Topological inflation with multiple winding

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(Received 8 January 1998; published 15 May 1998)

We analyze the core dynamics of critically coupled, superheavy gauge vortices in the (2+1)-dimensional Einstein-Abelian Higgs system. By numerically solving the Einstein and field equations for various values of the symmetry breaking scale, we identify the regime in which static solutions cease to exist and topological inflation begins. We explicitly include the topological winding of the vortices into the calculation and extract the dependence on the winding of the critical scale separating the static and inflating regimes. Extrapolation of our results suggests that topological inflation might occur within high winding strings formed at the grand unified scale. [S0556-2821(98)06112-8]

PACS number(s): 98.80.Cq

I. INTRODUCTION

Two of the most exciting possibilities suggested by modern particle cosmology are the production of topological defects and the existence of an inflationary stage in the early universe. In the last few years it has been pointed out independently by Vilenkin [1] and by Linde [2] that these two phenomena may be related. A necessary condition for inflation to occur is that the energy density of the universe be dominated by the vacuum energy of a homogeneous scalar field, called the ''inflaton.'' Correspondingly, a central feature of topological defects is that their cores are regions of spacetime in which a scalar field is forced to sit out of its vacuum manifold, taking a value corresponding to a false vacuum. Hence, the cores of topological defects are regions in which energy density is trapped in the form of the vacuum energy of a scalar field. Vilenkin and Linde realized that if the energy density trapped in a defect core and the core radius itself were large enough, this energy could be considered uniform (i.e. horizon sized), and the core would satisfy the conditions for inflation. This scenario is what is known as *topological inflation*. The advantage of this implementation is that, while traditionally the necessary conditions require the fine tuning of a scalar potential, here inflation is inevitable if the vacuum manifold of the theory at high temperatures satisfies a topological constraint. Thus, the question of initial conditions becomes one of topology.

If defects are produced when a scalar field gets a vacuum expectation value equal to η in the early universe, then an approximate criterion proposed by Vilenkin and Linde for topological inflation to occur is $\eta > m_p$, where m_p is the Planck mass. (Note that the energy density in the scalar field is proportional to $\lambda \eta^4$ which is assumed to be less than the Planck energy density.) The supposition that, under this condition, time-dependent defect solutions are inevitable is supported by the traditional solutions for the spacetime around, for example, a cosmic string. At symmetry breaking scales significantly below the Planck scale, the spacetime around a static cosmic string is conical [3], with a deficit angle proportional to η^2. However, as η increases to the order of the Planck mass, the deficit angle approaches 2π and static solutions cease to exist [4]. Recently, the criterion for topological inflation has been made more precise by numerical simulations of the spacetime structure around various defects: gauge monopoles [5], domain walls [6], and global monopoles [6,7]. These analyses all focus on defects with unit topological charge, and find that in that case, the criterion for topological inflation is

$$\eta > \eta_{cr} \equiv 0.33 m_p . \tag{1.1}$$

The aim of this paper is first to verify that the above criterion holds for the important case of gauge cosmic strings and, second, to investigate how the criteria for topological inflation depend on the topological charge of the defects considered.

Our motivations are twofold. First, for defects of unit winding, Eq. (1.1) implies that, to realize topological inflation, we must work very close to the Planck scale, at which our field theories may not be valid. Further, it is clear that topological inflation cannot occur at the grand unified theory (GUT) scale for unit winding defects. We hope that, when higher topological charges are included, these constraints will be alleviated and the value of η_{cr} will decrease. In fact, one might expect such behavior from considering the static string solutions, since the deficit angle we mentioned above is also dependent on the winding n, and so static solutions should cease to exist for lower values of η if $n>1$. The spacetime structure for such higher winding strings is the focus of this paper.

Our second motivation comes from models in which particles are described as solitons [8], and in particular from the *dual standard model* [9]. In this theory, all the standard model particles arise as monopoles of some bosonic field theory. In such a model it is natural to wonder what happens to matter at high densities when the core structure of the solitons becomes important. For example, stars can be seen as collections of huge numbers ($\sim 10^{57}$) of monopoles. If the monopoles become squeezed together tightly enough for a

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large region of the star to be in the false vacuum with high winding, might topological inflation occur?

