Nonlinear metric perturbations and production of primordial black holes

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We consider a simple inflationary model with a peculiarity in the form of a ''plateau'' in the inflaton potential. We use the formalism of a coarse-grained field in order to describe the production of metric perturbations h of an arbitrary amplitude, and obtain a non-Gaussian probability function for such metric perturbations. We associate the spatial regions having large perturbations $h\sim 1$ with the regions going to primordial black holes after inflation. We show that in our model the nonlinear effects can lead to overproduction of primordial black holes. [S0556-2821(98)04712-2]

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I. INTRODUCTION

Starting from the pioneering works by Zel'dovich and Novikov [1], and also by Hawking [2], the primordial black holes (PBH's) have been the subject of extensive ivestigations. The presence of PBH's may significantly influence the physical processes and effects in the Universe [such as nucleosynthesis, cosmic microwave background radiation (CMBR) spectral distortions, or distortions of γ-ray background radiation] due to the Hawking effect [3] and PBH's may be a component of dark matter (see, e.g., Refs. [4,5]). The formation of PBH's is determined by small scale, but large amplitude inhomogeneities in the early Universe, and the processes of PBH formation, evolution, and decay link the physical conditions of the early Universe with conditions in the radiation-dominated epoch and present-day cosmology. Even the very absence of PBH's may significantly constrain the models of the beginning of cosmological evolution.

Usually the processes of PBH formation are associated with production of the scalar mode of perturbations during inflation (see, e.g., Refs. [5–9]) or phase transitions in the early Universe [10]. In this paper we discuss the first possibility, which allows us to use the powerful and well-elaborated theory of instability of the expanding Universe for analysis of the conditions under which PBH's can form.

The theory of the generation of adiabatic perturbations during inflation started from pioneering papers [11–13]. It was established that the rms amplitude of metric perturbations δ_{rms} is connected with the parameters of inflationary theory by means of the relation

$$\delta_{\mathrm{rms}}=\frac{1}{2\pi}\frac{H^2}{|\dot{\phi}|}, \tag{1}$$

where H is the Hubble parameter and $\dot{\phi}$ is the velocity of the field evolving during inflation. To get PBH abundance in an observable amount, one should have $\delta_{\mathrm{rms}}\sim 10^{-2}-10^{-1}$ (see, e.g., Ref. [14]). On the other hand Cosmic Background Explorer (COBE) CMBR data, as well as the analysis of the large-scale structure formation constrain the amplitude of perturbations $\delta_{\mathrm{rms}}\sim 10^{-5}$ at superlarge scales. Therefore to get PBH's one should increase the amplitude of the perturbations by a factor 10^3-10^4 at small scales. Unfortunately this cannot be reached in the simplest inflationary models, since in these models δ_{rms} logarithmically grows with the increase in scale, and one should use nonstandard models having additional power at small scales to obtain a significant PBH amount.

Recently, several models of such type were proposed. For instance, Carr and Lidsey [6] proposed a toy model having a blue-type spectrum [the spectrum $\delta_{\mathrm{rms}}(k)\propto k^a$, where k is the wave number, and a is the spectral index] and investigated the constraint on the spectral index a associated with possible PBH formation in such a model. Linde [15] has shown that blue-type spectra can be naturally obtained in the two-field model of so-called hybrid inflation.

Another type of model having a spike in the power spectrum at some scale k_{bh} was proposed by Ivanov, Naselsky, and Novikov [5] (INN).[1] They considered a one-field inflationary model with inflaton ϕ and assumed that the potential $V(\phi)$ had a ''plateau'' region at some scale k_{bh} and a standard form (say, power-law form) outside the plateau region. The field ϕ slows down in the plateau region increasing the spectrum of perturbations at the scale k_{bh} according to Eq. (1). One can adjust the parameters of the plateau region to obtain the desired increase of the spectrum, and consequently the desired PBH amount. Garcia-Bellido *et al.* [8] and also Randall *et al.* [9] considered more realistic two-field models which had a saddle point in two-dimensional form of the potential $V(\phi,\psi)$. Like the one-field model, the evolution of the system of fields slows down near the saddle point increasing the spectrum power. Randall *et al.* pointed out that such models solve several fine-tuning problems of standard inflation, and therefore look very natural from the point of view of high-energy physics. Garcia-Bellido *et al.* carefully investigated the process of PBH formation in such models (see also recent work by Yokoyama [18]).

If the primordial black holes are not superlarge, they probably collapse during the radiation-dominated epoch of

[1]See also the papers by Hodges and Blumenthal, Hodges *et al.* [16] and Kates *et al.* [17], who employed similar models in the context of the large-scale structure formation theory.

the evolution of the Universe. This means that the amplitude h_* of the metric inhomogeneities inside the regions going to PBH's should be of order of unity to overcome the strong pressure forces during the collapse of the perturbed region [14]. These large amplitude metric inhomogeneities are assumed to be generated during inflation as rare events in the random field of the metric perturbations. Since the amplitude of the inhomogeneities h_* is rather large, the natural question is to what extent can we rely on the linear theory of perturbations which usually gives Gaussian probability distribution of PBH formation?

To answer this question we can apply the formalism of coarse-grained fields (introduced by Starobinsky [19]) as an alternative approach to the linear theory that can describe large amplitude deviations of the field and the metric from background quantities. According to this approach, the spatially inhomogeneous field $\phi(\vec{x},t)$ is divided into two parts: the large-scale part ϕ_{ls}, which consists of the modes with physical wavelengths $\lambda \propto ak^{-1}$ greater than some characteristic scale $\lambda_{c-g} \geqslant H^{-1}$, and the small scale part which consists of modes with $\lambda < \lambda_{c-g}$. During inflation, the physical wavelengths are stretched and new perturbations are added to ϕ_{ls}. This effect may be considered a new random force $f(t)$ in the equation of motion of the field ϕ_{ls}, and usually the dynamics of ϕ_{ls} is described in terms of a diffusion equation for the probability density $\Psi(\phi_{\mathrm{ls}},t)$. This equation was the subject of a number of works in connection with problems of quantum gravity and large-scale structure formation. Recently it was pointed out that this equation can be employed for calculations of the probability to find large amplitude peaks in the random distribution of field ϕ_{ls}, and it was mentioned that such an approach can be applied to the problem of PBH formation [20].

