

# Triangular mass matrices of quarks and Cabibbo-Kobayashi-Maskawa mixing

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Every nonsingular fermion mass matrix, by an appropriate unitary transformation of right-chiral fields, is equivalent to a triangular matrix. Using the freedom in choosing bases of right-chiral fields in the minimal standard model, reduction to triangular form allows for simple analytic expressions for the CKM matrix in terms of quark masses and a minimal set of parameters. Furthermore, diagonalization of the quark mass sectors can be shifted to one charge sector only, without losing the concise and economic triangular form. The corresponding effective triangular mass matrix is reconstructed, up to trivial phases, from the moduli of the CKM matrix elements, and vice versa, in a unique way. This reconstruction may also be relevant for the invariant measure of  $CP$  violation which we briefly discuss. [S0556-2821(98)03811-9]

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## I. INTRODUCTION

Although generation mixing, eventually, must be due to as yet unknown physics beyond the minimal standard model, its parametrization in terms of the Cabibbo-Kobayashi-Maskawa (CKM) matrix is quite restrictive and could, in fact, turn out to be inconsistent with precision measurements of weak decays and  $CP$ -violating amplitudes. With the restriction to three generations unitarity of the CKM matrix alone imposes (nonlinear) constraints on observables which may or may not be obeyed by experiment: see e.g. [1,2,3,4]. Further constraints are obtained if the mixing matrix is derived from the primordial mass matrices of up-type quarks and of down-type quarks. In adopting this latter strategy and before even invoking specific models of quark mass matrices, it is important to formulate the mass terms such that all redundant, unobservable, features are left out from the start. Only if this is achieved can one hope to sharpen the tests of compatibility of the CKM scheme for three generations with experiment. As we shall see below this means much more than the well-known eliminating irrelevant phases by redefinition of basis states.

In this paper we show that the essential information contained in a given, nonsingular quark mass matrix can be expressed in a particularly economic and concise way. Making use of the well-known freedom in choosing bases of right-chiral fields every nonsingular mass matrix is equivalent to a triangular matrix. Furthermore, simultaneous diagonalization of two charge sectors can be shifted to one of them without losing the simplicity of triangular matrices. We study examples for the cases of two and three generations, with and without additional model assumptions. Implications for invariants that describe  $CP$  violating observables are also touched.

In Sec. II we formulate and prove the decomposition theo-

rem that is central to our analysis. Section III contains an interpretation of triangular mass matrices in terms of quark representations and quark mixing. Making use of the decomposition theorem, we go one step further in Sec. IV, and shift diagonalization to one charge sector only, either to the customary down-sector or, equivalently, to the up-sector. This procedure is illustrated by an analytic example with two generations. The general case of three generations is treated in Sec. V which also gives explicit and analytic formulas for the entries of the CKM matrix. In Sec. VI we briefly discuss the rephasing invariant measure of  $CP$  violation which expresses this quantity in terms of observables only. In Sec. VII we perform the reconstruction of the effective (triangular) mass matrix in terms of the elements of the CKM matrix. The final section, Sec. VIII, summarizes our results and offers a few conclusions.

## II. REDUCTION OF NONSINGULAR MASS MATRICES TO TRIANGULAR FORM

In order to set the notation we start by recalling a few well-known facts about the relation of the quark mixing matrix in charged-current (CC) weak interactions to the mass matrices in the charge  $+2/3$  up-quark sector and the charge  $-1/3$  down-quark sector.

The minimal standard model describes CC weak interactions by purely left-handed currents, giving rise to the effective “ $V-A$ ” Lorentz structure at low energies and maximal parity violation. The right-chiral fields in the three-generation spinor field

$$\Psi = (u'_L, d'_L, u'_R, d'_R, c'_L, s'_L, c'_R, s'_R, t'_L, b'_L, t'_R, b'_R)^T \quad (1)$$

are inert to charged-current interactions and, being singlets with respect to the weak  $SU(2)$  structure group, are fixed only up to independent, unitary transformations  $U_R^{(u)}$  and  $U_R^{(d)}$  in the up- and down-charge sectors, respectively. In Eq. (1) the primes refer to the weak interaction states, and the mass eigenstates will be denoted by the same symbols

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without a prime. Thus, if the mass matrices of up- and down-quarks in the basis (1) are  $M^{(u)}$  and  $M^{(d)}$ , they are diagonalized by the bi-unitary transformations

$$V_L^{(u)} M^{(u)} V_R^{(u)\dagger} \quad \text{and} \quad V_L^{(d)} M^{(d)} V_R^{(d)\dagger}. \quad (2)$$

The Cabibbo-Kobayashi-Maskawa mixing matrix refers to CC interactions, hence to left-chiral fields only, and is given by the product

$$V^{(\text{CKM})} = V_L^{(u)} V_L^{(d)\dagger}. \quad (3)$$

The CKM mixing matrix is determined by the mass matrices in the charge  $+2/3$  and  $-1/3$  sectors and depends only on the unitary transformations acting on left-chiral fields, up to the well-known freedom in fixing the phases of its entries (see e.g. [5]). The unitaries acting on the right-chiral fields,  $V_R^{(u)}$  and  $V_R^{(d)}$ , remain unobservable. We shall make use of this well-known freedom below.

