Axial U(1) symmetry breaking and the second Weinberg sum rule

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A current-algebraic result due to Nieh is used to evaluate corrections to the second Weinberg sum rule due to the $U_A(1)$ symmetry-breaking effective interactions. The 't Hooft interaction produces the dominant part of the second sum rule breaking terms both in the flavor-singlet and in the flavor-octet channels, whereas the Veneziano-Witten interaction leaves the second sum rule intact. This is a manifestation of the second sum rule's sensitivity to violations of the Feynman–Gell-Mann–Zweig $U_L(6)\times U_R(6)$ chiral current algebra. These predictions are compared with extant experimental data, tentatively favoring the 't Hooft interaction. [S0556-2821(98)00911-4]

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Weinberg's celebrated first sum rule (WSR I)

$$\int_0^\infty \frac{ds}{s}\,[\rho^V(s)-\rho^A(s)]=f_\pi^2, \tag{1}$$

and second sum rule (WSR II) [1]

$$\int_0^\infty ds[\rho^V(s)-\rho^A(s)]=0, \tag{2}$$

for the difference of vector and axial vector spectral functions, have long been perceived as statements about the chiral symmetry of the underlying theory at asymptotically large momenta.[1] The first of the two sum rules is the better understood one [3]: it is believed to be valid in QCD and forms one of the foundations of Shifman-Vainshtein-Zakharov (SVZ) sum rules [4]. Moreover, no violation of the first sum rule has been reported to date.

The second sum rule, on the other hand, was first extended to three flavors [5] and then challenged on empirical grounds [6]. Two critical assessments [7,8] of the assumptions underlying this sum rule appeared in that early period, but seem to have passed largely unnoticed, with one significant exception [9]. Subsequently the second sum rule disappeared from further theoretical investigation until Bernard *et al.* [10] reexamined it from the standpoint of Wilson's operator product expansion in QCD. That work showed that, in the general case, the current quark mass induced corrections to the right-hand side of the sum rule do not vanish. These corrections were subsequently calculated within the SVZ sum rule approach and expressed in terms of current quark masses and the quark condensate [11]. One result of this paper is the precise form of the leading current quark mass induced nonchiral correction to the WSR II, based only on tried-and-true current algebraic methods. More importantly, we shall show that there is a larger contribution to WSR II breaking stemming from the $U_A(1)$ symmetry-breaking 't Hooft interaction.[2] Another candidate for the effective Lagrangian describing $U_A(1)$ symmetry breaking, the Veneziano-Witten (VW) interaction, leads to *no* corrections to the WSR II.

We shall use the spectral functions $\rho^{(V,A)}$ for the vector and axial vector two-point functions defined by the time-ordered current-current correlator

$$i\int d^4x e^{iq\cdot x}\langle 0|T\{J_\mu^a(x)J_\nu^b(0)\}|0\rangle=\tilde{\Pi}_{\mu\nu}^{ab}(q^2), \tag{3}$$

where the generic symbol $J_\mu^a(x)$ stands for either a vector V_μ^a or an axial vector current A_μ^a,

$$V_\mu^a(x)=\bar{\Psi}(x)\gamma_\mu\frac{\lambda^a}{2}\Psi(x),$$

$$A_\mu^a(q^2)=\bar{\Psi}(x)\gamma_\mu\gamma_5\frac{\lambda^a}{2}\Psi(x), \tag{4}$$

of flavor $a=0,...,8$, where λ^a are the Gell-Mann matrices. These currents form the Feynman–Gell-Mann–Zweig chiral $U_L(6)\times U_R(6)$ current algebra [13,14]. The spectral functions are given by

$$\rho_{V,A}^{ab}(q^2)=-\frac{1}{\pi}\,\mathrm{Im}\,\tilde{\Pi}_{V,A}^{ab}(q^2), \tag{5}$$

where $\tilde{\Pi}_{V,A}^{ab}(q^2)_{\mu\nu}=\tilde{\Pi}_{V,A}^{ab}(q^2)T_{\mu\nu}$ and $T_{\mu\nu}=(g_{\mu\nu}-q_\mu q_\nu/q^2)$.

It is believed that Weinberg's sum rules are statements about the asymptotic validity of the left- and right-handed charge densities' chiral algebra $SU_L(3)\times SU_R(3)$. Nieh [7] has managed to express the violation of the WSR II due to the Hamiltonian H as

[1] This theory need not be QCD: Analogues of these sum rules also play a role in technicolor models [2].