Here we would like to note that when studying the effects originating after the end of inflation, such as PBH formation, one should use the large scale part of the metric instead of the large scale part of the field. Contrary to the field ϕ_{ls}, the large scale part of the metric, namely, the ''inhomogeneous scale factor $a_{\mathrm{ls}}(\vec{x})$'' [see Eqs. (23),(24) for an exact definition] is the quantity conserved during the evolution outside the horizon, and this property allows us to connect the physical conditions during the inflation with the physical conditions during radiation-dominated epoch, when PBH's are formed. Moreover, the criterion for PBH formation can be directly formulated in terms of $a_{\mathrm{ls}}(\vec{x})$ (Refs. [21,22]). Therefore, the calculation of $a_{\mathrm{ls}}(\vec{x})$ gives us a tool to describe quantitatively the generation of nonlinear metric perturbations, and the evolution of these perturbations into PBH's.

In this paper we calculate the probability distribution function $\mathcal{P}[a_{\mathrm{ls}}(\vec{x})]$ in a model with an almost flat region in the inflaton potential. The main idea of our calculations has already been applied in the models of so-called stochastic inflation (see, e.g., Ref. [23], and references therein) and is very simple. When the field ϕ_{ls} evolves inside the plateau region it slows down, and the random kicks [described by the force $f(t)$] significantly influence its evolution. So, the trajectory of the field inside the plateau region becomes stochastic, and the time Δt that the field spends on the plateau, depends on the realization of the stochastic process. The total increase of the scale factor a_{ls} during the field evolution on the plateau, is obviously determined by Δt: $a_{\mathrm{ls}} \propto e^{H\Delta t}$. Since different regions of the Universe separated by distances greater than H^{-1} evolve independently, the increase of a_{ls} corresponding to different regions is determined by different realizations of the random process. Thus the scale factor a_{ls} varies from one region to another after the field passes the plateau, that is, the quantum effects generate the coordinate dependence of the scale factor. The shape of function $a_{\mathrm{ls}}(\vec{x})$ is conserved during the subsequent evolution of the Universe until the scale of inhomogeneity crosses the horizon the second time. At that time, in regions with a significant contrast of $a_{\mathrm{ls}}(\vec{x})$, primordial black holes are formed.

Using the approach described above we calculate the probability distribution function $\mathcal{P}[a_{\mathrm{ls}}(\vec{x})]$. With the help of a simple criterion of PBH formation we relate $\mathcal{P}[a_{\mathrm{ls}}(\vec{x})]$ to the probability of PBH formation. We show that in our case the nonlinear effects overproduce PBH's. Note that this result differs from what was claimed in Ref. [20]. In this paper the non-Gaussian probability function for the field ϕ_{ls} $\Psi(\phi_{\mathrm{ls}},t)$ was calculated for similar models of the inflaton potential, and it was mentioned that the function $\Psi(\phi_{\mathrm{ls}},t_{\mathrm{end}})$ taken at the moment of the end of inflation t_{end} can strongly underproduce the large fluctuation of the field ϕ_{ls}. However, as we mentioned above, the amplitude of the coarse-grained field is not conserved during its evolution out to the horizon, and therefore the statistics of the field fluctuations is not directly related to the statistics of the PBH formation. Although our result is very important qualitatively, it does not significantly change the estimate based on linear theory.

We use the simple one-field model, proposed by INN (see also Refs. [24,25]).[2] Because of the simplicity of this model the bulk of our results are obtained analytically. We hope that our approach provides a reasonable approximation to the case of more complicated two-field models. We are going to discuss these models in a future work.

The paper is organized as follows. We introduce our model and discuss the classical dynamics of the metric and field in Sec. II. In Sec. III we obtain an expression for $\mathcal{P}[a_{\mathrm{ls}}(\vec{x})]$. We consider the role of nonlinear effects on the statistics of PBH production in Sec. IV. We summarize our conclusions and discuss the applicability of our approach in Sec. V.

II. THE DYNAMICS OF CLASSICAL MODEL

In this section we consider the classical dynamics of spatially homogeneous parts of the metric and field in the simplest inflationary model with a single scalar field (inflaton) and with a peculiarity in the inflaton potential. In this case the system of dynamical equations contains only two dynamical variables —scale factor $a(t)$ and the spatially homo-

[2]Although the one-field model with a plateau is not natural from a high-energy physics point of view, it can be considered as an approximation to the more realistic, but technically more complicated, two-field models. Note that the saddle point in the effectively one-field potential can be obtained because of vacuum intersections of the inflaton with other fields (see, e.g., Ref. [23], and references therein).

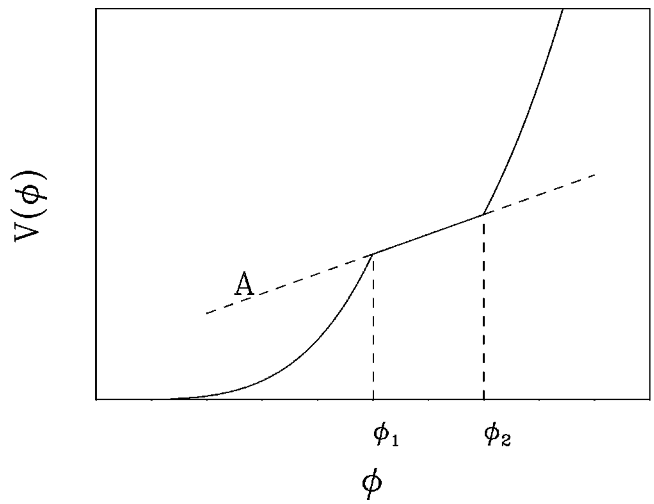

FIG. 1. The schematic picture of the inflaton potential with the peculiarity in the form of the plateau. The plateau region is contained between two field values ϕ_1, ϕ_2, and has a slope $A=(\partial/\partial\phi)V$. We assume the potential to have a power-law form outside the plateau region.

geneous part $\phi_0(t)$ of the field ϕ—and reduces to the Hamiltonian constraint equation

$$H^2=\frac{8\pi}{3}\left(V(\phi_0)+\frac{\dot{\phi}_0^2}{2}\right) \tag{2}$$

and to the equation of motion for field ϕ_0

$$\ddot{\phi}_0+3H\dot{\phi}_0+\frac{\partial}{\partial\phi}V(\phi_0)=0, \tag{3}$$

where $H=\dot{a}/a$, and other symbols have their usual meaning. We hereafter use the natural system of units.

We assume that the effective potential $V(\phi)$ has a small almost flat region (plateau) between some characteristic values of field ϕ_1 and ϕ_2 (see Fig. 1).