In the present work, fueled by the above considerations, we consider the simple example of an Abelian-Higgs vortex with winding n in $2+1$ spacetime dimensions. We do so because it is easier to work with multiple winding vortices than the analogous monopoles, although the problem of monopoles for $n>1$ is under consideration. In the next section we present the model and the equations of motion we solve. We give our initial conditions and describe how we expect solutions to behave in some asymptotic regimes. In Sec. III, we briefly discuss the implementation of the numerical algorithms we use to solve the equations and in Sec. IV we present our results. Section V contains a discussion of the results and their implications for topological inflation.

II. MODEL

Consider the Abelian Higgs model with a complex scalar field Φ and a $U(1)$ gauge field A_μ, coupled to gravity in $2+1$ spacetime dimensions. The action is

$$S=\int d^3x\sqrt{-g}\left(\frac{1}{16\pi G}\mathcal{R}+\mathcal{L}\right), \tag{2.1}$$

where $\mathcal{R}$ is the Ricci scalar and the Lagrangian density for the matter fields is

$$\mathcal{L}=(D_\mu\Phi)^*D^\mu\Phi-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}-V(\Phi). \tag{2.2}$$

Here, the covariant derivative is $D_\mu\Phi=(\nabla_\mu+i\epsilon A_\mu)\Phi$, the gauge field strength is $F_{\mu\nu}=\partial_\mu A_\nu-\partial_\nu A_\mu$, and the scalar potential is

$$V(\Phi)=\frac{\lambda}{4}(\Phi^*\Phi-\eta^2), \tag{2.3}$$

with λ, ϵ constants and η a mass scale.

In cylindrical polar coordinates (r,θ), we make the usual Nielsen-Olesen string *Ansatz* for the fields

$$\Phi(x)=f(r,t)e^{in\theta},$$

$$A_\theta=\frac{1}{\epsilon}[h(r,t)-n],$$

where the integer n is the winding number of the string. The metric *Ansatz* is

$$ds^2=dt^2-e^{B(r,t)}dr^2-e^{C(r,t)}r^2d\theta^2, \tag{2.4}$$

where B and C are functions of the radial coordinate and time.

These *Ansätze* lead to four Einstein equations and two field equations for a total of four unknown functions: B, C, f and h. Two of the six equations are first order in time derivatives and are the constraint equations. The four equations we solve are

$$e^B(-2\ddot{C}-\dot{C}^2)=\frac{32\pi}{m_p^2}\left[e^B\dot{f}^2+f'^2+\frac{e^{B-C}}{2\epsilon^2r^2}\dot{h}^2+\frac{e^{-C}}{2\epsilon^2r^2}h'^2-\frac{e^{B-C}}{r^2}h^2f^2-\frac{\lambda}{4}e^B(f^2-\eta^2)^2\right], \tag{2.5}$$

$$e^C(-2\ddot{B}-\dot{B}^2)=\frac{32\pi}{m_p^2}\left[e^C\dot{f}^2-e^{C-B}f'^2-\frac{1}{2\epsilon^2r^2}\dot{h}^2+\frac{e^{-B}}{2\epsilon^2r^2}h'^2+\frac{1}{r^2}h^2f^2-\frac{\lambda}{4}e^C(f^2-\eta^2)^2\right], \tag{2.6}$$

$$\ddot{f}-e^{-B}f''+\frac{e^{-C}}{r^2}h^2f+\frac{1}{2}(\dot{B}+\dot{C})\dot{f}-e^{-B}\left[\frac{1}{r}+\frac{1}{2}(B'+C')\right]f'+\frac{\lambda}{2}(f^2-\eta^2)f=0, \tag{2.7}$$

$$\ddot{h}-e^{-B}h''-\frac{1}{2}(\dot{C}-\dot{B})\dot{h}+\frac{e^{-B}}{2}(C'+B')h'+e^{-B}\frac{h'}{r}+2\epsilon^2f^2h=0, \tag{2.8}$$

where an overdot (prime) denotes a derivative with respect to time (r). At all times, we insist that the two additional constraint equations