In this section we show that the essential information that determines the CKM mixing matrix can be encoded in mass matrices of triangular form, either upper or lower triangular, provided these are not singular. For definiteness, in what follows we shall always choose lower triangle matrices:

$$T^{(u)} \quad \text{and} \quad T^{(d)}, \quad \text{with} \quad T_{ik}^{(u/d)} = 0 \quad \text{for all} \quad k > i.$$

The importance of this observation for the physics of CC weak interactions of quarks will be discussed below, in a rather general framework for generation mixing. The mathematical fact is based on the following lemma and decomposition theorem [6].

*Lemma:* Let  $M$  be an arbitrary, nonsingular matrix of dimension  $n$  and let  $H = MM^\dagger$ , so that  $H$  is a positively definite, Hermitian  $n \times n$  matrix. The matrix  $H$  can be represented in the form  $H = TT^\dagger$ , where  $T$  is a nonsingular, lower triangular, matrix of dimension  $n$ .

The lemma is trivially true in dimension 1. For  $n > 1$  it is proved by induction with respect to  $n$ ; see [6].

This lemma is used in proving the following theorem:

*Decomposition theorem.* Any nonsingular  $n \times n$  matrix  $M$  can be decomposed into the product of a nonsingular, lower triangular matrix  $T$  and a unitary matrix  $U$ ,

$$M = TU, \quad \text{with} \quad T_{ik} = 0 \quad \forall \quad k > i, \quad UU^\dagger = \mathbb{1}. \quad (4)$$

This decomposition is unique up to multiplication of  $U$  from the left by a diagonal unitary matrix  $W = \text{diag}(e^{i\omega_1}, \dots, e^{i\omega_n})$ .

*Proof.* By the lemma the Hermitian matrix  $H = MM^\dagger$  equals  $TT^\dagger$ , with  $T$  a lower triangle matrix. This being nonsingular one calculates  $U = T^{-1}M$  and proves  $U$  to be unitary: Indeed,  $MM^\dagger = TUU^\dagger T^\dagger = TT^\dagger$ . Multiplying by  $T^{-1}$  from the left and by  $(T^\dagger)^{-1}$  from the right,  $UU^\dagger = \mathbb{1}$ . Suppose now that there is more than one decomposition (4), say  $M = TU = T'U'$ , where  $T$  and  $T'$  are (lower) triangular. This means that  $T = T'W$ , where  $W = U'U^\dagger$  is unitary. The requirement that both  $T$  and  $T'$  be lower triangular fixes  $W$  to be diagonal.

Note that the decomposition theorem is equivalent to the Schmidt orthogonalization procedure. To see this we interpret the rows of the matrix  $M = \{m_{ik}\}$  as  $n$  linearly indepen-

dent vectors  $\mathbf{v}^{(i)} = (m_{i1}, \dots, m_{in})$ ,  $i = 1, \dots, n$ . Likewise, the rows of  $U = \{u_{ik}\}$  form a set of orthogonal unit vectors  $\mathbf{u}^{(i)} = (u_{i1}, \dots, u_{in})$ ,  $i = 1, \dots, n$ . Equation (4) then reads

$$\mathbf{v}^{(i)} = \sum_{k=1}^i T_{ik} \mathbf{u}^{(k)}, \quad i = 1, \dots, n.$$

These latter equations are useful in determining the matrix elements  $\{t_{ik}\}$  of  $T$  from those of  $M = \{m_{ik}\}$ . For example, in the case  $n = 3$  we have

$$|t_{11}|^2 = \sum_{k=1}^3 |m_{1k}|^2, \quad (5)$$

$$t_{21} = \sum_{k=1}^3 m_{1k}^* m_{2k} / t_{11}^*, \quad |t_{22}|^2 = \sum_{k=1}^3 |m_{2k}|^2 - |t_{21}|^2, \quad (6)$$

$$t_{31} = \sum_{k=1}^3 m_{1k}^* m_{3k} / t_{11}^*, \quad t_{32} = \left[ \sum_{k=1}^3 m_{2k}^* m_{3k} - t_{31} t_{21}^* \right] / t_{22}^*,$$

$$|t_{33}|^2 = \sum_{k=1}^3 |m_{3k}|^2 - |t_{31}|^2 - |t_{32}|^2. \quad (7)$$

Note that Eqs. (5)–(7) reflect the non-uniqueness of the unitary  $U$  in Eq. (4) noted above.

