factors in terms of the new Kasner parameters

$$\begin{aligned} a&=a_0' t^{p_1'},\\ b&=b_0' t^{p_2'},\\ c&=c_0' t^{p_3'},\\ e^{-\phi}&=d_0' t^{p_4'}, \end{aligned} \tag{3.79}$$

where

$$\begin{aligned} p_1'&=-\frac{p_1-M}{1+2p_1-M},\\ p_2'&=\frac{p_2+2p_1-M}{1+2p_1-M},\\ p_3'&=\frac{p_3+2p_1-M}{1+2p_1-M},\\ p_4'&=\frac{-M}{1+2p_1-M}=-M', \end{aligned} \tag{3.80}$$

and

$$\begin{aligned} \Lambda'&=a_0' b_0' c_0' d_0',\\ \eta&=(\Lambda')^{-1}\ln t+\text{const}, \\ \Lambda'&=(1+2p_1-M)\Lambda. \end{aligned} \tag{3.81}$$

If we take the axion field into account ($A\neq 0$) and assume that $a\gg b\gg c$ in Eqs. (3.23)–(3.25), then we obtain[9]

$$\begin{aligned} \alpha_{,\eta\eta}&=\frac{1}{2}(A^2-e^{4\alpha})e^{-2\phi},\\ \beta_{,\eta\eta}&=\frac{1}{2}(A^2+e^{4\alpha})e^{-2\phi},\\ \gamma_{,\eta\eta}&=\frac{1}{2}(A^2+e^{4\alpha})e^{-2\phi},\\ \phi_{,\eta\eta}&=A^2 e^{-2\phi}, \end{aligned} \tag{3.82}$$

with $\phi(\eta)$ given by Eq. (3.14). The Kasner solutions in terms of η-time are now given by [compare Eq. (3.18)]

$$\begin{aligned} \alpha(\eta)&=\frac{\Lambda}{2}(p_1+M)\eta+\text{const}\equiv\Lambda q_1\eta+\text{const},\\ \beta(\eta)&=\frac{\Lambda}{2}(p_2+M)\eta+\text{const}\equiv\Lambda q_1\eta+\text{const},\\ \gamma(\eta)&=\frac{\Lambda}{2}(p_3+M)\eta+\text{const}\equiv\Lambda q_1\eta+\text{const},\\ \phi(\eta)&=\Lambda M\eta+\text{const}. \end{aligned} \tag{3.83}$$

The solutions which fulfill the above initial conditions (3.83) for $\eta\to\infty$ ($p_1<0$) are

$$\alpha(\eta)=-\frac{1}{2}\ln\left(\frac{1}{p_1}\cosh p_1\eta\right)+\frac{1}{2}\phi(\eta), \tag{3.84}$$

$$\beta(\eta)=\frac{1}{2}\ln\left(\frac{1}{p_1}\cosh p_1\eta\right)+\frac{1}{2}(p_1+p_2)\eta+\frac{1}{2}\phi(\eta), \tag{3.85}$$

$$\gamma(\eta)=\frac{1}{2}\ln\left(\frac{1}{p_1}\cosh p_1\eta\right)+\frac{1}{2}(p_1+p_3)\eta+\frac{1}{2}\phi(\eta), \tag{3.86}$$

$$\phi(\eta)=\ln\left[\cosh M\eta+\sqrt{1-\frac{A^2}{M^2}}\sinh M\eta\right], \tag{3.87}$$

or, alternatively

$$\alpha(\eta)=-\frac{1}{2}\ln\left(\frac{1}{2q_1-M}\cosh(2q_1-M)\eta\right)+\frac{1}{2}\phi(\eta), \tag{3.88}$$

$$\beta(\eta)=\frac{1}{2}\ln\left(\frac{1}{2q_1-M}\cosh(2q_1-M)\eta\right)+(q_1+q_2-M)\eta+\frac{1}{2}\phi(\eta), \tag{3.89}$$

$$\gamma(\eta)=\frac{1}{2}\ln\left(\frac{1}{2q_1-M}\cosh(2q_1-M)\eta\right)+(q_1+q_3-M)\eta+\frac{1}{2}\phi(\eta), \tag{3.90}$$

with $\phi(\eta)$ unchanged.

One can easily check by putting these solutions into the constraint (3.19), that the condition (3.54) is satisfied, which in turn ensures that the conditions (3.50),(3.51) are satisfied. In particular, note that, for Eqs. (3.88)–(3.90), we need to replace $p_i's$ by $q_i's$. In the limit $\eta\to-\infty$, (that is, $t\to 0$), they approach the following forms:

$$\alpha(\eta)\sim-\frac{\Lambda}{2}(p_1-M)\eta,$$

[9] If $M>0$, then the term $e^{-2\phi}$ increases for $\eta\to-\infty$. If, in turn, $p_1<0$, $M>0$, and $2p_1-M>0$, then the term $e^{2(2p_1-M)\eta}$ decreases for $\eta\to-\infty$ and the whole picture is dominated by the axion term $1/2A^2e^{-2\phi}$. It follows that the field equations become isotropic $\alpha_{,\eta\eta}=\beta_{,\eta\eta}=\gamma_{,\eta\eta}=1/2\phi_{,\eta\eta}=1/2A^2e^{-2\phi}$. We see that the axion isotropises the model and chaos is impossible in such a case. It seems that the time-dependent axion field ansatz (cf. Appendix A) would allow chaos, but it is not admitted by the BIX geometry.

$$\beta(\eta)\sim\frac{\Lambda}{2}(p_2+2p_1-M)\eta,$$

$$\gamma(\eta)\sim\frac{\Lambda}{2}(p_3+2p_1-M)\eta,$$

$$\phi(\eta)\sim-\frac{\Lambda}{2}M\eta, \tag{3.91}$$

or

$$\alpha(\eta)\sim-\Lambda(q_1-M)\eta,$$

$$\beta(\eta)\sim\Lambda(q_2+2q_1-M)\eta,$$

$$\gamma(\eta)\sim\Lambda(q_3+2p_1-M)\eta,$$

$$\phi(\eta)\sim-\Lambda M\eta. \tag{3.92}$$

Having given the conditions (3.50),(3.51), one can express the indices p_2 and p_3 by using p_1 and M, i.e.,

$$p_2=\frac{1}{2}[(M+1-p_1) - \sqrt{-3p_1^2+2p_1(M+1)+1-M(M+2)}],$$

$$p_3=\frac{1}{2}[(M+1-p_1) + \sqrt{-3p_1^2+2p_1(M+1)+1-M(M+2)}]. \tag{3.93}$$

Since the expression under the square root should be non-negative, one can extract the restriction (3.56) on the permissible values of M. However, we are interested in knowing whether the curvature terms on the right-hand side of the field equations (3.23)–(3.25) really increase as $\eta\to-\infty$ ($t\to 0$). This would require either $a^4e^{-2\phi}$, $b^4e^{-2\phi}$, or $c^4e^{-2\phi}$ to increase if the transition to another Kasner epoch is to occur [19,21]. Since

