Fixed-charge ensembles and induced parity-breaking terms

C. D. Fosco*
Centro Atómico Bariloche, 8400 Bariloche, Argentina
(Received 22 October 1997; published 21 April 1998)

Recently derived results for the exact induced parity-breaking term in 2+1 dimensions at finite temperature are shown to be relevant to the determination of the free energy for fixed-charge ensembles. The partition functions for fixed total charge corresponding to massive fermions in the presence of Abelian and non-Abelian magnetic fields are discussed. We show that the presence of the induced Chern-Simons term manifests itself in that the free energy depends strongly on the relation between the external magnetic flux and the value of the fixed charge. [S0556-2821(98)06308-5]

PACS number(s): 11.10.Kk, 11.10.Wx, 11.30.Er

Quantum field theory in $2+1$ dimensions continues to be a subject of active research, because of its many distinctive properties, with no $(3+1)$-dimensional counterpart. Examples, provided by $(2+1)$-dimensional ("planar") models, are worth studying not only for purely theoretical reasons, but also because many important physical systems, or experimental situations, are indeed essentially planar, as in the well-known examples borrowed from condensed matter physics [1]. This also happens for some astrophysical objects whose configurations are approximately invariant under translations along one of the spatial dimensions, which renders the relevant dynamics two dimensional.

One of the more striking properties of $(2+1)$-dimensional physics is that it allows for the existence of fractional statistics, realized in terms of the so-called "anyons," namely, identical particles with neither bosonic nor fermionic statistics.

Closely related to fractional statistics is the fact that in $2+1$ dimensions a gauge field can be equipped with a gauge invariant and parity-breaking action with nontrivial topological properties, namely, the Chern-Simons action. This action, if not introduced *ab initio* in the model, may be induced dynamically by virtual matter-field processes [2]. The issue of the precise form of this induced action at finite temperature has been a long-standing problem, with obvious relevance for the applications. In some recent works, the exact expression for this induced term under some simplifying assumptions was derived for both the Abelian [3,4] and the non-Abelian cases [5]. This result was also rederived and generalized in [6].

In this Brief Report we shall show first that the configurations that have been studied in those references are precisely the ones needed in order to study the statistical mechanics of a fermion gas in a background magnetic field in the "fixed charge ensemble" [7], and then we will find the difference between the free energy for such an ensemble and the one corresponding to the canonical ensemble. We will also show that properties such as the behavior of this induced action under large gauge transformations find a natural and concrete realization here.

*Member of CONICET.

By a fixed-charge ensemble we mean one where the values of one or more conserved and compatible (i.e., mutually commuting) charges have a δ-like statistical weight. Namely, if the fixed value is, say, q, only configurations having that eigenvalue for the charge operator are summed up in the statistical average. This should be contrasted with the grand-canonical ensemble, where only the *average* of the charge is fixed, but there is indeed room for fluctuations around this mean value. We may illustrate this distinction by saying that the microcanonical ensemble is a particular case of a fixed-charge one, where the fixed charge is just the Hamiltonian.

The interest in this kind of ensemble stems from the fact that the experimental situation under study may very well correspond to it (as in an electrically insulated sample, for example). The predictions shall differ significantly from the ones of other ensembles for nonmacroscopic systems (results will of course agree in the thermodynamic limit, where all the fluctuations may be ignored). Illustrative examples of this kind of calculation are the color singlet calculation (for an SU(N) theory) of Ref. [8], and the fixed three-momentum ensemble of Ref. [9].

Our main idea in this Brief Report is that, as the Chern-Simons term provides a link between the magnetic field and the charge, it will strongly affect the statistical properties of a system in the presence of an external magnetic field, and in the fixed-charge ensemble. Moreover, we shall show that it is crucial to use the exact induced Chern-Simons term rather than the perturbative one in the derivation of this free energy.