In applying the decomposition theorem (4) to the mass matrix of the quark sector with charge  $q$ ,  $q = +\frac{2}{3}$  or  $-\frac{1}{3}$ , let  $\hat{M}^{(q)}$  be such that it connects right-chiral fields to conjugate left-chiral fields. The mass terms in the Lagrangian have the form

$$\mathcal{L}_{\text{mass}} = \sum_q \bar{\Psi} \hat{M}^{(q)} \Psi + \text{H.c.} \equiv \sum_q \overline{(\Psi)}_L \hat{M}^{(q)} (\Psi)_R + \text{H.c.}$$

Now, replacing  $\hat{M}^{(q)}$  by the product on the right-hand side (RHS) of Eq. (4), the unitary matrices are absorbed by a redefinition of the right-chiral fields

$$\begin{aligned} \{u_R'^{(n)}, n = 1, 2, 3\} &\equiv \{u_R', c_R', t_R'\}, \\ \{d_R'^{(n)}, n = 1, 2, 3\} &\equiv \{d_R', s_R', b_R'\}, \end{aligned}$$

so that the general mass Lagrangian becomes

$$\mathcal{L}_{\text{mass}} = \sum_{n,m=1}^3 \overline{u_L'^{(n)}} T_{nm}^{(u)} u_R'^{(m)} + \sum_{n,m=1}^3 \overline{d_L'^{(n)}} T_{nm}^{(d)} d_R'^{(m)} + \text{H.c.}, \quad (8)$$

where  $T^{(u)}$  and  $T^{(d)}$  are  $3 \times 3$  (lower) triangular.

### III. INTERPRETATION IN TERMS OF GENERATION MIXING

Having shown that the essential information on mass matrices in a given charge sector, by the decomposition theorem, is fully contained in their triangle form, we now pause to interpret this result in terms of the physics of generation mixing. For the sake of simplicity let us consider the case of two generations which, in fact, need not be identical replicas of each other. Let the fermions of the theory fall into two

irreducible representations of the structure group, say  $\rho_I$  and  $\rho_{II}$ , of dimensions  $n_1$  and  $n_2$ , respectively, and denote the two sets of basis states which span these representations by  $\Psi_I$  and  $\Psi_{II}$ . Let us assume that we are given an operator  $\hat{T}$  whose representation in the given basis has triangular form, viz.

$$\hat{T} = \begin{pmatrix} T_{11} & 0 \\ T_{21} & T_{22} \end{pmatrix},$$

where the diagonal blocks  $T_{11}$  and  $T_{22}$  are square matrices of dimension  $n_1$  and  $n_2$ , respectively, and  $T_{21}$  is an  $n_1 \times n_2$  off-diagonal block. A unitary transformation  $\hat{R} = \text{diag}(R_I, R_{II})$  of the bases will take the operator  $\hat{T}$  to

$$\hat{T}' = \hat{R} \hat{T} \hat{R}^\dagger = \begin{pmatrix} R_I T_{11} R_I^\dagger & 0 \\ R_{II} T_{21} R_I^\dagger & R_{II} T_{22} R_{II}^\dagger \end{pmatrix}.$$

Thus, the subspace spanned by  $\Psi_{II}$  is an invariant subspace: the subspace  $\Psi_I$  is not. In this situation the combined representation spanned by  $(\Psi_I, \Psi_{II})$  is said to be reducible but indecomposable, and is written as a semi-direct sum

$$\rho_I \oplus \rho_{II}. \quad (9)$$

This framework is completely general. It seems to us the most economic and natural parametrization of generation mixing. Indeed, after extracting all unphysical phases, by redefinition of quark fields, the triangular form contains the essential, minimal information on the mass matrix. This is particularly important if one wishes to obtain analytic expressions of the CKM matrix in terms of quark mass matrices and, conversely, if one sets out to reconstruct the latter from the observed mixing. In order to emphasize this point we note that the CKM matrix (3) is unchanged if the quark mass matrices  $M^{(u)}$  and  $M^{(d)}$  are both multiplied from the left by an arbitrary unitary matrix  $X$  such that

$$M'^{(u)} = X M^{(u)}, \quad M'^{(d)} = X M^{(d)}.$$

In the case of general, non-triangular mass matrices these formulas reflect the nine parameter freedom in reconstructing the mass matrices from the CKM matrix discussed by Kusenko [7]. If, on the other hand, we require  $M^{(q)}$  and  $M'^{(q)}$ , with  $q = u$  or  $d$ , to be triangular before and after the transformation, then  $X$  must be diagonal, its entries being pure phases [8]. There may, in fact, be good theoretical reasons for assuming the fermions of the standard model to fall into representations of the type (9). For example, representations of this type are characteristic of graded (or super) Lie algebras; cf. [9]. Clearly, if a multiplet of scalar Higgs fields appears multiplied with an operator acting on representations of this type and if the electrically neutral component of the Higgs field develops a nonzero vacuum expectation value, fermions of the same charge belonging to different subspaces, via their Yukawa couplings, acquire triangular mass matrices.

In the case of replication of generations the two terms in Eq. (9) are identical. All gauge interactions act within each of the diagonal blocks corresponding to the irreducible rep-

resentations  $\rho_1$  and  $\rho_2$ , while the off-diagonal block contains the physics that causes mixing via the mass matrices.