$$a^4e^{-2\phi}\propto t^{(2p_1-M)}=t^{(1+p_1-p_2-p_3)},$$

$$b^4e^{-2\phi}\propto t^{(2p_2-M)}=t^{(1+p_2-p_3-p_1)}, \tag{3.94}$$

$$c^4e^{-2\phi}\propto t^{(2p_3-M)}=t^{(1+p_3-p_1-p_2)},$$

we need one of the following three conditions to be satisfied (remember that we have assumed $p_1<0$, $M>0$):

$$2p_1-M=1+p_1-p_2-p_3<0,$$

$$2p_2-M=1+p_2-p_3-p_1<0, \tag{3.95}$$

$$2p_3-M=1+p_3-p_1-p_2<0.$$

The three conditions (3.95), with the help of Eq. (3.93), are equivalent to

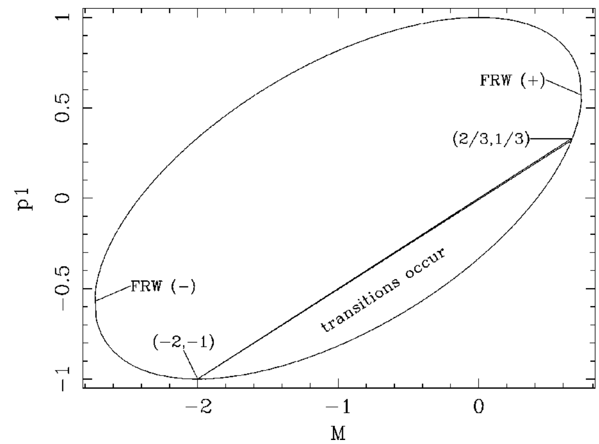

FIG. 3. The plot of the conditions (3.96),(3.97) for the transitions from one Kasner epoch to another to begin in terms of the first and the ''fourth'' Kasner indices p_1 and M. Obviously, isotropic FRW (+) and FRW (−) points $(-1+\sqrt{3},1/\sqrt{3})$ and $(-1-\sqrt{3},-1/\sqrt{3})$ respectively, are excluded. The transitions occur below the line $p=M/2$. From the picture one can see that in the vacuum general relativity case ($M=0$) the transitions are possible whenever $-1/3\leqslant p_1\leqslant 0$. This is in agreement with the standard calculations. However, in the dual case ($M=-2$) transitions do not occur at all.

$$p_1<\frac{M}{2}, \tag{3.96}$$

$$-3p_1^2+2p_1(M+1)+1-M(M+2)>0. \tag{3.97}$$

A plot of these conditions is given in Fig. 3. The last of these conditions, (3.97), provides bounds on the possible values of M if a transition to occur:

$$-2\leqslant M\leqslant\frac{2}{3}. \tag{3.98}$$

Now, we see that the regions where the Friedmann isotropic limit is possible (all the Kasner indices equal—this happens for $M=-1-\sqrt{3}$ and $M=-1+\sqrt{3}$) are excluded. One can always find the range of the indices for a transition from one Kasner epoch to another to occur in the string frame.

Instead of expressing the conditions for Kasner-type transitions in terms of p_1 and M, we can follow the pattern of [19] and write them in terms of p_1 and p_2. From the conditions (3.50),(3.51), we can write

$$p_3=M+1-p_1-p_2, \tag{3.99}$$

$$M=p_1+p_2-1\pm\sqrt{1-p_1^2-p_2^2}, \tag{3.100}$$

where the plus sign is for $M>-1$, $p_3>0$ and minus sign for $M<-1$, $p_3<0$. So, p_1 and p_2 must be such that (M real)

$$1-p_1^2-p_2^2\geqslant 0, \tag{3.101}$$

and one of the three conditions (3.95) must be satisfied, i.e., either

$$p_1^2+p_2^2-p_1p_2+p_1-p_2<0,$$

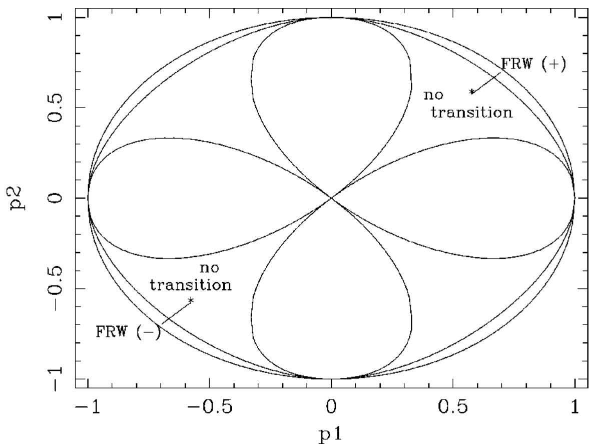

FIG. 4. The plot of the conditions (3.101)–(3.103) for the transitions from one Kasner epoch to another to begin, in terms of the two Kasner indices p_1 and p_2. As in Fig. 3 the isotropic FRW $(+)$ and FRW $(-)$ points, $(1/\sqrt{3},1/\sqrt{3})$ and $(-1/\sqrt{3},-1/\sqrt{3})$ respectively, together with the two neighboring regions, are excluded.

$$p_1^2+p_2^2-p_1p_2-p_1+p_2<0, \tag{3.102}$$

$$p_1^2+p_2^2+p_1p_2-p_1-p_2>0,$$

for $p_3>0$, or

$$\begin{aligned} &p_1^2+p_2^2-p_1p_2+p_1-p_2<0,\\ &p_1^2+p_2^2-p_1p_2-p_1+p_2<0,\\ &p_1^2+p_2^2+p_1p_2+p_1+p_2>0, \end{aligned} \tag{3.103}$$

for $p_3<0$.

The plot of these conditions is given in Fig. 4.

In summary, we have been able to determine the range of values that can be taken by the Kasner indices in the string frame and have proposed a parametrization which describes the evolution of these indices. Also, we have determined the values of the Kasner indices (see Figs. 3 and 4) for which the spacetime oscillations can really take place. However, now we have to determine whether the oscillations can be stopped once they have started as $t\to 0$. Before we come to this in Sec. V we first discuss some relations between the Kasner indices and duality.

IV. EXCHANGE OF KASNER INDICES AND DUALITY

The low-energy-effective-action equations (2.1)–(2.3) exhibit continuous global $O(d,d)$ symmetry (d is the number of spatial dimensions) which is an example of T-duality within the string theory [23]. It differs from S-duality or the $SL(2,R)$ invariance of superstring models, mentioned in Sec. III (see e.g. [26]). For the class of homogeneous models under consideration T-duality is a global $O(3,3)$ invariance under which (where $\overline{\phi}$ is the so-called shifted dilaton field)

$$\mathbf{M}\to\mathbf{M}'=\mathbf{\Omega}^T\mathbf{M}\mathbf{\Omega}, \quad \overline{\phi}\equiv\phi-\ln\sqrt{\det\ \mathbf{G}}\to\overline{\phi}. \tag{4.1}$$

Here, $\mathbf{\Omega}$ is 6×6 constant matrix satisfying

$$\mathbf{\Omega}^T\mathbf{\Pi}\mathbf{\Omega}=\mathbf{\Pi}, \qquad \mathbf{\Pi}=\begin{pmatrix}\mathbf{0} & \mathbf{1}\\ \mathbf{1} & \mathbf{0}\end{pmatrix}, \tag{4.2}$$