The potential is also assumed to be proportional to ϕ^4 outside the ''plateau'' region:

$$V(\phi)=\frac{\lambda\phi^4}{4} \tag{4}$$

at $\phi<\phi_1$,

$$V(\phi)=V(\phi_1)+A(\phi-\phi_1) \tag{5}$$

at $\phi_1<\phi<\phi_2$, and

$$V(\phi)=\frac{\tilde{\lambda}\phi^4}{4} \tag{6}$$

at $\phi>\phi_2$. Here $V(\phi_1)=\lambda\phi_1^4/4$, $\tilde{\lambda}=\lambda(\phi_1/\phi_2)^4+4A(\phi_2-\phi_1)/\phi_1^4$. As we will see below the size of the flat region is very small $\Delta\phi=\phi_2-\phi_1\ll\phi$, $A(\phi_2-\phi_1)/V(\phi_1)\ll 1$ so we can set $\lambda\approx\tilde{\lambda}$. At sufficiently large values of $\phi_0>1$ the kinetic term in Eq. (2) is negligible in comparison with the potential term

$$\frac{\dot{\phi}_0^2}{2}\ll V(\phi_0), \tag{7}$$

and Eq. (2) reduces to an algebraical relation between H and ϕ_0 (so-called slow-roll approximation):

$$H=\sqrt{\frac{8\pi}{3}V(\phi_0)}. \tag{8}$$

From Eq. (8) it follows that the Universe expands quasiexponentially ($H\approx$const and $a\propto e^{Ht}$) at $\phi_0>1$.

It can also be easily shown that outside the plateau region the field moves with large friction at $\phi_0>1$, so

$$|\ddot{\phi}_0|\ll|3H\dot{\phi}_0|. \tag{9}$$

The friction dominated condition (9) helps to simplify the integration of the system (2),(3). Integrating Eqs. (2),(3) with the help of inequalities (7),(9) at $\phi_0>\phi_2$, we have

$$\phi_0(t)=\tilde{\phi}_0\exp-\left[\left(\sqrt{\frac{\tilde{\lambda}}{6\pi}}t\right)\right], \tag{10}$$

and

$$a(\tilde{\phi}_0)=a_0\exp[N(\tilde{\phi}_0)-N(\phi_0)], \tag{11}$$

where $\tilde{\phi}_0$ and a_0 are some initial values of the field and scale factor:

$$N(\phi_0)=\int_{\phi_2}^{\phi_0}Hdt=\pi(\phi_0^2-\phi_2^2) \tag{12}$$

is the number of e folds of the scale factor during the field rolling down starting from some initial value of ϕ and down to the field ϕ_2. Similar formulas hold at $\phi_{\mathrm{end}}<\phi_0<\phi_1$:

$$\phi_0(t)=\phi_1\exp\left(-\sqrt{\frac{\lambda}{6\pi}}(t-t_1)\right), \tag{13}$$

$$a(\phi_0)=a_1\exp[N_{\mathrm{end}}(\phi_1)-N_{\mathrm{end}}(\phi)], \tag{14}$$

where $\phi_0(t_1)=\phi_1$, $a_1=a(t_1)$, and $N_{\mathrm{end}}(\phi_0)$ is the number of e folds up to the end of inflation: $N_{\mathrm{end}}(\phi_0)=\pi(\phi_0^2-\phi_{\mathrm{end}}^2)$, where we assume that inflation ends at a standard (for $\lambda\phi^4$ theory) value of $\phi_{\mathrm{end}}=1/\sqrt{2\pi}$. Note that $N_{\mathrm{end}}(\phi_1)$ should be rather large. For example, to get a feature in the spectrum at scales corresponding to the solar mass, we should have $N_{\mathrm{end}}(\phi_1)\sim 30$. Therefore, the value of ϕ_1 should be greater than unity [$\phi_1\sim 3$ for $N_{\mathrm{end}}(\phi_1)\sim 30$].

Now let us consider the dynamics of inflaton in the plateau region $\phi_1<\phi_0<\phi_2$. In this region Eq. (3) is simplified to

$$\ddot{\phi}_0+3H_0\dot{\phi}_0+A=0, \tag{15}$$

where $H_0=\sqrt{(8\pi/3)V_0}$. The solution of Eq. (15) can be written as

$$\phi_0 = \phi_2 + \frac{1}{3H_0}\dot{\phi}_{\rm in}(1-e^{-3H_0 t}) - \frac{At}{3H_0}$$
$$= \phi_2 - \frac{1}{6\pi\phi_2}(1-e^{-3H_0 t}) - \frac{At}{3H_0} \quad (16)$$

and for the field velocity we have

$$\dot{\phi}_0 = \dot{\phi}_{\rm in} e^{-3H_0 t} - \frac{A}{3H_0}, \quad (17)$$

where $\dot{\phi}_{\rm in} = \dot{\phi}_0|_{\phi_0=\phi_2} = -(1/3H_0)(\partial/\partial\phi)V(\phi_2) = -\sqrt{\tilde{\lambda}/6\pi}\phi_2$ is the field velocity at moment $t=0$ of entrance of the field in the plateau region. The second term in Eq. (16) and the first term in Eq. (17) are due to inertial influence of initial velocity $\dot{\phi}_{\rm in}$, and the last terms in both equations are due to the nonzero slope of the potential in the plateau region. The evolution of the field in the plateau region can be divided into two stages. At the first stage the field evolves mainly due to the inertial term, and velocity exponentially decreases with time. After some characteristic time t_* the nonzero slope of potential A starts to determine the evolution, the velocity tends to the constant value $\dot{\phi}_{fd} = -A/3H_0$, and the field amplitude starts to decrease linearly with time. The time t_* can be estimated by equating the inertial and potential terms in Eq. (16), and is determined by the condition $3H_0 t_* e^{3H_0 t_*} = B/A$, where $B = (\partial/\partial\phi)V(\phi_0=\phi_1)$. As we discussed in the Introduction, the spectrum amplitude is inversely proportional to the field velocity $[\delta_{\rm rms} \approx (1/2\pi)(H^2/|\dot{\phi}|)]$, therefore we need to slow down the velocity approximately by $\sim 10^3 - 10^4$ times to get the increase of the spectrum amplitude from the initial value $\delta_{\rm rms}({\rm in}) = (1/2\pi)(H^3/B) \sim 10^{-5}$ up to the typical one for PBH production $\delta_{\rm rms} \sim 10^{-2} - 10^{-1}$. For that, we should fix the ''amplification'' parameter $\alpha = B/A \sim 10^3 - 10^4$.