For instance, with this interpretation in mind, it seems natural to assume the diagonal blocks to be the same for each generation. This is equivalent to saying that if electroweak interactions were switched off quark masses in a given charge sector would be degenerate. As shown in [10] and [11], in the case of two generations, this assumption fixes the Cabibbo angle in terms of the quark masses. With  $m_1, \mu_1$  denoting the masses of the first generation, say  $m_1 \equiv m_u$  and  $\mu_1 \equiv m_d$ , and  $m_2, \mu_2$  denoting the masses of the second generation, say  $m_2 \equiv m_c$  and  $\mu_2 \equiv m_s$ , one obtains

$$\cos \theta = \frac{\sqrt{m_1 \mu_1} + \sqrt{m_2 \mu_2}}{\sqrt{(m_1 + m_2)(\mu_1 + \mu_2)}}. \quad (10)$$

In the case of three generations, as worked out in [11], the same assumption leads to an analytic expression of the CKM matrix (3) in terms of the quark masses and a few parameters which has a remarkable similarity to established phenomenological forms [12].

#### IV. SHIFTING DIAGONALIZATION TO ONE SECTOR ONLY

In computing the CKM matrix (3) one may proceed by independent diagonalization of the triangular matrices  $T^{(u)}$  and  $T^{(d)}$  of Eq. (8), by means of bi-unitary transformations in each charge sector: cf. Eq. (2). As an alternative to this tedious calculation [11] we now show that diagonalization can be shifted to one of the charge sectors only, whose effective mass matrix again has triangular form. This is the content of the following theorem.

*Shift theorem.* Given two nonsingular triangle matrices relating right-chiral to left-chiral fermion fields,  $T^{(q_1)}$  and  $T^{(q_2)}$ , in the charge sectors  $q_1$  and  $q_2 = q_1 \pm 1$ , respectively, and a bi-unitary transformation which diagonalizes  $T^{(q_1)}$ , viz.

$$V_L^{(q_1)} T^{(q_1)} V_R^{(q_1)\dagger} = T^{(q_1)}. \quad (11)$$

If the same bi-unitary transformation is simultaneously applied to  $T^{(q_2)}$ , the charge changing current relating the left-chiral fermion fields remains unchanged. Furthermore, by an additional unitary transformation of the right-chiral fields of charge  $q_2$ , the transformed matrix can again be cast into triangular form:

$$V_L^{(q_1)} T^{(q_2)} V_R^{(q_1)\dagger} = \mathcal{T} \mathcal{U}. \quad (12)$$

The first part of the statement is fairly obvious and well known: if the same transformation  $V_L^{(q_1)}$  is applied to the left-chiral fields of both charges, the matrix elements  $\overline{\psi^{(q_1)} I_\pm \psi^{(q_2)}}$  of isospin raising and lowering operators do not change. The second part is a consequence of the decomposition theorem. Here, in fact, the right-hand factor  $V_R^{(q_1)\dagger}$  is irrelevant because it may be absorbed in the unitary  $\mathcal{U}$  acting on the right-chiral fields. Note that the right factor of the bi-unitary transformation (11) follows from the left factor:

$$V_R^{(q_1)} = T^{(q_1)} V_L^{(q_1)} (T^{(q_1)})^{-1}{}^\dagger.$$

We illustrate the shift theorem for the example of two quark generations for which the basis (1) reduces to

$$\Psi = (u'_L, d'_L, u'_R, d'_R, c'_L, s'_L, c'_R, s'_R)^T.$$

The mass matrices are

$$T^{(u)} = \begin{pmatrix} \alpha^{(u)} & 0 \\ \kappa^{(u)} & \beta^{(u)} \end{pmatrix}, \quad T^{(d)} = \begin{pmatrix} \alpha^{(d)} & 0 \\ \kappa^{(d)} & \beta^{(d)} \end{pmatrix},$$

where the parameters may be chosen real, without loss of generality. In this case they are related to the quark masses by

$$\alpha^{(u)} \beta^{(u)} = m_u m_c, \quad \alpha^{(u)2} + \beta^{(u)2} + \kappa^{(u)2} = m_u^2 + m_c^2$$

(and analogous relations for the parameters  $\alpha^{(d)}, \beta^{(d)}, \kappa^{(d)}$ , in terms of  $m_d$  and  $m_s$ ). Then

$$V_L^{(u)} = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}, \quad V_R^{(u)} = \frac{1}{\alpha^{(u)}} \begin{pmatrix} m_u a & -m_c b \\ -m_c b & -m_u a \end{pmatrix},$$

where  $a$  and  $b$  are given by

$$a = \sqrt{\frac{m_c^2 - \alpha^{(u)2}}{m_c^2 - m_u^2}}, \quad b = \sqrt{1 - a^2}.$$

[We note in passing that signs were chosen such that  $T^{(u)} = \text{diag}(m_u, -m_c)$ .] In the basis given above the step operators of weak isospin are represented by

$$\hat{I}_\pm = \frac{1}{2} \left( \begin{array}{cc|cc} \tau_\pm & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & \tau_\pm & 0 \\ 0 & 0 & 0 & 0 \end{array} \right),$$

where the entries are  $2 \times 2$  block matrices,  $\tau_i$  being Pauli matrices. The transformation  $V_L^{(u)}$ , when applied to both charge sectors, reads, in this basis,

$$\hat{V}_L^{(u)} = \left( \begin{array}{cc|cc} a\mathbb{1} & 0 & -b\mathbb{1} & 0 \\ 0 & \mathbb{1} & 0 & 0 \\ \hline b\mathbb{1} & 0 & a\mathbb{1} & 0 \\ 0 & 0 & 0 & \mathbb{1} \end{array} \right),$$

which obviously commutes with  $\hat{I}_\pm$ .