$\mathbf{1}$ is the 3×3 identity matrix, and

$$\mathbf{M}\equiv\begin{pmatrix}\mathbf{G}^{-1} & -\mathbf{G}^{-1}\mathbf{B}\\ \mathbf{B}\mathbf{G}^{-1} & \mathbf{G}-\mathbf{B}\mathbf{G}^{-1}\mathbf{B}\end{pmatrix}, \tag{4.3}$$

where $\mathbf{G}=g_{ij}(t)$ and $\mathbf{B}=B_{ij}(t)$ are 3×3 matrices. Any 6×6 constant matrix $\mathbf{\Omega}$ obeying Eq. (4.2) generates new solutions $\mathbf{M}'$ from the original set $\mathbf{M}$. Notice that for full $O(3,3)$ symmetry both $\mathbf{G}$ and $\mathbf{B}$ have to be functions of time. This is especially important for the antisymmetric tensor potential B_{ij} which is time-dependent and so leads to a space-dependent pseudoscalar axion field, h [see Eq. (A40) of the Appendix]. This is called the ''elementary ansatz'' and is only admissible in a restricted class of metrics of spatially homogeneous metrics [3,4]. As we prove in the Appendix, the elementary ansatz is not compatible with (even axisymmetric) Bianchi type IX geometry. Thus, because of our homogeneous ansatz (2.25) [or Eq. (A43) in terms of pseudoscalar axion field], the full $O(d,d)$ symmetry is broken and it can only be recovered if the antisymmetric tensor field B_{ij} (or axion h) vanishes. If this happens, we can make a choice $\mathbf{\Omega}=\mathbf{\Pi}$ and consider the ''scale factor duality'' where

$$\begin{aligned} a'^2&\to\frac{1}{a^2},\\ b'^2&\to\frac{1}{b^2},\\ c'^2&\to\frac{1}{c^2},\\ \phi'&\to\phi-2\ln abc. \end{aligned} \tag{4.4}$$

It is useful to define the logarithm of an average scale factor $\overline{\beta}$ and the so-called shifted dilaton $\overline{\phi}$ defined by [14,15]

$$\begin{aligned} \overline{\beta_i}&=\frac{1}{\sqrt{3}}\ln a_i,\\ \overline{\beta}&=\frac{1}{\sqrt{3}}\ln(abc),\\ \overline{\phi}&=\phi-\sqrt{3}\overline{\beta}. \end{aligned} \tag{4.5}$$

Using Eqs. (3.40)–(3.45) we have the relations (4.5) in terms of Kasner indices: i.e.,

$$\begin{aligned} \overline{\phi}&=\phi-\ln\Lambda-(M+1)\ln t,\\ \overline{\beta_i}&=\frac{1}{\sqrt{3}}(\ln a_{0i}+p_i\ln t),\\ \overline{\beta}&=\frac{1}{\sqrt{3}}[\ln\Lambda+(M+1)\ln t], \end{aligned} \tag{4.6}$$

where $a_{0i}=\{a_0,b_0,c_0\}$. In these variables the duality symmetry is just expressed by

$$\overline{\beta_i(t)}=-\overline{\beta_i(t)},$$
$$\overline{\beta}(t)=-\overline{\beta}(t), \tag{4.7}$$
$$\overline{\phi}(t)=\overline{\phi}(t);$$

or, in terms of Kasner indices, it reads $p_i\to -p_i$, $M+1 \to -(M+1)$. After inclusion of time symmetry we have [15]

$$\overline{\beta_i(t)}=-\overline{\beta_i(-t)},$$
$$\overline{\beta}(t)=-\overline{\beta}(-t), \tag{4.8}$$
$$\overline{\phi}(t)=\overline{\phi}(-t).$$

In the isotropic case $p_i=\pm 1/\sqrt{3}$ and we recover exactly the case given in Ref. [15].

The same relations can be written down using the time coordinate η instead of t. Using the exact expressions, (3.76), for Kasner solutions we have

$$\overline{\beta}=\frac{\Lambda}{\sqrt{3}}(M+1)\,\eta,$$
$$\overline{\phi}=\phi-\sqrt{3}\overline{\beta}-\Lambda\,\eta, \tag{4.9}$$
$$\overline{\beta}_i=\frac{\Lambda}{\sqrt{3}}p_i\,\eta.$$

So, one could relate these by duality symmetries

$$\overline{\phi}(\eta)\to\overline{\phi}(-\eta), \tag{4.10}$$

$$\overline{\beta}(\eta)\to-\overline{\beta}(-\eta), \tag{4.11}$$

for $\eta\to\pm\infty$ respectively, using chaotic changes $p_i\to -p_i$.

In our analysis we have used the standard procedure of assuming that the mixmaster model is well described by a sequence of Kasner-to-Kasner transitions. This assumption is an analogue of that of steep walls in the Hamiltonian approach. Numerical studies of mixmaster models show it to be a good approximation even in the presence of chaotic behavior. We do not find chaotic behavior and so the approximation should be better over long periods of evolution. We note also that the approximations made ($a\gg b\gg c$) to study single Kasner-to-Kasner transitions reduce the equations to these of the axisymmetric case. This describes a single Kasner-to-Kasner transition. We therefore expect the duality relationships characterizing Kasner-to-Kasner transitions to provide good approximations to the properties of the exact mixmaster behavior and we do not see any reason to consider non-Abelian dualities of this exact model [24]. We do not know whether the string Bianchi type IX model is integrable in general.

V. HAMILTONIAN APPROACH TO BIANCHI IX STRING MODELS

In this section we formulate a generalized Kasner model in Hamiltonian formalism as in Refs. [17,19,22] in order to discuss the conditions for an infinite sequence of scatterings to occur against the walls of the curvature potential. As in the previous sections we discuss the problem in both the Einstein and the string frames. We also introduce the so-called B-frame, or axion frame [26], in which axion is minimally coupled.

A. Einstein frame

We introduce the following standard parametrization for the Einstein-frame scale factors:

$$\tilde{a}=e^{\tilde{\alpha}+\psi_{+}+\sqrt{3}\psi_{-}},$$
$$\tilde{b}=e^{\tilde{\alpha}+\psi_{+}-\sqrt{3}\psi_{-}}, \tag{5.1}$$
$$\tilde{c}=e^{\tilde{\alpha}-2\psi_{-}},$$

and we define the potential, which describes the spatial curvature anisotropy (2.21) felt by scale factors in the Einstein-frame by

$$\tilde{V}(\psi_{\pm})=e^{-2\tilde{\alpha}}V(\psi_{\pm}), \tag{5.2}$$

where

$$V(\psi_{\pm})=\frac{1}{2}[e^{-8\psi_{+}}+2e^{4\psi_{+}}(\cosh 4\sqrt{3}\psi_{-}-1)$$
$$-4e^{-2\psi_{+}}\cosh 2\sqrt{3}\psi_{-}]. \tag{5.3}$$

Using Eqs. (5.1),(5.2), Eqs. (3.7)–(3.10) read as

$$\tilde{\alpha}'^2=\psi_{+}'^2+\psi_{-}'^2+\frac{1}{12}\phi'^2+\frac{1}{12}A^2e^{-2\phi-6\tilde{\alpha}}+\frac{1}{6}e^{-2\tilde{\alpha}}V(\psi_{\pm}). \tag{5.4}$$