$$\dot{B}\dot{C}+e^{-B}\left(-2C''-C'^2+B'C'-4\frac{C'}{r}+2\frac{B'}{r}\right)=\frac{32\pi}{m_p^2}\left[\dot{f}^2+e^{-B}f'^2+\frac{e^{-C}}{2\epsilon^2r^2}\dot{h}^2+\frac{e^{-B-C}}{2\epsilon^2r^2}h'^2+\frac{e^{-C}}{r^2}h^2f^2+\frac{\lambda}{4}(f^2-\eta^2)^2\right] \tag{2.9}$$

and

$$-2\dot{C}'-\dot{C}C'+\dot{B}C'-2\frac{\dot{C}}{r}+2\frac{\dot{B}}{r}=\frac{32\pi}{m_p^2}\left[2\dot{f}f'+\frac{e^{-C}}{\epsilon^2r^2}\dot{h}h'\right] \tag{2.10}$$

be satisfied.

Now consider the initial conditions for these equations. We begin with a cylindrically symmetric string configuration for the fields f and h, which is initially static. We define the metric to be initially flat but with non-vanishing first time derivatives. For the metric, these conditions are simply implemented as

$$B(r,0)=C(r,0)=0, \tag{2.11}$$

with $\dot{B}(r,0)$ and $\dot{C}(r,0)$ obtained from the constraint equation (2.9). For the fields, however, we need the initial profile functions $f(r,0)$ and $h(r,0)$. To simplify this process, we work in the *Bogomoln'yi limit* defined by

$$\beta \equiv \frac{\lambda}{2\epsilon^2} = 1. \tag{2.12}$$

In this case, there are no forces between static strings and the energy saturates a topological bound. Since we are primarily concerned with gravitational effects, we do not expect results for $\beta \neq 1$ to be significantly different. In the Bogomoln'yi limit, the static field equations reduce to the two first order equations

$$f' + e^{(B-C)/2}\frac{hf}{r} = 0, \tag{2.13}$$

$$h' + \frac{\lambda}{2} r e^{(B+C)/2}(f^2 - \eta^2) = 0, \tag{2.14}$$

which can be solved numerically.

Our procedure is as follows. We first complete our initial conditions by solving Eqs. (2.13), (2.14) subject to the boundary conditions

$$\lim_{r\to\infty} f(r) = \eta$$

$$h(0) = -n. \tag{2.15}$$

We then solve Eqs. (2.5), (2.6), (2.7), (2.8) with the initial conditions we have just described. Throughout the evolution we verify that the constraint equations (2.9), (2.10) are satisfied at each step as a check of our numerical scheme.

For a given topological charge n, this procedure is performed over a range of values of the symmetry breaking scale η. We define a solution exhibiting topological inflation to be one for which the total physical volume, $V_*(t)$, in the core of the defect is increasing exponentially. We define $V_*(t)$ by

$$V_*(t) \equiv 2\pi \int_0^{r_*(t)} dr r \exp\left[\frac{B(r,t)+C(r,t)}{2}\right], \tag{2.16}$$

where the core radius, $r_*(t)$, is defined by

$$f[r_*(t)] \equiv \frac{\eta}{2}. \tag{2.17}$$

Determining the functional form of $\eta_{cr}(n)$ is the central result of this paper.

It is a useful check of our results that one may simply derive an upper bound for the expansion rate in the core. Assume that inflation occurs and that the metric components B and C become very large compared to other fields in the core of the defect. Further, assume that only the vacuum energy of the scalar field is important in the core, i.e. that $\phi = 0$ with no derivatives important there. In this approximation, the equations of motion for the metric simplify dramatically and are easily solved to give

$$B(0,t) \sim C(0,t) \sim (8\pi\lambda)^{1/2}\left(\frac{\eta}{m_p}\right)^2 t. \tag{2.18}$$