To continue with this example let us introduce the assumption, mentioned above, of choosing the diagonal blocks to be the same in the two generations. This implies  $\alpha^{(u)} = \beta^{(u)} = \sqrt{m_u m_c}$ ,  $\kappa^{(u)} = m_c - m_u$ ,  $\alpha^{(d)} = \beta^{(d)} = \sqrt{m_d m_s}$ , and  $\kappa^{(d)} = m_s - m_d$ . The matrix elements  $T = \{t_{ik}\}$  are then found to be

$$t_{11} = \frac{1}{\sqrt{m_u + m_c}} \{ \sqrt{m_d m_s} \Delta \Sigma + m_d \sqrt{m_u m_d} \Sigma - m_s \sqrt{m_u m_s} \Delta \}^{1/2}, \quad (13)$$

$$t_{21} = \frac{m_s - m_d}{m_u + m_c} \Delta \Sigma / t_{11}, \quad t_{22} = -m_d m_s / t_{11}. \quad (14)$$

In these formulas we have fixed phases such that all entries are real and that  $t_{22}$  is negative. The symbols  $\Delta$  and  $\Sigma$  stand for

$$\Sigma = \sqrt{m_c m_s} + \sqrt{m_u m_d}, \quad \Delta = \sqrt{m_c m_d} - \sqrt{m_u m_s}. \quad (15)$$

It is now straightforward to determine the single unitary matrix which diagonalizes the matrix  $(TT^\dagger)$  and to confirm that this is the Cabibbo matrix with  $\theta$  as given by Eq. (10). One finds

$$V^C = \frac{1}{N} \begin{pmatrix} \Sigma & -\Delta \\ \Delta & \Sigma \end{pmatrix},$$

where  $N = \sqrt{(m_u + m_c)(m_d + m_s)}$ ,  $\Sigma$  and  $\Delta$  being defined in Eq. (15) above. This result is equivalent to the formula (10).

Clearly, the procedure is symmetric in the two charge sectors. The diagonalization may as well be shifted to the charge sector  $(q_1)$ . In the example given this is equivalent to interchanging

$$m_u \leftrightarrow m_d, \quad m_c \leftrightarrow m_s.$$

## V. CASE OF THREE GENERATIONS

Suppose that in the case of three generations we shift diagonalization to the up- or the down-sector, as described in the shift theorem and in Eq. (12). For the sake of clarity we write the effective mass matrix  $T = \{t_{ik}\}$  in terms of moduli and phases of its entries as follows:

$$T = \begin{pmatrix} t_{11} & 0 & 0 \\ t_{21} & t_{22} & 0 \\ t_{31} & t_{32} & t_{33} \end{pmatrix} \equiv \begin{pmatrix} \alpha e^{i\varphi_\alpha} & 0 & 0 \\ \kappa_1 e^{i\varphi_1} & \beta e^{i\varphi_\beta} & 0 \\ \kappa_3 e^{i\varphi_3} & \kappa_2 e^{i\varphi_2} & \gamma e^{i\varphi_\gamma} \end{pmatrix}, \quad (16)$$

a notation that is consistent with the one employed in the example discussed in Sec. IV.

For definiteness let us shift the analysis to the down-sector in which case the matrix  $(TT^\dagger)$  has eigenvalues  $\{m_d^2, m_s^2, m_b^2\}$ . From its characteristic polynomial we obtain the following equations:

$$m_d^2 m_s^2 m_b^2 = \alpha^2 \beta^2 \gamma^2, \quad (17)$$

$$m_d^2 m_s^2 + m_d^2 m_b^2 + m_s^2 m_b^2 = \alpha^2 \beta^2 + \beta^2 \gamma^2 + \gamma^2 \alpha^2 + \alpha^2 \kappa_2^2 + \beta^2 \kappa_3^2 + \gamma^2 \kappa_1^2 + \kappa_1^2 \kappa_2^2 - 2\beta \kappa_1 \kappa_2 \kappa_3 \times \cos(\varphi_\beta - \varphi_1 + \varphi_3 - \varphi_2), \quad (18)$$

$$m_d^2 + m_s^2 + m_b^2 = \alpha^2 + \beta^2 + \gamma^2 + \kappa_1^2 + \kappa_2^2 + \kappa_3^2. \quad (19)$$

The decomposition theorem (4) tells us that the matrix  $\mathcal{T}$ , Eq. (12), is determined up to multiplication by  $W = \text{diag}(e^{i\omega_1}, e^{i\omega_2}, e^{i\omega_3})$  from the right, where the  $\omega_i$  are arbitrary. Under this substitution relations (17) and (19) are trivially invariant; relation (18) is also invariant because the argument of the cosine is unchanged.