This is the Hamiltonian constraint. The Einstein-frame action in terms of the scale factors (5.1), after integrating out spatial variables, is given by (compare [25])

$$S=\int d\tilde{t}e^{3\tilde{\alpha}}\Big[-6\tilde{\alpha}'^2+6\psi_{+}'^2+6\psi_{-}'^2+\frac{1}{2}\phi'^2$$
$$+\frac{1}{2}A^2e^{2\phi}\sigma'^2+e^{-2\tilde{\alpha}}V(\psi_{\pm})\Big], \tag{5.5}$$

and the conjugate momenta are

$$\pi_{\tilde{\alpha}}=-12\tilde{\alpha}'e^{3\tilde{\alpha}},$$
$$\pi_{+}=12\psi_{+}'e^{3\tilde{\alpha}},$$
$$\pi_{-}=12\psi_{-}'e^{3\tilde{\alpha}}, \tag{5.6}$$
$$\pi_{\phi}=\phi'e^{3\tilde{\alpha}},$$
$$\pi_{\sigma}=\sigma'e^{2\phi+3\tilde{\alpha}}=\text{const}=A,$$

so the Hamiltonian is

$$H=-\frac{\pi_\alpha^2}{24}+\frac{\pi_+^2}{24}+\frac{\pi_-^2}{24}+\frac{\pi_\phi^2}{2}+\frac{\pi_\sigma^2}{2}+e^{4\tilde{\alpha}}V(\psi_\pm). \quad (5.7)$$

Now, we follow the standard discussion of the potential walls

$$\bar{V}=e^{4\tilde{\alpha}}V(\psi_\pm) \quad (5.8)$$

being hit by a particle moving in the potential well (see Ref. [22]). In the region $\psi_+\ll -1$ and $\psi_-\approx 0$, the approximate distance from the origin of coordinates ψ_+ and ψ_- to the wall is given by

$$D=-\frac{1}{2}\tilde{\alpha}, \quad (5.9)$$

while the maximum apparent velocity of this wall is

$$v_{max}=\tilde{\alpha}'. \quad (5.10)$$

The velocity of a particle moving against the walls is

$$v_p=\sqrt{\psi_+'^2+\psi_-'^2}, \quad (5.11)$$

and it will not be scattered infinitely many times if there is some region of the potential which the particle enters and from which it cannot catch up with the wall, i.e., if

$$v_p=\sqrt{\psi_+'^2+\psi_-'^2}<\tilde{\alpha}' \approx\sqrt{\psi_+'^2+\psi_-'^2+(1/12)\phi'^2+(1/12)A^2e^{-2\phi-6\tilde{\alpha}}}. \quad (5.12)$$

Clearly, this condition is fulfilled in every case unless $\phi' =A=0$ (no dilaton and axion–that is, the general relativity vacuum regime), which reflects the fact that a particle cannot be scattered infinitely many times and that *there is no chaos in the Einstein frame*. This result is expected since in the Einstein frame both dilaton and axion fields behave as stiff fluids with the equation of state $p=\varrho$. A numerical discussion of 10-dimensional axion-dilaton low-energy effective-action models in the Einstein frame where nine dimensions were split into three isotropic 3-dimensional spaces leading effectively to our anisotropic 4-dimensional model, was also given in Ref. [27] with the same final conclusion about the non-existence of chaos.

B. String frame and axion frame

In the string frame we can use the same parametrization as in Eq. (5.1), but we just drop the tildes. The potential (5.2) can also be used without tildes. In that parametrization Eq. (3.5) becomes

$$\dot{\alpha}^2=\dot{\psi}_+^2+\dot{\psi}_-^2-\frac{1}{6}\dot{\phi}^2+\frac{1}{6}e^{-2\alpha}V(\psi_\pm)+\frac{1}{12}A^2e^{-6\alpha}+\dot{\alpha}\dot{\phi}. \quad (5.13)$$

After applying the variables $\bar{\beta}$ and $\bar{\phi}$ defined by Eq. (4.5) we can remove the $\dot{\phi}\dot{\alpha}$ term, obtaining

$$\dot{\bar{\phi}}^2=\dot{\bar{\beta}}^2+6\dot{\psi}_+^2+6\dot{\psi}_-^2+e^{-2(\bar{\beta}/\sqrt{3})}V(\psi_\pm)+\frac{1}{2}A^2e^{-2\sqrt{3}\bar{\beta}}. \quad (5.14)$$

Following the analysis given in [15], we apply a new time coordinate τ defined as

$$dt=d\tau e^{-\bar{\phi}}, \quad (5.15)$$

and define a new variable, y, which is the logarithm of an averaged scale factor in the conformally related axion frame (or B-frame) in which the axion is minimally coupled [26]. This is given by

$$y\equiv\sqrt{3}\bar{\phi}+\bar{\beta}, \quad (5.16)$$

and brings Eq. (5.14) to the form $[(\ldots)_\tau=d(\ldots)/d\tau]$

$$y_\tau^2=\phi_\tau^2+12\psi_{+\tau}^2+12\psi_{-\tau}^2+A^2e^{-2\phi}+2e^{-(2/\sqrt{3})y}V(\psi_\pm). \quad (5.17)$$

Equation (5.17) is, in fact, the Hamiltonian constraint obtained from the action ($\lambda_s^2=8\pi G$)

$$S=\frac{\lambda_s}{4}\int d\tau[\phi_\tau^2-y_\tau^2+12(\psi_{+\tau}^2+\psi_{-\tau}^2) -A^2e^{-2\phi}-2e^{-2(\sqrt{3}y)}V(\psi_\pm)]. \quad (5.18)$$

The canonical momenta are then

$$\pi_\phi=\frac{\lambda_s}{2}\phi_\tau,$$

$$\pi_y=-\frac{\lambda_s}{2}y_\tau,$$

$$\pi_+=6\lambda_s\psi_{+\tau},$$

$$\pi_-=6\lambda_s\psi_{-\tau}, \quad (5.19)$$

and the Hamiltonian is just

$$H=\frac{1}{\lambda_s}\left[\pi_\phi^2-\pi_y^2+\frac{1}{12}(\pi_+^2+\pi_-^2) +\frac{\lambda_s^2}{4}A^2e^{-2\phi}+\frac{\lambda_s^2}{2}e^{-(2/\sqrt{3})y}V(\psi^\pm)\right]. \quad (5.20)$$

Following the analysis of the previous section, V A, we see that the maximum apparent velocity of the wall at $\psi_+\ll -1$, $\psi_-\approx 0$ is given by

$$v_{max}=\frac{1}{2\sqrt{3}}y_\tau, \quad (5.21)$$

and the condition for chaotic scatterings to cease is just that

$$v_p=\sqrt{\psi_{+\tau}^2+\psi_{-\tau}^2}<\frac{1}{2\sqrt{3}}y_\tau$$

$$\approx\sqrt{\psi_{+\tau}^2+\psi_{-\tau}^2+\frac{1}{12}\phi_\tau^2+\frac{1}{12}A^2e^{-2\phi}}\,, \tag{5.22}$$

This is clearly fulfilled except in the general relativity case where $\phi=A=0$ (i.e., no axion and dilaton fields). This gives our final conclusion that *there is no chaos in BIX string cosmology in the string or axion frames*.