The partition function $\mathcal{Z}_q$ corresponding to the ensemble with fixed charge q, at a given temperature $T=1/\beta$, for a system described by a quantum Hamiltonian H, and having a conserved additive charge Q ($[H,Q]=0$), is

$$\mathcal{Z}_q=\int_{-\pi}^{\pi}\frac{d\theta}{2\pi}e^{-i\theta q}\mathcal{Z}_\theta, \tag{1}$$

where

$$\mathcal{Z}_\theta=\mathrm{Tr}e^{-\beta H+i\theta Q}. \tag{2}$$

We are assuming the normalization of Q is such that its eigenvalues are just integer numbers. Note that $\mathcal{Z}_\theta$ is formally equivalent to the grand canonical partition function for a system with an *imaginary* chemical potential θ. If the trace in Eq. (2) is evaluated using a complete set of simultaneous eigenstates of H and Q, then it follows immediately that Eq. (1) will only pick up contributions from quantum states with eigenvalue q for Q. Also by using this complete set one sees that $\mathcal{Z}_\theta$ is a periodic function of θ, with period 2π. Of course, this is closely related to the assumption that particles in the physical spectrum have integer charge. We shall see how this fact turns out to be important for the application to the $(2+1)$-dimensional case, where this periodicity is tantamount to gauge invariance under large gauge transformations.

Alternatively, definition (1) can be justified by noting that

$$P_q=\int_{-\pi}^{\pi}\frac{d\theta}{2\pi}e^{-i\theta(q-Q)} \tag{3}$$

is a projector onto charge-q states. In a fixed-charge ensemble, the fixed charge does not fluctuate at all, as can be shown explicitly by noting that the averages (denoted $\langle\cdots\rangle_q$) of the powers of Q may be written as

$$\langle Q^n\rangle_q=(-i)^n\mathcal{Z}_q^{-1}\int_{-\pi}^{\pi}\frac{d\theta}{2\pi}e^{-i\theta q}\frac{\partial^n}{(\partial\theta)^n}\mathcal{Z}_\theta=q^n, \tag{4}$$

where the periodicity of $\mathcal{Z}_\theta$ has been used in order to ignore terms in the integration by parts. We want to construct the partition function $\mathcal{Z}_q(A)$ for the case of a fermionic field in $2+1$ dimensions in the presence of an external magnetic field (here A is the vector potential corresponding to the magnetic field). From the analogy between $\mathcal{Z}_\theta(A)$ and the partition function in the presence of an imaginary chemical potential, we immediately obtain the path-integral representation

$$\mathcal{Z}_\theta(A)=\int\mathcal{D}\bar\psi\mathcal{D}\psi\exp\left\{-\int_0^\beta d\tau\int d^2x\,\bar\psi(\tau,x)\times\left[\gamma_jD_j+M+\gamma_3\left(\partial_\tau-i\frac{\theta}{\beta}\right)\right]\psi(\tau,x)\right\}, \tag{5}$$

where $D_j=\partial_j+ieA_j(x)$, and the notation and conventions are identical to the ones used in [4,5].

It should now become evident that (5) corresponds to exactly the same kind of configuration considered in [4,5], if one makes the identification $\tilde{A}_3=-\theta/e\beta$. Periodicity in θ for Eq. (5) is equivalent to invariance under large gauge transformations, after this identification is made.

We now separate $\mathcal{Z}_\theta$ into its phase and its modulus, which are given by the exponentials of the parity-breaking and parity-conserving parts of the effective action, respectively,

$$\mathcal{Z}_\theta=e^{-\Gamma_{\text{odd}}(A)}\times e^{-\Gamma_{\text{even}}(A)}. \tag{6}$$

We know from [3–5] that, for this kind of configuration, Γ_{odd} can be exactly evaluated, and moreover that its periodicity may be assured if the parity anomaly is properly taken into account. As we have assumed that the ensemble corresponds to an *integer* charge q, periodicity of $\mathcal{Z}_\theta$ is required. We shall later on discuss the nonperiodic ‘‘gauge anomalous’’ $\mathcal{Z}_\theta$.