Thus, any inflationary behavior we observe should have an associated Hubble constant H that satisfies

$$H < (8\pi\lambda)^{1/2}\left(\frac{\eta}{m_p}\right)^2. \tag{2.19}$$

Our intuition for believing that higher topological charges will alleviate the high symmetry breaking scales required for topological inflation comes from two sources. First, consider the asymptotic form of the metric for static cosmic string solutions (in 2+1 dimensions)

$$ds_{2+1}^2 = dt^2 - dr^2 - r^2 d\tilde{\theta}^2, \tag{2.20}$$

where $\tilde{\theta}$ is the angle in a locally Minkowski but globally conical spatial section, taking values in the range

$$0 \leq \tilde{\theta} < 2\pi\left(1 - 4|n|\frac{\eta^2}{m_p^2}\right). \tag{2.21}$$

For strings of unit winding, this metric is applicable as long as the deficit angle is less than 2π, that is, for symmetry breaking scales $\eta \ll m_p$. However, for higher winding, static solutions cease to exist for

$$4|n|\left(\frac{\eta}{m_p}\right)^2 < 1. \tag{2.22}$$

Thus, we expect that asymptotically static solutions become impossible at a lower critical symmetry breaking scale for defects with multiple windings and we might guess that the critical value of η at which static solutions cease to exist falls off as $1/\sqrt{n}$. It is natural to wonder if the same is true in the core of these defects, although, of course, the absence of static solutions does not guarantee that the core will inflate.

Second, a perturbative analysis of the matter fields around the center of such defects demonstrates that defects with $|n|>1$ have a wider core and higher energy density than the corresponding unit charge configurations. Both these effects suggest that topological inflation might be more easily achieved in high winding defects. Unfortunately, it does not seem possible to quantitatively understand the effect of multiple windings on topological inflation with an analytic approach. Thus, here we have solved the system numerically, in the spirit of other authors in the unit winding case [7,5,6].

III. NUMERICAL IMPLEMENTATION AND RESULTS

In this section, we present the results of our numerical simulations of the (2+1)-dimensional Einstein-Abelian-Higgs system. The system of non-linear partial differential equations we study is non-trivial to solve numerically. Therefore, before we present our results, let us briefly discuss the numerical techniques we use, in the hope that this discussion will help others investigating similar problems.

A. Numerical implementation

There are two stages to solving the equations. The first is to generate the field configurations that will serve as the initial values for the time dependent evolution equations, and the second is to integrate the partial differential equations that describe the time dependence of the fields. To attack the former, we observe that the Bogomoln'yi equations (2.13) and (2.14) have asymptotic solution

$$\lim_{r\to 0} f(r) \sim f_0 r^n, \tag{3.1}$$

where f_0 is a constant of integration. This constant can be used as a free parameter in a shooting method solution to the boundary value problem. We find that shooting is effective in generating accurate solutions out to a radius of 10η, sufficient for both fields to reach their $r\to\infty$ asymptotic values for the windings we consider.

With the initial conditions in hand, we must now consider how to integrate the time dependent partial differential equations. Typically one replaces derivatives with finite difference approximations. For a generic variable $y(r,t)$, one solves for values on a lattice $y(r,t)\to y_i^j$, where subscripts indicate the position in the space lattice and superscripts indicate the location in the time lattice. Derivatives are replaced with finite difference approximations, e.g.

$$\ddot{y} \approx \frac{y_i^{j+1} - 2y_i^j + y_i^{j-1}}{dt^2}$$

$$y'' \approx \frac{y_{i+1}^j - 2y_i^j + y_{i-1}^j}{dr^2}$$

$$\vdots$$

which are second order approximations in dt and dr. However, there is in general no guarantee that a particular differencing scheme is stable. That is to say, for poor schemes the result from integrating the difference equations may diverge exponentially from the true solution. To test the differencing methods, we may use the stability analysis for linear equations which is covered in any good reference on partial differential equations (see e.g. [10], Chap. 19, and references therein). The results of such an analysis also provide good intuition when dealing with the non-linear equations for the metric and fields we are considering. For our equations, we find that when solving for f and h, stability is assured if the spatial derivatives of these fields are evaluated implicitly. That is, for the jth time step we evaluate the second spatial derivative of f as

$$f'' = (y_{i+1}^{j+1} - 2y_i^{j+1} + y_{i-1}^{j+1})/dr^2, \tag{3.2}$$

and similarly for the others. At each time step, the implicit scheme gives us a set of coupled linear equations for the fields f_i^{j+1} and h_i^{j+1}, solving which reduces to inverting a tri-diagonal matrix, a standard problem in linear algebra. We also find that care is needed when evaluating the metric equations (2.5) and (2.6). Both can be written in the form