It is not surprising that the physical behavior should be similar in every frame [8], so that if there is no chaos in the Einstein frame there should not be chaos in any other frame. String theory appears to impose too much symmetry through its duality invariances for chaos to appear.[10]

VI. DISCUSSION

In this paper we have carried out a detailed analysis of the spatially homogeneous universes of Bianchi type IX in the context of low-energy string theory. These universes are of special interest in relativistic cosmology because they display chaotic behavior in vacuum and in the presence of fluids with $p<\rho$. They were originally termed ''mixmaster'' universes by Misner because they offered the possibility for light to travel all the way around the universe in different directions. Moreover, they are the most general closed universes which are spatially homogeneous and may be closely related to parts of the general solution of Einstein's equations in the neighborhood of a strong curvature singularity of the sort that characterizes the initial state of general relativistic cosmological models. This behavior has also been extensively investigated because it is of intrinsic mathematical interest. It is also known that its occurrence in general relativity depends upon the dimensionality of space. We investigated the string cosmological equations for the type IX metric. We found that chaotic behavior does not occur in string cosmology in either the Einstein or the string or axion frames. While it is possible for finite sequences of oscillations to occur in the scale factors' evolution on approach to $t=0$, these oscillations cannot continue indefinitely. They inevitably terminate in a state in which all the three orthogonal scale factors decrease with decreasing time monotonically on approach to the initial singularity. We investigated the detailed sequences of evolutionary changes that can take place in the evolution during the finite sequences of oscillations between epochs which are well approximated by Kasner universes. We found that the duality symmetry required of the string evolution introduced new invariances for the possible changes in the Kasner parameters in addition to those which characterize the Kasner-to-Kasner cycles of oscillations. The requirements of duality invariance on the evolution of the metric appear to be so constraining that chaotic behavior is excluded. We have obtained these results in two complementary ways: by direct matching of asymptotic expansions to the solutions of the sytem of non-linear ordinary differential equations of string cosmology and by use of the Hamiltonian formulation of cosmology. In the Hamiltonian picture the evolution of the type IX string cosmology is represented as the motion of a ''universe point'' inside a potential that is open only along three narrow channels. The walls of this ''almost closed'' potential of Hénon-Heiles type expand outwards as the singularity is reached. Whereas in the vacuum models of general relativity, the universe point always catches the walls and bounces chaotically around within the potential, in string theory the universe point need never catch the walls. If it is moving towards a wall at a very oblique angle then the normal component of its velocity towards the wall can become too small for it ever to catch the wall. In general, we find that this situation always arises after a finite number of collisions have occurred in the Mixmaster string cosmology. The resulting asymptotic state is therefore similar to that in a model with no potential walls at all; that is, to the Bianchi type I or Kasner universe.

[10]Chaos has been studied in other related situations in Ref. [28].

ACKNOWLEDGMENTS

M.P.D. wishes to thank Amithaba Lahiri and David Wands for useful discussions, Malcolm MacCallum for helpful correspondence, and the NATO/Royal Society for support while at the University of Sussex. M.P.D. was also partially supported by the Polish Research Committee (KBN) grant No 2 PO3B 196 10. J.D.B. was supported by the PPARC.

APPENDIX A: RICCI TENSOR IN A COORDINATE FRAME

In this appendix we discuss a possibility of admittinig a time-dependent antisymmetric tensor potential $B_{\mu\nu}=B_{\mu\nu}(t)$ as given in Ref. [3] to the axisymmetric Bianchi type IX model. It has been proven [4] that such a potential cannot be admitted to a general (i.e. non-axisymmetric case). However, to achieve this, we first give the components of Ricci tensor for the axisymmetric Bianchi type IX model in terms of coordinates rather than in orthonormal frames of Sec. II. For the sake of generality, we start with a general metric.

The metric (2.9) in a coordinate frame has the following components:

$$g_{\overline{00}}=1, \tag{A1}$$

$$g_{\overline{11}}=-c^2(t), \tag{A2}$$

$$g_{\overline{22}}=-[a^2(t)\cos^2\psi+b^2(t)\sin^2\psi], \tag{A3}$$

$$g_{\overline{33}}=-\sin^2\theta[a^2(t)\sin^2\psi+b^2(t)\cos^2\psi]-c^2(t)\cos^2\theta, \tag{A4}$$

$$g_{\overline{13}}=-c^2(t)\cos\theta, \tag{A5}$$

$$g_{\overline{23}}=-\sin\psi\cos\psi\sin\theta[a^2(t)-b^2(t)]. \tag{A6}$$

The Ricci tensor components in the coordinate frame can be calculated using the relations [6] [cf. Eqs. (2.9)–(2.12)]

$$R_{\alpha\beta}=e^j_\alpha e^k_\beta R_{jk}\,, \tag{A7}$$

$$R^\beta_\alpha=e^j_\alpha e^\beta_k R^k_j\,, \tag{A8}$$

where $R_{\alpha\beta}$ is the Ricci tensor in the coordinate frame while R_{jk} is the Ricci tensor in the orthonormal frame (correspondingly $\alpha,\beta=\overline{0},\overline{1},\overline{2},\overline{3}$ are the coordinate frame indices and $i,j=0,1,2,3$ are the orthonormal frame indices). Thus, 1,2,3 refer to $\sigma^1,\sigma^2,\sigma^3$, while $\overline{1},\overline{2},\overline{3}$ refer to ψ,θ,φ respectively. According to Eqs. (2.10)–(2.12),

$$e_{\overline{2}}^{1}=a\cos\psi,$$

$$e_{\overline{3}}^{1}=a\sin\psi\sin\theta,$$

$$e_{\overline{2}}^{2}=b\sin\psi, \tag{A9}$$

$$e_{\overline{3}}^{2}=-b\cos\psi\sin\theta,$$

$$e_{\overline{1}}^{3}=c,$$

$$e_{\overline{3}}^{3}=c\cos\theta,$$

and

$$e_{1}^{\overline{1}}=-\frac{1}{a}\sin\psi\cot\theta, \tag{A10}$$

$$e_{2}^{\overline{1}}=\frac{1}{b}\cos\psi\cot\theta,$$

$$e_{3}^{\overline{1}}=\frac{1}{c},$$

$$e_{1}^{\overline{2}}=\frac{1}{a}\cos\psi, \tag{A11}$$

$$e_{2}^{\overline{2}}=\frac{1}{b}\sin\psi,$$

$$e_{1}^{\overline{3}}=\frac{1}{a}\,\frac{\sin\psi}{\sin\theta},$$

$$e_{2}^{\overline{3}}=-\frac{1}{b}\,\frac{\cos\psi}{\sin\theta}.$$

Then, the Ricci tensor components in the coordinate frame are given by

$$R_{\overline{11}}=c^2R_{33}, \tag{A12}$$

$$R_{\overline{22}}=a^2\cos^2\psi R_{11}+b^2\sin^2\psi R_{22}, \tag{A13}$$

$$R_{\overline{33}}=a^2\sin^2\psi\sin^2\theta R_{11}+b^2\cos^2\psi\sin^2\theta R_{22}+c^2\cos^2\theta R_{33}, \tag{A14}$$