Our model has two possible limiting variants depending on the relation between time t_c of the crossing of the plateau region by the field ϕ_0 $[\phi_0(t_c) = \phi_1]$ and t_*. If $t_c \approx t_*$ the field crosses the plateau mainly due to inertia. In this case the parameter α determines the number of e folds during plateau crossing $\delta N \approx H_0 t_c \approx \frac{1}{3}\ln \alpha \approx 2.3$, and therefore the width of the produced bump in the spectrum remains small and fixed. A model of similar type was discussed by INN. Here we consider another possible case $t_c > t_*$, where the field spends some time on the plateau, evolving in the friction-dominated approximation. In this case the width of the spectrum is determined by the value of t_c, which is the free parameter of our model. Instead of t_c we will parametrize our model by the quantity γ—the ratio of wave numbers, corresponding to the fields ϕ_1, ϕ_2, respectively, $t_c = H_0^{-1} \ln \gamma$. The parameter γ cannot be too small $\gamma > \alpha^{1/3}$ and we take $\gamma \approx 10^3$ in the estimations. If γ is not extremely large $\ln \gamma \ll N(\phi_1)$, the size of the plateau $\Delta\phi_0 = \phi_2 - \phi_1$ is of order of typical size $\Delta\phi_* = B/9H^2$. The typical relative size of the plateau is very small:

$$\frac{\Delta\phi_0}{\phi_0} = \frac{1}{6\pi\phi_0^2} \approx \frac{1}{6N(\phi_1)} \approx 0.0055. \quad (18)$$

Thus, the correction due to the presence of the plateau practically does not influence the dynamics of the field outside plateau region and we can set $\lambda = \tilde{\lambda}$. On the other hand, the size of plateau is much greater than H_0 — the typical size of quantum fluctuations, $\Delta\phi_* = H_0/6\pi\delta_{\rm rms}({\rm in}) \sim 10^5 H_0$.

Typically, the estimate $\Delta\phi_0/\phi_0 \ll 1$ holds for arbitrary power-low potentials $V(\phi) \propto \phi^p$ provided power p is not very large. However, the opposite limiting case is also possible. For example, Bullock and Primack [20] proposed a potential of the form

$$V(\phi) = \lambda_{\rm BP}[1 + \arctan(\phi)], \quad \phi > 0,$$
$$V(\phi) = \lambda_{\rm BP}(1 + 4\times 10^{33}\phi^{21}), \quad \phi < 0, \quad (19)$$

where the constant $\lambda_{\rm BP} = 6\times 10^{-10}$ is chosen to normalize the large-scale part of spectrum to the rms amplitude $\approx 3\times 10^{-5}$. The flat region in this potential starts from $\phi = 0$ and ends at $\phi = -1.23\times 10^{-2}$, and inflation ends itself at $\phi = \phi_{\rm end} = -1.55\times 10^{-2}$. It was mentioned by Bullock and Primack that this potential leads to strongly non-Gaussian statistics of field perturbations.

III. NONLINEAR METRIC PERTURBATIONS FROM THE QUANTUM DYNAMICS OF COARSE-GRAINED FIELD

It is well known that there are two equivalent ways to describe an inhomogeneous Universe. The first way is to consider inhomogeneities as small corrections to the homogeneous space-time and study them in the framework of linear theory of perturbations. Another approach splits the metric and the field into a large-scale part (coarse grained over some scale greater than horizon scale) and a small-scale part. During inflation, the dynamical equations for coarse-grained field $\phi_{\rm ls}$ and coarse-grained scale-factor $a_{\rm ls}$ are equivalent to Eqs. (3),(8) provided the quantum effects are switched off. The quantum effects continuously produce new inhomogeneities of random amplitude with scales greater than the scale of coarse-graining. These inhomogeneities should be added to $\phi_{\rm ls}$ and $a_{\rm ls}$ and effectively this leads to the presence of a stochastic force term in the equations of motion. Therefore, the dynamics of coarse-grained variables can be described in terms of the distribution functions of $\phi_{\rm ls}$ and $a_{\rm ls}$, and in principal these distribution functions can provide the same information as the power spectrum of perturbations, and furthermore the coarse-grained formalism gives a tool for the description of the metric perturbations with amplitude greater than 1.

The effective dynamical equation for the field $\phi_{\rm ls}$ has the form [19][3]

$$\ddot{\phi}_{\rm ls} + 3H_{\rm ls}\dot{\phi}_{\rm ls} + \frac{\partial}{\partial\phi}V(\phi_{\rm ls}) = D^{1/2}f(t), \quad (20)$$

where $D = 9H_{\rm ls}^5/(2\pi)^2$, and $f(t)$ is a delta-correlated random force $\langle f(t_1)f(t_2)\rangle = \delta(t_1 - t_2)$. The equation for coarse-grained scale factor $a_{\rm ls}$ remains unchanged:

[3]See also Ref. [26], and references therein.

$$H_{\rm ls}=\sqrt{\frac{8\pi}{3}V(\phi_{\rm ls})}. \tag{21}$$

The solution of the set of Eqs. (20),(21) is an extremely difficult problem, and can be done under some additional simplifying assumptions. For example, if we choose the featureless potential, and consider the friction-dominated solutions of Eq. (20), we can obtain the solutions describing a self-reproduced inflationary Universe (provided the stochastic term in Eq. (20) dominates the potential term, see, for example, Linde [23]). In our case we cannot use the friction-dominated condition in the beginning of the field evolution inside the plateau region. However, we can adopt other simplifying assumptions: first we can set $H_{\rm ls}=H_0=$const inside and near the plateau region, and second, we can omit the stochastic term in Eq. (20) outside the plateau region, assuming the field moves along the classical trajectory there. Under these assumptions the statistics of the scale factor $a_{\rm ls}$ is totally determined by the time Δt that field $\phi_{\rm ls}$ spends in the plateau region

$$\Delta N=\ln(a_{\rm out}/a_{\rm in})=H_0\Delta t, \tag{22}$$

where $a_{\rm in}$ is the value of the scale factor at the time $t=0$ of the entrance of the field into the plateau region, and $a_{\rm out}$ corresponds to the moment Δt when the field leaves the plateau region. To see this let us consider the evolution of the scale factor $a_{\rm ls}$ in the comoving coordinate system. Outside the horizon the hypersurfaces of constant comoving time $t_{\rm com}$ practically coincide with hypersurfaces of constant energy density $\epsilon=$const. On the other hand, the field $\phi_{\rm ls}$ evolves slowly during inflation and hypersurfaces of constant energy density are close to hypersurfaces $\phi_{\rm ls}=$const, and therefore we can put $a_{\rm ls}(t_{\rm com})=a_{\rm ls}(\phi_{\rm ls})$. After the field passes the plateau region, the evolution of $a_{\rm ls}(\phi_{\rm ls})$ can be described by the standard expression (14), so we have