$$\ddot{C} + \dot{C}^2 = F, \tag{3.3}$$

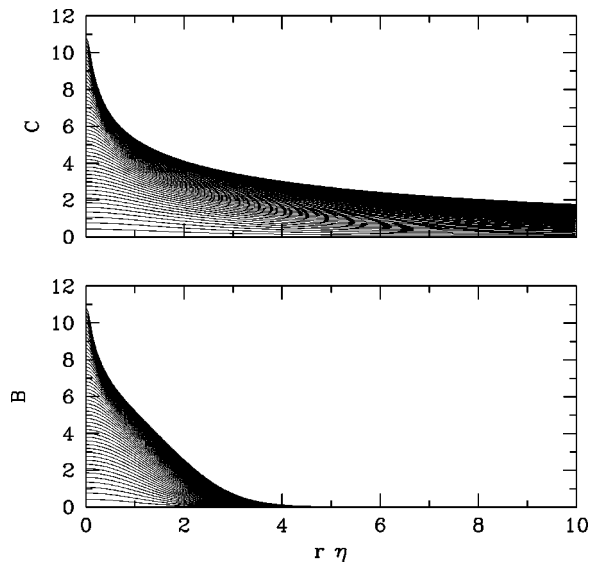

FIG. 1. The metric fields $B(r)$ and $C(r)$ for an $n=1$ string with symmetry breaking scale $\eta = 0.2m_p > \eta_{cr}$. The functions are plotted at equal time steps with the higher amplitude curves occurring at later times.

where F just represents the terms on the right hand side of the equation. Treating $\dot{C}$ as an independent variable such that $\dot{C}_i^{j+1} = \dot{C}_i^{j-1} + 2dt\dot{C}_i^j$, we found that it is necessary to evaluate $\dot{C}$ implicitly for Eq. (3.3) to be stable, i.e.

$$\frac{\dot{C}_i^{j+1} - \dot{C}_i^{j-1}}{dt} + (\dot{C}_i^{j+1})^2 = F_i^j. \tag{3.4}$$

The above quadratic has two solutions, but one may easily obtain the right one by taking the limit as $dt\to 0$ and noting that, for the correct root, $\dot{C}_i^{j+1} - \dot{C}_i^{j-1}$ should vanish. These suggestions worked well for us, although they do not represent the only stable differencing schemes and they may not generalize to other similar problems.

B. Results

As we mentioned in Sec. II, our strategy was to evolve the initial configurations for a given value of the winding n, for various values of the scale η. As an example, in Fig. 1 we show the metric fields $B(r,t)$ and $C(r,t)$ for an $n=1$ string with symmetry breaking scale $\eta = 0.2m_p > \eta_{cr}$. For comparison, in Fig. 2 we show the same fields for an $n=5$ string with $\eta = 0.07 > \eta_{cr}$. These plots demonstrate that the metric fields grow exponentially in the core but that the core size decreases exponentially. It is the competition between these two effects that determines whether inflation occurs or not.

In both cases inflation is occurring in the core of the defect, although this is not clear until we apply our criterion that the total volume of physical space be increasing exponentially in the core. Note that, as an artifact of our initial conditions, there is an initial period of time during which the system relaxes to its final state.

To illustrate how the criterion is applied, Fig. 3 shows $\dot{V}/V$ as a function of time t for two values of η, one for which the core inflates, and the other for which it does not,