$$R_{\overline{13}}=c^2\cos\theta R_{33}, \tag{A15}$$

$$R_{\overline{23}}=\sin\psi\cos\psi\sin\theta(a^2R_{11}-b^2R_{22}), \tag{A16}$$

$$R_{\overline{12}}=0, \tag{A17}$$

and

$$R_{\overline{1}}^{\overline{1}}=R_3^3, \tag{A18}$$

$$R_{\overline{2}}^{\overline{2}}=\cos^2\psi R_1^1+\sin^2\psi R_2^2, \tag{A19}$$

$$R_{\overline{3}}^{\overline{3}}=\sin^2\psi R_1^1+\cos^2\psi R_2^2, \tag{A20}$$

$$R_{\overline{3}}^{\overline{1}}=[-\sin^2\psi R_1^1-\cos^2\psi R_2^2+R_3^3]\cos\theta, \tag{A21}$$

$$R_{\overline{3}}^{\overline{2}}=\sin\psi\cos\psi\sin\theta(R_1^1-R_2^2), \tag{A22}$$

$$R_{\overline{2}}^{\overline{3}}=\frac{\sin\psi\cos\psi}{\sin\theta}(R_1^1-R_2^2), \tag{A23}$$

$$R_{\overline{2}}^{\overline{1}}=-\sin\psi\cos\psi\cot\theta(R_1^1-R_2^2), \tag{A24}$$

$$R_{\overline{1}}^{\overline{3}}=0, \tag{A25}$$

$$R_{\overline{1}}^{\overline{2}}=0. \tag{A26}$$

If two axes are the same $[a(t)=b(t)]$ the metric (2.9) [or its components given by Eqs. (A1)–(A6)] simplifies to [11,12]

$$ds^2=dt^2-c^2(d\psi+\cos\theta d\varphi)^2-a^2(d\theta^2+\sin^2\theta d\varphi^2). \tag{A27}$$

The nonvanishing Christoffel symbols for the metric (A27) are

$$\Gamma_{0\overline{1}}^{\overline{1}}=\frac{\dot c}{c},\quad \Gamma_{0\overline{2}}^{\overline{2}}=\frac{\dot a}{a},\quad \Gamma_{0\overline{3}}^{\overline{3}}=\frac{\dot a}{a},$$

$$\Gamma_{0\overline{3}}^{\overline{1}}=\left(\frac{\dot c}{c}-\frac{\dot a}{a}\right)\cos\theta,\quad \Gamma_{\overline{11}}^{\overline{0}}=\dot c c,\quad \Gamma_{\overline{22}}^{\overline{0}}=\dot a a,$$

$$\Gamma_{\overline{33}}^{\overline{0}}=\dot c c\cos^2\theta+\dot a a\sin^2\theta,\quad \Gamma_{\overline{13}}^{\overline{0}}=\dot c c\cos\theta,$$

$$\Gamma_{\overline{21}}^{\overline{1}}=\frac{1}{2}\,\frac{c^2}{a^2}\cot\theta,\quad \Gamma_{\overline{23}}^{\overline{1}}=\frac{1}{2\sin\theta}\left(\frac{c^2-a^2}{a^2}\cos^2\theta-1\right), \tag{A28}$$

$$\Gamma_{\overline{31}}^{\overline{2}}=\frac{1}{2}\,\frac{c^2}{a^2}\sin\theta,\quad \Gamma_{\overline{33}}^{\overline{2}}=\sin\theta\cos\theta\frac{c^2-a^2}{a^2},$$

$$\Gamma_{\overline{12}}^{\overline{3}}=-\frac{1}{2}\,\frac{c^2}{a^2}\,\frac{1}{\sin\theta},\quad \Gamma_{\overline{23}}^{\overline{3}}=\cot\theta\left(1-\frac{1}{2}\,\frac{c^2}{a^2}\right).$$

The gradients of the dilaton calculated with respect to the metric (A27) are given by

$$\nabla_{\overline{0}}\nabla^{\overline{0}}\phi=\ddot\phi, \tag{A29}$$

$$\nabla_{\overline{1}}\nabla^{\overline{1}}\phi=\frac{\dot c}{c}\dot\phi, \tag{A30}$$

$$\nabla_{\bar{2}}\nabla^{\bar{2}}\phi=\nabla_{\bar{3}}\nabla^{\bar{3}}=\frac{\dot{a}}{a}\dot{\phi}, \tag{A31}$$

$$\nabla_{\bar{3}}\nabla^{\bar{1}}\phi=\dot{\phi}\left(\frac{\dot{c}}{c}-\frac{\dot{a}}{a}\right)\cos\theta, \tag{A32}$$

and

$$\nabla_{\bar{0}}\phi\nabla^{\bar{0}}\phi=\dot{\phi}^2. \tag{A33}$$

The nonzero components of the Ricci tensor in a coordinate frame for the metric (A27) can be obtained from Eqs. (A12)–(A26) by putting $R^1_1=R^2_2$ and using Eqs. (2.17)–(2.20) for $a=b$ or directly using the Christoffel symbols. This gives

$$R_{\overline{11}}=\ddot{c}c+2\dot{c}c\frac{\dot{a}}{a}+\frac{1}{2}\frac{c^4}{a^4},\quad R_{\overline{13}}=R_{\overline{11}}\cos\theta,$$

$$R_{\overline{22}}=\ddot{a}a+\dot{a}^2+\dot{a}a\frac{\dot{c}}{c}+1-\frac{1}{2}\frac{c^2}{a^2},\quad R_{\overline{00}}=-\frac{\ddot{c}}{c}-2\frac{\ddot{a}}{a}, \tag{A34}$$

$$R_{\overline{33}}=R_{\overline{11}}\cos^2\theta+R_{\overline{22}}\sin^2\theta,$$

so

$$-R^{\bar{0}}_{\bar{0}}=\frac{\ddot{c}}{c}+2\frac{\ddot{a}}{a}, \tag{A35}$$

$$-R^{\bar{1}}_{\bar{1}}=\frac{\ddot{c}}{c}+2\frac{\dot{a}}{a}\frac{\dot{c}}{c}+\frac{1}{2}\frac{c^2}{a^4}, \tag{A36}$$

$$-R^{\bar{2}}_{\bar{2}}=-R^{\bar{3}}_{\bar{3}}=\frac{\ddot{a}}{a}+\frac{\dot{a}^2}{a^2}+\frac{\dot{a}}{a}\frac{\dot{c}}{c}+\frac{1}{a^2}-\frac{1}{2}\frac{c^2}{a^4}, \tag{A37}$$

$$R^{\bar{1}}_{\bar{3}}=\left(\frac{\ddot{a}}{a}-\frac{\ddot{c}}{c}-\frac{\dot{a}}{a}\frac{\dot{c}}{c}+\frac{\dot{a}^2}{a^2}+\frac{1}{a^2}-\frac{c^2}{a^4}\right)\cos\theta, \tag{A38}$$

and the Ricci scalar reads

$$R=-2\frac{\ddot{c}}{c}-4\frac{\ddot{a}}{a}-4\frac{\dot{a}}{a}\frac{\dot{c}}{c}-2\frac{\dot{a}^2}{a^2}-2\frac{1}{a^2}+\frac{1}{2}\frac{c^2}{a^4}. \tag{A39}$$

Note that in the coordinate frame $R^{\bar{2}}_{\bar{2}}=R^{\bar{3}}_{\bar{3}}$ while in the orthonormal frame [cf. Eqs. (2.18),(2.19)] $R^1_1=R^2_2$. This is reasonable, since the metric tensor is given by Eq. (A27), for the former case, and by Eq. (2.9) with $a=b$ for the latter case, where the indices refer to the orthonormal basis rather than to the chosen coordinates.