Having shifted the diagonalization to the down-sector, the CKM matrix (3) is given by the Hermitian conjugate of the unitary matrix that diagonalizes  $(\mathcal{T}\mathcal{T}^\dagger)$ . Adapting our earlier results (cf. Ref. [11]) to the present situation, we obtain the following analytic expressions for the CKM matrix in terms of the entries of  $\mathcal{T}$ , i.e. of the parameters  $\alpha, \dots, \varphi_\gamma$  of Eq. (16):

$$V^{(\text{CKM})} \equiv \begin{pmatrix} A e^{i\phi_A} & B e^{i\phi_B} & C e^{i\phi_C} \\ D e^{i\phi_D} & E e^{i\phi_E} & F e^{i\phi_F} \\ G e^{i\phi_G} & H e^{i\phi_H} & I e^{i\phi_I} \end{pmatrix} \quad (20)$$

$$= \begin{pmatrix} f(m_d)/N_d & f(m_s)/N_s & f(m_b)/N_b \\ g(m_d)/N_d & g(m_s)/N_s & g(m_b)/N_b \\ h(m_d)/N_d & h(m_s)/N_s & h(m_b)/N_b \end{pmatrix}, \quad (21)$$

where the functions  $f, g, h$ , and the normalization factors are given by

$$f(m_i) = \alpha\beta\kappa_1\kappa_2 e^{-i(\varphi_1 + \varphi_2 - \varphi_\alpha - \varphi_\beta)} - \alpha\kappa_3(\beta^2 - m_i^2) e^{-i(\varphi_3 - \varphi_\alpha)}, \quad (22)$$

$$g(m_i) = m_i^2 \kappa_1 \kappa_3 e^{-i(\varphi_3 - \varphi_1)} - \beta\kappa_2(\alpha^2 - m_i^2) e^{-i(\varphi_2 - \varphi_\beta)}, \quad (23)$$

$$h(m_i) = (\alpha^2 - m_i^2)(\beta^2 - m_i^2) - \kappa_1^2 m_i^2, \quad (24)$$

with  $m_i = m_d$ , or  $m_s$ , or  $m_b$ ,

$$N_d = \{[(\alpha^2 - m_d^2)(\beta^2 - m_d^2) - m_d^2 \kappa_1^2](m_b^2 - m_d^2)(m_s^2 - m_d^2)\}^{1/2}, \quad (25)$$

and with  $N_s$  and  $N_b$  obtained from Eq. (25) by cyclic permutation of  $(m_d, m_s, m_b)$ . The first equation, Eq. (20), is only meant to express the matrix elements of  $V^{(\text{CKM})}$  in terms of their moduli and their phases while Eq. (21) gives our explicit results in terms of  $\mathcal{T}$ . So, for instance, reality and the sign of Eq. (24) imply  $\phi_G = \phi_I = 0$  and  $\phi_H = \pi$ . Of course, the results fulfill all relations such as  $C^2 = 1 - A^2 - B^2$  etc. which follow from unitarity.

Finally, we recall that one may equally well shift diagonalization to the up-sector in which case  $(m_d, m_s, m_b)$  are replaced by  $(m_u, m_c, m_t)$ , while the parameters in Eq. (16) take different values because in determining Eq. (12) the charge sectors are interchanged. In this case the CKM matrix (3) is given by the unitary matrix that diagonalizes  $(\mathcal{T}\mathcal{T}^\dagger)$  (not its Hermitian conjugate).

## VI. CP-MEASURE AS A FUNCTION OF OBSERVABLES

As is well known the following nine quantities are rephasing invariants, i.e. are independent of the specific parametrization of the CKM matrix one chooses [13]:

$$\Delta_{i\alpha} = V_{j\beta}^{(\text{CKM})} V_{k\gamma}^{(\text{CKM})} V_{j\gamma}^{(\text{CKM})*} V_{k\beta}^{(\text{CKM})*}, \{i, j, k\}, \{\alpha, \beta, \gamma\} \in \{1, 2, 3\} \text{ cyclic.}$$

In particular, unitarity of  $V^{(\text{CKM})}$  implies that they all have the same imaginary part, (cf. [2,3,4]),

$$\mathcal{J} = \text{Im } \Delta_{i\alpha}, \quad (26)$$

which is a parametrization independent measure of the amount of  $CP$  violation in the standard model with three generations. We find it useful to call this quantity the  $CP$ -measure. It was shown some time ago that this quantity can be expressed in terms of moduli of CKM matrix elements [14]. In the notation of Eq. (20) it reads<sup>1</sup>

$$\mathcal{J} = \frac{1}{2} \{4A^2 B^2 D^2 E^2 - [A^2 E^2 + B^2 D^2 - (A^2 + B^2 + D^2 + E^2) + 1]^2\}^{1/2}. \quad (27)$$

It expresses the strength of  $CP$  violation in terms of the observable quantities  $A \equiv |V_{ud}|$ ,  $B \equiv |V_{us}|$ ,  $D \equiv |V_{cd}|$ , and  $E \equiv |V_{cs}|$ .