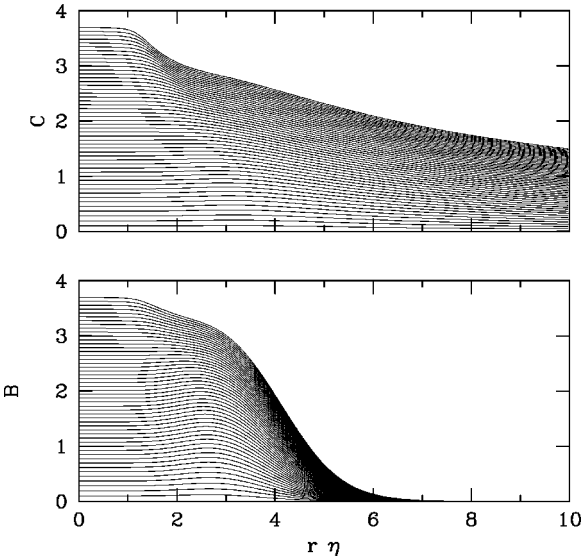

FIG. 2. The metric fields $B(r)$ and $C(r)$ for an $n=5$ string with symmetry breaking scale $\eta=0.07m_p>\eta_{cr}$. The functions are plotted at equal time steps with the higher amplitude curves occurring at later times.

for two cases of the topological winding, $n=1$ and $n=5$. It is the qualitative difference between these two classes of scales that allows us to home in on $\eta_{cr}\simeq 0.16$ for $n=1$ and $\eta_{cr}\simeq 0.06$ for $n=5$. In the $n=1$ case, the lower curve appears to turn up at late times. We believe this to be due to our rigid definition of the core radius, which does not take account of oscillations that appear in the matter fields. However, finite-size effects in the simulations limit our ability to test this.

Finally, in Fig. 4 we show the relationship between η_{cr} and the topological charge n on a log-log plot. The points are best fit by a linear relationship

$$\eta_{cr}\simeq \alpha n^p, \tag{3.5}$$

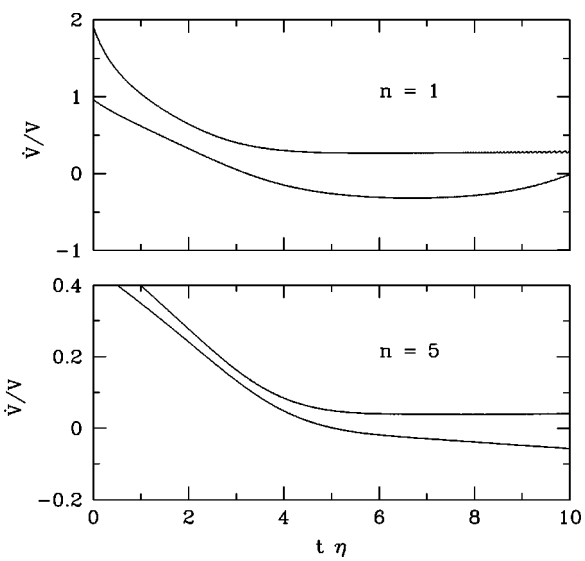

FIG. 3. $\dot{V}/V$ as a function of time t for two values of η, one for which the core inflates (upper curve in each figure), and the other for which it does not (lower curve in each figure), for two cases of the topological winding, $n=1$ and $n=5$.

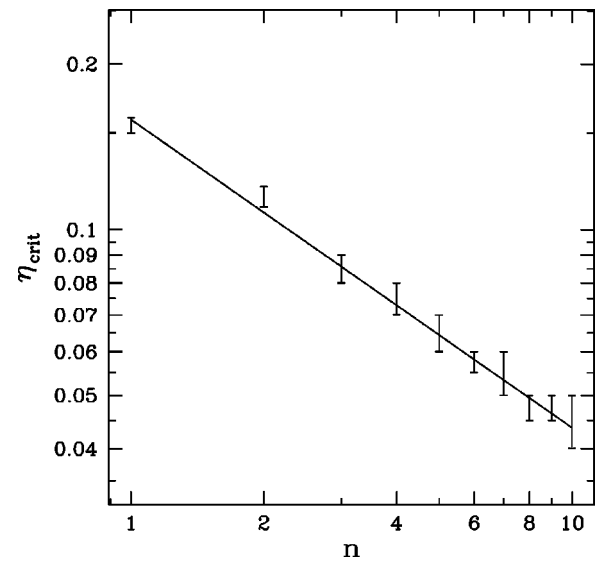

FIG. 4. A log-log plot of η_{cr}, in units of m_p, versus the topological charge n. The ''error bars'' denote the ranges within which we could numerically bracket η_{cr}. Their differing sizes reflect our initial trial and error guesses for the bracketing.

where

$$\alpha=0.16, \quad p=-0.56. \tag{3.6}$$

This is in excellent agreement with the naive estimate $p=-0.5$ obtained by analyzing the point at which the static asymptotic metric ceases to exist.