Now we convert our notation in terms of the three-index torsion field H to the notation given in [3] using the pseudoscalar torsion field h. Following [3] we define ($\alpha,\beta,\mu,\nu=\bar{0},\bar{1},\bar{2},\bar{2}$)

$$H^{\mu\nu\alpha}=e^{\phi}\epsilon^{\mu\nu\alpha\beta}h_{,\beta}. \tag{A40}$$

The equation of motion for h-field obtained via integrability conditions is then

$$\nabla^{\mu}\nabla_{\mu}h+\nabla^{\mu}\phi\nabla_{\mu}h=0. \tag{A41}$$

One can easily see that for the time-independent antisymmetric tensor potential $B_{\mu\nu}=B_{\mu\nu}(x)$ the h field can only depend on time and Eq. (A41) reads as

$$\ddot{h}+\left(\frac{\dot{a}}{a}+\frac{\dot{b}}{b}+\frac{\dot{c}}{c}\right)\dot{h}+\dot{\phi}\dot{h}=0, \tag{A42}$$

which integrates to give

$$\dot{h}=-A\frac{e^{-\phi}}{abc}, \tag{A43}$$

so from Eq. (A40) we have

$$H^{\overline{123}}=-\frac{A}{a^2b^2c^2\sin\theta}, \tag{A44}$$

or

$$H_{\overline{123}}=A\sin\theta, \tag{A45}$$

as required by Eq. (2.25) and $H^2=-6A^2/a^2b^2c^2$. With the H field chosen as above the equation of motion (2.3) is easily fulfilled. There is also a trivial solution of Eq. (A42), $\dot{h}=0$, but it corresponds to a constant torsion field. For the time-independent pseudoscalar axion field $h=h(x)$ [time-dependent antisymmetric tensor potential $B_{\mu\nu}=B_{\mu\nu}(t)$] the equation of motion (A41) reads

$$\partial^{\mu}\partial_{\mu}h+\Gamma^{\mu}_{\rho\mu}\partial^{\rho}h=0, \tag{A46}$$

which for the metric (2.9) reads as

$$g^{\overline{11}}\partial^2_{\bar{1}}h+g^{\overline{22}}\partial^2_{\bar{2}}h+g^{\overline{33}}\partial^2_{\bar{3}}h+g^{\overline{13}}\partial_{\bar{1}}\partial_{\bar{3}}h+g^{\overline{22}}\cot\theta\,\partial_{\bar{2}}h=0. \tag{A47}$$

For simplicity, let us introduce

$$(H^{\nu}_{\mu})^2=H_{\mu\alpha\beta}H^{\nu\alpha\beta}=-2e^{2\phi}(\delta^{\nu}_{\mu}g^{\rho\varepsilon}-\delta^{\rho}_{\mu}g^{\nu\varepsilon})\partial_{\varepsilon}h\,\partial_{\rho}h, \tag{A48}$$

$$H^2=H_{\mu\alpha\beta}H^{\mu\alpha\beta}. \tag{A49}$$

The non-zero components of these quantities (the energy-momentum tensor) which are used in the field equations (2.1)–(2.3) for the metric components (A27) are given by

$$(H^{\bar{0}}_{0})^2=-2e^{2\phi}(g^{\overline{11}}\partial_{\bar{1}}h\,\partial_{\bar{1}}h+2g^{\overline{13}}\partial_{\bar{1}}h\,\partial_{\bar{3}}h+g^{\overline{22}}\partial_{\bar{2}}h\,\partial_{\bar{2}}h+g^{\overline{33}}\partial_{\bar{3}}h\,\partial_{\bar{3}}h), \tag{A50}$$

$$(H^{\bar{1}}_{1})^2=-2e^{2\phi}(g^{\overline{13}}\partial_{\bar{1}}h\,\partial_{\bar{3}}h+g^{\overline{22}}\partial_{\bar{2}}h\,\partial_{\bar{2}}h+g^{\overline{33}}\partial_{\bar{3}}h\,\partial_{\bar{3}}h), \tag{A51}$$

$$(H^{\bar{2}}_{\bar{2}})^2 = -2e^{2\phi}(g^{\overline{11}}\partial_{\bar{1}}h\partial_{\bar{1}}h + 2g^{\overline{13}}\partial_{\bar{1}}h\partial_{\bar{3}}h + g^{\overline{33}}\partial_{\bar{3}}h\partial_{\bar{3}}h), \tag{A52}$$

$$(H^{\bar{3}}_{\bar{3}})^2 = -2e^{2\phi}(g^{\overline{11}}\partial_{\bar{1}}h\partial_{\bar{1}}h + g^{\overline{13}}\partial_{\bar{1}}h\partial_{\bar{3}}h + g^{\overline{22}}\partial_{\bar{2}}h\partial_{\bar{2}}h), \tag{A53}$$

$$(H^{\bar{1}}_{\bar{3}})^2 = 2e^{2\phi}(g^{\overline{11}}\partial_{\bar{1}}h\partial_{\bar{3}}h + g^{\overline{13}}\partial_{\bar{3}}h\partial_{\bar{3}}h), \tag{A54}$$

and

$$H^2 = -6e^{2\phi}(g^{\overline{11}}\partial_{\bar{1}}h\partial_{\bar{1}}h + 2g^{\overline{13}}\partial_{\bar{1}}h\partial_{\bar{3}}h + g^{\overline{22}}\partial_{\bar{2}}h\partial_{\bar{2}}h + g^{\overline{33}}\partial_{\bar{3}}h\partial_{\bar{3}}h). \tag{A55}$$

With the choice

$$\partial_{\bar{3}}h = E\sin\theta \neq 0 \qquad \partial_{\bar{1}}h = \partial_{\bar{2}}h = 0, \tag{A56}$$

the field equations remain homogeneous and

$$(H^{\bar{0}}_{\bar{0}})^2 = (H^{\bar{1}}_{\bar{1}})^2 = (H^{\bar{2}}_{\bar{2}})^2 = 2\frac{E^2e^{2\phi}}{a^2}, \tag{A57}$$

$$(H^{\bar{3}}_{\bar{3}})^2 = 0, \tag{A58}$$

$$(H^{\bar{1}}_{\bar{3}})^2 = 2\frac{E^2e^{2\phi}}{a^2}\cos\theta, \tag{A59}$$

$$H^2 = 6\frac{E^2e^{2\phi}}{a^2} \tag{A60}$$

and the equation of motion (A47) becomes

$$g^{\overline{33}}\partial_{\varphi}(E\sin\theta) = 0, \tag{A61}$$

and it is satisfied. Finally, from Eq. (A40), we have

$$H^{\overline{012}} = \frac{Ee^{\phi}}{a^2c}, \tag{A62}$$

$$H_{\overline{012}} = Ece^{\phi}, \tag{A63}$$

and $H^2 = 6Ee^{2\phi}/a^2$. This obeys the axion equation of motion (2.3) but is in contradiction with the axisymmetry condition for the Ricci components (A37) since $R^2_2 = R^3_3$ there while $(H^{\bar{3}}_{\bar{3}})^2 = 0$ and $(H^{\bar{2}}_{\bar{2}})^2 \neq 0$ here.