From Eq. (27) the following symmetries of  $\mathcal{J}$  are evident:

- (i)  $\mathcal{J}$  is invariant under the exchange  $B \leftrightarrow D$ . This property reflects our earlier remark that diagonalization may equivalently be shifted to the  $up$ -sector in which case the CKM matrix equals the diagonalization matrix, not its Hermitian conjugate.
- (ii) Simultaneous interchange  $A \leftrightarrow B$  and  $D \leftrightarrow E$  leaves  $\mathcal{J}$  invariant.

Combining the symmetries (i) and (ii) one sees that the simultaneous interchange  $A \leftrightarrow D$  and  $B \leftrightarrow E$  is also a symmetry. Finally, by combining all three of these one shows that  $\mathcal{J}$  is also invariant under  $A \leftrightarrow E$ .

It is easy to verify that  $\mathcal{J}$  vanishes, as it should, whenever one of the three generations decouples from the other two. For example, if the first generation decouples, we have  $A = 1$ , and hence  $B = D = 0$  and  $\mathcal{J} = 0$ .

For given values of the moduli  $A$ ,  $B$ , and  $E$ , the  $CP$ -measure is defined only for values of the modulus  $D$  in the interval  $(D_1, D_2)$ , where

$$D_{1,2} = \{ABE \mp \sqrt{(1-A^2-B^2)(1-B^2-E^2)}\} / (1-B^2). \quad (28)$$

$\mathcal{J}$  vanishes at these boundary points. It assumes its maximal value at

$$D_0 = \sqrt{A^2 B^2 E^2 + (1-A^2-B^2)(1-B^2-E^2)} / (1-B^2), \quad (29)$$

at which point the  $CP$ -measure takes the value

<sup>1</sup>In principle, the  $CP$ -measure is plus or minus the expression on the RHS of Eq. (27). The data seem to indicate that  $\mathcal{J}$  is positive, hence our choice of this sign.

$$\mathcal{J}(D_0) = \frac{1}{1-B^2} \sqrt{A^2 B^2 E^2 (1-A^2-B^2)(1-B^2-E^2)}. \quad (30)$$

In these formulas the moduli can be expressed in terms of quark masses and the parameters of  $\mathcal{T}$ , Eq. (16), by means of our formulas (21)–(25) above. Alternatively, they may be taken from experiment, as in the following example.

According to the minireport of the Particle Data Group [15] an overall best fit to the data allows for values of the magnitude  $A$  of the matrix element  $V_{ud}$  between 0.9745 and 0.9757, i.e. within an interval of width 0.0012. Similarly,  $B$  lies between 0.219 and 0.224,  $D$  lies between 0.218 and 0.224, and  $E$  lies between 0.9736 and 0.9750. The uncertainty of  $D$  being the largest was the reason why we solved our formula (27) for the  $CP$ -measure in terms of that quantity. Evidently, any other choice is possible. It is amusing to note that if we take the central values provided by the best fit, i.e.  $A=0.9751$ ,  $B=0.2215$ ,  $E=0.9743$ , we obtain  $D_0=0.2213$  for the point at which  $\mathcal{J}$  is maximal [cf. Eq. (29)], a value that happens to fall in the center of the allowed interval for  $D$ .

## VII. RECONSTRUCTION OF THE EFFECTIVE MASS MATRIX

In this section we show how to reconstruct the effective, triangular mass matrix  $\mathcal{T}$ , Eq. (16), from the CKM mixing matrix. This reconstruction is essentially unique, except for

trivial redefinitions of phases irrelevant for physics, because the triangular form of the effective mass matrix contains no redundant information. For the sake of definiteness we again assume that diagonalization is shifted, by the shift theorem (12), to the down-sector, such that the up mass sector is already diagonal while the down-sector has the effective, triangular form (16). We repeat, however, that the procedure is completely symmetric in the two charge sectors, and that the case of a nondiagonal, effective mass matrix in the up-sector is obtained from our formulas by simple and obvious modifications.

By absorbing redundant phases into the base states (see [11]), one finds that the physically relevant information coded by  $\mathcal{T}$  is contained in seven real parameters, viz.

$$\alpha, \beta, \gamma, \kappa_1, \kappa_2, \kappa_3, \Phi = \varphi_\beta - \varphi_1 + \varphi_3 - \varphi_2, \quad (31)$$

the first six of which can be chosen positive. Making use of Eqs. (17)–(19) that follow from the characteristic polynomial, we are left with four parameters. These will be determined from the CKM matrix as follows. From Eqs. (12), (2), and (3) we have

$$\begin{aligned} \mathcal{T}\mathcal{T}^\dagger &= V_L^{(u)} V_L^{(d)\dagger} \text{diag}(m_d^2, m_s^2, m_b^2) V_L^{(d)} V_L^{(u)\dagger} \\ &= V^{(\text{CKM})} \text{diag}(m_d^2, m_s^2, m_b^2) V^{(\text{CKM})\dagger}. \end{aligned} \quad (32)$$