IV. CONCLUSIONS AND DISCUSSION

We have analyzed the onset of topological inflation in the cores of cosmic string solutions to the Einstein-Abelian-Higgs system in 2+1 dimensions. For a soliton in a given sector of topological charge n, inflation occurs in the defect core if the symmetry breaking scale η is greater than a critical value $\eta_{cr}(n)$. The functional dependence of η_{cr} on the winding n was determined numerically and was found to be monotonically decreasing roughly, though not exactly, as $1/\sqrt{n}$. This result supports the intuition about defects with multiple windings gained from the asymptotic metric of static solutions and from perturbative analyses of the core fields. If our results can be extrapolated to very large n, and if strings with such a high winding form in phase transitions, then it is possible that topological inflation could occur in GUT scale defects.

The present analysis is especially relevant in theories in which particles are viewed as solitons. In these theories one would expect that, at high densities, the solitonic nature of particles would become important. Our results then show that, provided the number of particles is large enough, the tightly squeezed state of particles can start inflating. This may be relevant for the gravitational collapse of stars since the number of particles in a star is of order 10^{57}. In the context of the dual standard model, all particles correspond to magnetic monopoles and so we would expect the present considerations to apply there also. However, in this model, baryon number is not a conserved quantity and it is possible that the star evaporates before inflation can set in, much like

in the scenarios recently considered in [11,12].

A number of related investigations are suggested by our analysis. First, is it possible that the collision of a monopole and an antimonopole can result in an inflating region? Some years ago, Farhi, Guth and Guven [13] considered the possibility of creating a universe in particle collisions. Is their (negative) conclusion applicable even in soliton collisions? Second, we have only considered strings at critical coupling. For different choices of couplings, the strings could attract or repel each other. How do our results depend on the coupling constants? Does the instability of higher winding strings to decay into those of lower winding come into play and terminate topological inflation after a certain number of e-folds? We hope to return to some of these questions in future investigations.

ACKNOWLEDGMENTS

We would like to thank Inyong Cho, Martin Lemoine, Glenn Starkman and Alex Vilenkin for helpful discussions. A.D. and T.V. are thankful to the U.S. Department of Energy (DOE) for support. The work of M.T. was supported by the DOE, the National Science Foundation, and by funds provided by Case Western Reserve University.

[1] A. Vilenkin, Phys. Rev. Lett. **72**, 3137 (1994).
[2] A. Linde, Phys. Lett. B **327**, 208 (1994).
[3] B. Linet, Phys. Lett. A **124**, 240 (1987).
[4] M. E. Ortiz, Phys. Rev. D **43**, 2521 (1991).
[5] N. Sakai, Phys. Rev. D **54**, 1548 (1996).
[6] N. Sakai, H. Shinkai, T. Tachizawa, and K. Maeda, Phys. Rev. D **53**, 655 (1996); **54**, 2981(E) (1996).
[7] I. Cho and A. Vilenkin, Phys. Rev. D **56**, 7621 (1997).
[8] T. H. R. Skyrme, Proc. R. Soc. London **A247**, 260 (1958); **A262**, 237 (1961); Y. Nambu, Phys. Rev. D **10**, 4262 (1974).
[9] T. Vachaspati, Phys. Rev. Lett. **76**, 188 (1996); H. Liu and T. Vachaspati, Phys. Rev. D **56**, 1300 (1997).
[10] W. H. Press *et al.*, *Numerical Recipes in C: The Art of Scientific Computing Second Edition* (Cambridge University Press, Cambridge, England, 1992).
[11] U.-L. Pen, A. Loeb, and N. Turok, astro-ph/9712178.
[12] A. Kusenko, M. Shaposhnikov, P. G. Tinyakov, and I. I. Tkachev, Phys. Lett. B **423**, 104 (1998).
[13] E. Farhi and A. H. Guth, Phys. Lett. B **183**, 149 (1987).