$$a_{\rm ls}(\phi_{\rm ls})=a_{\rm in}\exp\left[\pi(\phi_1^2-\phi_{\rm ls}^2)+\Delta N\right], \tag{23}$$

where ΔN is nearly constant inside the coarse-grained regions with comoving scale $\lambda_{c-g}\approx a_{\rm out}H_0^{-1}$, but changes from one region to the other. Thus, the metric outside the horizon has the quasi-isotropic form

$$ds^2=dt^2-a_{\rm ls}^2(\phi_0)a_{\rm ls}(\vec{x})\,\delta_j^i dx_i dx^j, \tag{24}$$

where we represent the scale factor $a_{\rm ls}(\phi_{\rm ls})$ as a multiplication of two factors: $a(\phi_0)$ and $a_{\rm ls}(\vec{x})\equiv e^{\Delta N}$. Here $a_{\rm ls}(\phi_0)$ and $\phi_0(t)$ are determined by the classical equations (13),(14), and the spatial coordinates $\vec{x}$ are coarse grained over regions with scale λ_{c-g}. To estimate the change of metric from one region to another quantitatively, we introduce the definition of nonlinear metric perturbation

$$h\equiv\frac{a_{\rm ls}(\phi_{\rm ls})-a(\phi_0)}{a(\phi_0)}=\exp\, H_0(\Delta t-t_c)-1 \tag{25}$$

[we remind the reader that $t_c=H_0^{-1}\ln\gamma$ is the time which the field spends in the plateau region moving along the classical trajectory when the stochastic term in Eq. (20) is switched off]. Note that in the limit of small $h\ll 1$, the metric assumes the form

$$ds^2=dt^2-a^2(\phi_0)[1+2h(\vec{x})]\,\delta_j^i dx_i dx^j, \tag{26}$$

and the definition (25) is reduced to the standard expression for the growing mode of adiabatic perturbation outside the horizon. Namely, in this case h reduces to the gauge-independent quantities, introduced by a number of authors [11–13,27] up to a constant factor. The variables (25),(26) do not depend on time outside the horizon. Therefore, the use of these variables is very convenient to match the perturbations, generated during inflation with the perturbations, crossing horizon at the normal stage of the Universe evolution. As one can see from the Eq. (25) the metric perturbations are determined by stochastic variable Δt and the distribution of Δt must follow from the solution of Eq. (20). Note that the definition of nonlinear metric perturbations should be taken with caution. In principal, one can use another definition related to Eq. (25) by some nonlinear transformation, and having the same limit (26) in the case of small h. For example, Bond and Salopek [28] used the quantity $\tilde{h}=\ln[a_{\rm ls}(\phi_{\rm ls})/a(\phi_0)]$ to define nonlinear metric perturbations. However, the criterion for PBH formation can be directly expressed in terms of the quantity (25) (see next section), and therefore this quantity is the most natural variable for our purposes.

Although the assumption of constant H_0 greatly simplifies the problem it is still rather complicated for a simple analytical treatment.[4] For further progress we have to make some additional assumptions. We will consider below a plateau region of sufficiently large size. For this case the field approaches the end of the plateau in the friction-dominated approximation, which greatly simplifies the treatment of diffusion processes. To estimate the relevance of the friction-dominated approximation we should compare the time t_c and the time $t_*\sim\ln(\alpha)$ of the decay of the inertial term $\ddot{\phi}$ in Eqs. (15)–(17),(20). If $t_c>t_*$ and therefore $\gamma\gg\alpha^{1/3}$, the inertial term in these equations can be neglected at $t_*<t<t_c$. In this regime the solution of the classical equation of motion (15) has the form

$$\phi_0(\tau)\approx\phi_2-a\tau, \tag{27}$$

and Eq. (20) becomes

$$\frac{d\phi_{\rm ls}}{d\tau}+a=d^{1/2}f(\tau), \tag{28}$$

where $\beta=3H_0$, and we introduce the dimensionless time $\tau=\beta t$, $a=A/\beta^2$, and $d=D/2\beta^3=H_0^2/24\pi^2$. The stochastic

[4]In this case our problem is reduced to the first-passage problem for the one-dimensional Fokker-Planck (Kramers) equation, associated with Eq. (20) [29]. The general solution of this problem demands too much formalism [30], and is not considered here. Note, however, that the simple asymptotic estimates are still possible in this case [30,31].

equation (28) is associated with a simple diffusion type equation, describing the evolution of position probability distribution $\Psi(\tau,\phi)$:

$$\frac{\partial\Psi}{\partial\tau}=d\frac{\partial^2}{\partial\phi^2}\Psi+a\frac{\partial}{\partial\phi}\Psi. \tag{29}$$

Now we assume that the distribution Ψ is not spread out sufficiently before $\tau_*=\beta t_*$ and take the δ-distributed Ψ function at the moment $\tau=\tau_*$ as the initial condition for our problem,

$$\Psi(\tau_*)=\delta(\phi_{\rm ls}-\phi_*), \tag{30}$$

where $\phi_*=\Delta\phi-a\tau_*$ is the value of the field corresponding to the beginning of the friction-dominated part of the plateau region.[5]

Together with initial condition (30) we should specify the boundary condition at $\phi_{\rm ls}=\phi_1$. This condition depends on the form of the transition layer between the plateau region and the part of the potential with steep slope $(\partial/\partial\phi)V(\phi)=B$. We assume this transition to be sharp, and therefore set the condition of the absorbing wall at the downstream point $\phi_{\rm ls}=\phi_1$:

$$\Psi(\phi_1,\tau)=0. \tag{31}$$

Note that this boundary condition was used by Aryal and Vilenkin [32] for an analysis of stochastic inflation in the theory with top-hat potentials. In that paper it was shown that the more reasonable smooth transitions between the flat and steep regions of the potential are unlikely to significantly modify the resulting distribution.

In our case the probability density $P(\tau)$ of time τ relates to the solution of Eq. (27) as

$$\mathcal{P}(\tau)=S|_{\phi_{\rm ls}=\phi_1}=d\frac{\partial}{\partial\phi}\Psi, \tag{32}$$

where we define by S the probability current $S=d(\partial/\partial\phi)\Psi+a\Psi$. The conservation of the probability current allows us to estimate the correction term to Eq. (32) due to nonzero $\Psi(\phi_1)$. Assuming that field moves along the classical trajectory after $\phi_{\rm ls}=\phi_1$, we have $S(-\phi_1)\approx B/\beta^2\Psi\approx S(+\phi_1)\approx d(\partial/\partial\phi)\Psi$. Therefore the correction to the expression (32) is $\beta^2a/B=\alpha^{-1}\sim10^{-3}-10^{-4}$ times smaller than the leading term.