[2] A first indication of the latter was observed in a two-flavor effective chiral quark model calculation [12], which model contains another term breaking the WSR II that is absent from QCD.

$$\delta_{ij}\int_0^\infty ds[\rho_V^{ab}(s)-\rho_A^{ab}(s)]$$
$$=\int d^3x\langle 0|[[H,A_i^a(0,\mathbf{x})],A_j^b(0)]|0\rangle$$
$$-\int d^3x\langle 0|[[H,V_i^a(0,\mathbf{x})],V_j^b(0)]|0\rangle, \qquad (6)$$

where $(i,j=1,2,3)$, using the P_∞ method. An alternative derivation of Eq. (6) can be produced using canonical methods along the lines of Ref. [8], although that specific result was not derived in that paper. Such a derivation shows that Nieh's result (6) is based on two assumptions: (a) validity of the Källén-Lehmann representation for the current-current correlators, and (b) validity of the Heisenberg equations of motion. Equation (6) shows that the second sum rule actually tests the commutators of the *spatial current components* and the Hamiltonian, i.e., the invariance of the theory under $U_L(6)\times U_R(6)$ current algebra transformations, rather than the usual $U_L(3)\times U_R(3)$ chiral charge algebra, which it contains as a subalgebra.

There are three sources of $U_L(3)\times U_R(3)$ symmetry breaking in QCD: (i) the current quark masses; (ii) 't Hooft's $U_A(1)$ symmetry-breaking low-energy effective interaction approximating instanton-induced effects in QCD; and (iii) the electroweak (EW) interactions. In the following we shall examine only the first two, while disregarding the third one (EW), due to its small size. We find that *the 't Hooft interaction induces a WSR II violation that is common to all nine flavor channels*. If this effect can be empirically confirmed and separated from other WSR II violations in the data, it would constitute a significant piece of new evidence supporting 't Hooft's interaction, since the competing Veneziano-Witten interaction [15] turns out *not* to have any effect on the WSR II at all.

Inserting the current quark mass Hamiltonian $H_{\chi SB}(0)=\int d^3z\overline{\Psi}(z)M_q^0\Psi(z)$ into Eq. (2) we find

$$\int_0^\infty ds[\rho_V^{ab}(s)-\rho_A^{ab}(s)]$$
$$=\int d^3x\langle 0|[[H_{\chi SB},A_i^a(0,\mathbf{x})],A_j^b(0)]|0\rangle$$
$$-\int d^3x\langle 0|[[H_{\chi SB},V_i^a(0,\mathbf{x})],V_j^b(0)]|0\rangle$$
$$=-\langle 0|\overline{\Psi}\left\{\left\{M_q^0,\frac{\lambda^a}{2}\right\},\frac{\lambda^b}{2}\right\}\Psi|0\rangle. \qquad (7)$$

The expression on the right-hand side of Eq. (7) is the same as the one entering the Gell-Mann–Oakes–Renner (GMOR) formula

$$(f_{\rm ps}m_{\rm ps}^2f_{\rm ps})_{ab}=-\langle 0|[Q_5^a,[Q_5^b,\overline{\Psi}M_q^0\Psi]]|0\rangle$$
$$=-\langle 0|\overline{\Psi}\left\{\left\{M_q^0,\frac{\lambda^a}{2}\right\},\frac{\lambda^b}{2}\right\}\Psi|0\rangle, \qquad (8)$$

relating the pseudoscalar (ps) meson masses and decay constants to the above vacuum matrix elements. This leads to the following form of the WSR II:

$$\int_0^\infty ds[\rho_V^{ab}(s)-\rho_A^{ab}(s)]=\delta^{ab}m_a^2f_a^2$$
$$=\begin{cases}\frac{1}{3}[2m_K^2f_K^2+m_\pi^2f_\pi^2], & a,b\in(0),\\ m_\pi^2f_\pi^2, & a,b\in(1,2,3),\\ m_K^2f_K^2, & a,b\in(4,\ldots,7),\\ \frac{1}{3}[4m_K^2f_K^2-m_\pi^2f_\pi^2], & a,b\in(8).\end{cases} \qquad (9)$$

Note that some of these terms differ by orders of magnitude: the largest correction is in the $(a,b=8)$ channel, the rest of the strange $(a,b=0,4,\ldots,7)$ channels are comparable in size, whereas the isovector channels $(a,b=1,2,3)$ are almost 40 times smaller. Such large differences indicate that *two precise measurements, one in the isovector and one in the ''strange'' sector, would critically test this prediction* of flavor dependence. Since considerable amount of data already exist in the isovector channel [16], one must strongly encourage measurement of the WSR II in at least one of the ''strange'' channels [17].