Denoting the moduli of CKM matrix elements as in Eq. (20) one derives the following expressions from Eq. (32):

$$\alpha^2 = m_d^2 A^2 + m_s^2 B^2 + m_b^2 (1 - A^2 - B^2), \quad (33)$$

$$\begin{aligned} \alpha^2 \kappa_1^2 &= m_d^4 A^2 D^2 + m_s^4 B^2 E^2 + m_b^4 (1 - A^2 - B^2)(1 - D^2 - E^2) + (m_d^2 m_s^2 - m_d^2 m_b^2 - m_s^2 m_b^2) \\ &\quad \times (1 - A^2 - B^2 - D^2 - E^2 + A^2 E^2 + B^2 D^2) - 2m_b^2 (m_d^2 A^2 D^2 + m_s^2 B^2 E^2), \end{aligned} \quad (34)$$

$$\begin{aligned} \alpha^2 \kappa_3^2 &= m_d^4 A^2 (1 - A^2 - D^2) + m_s^4 B^2 (1 - B^2 - E^2) + m_b^4 (1 - A^2 - B^2)(A^2 + B^2 + D^2 + E^2 - 1) \\ &\quad - (m_d^2 m_s^2 - m_d^2 m_b^2 - m_s^2 m_b^2)(1 - A^2 - B^2 - D^2 - E^2 + A^2 E^2 + B^2 D^2 + 2A^2 B^2) + 2m_b^2 [m_d^2 A^2 (A^2 + D^2 - 1) \\ &\quad + m_s^2 B^2 (B^2 + E^2 - 1)], \end{aligned} \quad (35)$$

$$\beta^2 + \kappa_1^2 = m_d^2 D^2 + m_s^2 E^2 + m_b^2 (1 - D^2 - E^2). \quad (36)$$

These equations, together with Eqs. (17)–(19), are sufficient to calculate the set (31), once the moduli of the CKM mixing matrix and the quark masses are given. Thus, we obtain explicit and unambiguous expressions for the parameters (31) which determine the effective mass matrix  $\mathcal{T}$ , in terms of observables only.

If one of the charge sectors, say the up-sector, is diagonal from the start, the problem of reconstructing the mass matrix from the data is completely solved. If the two sectors are treated more symmetrically and if the mass matrices are nondiagonal in either charge sector, one might wish to go one step further by trying to reconstruct the original nondiagonal, triangular mass matrices  $T^{(u)}$  and  $T^{(d)}$  from the effective matrix  $\mathcal{T}$ . A promising example would be the physically in-

teresting case mentioned above, where these matrices have equal entries in the main diagonal. Although we have the necessary analytic formulas at our disposal, [cf. Eqs. (5)–(7)], this reconstruction is rather lengthy and tedious, and we leave it to later investigation.

## VIII. SUMMARY AND CONCLUSIONS

In the minimal standard model right-chiral fields do not participate in the charged current weak interaction and, as far as the interactions with vector bosons are concerned, the model is immune against base transformations of right-chiral

fields. Making use of this freedom in the choice of bases for right-chiral quark fields we showed that every nonsingular mass matrix is equivalent to a triangular matrix whose entries are calculated in Eqs. (5)–(7). In contrast with a more general form of the mass matrix (in a given charge sector), the equivalent triangular form is optimized in the sense that it eliminates all redundant parameters and exhibits in a simple and transparent manner the remaining freedom. In fact, the triangular form is suggestive and natural if the quark generations fall into representations of “semi-sum” type, i.e. which are reducible but indecomposable [cf. Eq. (9)]. Such representations are typical for super Lie algebras and have been discussed in the context of electroweak interactions and non-commutative geometry [10,11,16].

We then showed that even if both charge sectors, up and down, initially have nondiagonal, nonsingular mass sectors, diagonalization can be shifted to one charge sector only, the resulting effective mass matrix having again triangular form. Once the latter is known, the elements of the CKM matrix

can be calculated analytically [cf. Eqs. (20)–(25)]. In turn, the effective, triangular mass matrix is reconstructed analytically from the moduli of the CKM matrix elements [cf. Eqs. (32)–(36)]. This reconstruction is unique up to trivial phase redefinitions. Because the procedure is independent of any specific parametrization of the CKM matrix, but mainly because it is economic, concise and transparent, we strongly advocate the use of triangular matrices in describing quark and lepton mass sectors.

Analytic expressions of CKM matrix elements in terms of quark mass matrices may also be relevant for the invariant measure for  $CP$  violation, Eq. (27) and [14], which expresses it in terms of observable moduli.

We illustrate our procedure by simple examples for two and three generations. A more detailed numerical analysis including the available experimental data with their error bars is left for a future investigation. The application to charged lepton and neutrino masses, and the consequences for neutrino oscillations, is worked out in [16].

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