The conditions (30),(31) determine the solution of Eq. (29). This solution can be found by standard methods of the theory of diffusion equations (see, e.g., Ref. [33]), and in our case has the form

$$\begin{aligned}\Psi(\phi,\tau)=&\frac{1}{\sqrt{4\pi d(\tau-\tau_*)}}\\&\times\exp\left\{-\frac{1}{4d(\tau-\tau_*)}[\phi-\phi_*+a(\tau-\tau_*)]^2\right\}\\&\times\left(1-\exp\left\{-\frac{1}{d(\tau-\tau_*)}(\phi-\phi_1)\right.\right.\\&\left.\left.\times(\phi_*-\phi_1)\right\}\right).\end{aligned} \tag{33}$$

Substituting Eq. (33) into Eq. (32) we find the explicit expression for $\mathcal{P}(\tau)$:

$$\begin{aligned}\mathcal{P}(\tau)=&\frac{1}{\sqrt{4\pi d(\tau-\tau_*)}}\left(\frac{\phi_*-\phi_1}{\tau-\tau_*}\right)\\&\times\exp\left\{-\frac{1}{4d(\tau-\tau_*)}[\phi_1-\phi_*+a(\tau-\tau_*)]^2\right\}.\end{aligned} \tag{34}$$

The expression for the probability distribution of the metric can be readily obtained from Eq. (34). Using Eqs. (22)–(25) to express the time τ in terms of h, taking into account Eq. (27) and the definitions of a, d, and assuming $a>0$, we obtain

$$\mathcal{P}(h)=\frac{1}{\sqrt{2\pi\delta_{\rm pl}^2}}\frac{N_{\rm cl}}{N_{\rm st}^{3/2}}\frac{dN_{\rm st}}{dh}\exp\left\{-\frac{(N_{\rm st}-N_{\rm cl})^2}{2\delta_{\rm pl}^2N_{\rm st}}\right\}, \tag{35}$$

where $\delta_{\rm pl}=3H_0^3/2\pi A=\alpha\delta_{\rm rms}(\text{in})$ is the standard metric amplitude calculated for the plateau parameters, and

$$N_{\rm cl}=\ln\,\gamma-\tau_*/3,\quad N_{\rm st}=\ln\,(1+h)+N_{\rm cl} \tag{36}$$

are the numbers of e folds for the classical path $\phi_0(t)$ and for a random path $\phi_{\rm ls}(t)$, which start at $\phi_*=\phi(t_*)$ and end at ϕ_1.

When the perturbations are small $N_{\rm st}-N_{\rm cl}\approx h\ll1$, the distribution (35) has a standard Gaussian form:

$$\mathcal{P}(h)=P_G(h)=\frac{1}{\sqrt{2\pi\delta_{\rm pl}^2N_{\rm cl}}}\exp\left\{-\frac{h^2}{2\delta_{\rm pl}^2N_{\rm cl}}\right\}, \tag{37}$$

and in the opposite case of very large metric perturbations $h\gg1$ and $N_{\rm st}\sim\ln h>N_{\rm cl}$ the distribution $\mathcal{P}(h)$ deviates sharply from the Gaussian law and has the power-law form

$$\mathcal{P}(h)\propto h^{3/2+\delta_{\rm pl}^{-2}/4}. \tag{38}$$

As seen from Eqs. (35)–(38), the non-Gaussian effects overproduce the metric perturbations of high amplitude in our model. To understand this fact, let us discuss the origin of non-Gaussian effects in our model. There are two sources for such effects. First, note that the ‘‘effective dispersion’’ $\sigma_{\rm eff}^2=\delta_{\rm pl}^2N_{\rm st}$ in Eq. (35) depends itself on the value of the stochastic variable $N_{\rm st}$. Qualitatively, it can be explained as follows. In linear theory the dispersion $\sigma^2=\delta_{\rm pl}^2N_{\rm cl}$ is propor-

[5]The estimates show that the characteristic width of $\Psi(\tau_*)$ is of order of H and much less than the size of the friction-dominated region $\phi_*-\phi_1$.

tional to the time spent by the classical background field ϕ_0 on the plateau. In nonlinear theory the coarse-grained field $\phi_{ls}(t)$ plays the role of background field, and therefore the distribution of the family of neighbors to $\phi=\phi_{ls}(t)$ paths should be described in terms of the probability distribution with dispersion σ^2_{eff}, which is proportional to the time spent by field ϕ_{ls} on the plateau. Second, the amplitude of large metric perturbations h depends on N_{st} exponentially ($h\sim e^{N_{ls}}$), so an order of magnitude increase of N_{st} leads to the exponential increase of h. Obviously, these two effects increase the probability of large amplitude metric perturbations.

IV. PROBABILITY OF BLACK HOLE FORMATION

Although the distribution (35) provides very important information about the geometry of the spatial part of the metric outside horizon, it cannot be directly applied to the estimates of PBH formation. Indeed, the distribution (35) is formed by the field inhomogeneities with wave numbers k in the range $(\Delta k=[k_{min}\approx a_{in}H_0<k<k_{max}\approx a_{out}H_0])$. The process of PBH formation is determined mainly by the field modes with wave numbers $(\delta k\approx k_{bh}\ll\Delta k)$, where k_{bh} is the typical PBH wave number. The modes with $k<k_{bh}$ compose the large-scale background part of the metric at the moment of PBH formation, and do not influence the formation of PBH's significantly. The modes with $k>k_{bh}$ lead to high-frequency modulation of the perturbation with $k\sim k_{bh}$, which is also unimportant, provided the mode with $k\sim k_{bh}$ crosses the horizon the second time at the radiation-dominated epoch. Therefore, in order to obtain the probability of PBH formation, we should subtract the contribution of the large-scale and small-scale metric perturbations.