Take the effective 't Hooft quark self-interaction, which is a low-energy approximation to the instanton-induced effects in QCD, for three light flavors,[3]

$$\mathcal{L}_{\rm tH}^{(6)}=K[\det_f\left(\overline{\Psi}(1+\gamma_5)\Psi\right)+\det_f\left(\overline{\Psi}(1-\gamma_5)\Psi\right)]=-\mathcal{H}_{\rm tH}^{(6)} \qquad (10)$$

and insert it into the double commutators in Eq. (6) as $\mathcal{H}_{\chi SB}=\mathcal{H}_{\rm tH}^{(6)}$; we find

$$\int d^3x\langle 0|[[H_{\rm tH}^{(6)},A_i^a(0,\mathbf{x})],A_j^b(0)]|0\rangle$$
$$-\int d^3x\langle 0|[[H_{\rm tH}^{(6)},V_i^a(0,\mathbf{x})],V_j^b(0)]|0\rangle$$
$$=-6\langle 0|\mathcal{H}_{\rm tH}^{(6)}|0\rangle\delta_{ij}\delta^{ab}. \qquad (11)$$

The interacting ground state (''vacuum'') expectation value of the 't Hooft interaction is related to the 't Hooft mass with three light flavors via [15]

$$m_{\rm tH}^2f_0^2=6\langle 0|\mathcal{L}_{\rm tH}^{(6)}|0\rangle=-12K\langle\overline{q}q\rangle_0^3+O(1/N_C), \qquad (12)$$

where we made the ''factorization hypothesis'' in the second line, i.e., we assumed that the vacuum expectation value (VEV) of the operator product is saturated by the product of the individual operator VEV's. Further, we have assumed good parity and SU(3) symmetry of the nonperturbative vacuum, i.e., $\langle\overline{\Psi}\lambda_3\Psi\rangle_0=\langle\overline{\Psi}\lambda_8\Psi\rangle_0=0$. $O(1/N_C)$ serves to

[3]''Light'' flavors are defined by comparison of the current quark mass induced ps meson masses, e.g., of $m_K\simeq 0.5$ GeV for strange quarks, or $m_D\simeq 1.9$ GeV for charmed ones, with the 't Hooft mass $m_{\rm tH}\simeq 0.85$ GeV (with three flavors), see Sec. IV C of the second reference in [15]. This leads to the categorization of up, down, and strange quarks as light, and the rest as heavy.

remind one that we have neglected *all* $1/N_C$ suppressed terms, not just the ones correcting factorization. The empirical value of $f_0^2 m_{\rm tH}^2$ is determined as $(300\,{\rm MeV})^4$ from the Ref. [15] result

$$f_0^2 m_{\rm tH}^2 = f_{\eta'}^2 m_{\eta'}^2 + f_\eta^2 m_\eta^2 - f_K^2 (m_{K^+}^2 + m_{K^0}^2) + f_\pi^2 (m_{\pi^+}^2 - m_{\pi^0}^2). \tag{13}$$

This leads to

$$\int d^3x \langle 0 | [[H_{\rm tH}^{(6)}, A_i^a(0,\mathbf{x})], A_j^b(0)] | 0 \rangle - \int d^3x \langle 0 | [[H_{\rm tH}^{(6)}, V_i^a(0,\mathbf{x})], V_j^b(0)] | 0 \rangle = \delta^{ab} f_0^2 m_{\rm tH}^2 . \tag{14}$$

Note that this result holds for *all* $a,b = 0,\ldots,8$, i.e., not only in the flavor-singlet channel $(a,b=0)$, as one might have initially expected, but also in the flavor-octet channels $(a,b = 1,\ldots,8)$. The latter is something of a surprise, since we have come to expect its influence only in the flavor-singlet ps and scalar channels [15]. The resolution of this ''puzzle'' lies in the fact that here one is sensitive to the violation of the $U_L(6)\times U_R(6)$ *current* algebra, rather than that of the (usual) $SU_L(3)\times SU_R(3)$ algebra of chiral *charges*, and that the 't Hooft interaction violates the $U_L(6)\times U_R(6)$ symmetry. Adding now Eq. (7) to the right-hand side of Eq. (14) we find

$$\int_0^\infty ds [\rho_V^{ab}(s) - \rho_A^{ab}(s)] = \delta^{ab} f_0^2 m_{\rm tH}^2 + f_a m_{ab}^2 f_b . \tag{15}$$

There is, however, another way of effectively breaking the $U_A(1)$ symmetry with quark degrees of freedom: the Veneziano-Witten effective quark interaction (for original references and discussion, see Ref. [15])

$$\mathcal{L}_{\rm VW}^{(12)} = K' \{ {\rm det}_f [\overline{\Psi}(1+\gamma_5)\Psi] - {\rm det}_f [\overline{\Psi}(1-\gamma_5)\Psi] \}^2 . \tag{16}$$