A possible ansatz which would satisfy the axisymmetry condition would be

$$\partial_{\bar{1}}h = \frac{Ba(t)\sin\theta}{\sqrt{a(t)^2\sin^2\theta + c(t)^2\cos^2\theta}}, \tag{A64}$$

but it leads to both time and space dependences of the pseudoscalar axion field, $h = h(t,\psi,\theta)$, and does not satisfy the equation of motion (A41) [nor Eq. (A47) which is obtained for $h = h(x)$].

One could also try to add

$$\partial_{\bar{2}}h = D = \text{const}, \tag{A65}$$

to the nonzero component (A56), but again this does not satisfy the equation (A47).

The final conclusion is that one is not able to impose the axion field even in axisymmetric BIX models despite the fact that there is a distinguished direction in the model (which is different from electromagnetic field case—see [11]). The reason seems to be that even in the axisymmetric case there is still $SO(3)$ symmetry group present and we are only adding an additional symmetry $SO(2)$ which does not cancel the former one, giving the total symmetry $SO(3) \otimes SO(2)$ rather than just $SO(2)$.

[1] E. S. Fradkin and A. A. Tseytlin, Nucl. Phys. **B261**, 1 (1985); C. G. Callan, D. Friedan, E. J. Martinec, and M. J. Perry, *ibid.* **B262**, 593 (1985).

[2] M. Mueller, Nucl. Phys. **B37**, 37 (1990); G. Veneziano, Phys. Lett. B **265**, 287 (1991); M. Gasperini and G. Veneziano, Mod. Phys. Lett. A **39**, 3701 (1993); M. Gasperini and G. Veneziano, Astrophys. **1**, 317 (1993); E. J. Copeland, A. Lahiri, and D. Wands, Phys. Rev. D **50**, 4868 (1994); M. Gasperini and R. Ricci, Class. Quantum Grav. **12**, 677 (1995); A. Saaryan, Astrophys. **38**, 164 (1995); M. Gasperini and G. Veneziano, Phys. Rev. D **50**, 2519 (1994).

[3] E. J. Copeland, A. Lahiri, and D. Wands, Phys. Rev. D **51**, 1569 (1995).

[4] J. D. Barrow and K. E. Kunze, Phys. Rev. D **55**, 623 (1997); J. D. Barrow and M. P. Dąbrowski, *ibid.* **55**, 630 (1997).

[5] N. A. Batakis and A. Kehagias, Nucl. Phys. **B449**, 248 (1995); J. P. Mimóso and D. Wands, Phys. Rev. D **52**, 5612 (1995).

[6] V. A. Belinskii, E. M. Lifshitz, and I. M. Khalatnikov, Sov. Phys. Usp. **102**, 745 (1971); L. D. Landau and E. M. Lifshitz, *The Classical Theory of Fields* (Pergamon, Oxford, 1975); J. D. Barrow, Phys. Rev. Lett. **46**, 963 (1981); Phys. Rep. **85**, 1 (1982); D. Chernoff and J. D. Barrow, Phys. Rev. Lett. **50**, 134 (1983); J. D. Barrow, in *Classical General Relativity*, edited by W. Bonnor, J. Islam, and M.A.H. MacCallum (Cambridge University Press, Cambridge, England, 1984), p. 25.

[7] V. A. Belinskii and I. M. Khalatnikov, Sov. Phys. JETP **36**, 591 (1973).

[8] G. Veneziano, Phys. Lett. B **406**, 297 (1997); A. Buonanno, K. A. Meissner, C. Ungarelli, and G. Veneziano, Phys. Rev. D **57**, 2543 (1998); A. Feinstein, R. Lazkoz, and M. A. Vázquez-Mozo, *ibid.* **56**, 5166 (1997).

[9] J. D. Barrow and K. E. Kunze, Phys. Rev. D **56**, 741 (1997).

[10] B. A. Campbell, A. Linde, and K. A. Olive, Nucl. Phys. **B355**, 146 (1991); Eq. (7).

[11] D. R. Brill, Phys. Rev. **133**, B845 (1964).

[12] J. D. Barrow, Nature (London) **272**, 211 (1978).

[13] J. D. Barrow, Nucl. Phys. **B296**, 697 (1988).
[14] R. Easther, K. Maeda, and D. Wands, Phys. Rev. D **53**, 4247 (1996).
[15] M. Gasperini and G. Veneziano, Gen. Relativ. Gravit. **28**, 1301 (1996).
[16] J. Wainwright and L. Hsu, Class. Quantum Grav. **6**, 1409 (1989); J. Wainwright, in *Dynamical Systems in Cosmology*, edited by J. Wainwright and G. Ellis (Cambridge University Press, Cambridge, England, 1997), p. 132.
[17] M. P. Ryan and L. C. Shepley, *Homogeneous Relativistic Cosmologies* (Princeton University Press, Princeton, NJ, 1975).
[18] J. S. R. Chisholm and J. McEwan, Proc. R. Soc. London **A336**, 421 (1974); G. A. Baker, *Essentials of Padé Approximants* (Academic, New York, 1975).
[19] P. Halpern, Phys. Rev. D **33**, 354 (1986).
[20] J. D. Barrow and J. Stein-Schabes, Phys. Rev. D **32**, 1595 (1985).
[21] H. Ishihara, Prog. Theor. Phys. **74**, 354 (1986).
[22] T. Furosawa and A. Hosoya, Prog. Theor. Phys. **73**, 467 (1985).
[23] K. A. Meissner and G. Veneziano, Phys. Lett. B **267**, 33 (1991); Mod. Phys. Lett. A **6**, 3397 (1991); K. A. Meissner, Phys. Lett. B **392**, 298 (1997).
[24] M. Gasperini, R. Ricci, and G. Veneziano, Phys. Lett. B **319**, 438 (1993).
[25] J. E. Lidsey, ''On the Cosmology and Symmetry of Dilaton-Axion Gravity,'' e-Print gr-qc/9609063.
[26] E. J. Copeland, R. Easther, and D. Wands, Phys. Rev. D **56**, 874 (1997).
[27] A. R. Liddle, R. G. Moorhouse, and A. B. Henriques, Nucl. Phys. **B311**, 719 (1988).
[28] D. V. Gal'tsov and M. S. Volkov, Phys. Lett. B **256**, 17 (1991); I. Ya. Aref'eva, P. B. Medvedev, O. A. Rytchkov, and I. V. Volovich, ''Chaos in M(atrix) Theory,'' e-Print: hep-th/9710032; J. D. Barrow and J. Levin, Phys. Rev. Lett. **80**, 656 (1998).