In general it is very difficult to separate the perturbations of a given scale in the framework of the nonlinear approach. However, we can estimate the probability density of the perturbations, corresponding to the smallest scale $k_{bh}\approx a_{out}H_0$.[6] For that we simply put $N_{cl}=1$ in Eqs. (35),(36), assuming that the random process starts when the mode with wave number $k_1=e^{-1}a_{out}H_0$ crosses the horizon. This procedure automatically subtracts the large-scale contribution of modes with $k<k_1$. The small-scale contribution is also absent due to our absorbing boundary condition. We have

$$\mathcal{P}(h)=\frac{1}{\sqrt{2\pi\delta^2_{pl}}}\frac{1}{(x+1)^{3/2}}\exp\left\{-\frac{x^2}{2\delta^2_{pl}(x+1)}\right\} \qquad (39)$$

from Eq. (35), where $x=\ln(1+h)$, and in the limit of small h we obtain again the Gaussian distribution

$$\mathcal{P}(h)\approx\mathcal{P}_G(h)=\frac{1}{\sqrt{2\pi\delta^2_{pl}}}\exp\left\{-\frac{h^2}{2\delta^2_{pl}}\right\}. \qquad (40)$$

The distribution (39) has nonzero first momentum $M_1=\int_{-1}^{\infty}dh\,h\mathcal{P}(h)=\frac{3}{2}\delta_{pl}$ (the lower limit of the integration should be -1, since the metric perturbations with $h<-1$ are cut off). The contribution of M_1 should be added to the background part of the metric, and further we will use the renormalized metric perturbation $h_r=h-\frac{3}{2}\delta_{pl}$ instead of h. The probability to find the metric perturbations h_r with amplitude greater than some threshold value h_* $P(h_*)=\int_{h_*}^{\infty}dh\,\mathcal{P}(h)$ can be estimated as

$$P(h_*)\approx\frac{1}{\sqrt{2\pi}}\left(\frac{2\delta_{pl}(x_*+1)^{1/2}}{x_*(x_*+2)}\right)\exp\left\{-\frac{x_*^2}{2\delta^2_{pl}(x_*+1)}\right\}, \qquad (41)$$

where $x_*=\ln(1+\frac{3}{2}\delta_{pl}+h_*)$, and we assume $h_*\gg\delta_{pl}$. The same quantity, but calculated for the Gaussian distribution takes the well-known form

$$P_G(h)\approx\frac{1}{\sqrt{2\pi}}\frac{\delta_{pl}}{h_*}\exp\left\{-\frac{h_*^2}{2\delta^2_{pl}}\right\}. \qquad (42)$$

The observed quantities (such as, e.g., the matter density of PBH's in different cosmological epochs) can be easily expressed in terms of the probability $P(h_*)$, provided the mass of the PBH's and some criterion for PBH formation are fixed. In our case the criterion for PBH formation should give the information about the threshold value h_*. Since this criterion plays a very important role, let us discuss it in some detail. First let us note that PBH's are formed from high amplitude peaks in the density distribution which are approximately spherically symmetric (see, e.g., Ref. [34]). It can also be easily shown that the maxima in the matter density correspond to the maxima in the function $a_{ls}(\vec{x})$. The form of $a_{ls}(\vec{x})$ totally specifies the number of regions going to PBH's as well as dynamics of the collapsing regions. Therefore we formulate the criterion of PBH formation in terms of conditions imposed on the function $a_{ls}(\vec{x})$.

The first criterion was formulated by Carr in his seminal paper [14]. It was shown that an overdense region forms a PBH if the density contrast at the horizon scale $\delta\rho/\rho$ lies approximately within the limits $\frac{1}{3}<\delta\rho/\rho<1$. The first part of this inequality tells us that the overdense region should stop expansion before the scale of the region crosses the sound horizon. The second part requires that the overdense region not collapse before crossing the causal horizon, and consequently the perturbation does not produce a closed world separated from the rest of the Universe. Then the criterion for PBH formation was improved by Nadegin, Novikov, and Polnarev [21] (NNP), and also by Biknell and Henriksen [22] with the help of numerical computations. The initial condition used by NNP was chosen as a nonlinear metric perturbation having the form of a part of the closed Friedman Universe matched with the spatially flat Universe through an intermediate layer of negative density perturbation. The conditions for PBH formation depend on the size of this part (i.e., the amplitude of the perturbation), as well as on the size of the matching layer. The smaller the matching layer is, the larger the pressure gradients needed to prevent collapse will be. Therefore, the amplitude of the perturbation

[6] In this connection, let us note that the black holes of smallest mass should give the major contribution to the present fraction of black holes, provided the PBH spectrum is flat (Carr [14]).

forming a PBH must be greater in the case of a narrow intermediate layer. In terms of our function $a(\vec{x})$ the NNP criterion reads

$$h_{*} \equiv \frac{a_{+}}{a_{-}} - 1 > 0.75 - 0.9, \tag{43}$$

where a_{+} is the value of $a(\vec{x})$ at the maximum of the perturbation and a_{-} is the same quantity outside the perturbed region.[7] The first number on the right-hand side of Eq. (43) corresponds to the matching layer of a size comparable with the size of the overdense region, and the second number corresponds to the narrow matching layer. Assuming the matching layer is sufficiently large we take $h_{*} = 0.75$ as a criterion of PBH formation.

Once the criterion is specified, we can link the desired PBH abundance $\beta(M_{\mathrm{PBH}}) \approx P(h_{*\mathrm{PBH}})$ with the parameters of our model. For instance, consider a model having a matter density of PBH's equal to the critical one (the density parameter $\Omega_{\mathrm{PBH}} = 1$). In this model we have [3,6]

$$\beta(M) = 10^{-8} \left(\frac{M}{M_{\odot}} \right)^{1/2}. \tag{44}$$

Equating the expression (44) to the probability function (39), we have the equation determining the amplitude $\delta_{1\mathrm{pl}}$ required for PBH abundance (44) as a function of M_{PBH},

$$P(h_{*\mathrm{PBH}}, \delta_{1\mathrm{pl}}) = \beta(M_{\mathrm{PBH}}), \tag{45}$$

and equating the expressions (42) and (44) we obtain the analogous equation for determining the reference amplitude $\delta_{2\mathrm{pl}}$ when the non-Gaussian effects are switched off. The solution of these equations is given in Fig. 2.

One can see from this figure that the quantities $\delta_{1\mathrm{pl}}$ and $\delta_{2\mathrm{pl}}$ increase with increasing M_{PBH} and $\delta_{1\mathrm{pl}}$ is always smaller than $\delta_{2\mathrm{pl}}$. This means that non-Gaussian effects overproduce PBH's in our model [at least when the simple criterion (43) is used], and the slope of the potential can be steeper than that required in the Gaussian case. Typically, the ratio $\delta_{2\mathrm{pl}}/\delta_{1\mathrm{pl}}$ is about 1.5. Say, for the case of $M_{\mathrm{PBH}} = M_{\odot}$, we have $\delta_{1\mathrm{pl}}(M_{\odot}) \approx 0.089$ and $\delta_{2\mathrm{pl}}(M_{\odot}) \approx 0.134$. We plot the probability function $\mathcal{P}(h)$ for $\delta_{1\mathrm{pl}}(M_{\odot}) = 0.089$ in Fig. 3.