Insert this into the double commutators in Eq. (6); direct calculation yields *zero*, to leading order in $1/N_C$,

$$\int d^3x \langle 0 | [[H_{\rm VW}^{(12)}, A_i^a(0,\mathbf{x})], A_j^b(0)] | 0 \rangle - \int d^3x \langle 0 | [[H_{\rm VW}^{(12)}, V_i^a(0,\mathbf{x})], V_j^b(0)] | 0 \rangle = 0 + O(1/N_C), \tag{17}$$

thus leading to purely current quark mass induced corrections to the WSR II:

$$\int_0^\infty ds [\rho_V^{ab}(s) - \rho_A^{ab}(s)] = f_a m_{ab}^2 f_b , \tag{18}$$

in the Veneziano-Witten model. Hence we see that there is a significant difference (of an order of magnitude) between these two models of $U_A(1)$ symmetry breaking in the isovector channel $(a,b=1,2,3)$. Only one precise measurement in

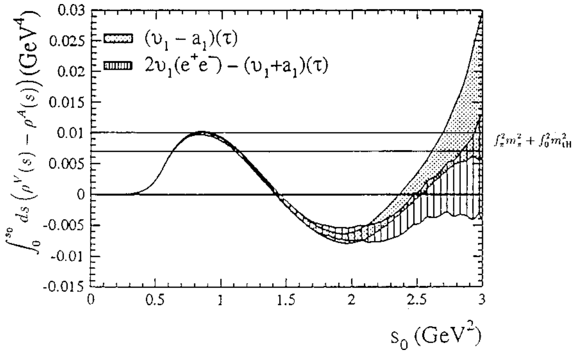

FIG. 1. The second Weinberg sum rule as a function of the upper integration bound s_0. The two horizontal lines represent the bounds of our prediction $f_\pi^2 m_\pi^2 + f_0^2 m_{\rm tH}^2$ in the asymptotic limit $s_0 \to \infty$, the width being due to uncertainties in $f_\eta, f_{\eta'}$. Dotted and hatched areas are the error bands extracted from the τ-decay and the $e^+e^- + \tau$ data, respectively (data from Ref. [16]). For $m_{\eta_c}^2 \simeq 9\,{\rm GeV}^2 \leqslant s_0$ one must recalculate $m_{\rm tH}^2$ with four flavors. The Veneziano-Witten model prediction $f_\pi^2 m_\pi^2$ is too small to be distinguished from zero on this figure.

the isovector channel should be sufficient to discriminate between these two models, under the *proviso* that no significant new corrections (beyond the aforementioned two) exist.

Recent data obtained from hadronic τ decays by the ALEPH Collaboration have been analyzed [16] with a view to testing the Weinberg sum rules. The first three moments of the isovector (charge-changing components $a,b=1,2$) spectral function $V-A$ difference have been evaluated as functions of the upper energy squared cutoff s_0; for the WSR II see the dotted area in Fig. 1. This figure also shows a linear combination of the τ-decay data and those from e^+e^- experiments (vertically hatched). Although it is manifest that the WSR II has *not* reached saturation at presently accessible energies, we may nevertheless draw the following conclusions: (i) the current quark mass contribution, which also equals the total prediction in the Veneziano-Witten model, is negligible as compared with that from the 't Hooft term;[4] and (ii) only the 't Hooft interaction leads to the observed order of magnitude of WSR II, and, it is in agreement, perhaps fortuitously, with all of the presently available data, i.e., with *both* the τ decay and the e^+e^- experiments. This appears as a piece of evidence in favor of the 't Hooft interaction over the Veneziano-Witten one.

It is curious that despite their obvious similarity, the two

[4]Nason and Palassini [18] have studied hadronic τ decays in QCD with instantons explicitly taken into account. They found relative corrections of at most 3%, i.e., substantially smaller than ours. There is a number of potential explanations of this fact: (i) the fact that four different moments of the spectral function enter the τ decay rate, not all of which are very sensitive to instantons; (ii) the integrals entering the τ decay rate extend over a finite range rather than an infinite one in the WSR II; (iii) the local 't Hooft interaction is only a low-energy approximation to the instanton effects in QCD; (iv) dependence of the 't Hooft mass on the lower cutoff in explicit instanton calculations (this dependence is subsumed in the coupling constant K in the effective interaction approach); and possibly other, presently unknown, causes.

Weinberg sum rules test two different sectors of the current algebra: WSR I checks the identity of the vacuum expectation values of the isoscalar vector and axial Schwinger terms [3], whereas WSR II checks the $U_L(6)\times U_R(6)$ current algebra. One also cannot help but wonder what prevented a more timely application of the Nieh formula, especially in view of the fact that the said result was discussed in an authoritative review [9]. That is, however, a question for historians of science.

In conclusion, we have applied Nieh's formula to the quark mass term, the 't Hooft and the Veneziano-Witten interaction Hamiltonians. The measured values of the isovector WSR II are of the same magnitude as those estimated from the quark mass plus the 't Hooft interaction. Novel experimental methods seem necessary to extend the measurements of the vector and axial spectral densities to higher energies, so as to approach the asymptotic values of the sum rules more closely, and to measure the spectral functions in other flavor channels.

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