In this figure, we also plot the Gaussian probability function $\mathcal{P}_G(h)$ for $\delta_{2\mathrm{pl}}(M_{\odot}) = 0.134$ (dashed line) and the same quantity for $\delta_{1\mathrm{pl}}(M_{\odot}) = 0.089$ (dotted line). Comparing the curves that correspond to the same PBH abundance, we see that the non-Gaussian curve is flatter having larger values of $\mathcal{P}(h)$ at large h. The values of the Gaussian curve with the same plateau parameter $\delta_{1\mathrm{pl}}(M_{\odot})$ is smaller by many orders of magnitude than the values of the non-Gaussian curve in the case of large h.

Finally, let us note that the non-Gaussian effects do not significantly modify the estimates based on Gaussian theory.

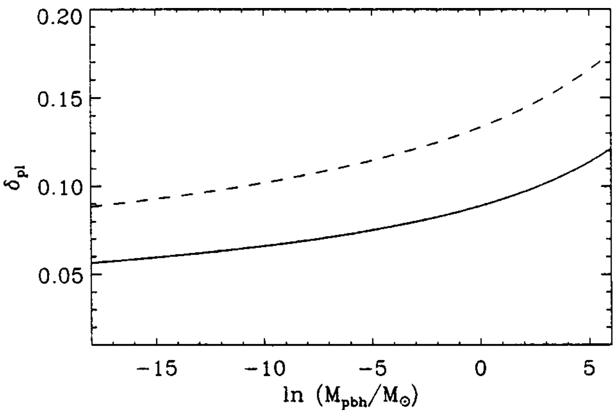

FIG. 2. We plot the dependence of plateau parameter δ_{pl} on PBH's mass M_{PBH} assuming that the PBH's abundance is given by Eq. (44). The solid line represents the solution of Eq. (45) (i.e, we calculate δ_{pl} taking into account the non-Gaussian effects in this case). The dashed line represents δ_{pl} calculated in the standard Gaussian theory. The PBH masses lie in the range $10^{-18} M_{\odot} < M_{\mathrm{PBH}} < 10^{6} M_{\mathrm{PBH}}$. The PBH's of mass $10^{-18} M_{\odot} \sim 10^{15} g$ should be evaporated at the present time. Actually, the abundance of these PBH's is constrained much larger than is assumed in our calculations.

As we have seen, the ambiguity in the choice of the plateau slope due to these effects is about 1.5. This ambiguity seems to be less than the ambiguity in other parameters and can be obviously absorbed by a small change of the potential slope.

V. DISCUSSION

We demonstrated that non-Gaussian effects related to the dynamics of the coarse-grained field (inflaton) and to the evolution of the large-scale part of the metric overproduce large-amplitude inhomogeneities of the metric compared to the prediction of the Gaussian (linear) theory of perturbations. We derived an analytical expression for the non-

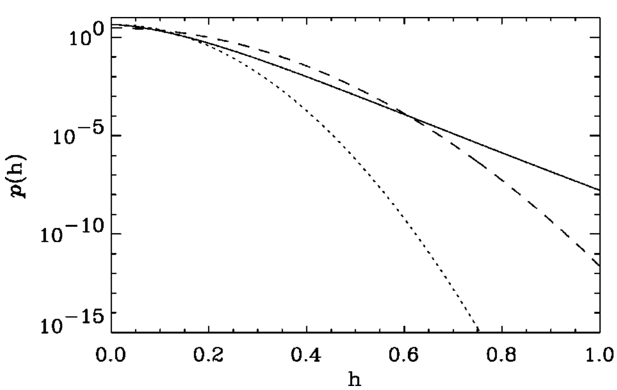

FIG. 3. The dependence of probability density $\mathcal{P}(h)$ on the metric amplitude h. The non-Gaussian curve (solid line) is calculated with help of Eq. (39) assuming PBH abundance $\beta(M_{\odot}) \approx 10^{-8}$. That gives $\delta_{1\mathrm{pl}}(M_{\odot}) \approx 0.089$. The dashed line is the reference Gaussian probability density calculated for the same abundance. For that curve we have $\delta_{2\mathrm{pl}}(M_{\odot}) \approx 0.134$. The dotted curve represents the Gaussian distribution taken with $\delta_{1\mathrm{pl}}(M_{\odot}) \approx 0.089$. This distribution strongly underproduces PBH's, and in this case we have $\beta \sim 10^{-17}$.

[7]In linear theory the density perturbation at the horizon scale is related to the metric perturbation by $\delta\rho/\rho = (4/9)h$ (see, e.g., Ref. [27]). Therefore the estimate (43) is in agreement with Carr's result.

Gaussian probability distribution for nonlinear metric perturbations, and estimated the influence of nonlinear effects on the probability of primordial black hole formation. We used the simple single-field inflationary model with a peculiarity in the form of the flat region in inflaton potential $V(\phi)$ and a power-law slope of the potential outside the peculiarity region. The key point of our approach is in the use of inhomogeneous coarse-grained metric function $a(\vec{x})$ instead of the coarse-grained field $\phi_{\rm ls}$ as a basic quantity. This allowed us to match the physical condition of production of inhomogeneities during inflation with the ''observable'' quantities.

Our results can be considered as semiqualitative only. The uncertainties come from the phenomenological character of our inflationary model as well as from the oversimplified treatment of the process of PBH formation. The uncertainties related to the choice of parameters of the inflationary model are mainly due to the unknown form of the potential between the steep and flat regions, and also due to our friction-dominated assumption in the consideration of the stochastic process. These uncertainties can be eliminated with the help of numerical simulations of stochastic processes in more realistic models of inflation. The ambiguities concerning the criterion of PBH formation are mainly due to the one-point treatment of this process. Actually, PBH formation is nonlocal, and the dynamics of collapsing region depends strongly on the form of the spatial profile of the density perturbation (see, e.g., Refs. [22,35] for discussions of this point). The form of the spatial profile can be studied by means of n-point correlation functions of the coarse-grained metric and field. Unfortunately, the formalism of n-point correlation functions is still not elaborated (see, however, Ref. [36] for the first discussion). Note that the influence of the spatial profile of the collapsing region can probably be taken into account by a redefinition of the threshold value h_*, and this value might be effectively less. In this case the role of nonlinear effects would be lessened.

Finally we would like to note that the form of the distribution (35) does not depend explicitly on the specific parameters of our model. This allows us to suppose that similar distributions can be obtained in more complicated models, say, in the two-field models proposed in Refs. [8,9]. We are going to check this very interesting assumption in our future work.

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