The result for Γ_{odd}, including the parity anomaly piece is [4,5]

$$\Gamma_{\text{odd}}(\theta,A)=i\frac{e}{2\pi}\frac{M}{|M|}\times\Phi\left\{\arctan\left[\tanh\left(\frac{\beta|M|}{2}\right)\tan\left(\frac{\theta}{2}\right)\right]-\frac{1}{2}\theta\right\}, \tag{7}$$

where $\Phi=\int d^2x\,\epsilon_{jk}\partial_jA_k$ is the static magnetic flux, and the branch of the arctan is chosen according to the value of θ. The even part of Γ cannot be found exactly, but fortunately there is a well-defined regime where its dependence on θ can be safely ignored. This is the case when $\beta|M|\gg1$, as can be checked explicitly in the calculation of [6], which yields the leading parity conserving contribution to Γ. For example, in a smooth gauge field configuration (though the same holds true without this assumption),

$$\Gamma_{\text{eveb}}(\theta,A_j)\simeq\Gamma^{(2)}(0,A_j)+\frac{e^2\beta}{48\pi M}\times\frac{\tanh(\beta M/2)}{\cos^2(e\beta\tilde{A}_3/2)+\tanh^2(\beta M/2)\sin^2(e\beta\tilde{A}_3/2)}\times\int d^2x\,F_{jk}F_{jk}, \tag{8}$$

where it becomes evident that dependence on $\tilde{A}_3$ (and hence on θ) is exponentially suppressed for large $\beta|M|$. A more complete analysis shows that it is not even necessary to have $\beta|M|\gg1$, but already for $\beta|M|$ of order 1 the dependence on $\tilde{A}_3$ can be ignored. Ignoring thus the θ dependence of Γ_{even},

$$\Gamma_{\text{even}}(\theta,A_j)\simeq\Gamma_{\text{even}}(0,A_j)=\Gamma(0,A_j), \tag{9}$$

where the last equality proceeds from the fact that there is no odd part for $\theta=0$. We can then take the even contribution out of the integral over θ, obtaining

$$\frac{\mathcal{Z}_q(A)}{\mathcal{Z}(A)}\simeq\int_{-\pi}^{\pi}\frac{d\theta}{2\pi}e^{-i\theta q-\Gamma_{\text{odd}}(\theta,A)}. \tag{10}$$

Note that in the last expression $\mathcal{Z}(A)\equiv\exp[-\Gamma(0,A_j)]$ is the partition function in the presence of a magnetic field in the *canonical* ensemble. This shows that the specific properties of the fixed charge ensemble when $\beta|M|$ is large are determined by Γ_{odd}. Equivalently, in terms of the respective free energies $F\equiv-(1/\beta)\ln Z$,

$$F_q-F\simeq-\frac{1}{\beta}\ln\left\{\int_{-\pi}^{\pi}\frac{d\theta}{2\pi}e^{-i\theta q-\Gamma_{\text{odd}}(\theta,A)}\right\}. \tag{11}$$

Now we can consider the behavior of Eq. (10) for different limits: When $\beta M\to\infty$, as the parity anomaly term cancels the induced term coming from the explicit parity-breaking mass M, so that Γ_{odd} tends to zero. This means that, when $\beta M\to\infty$,

$$F_q \simeq F \rightarrow -\frac{1}{\beta}\log[\delta_{q,0}]. \tag{12}$$

The meaning of this equation is clear, ensembles with nonzero charge are separated by an infinite free energy barrier, and only the zero charge one is physically possible.

When $\beta|M|$ is large but not necessarily zero, ensembles with $q\neq 0$ are possible, and we shall discuss them now. We first note that, due to Parseval's identity, as $\Gamma_{\rm odd}$ is purely imaginary, we have the sum rule

$$1=\sum_{n=-\infty}^{n=+\infty}\left|\frac{\mathcal{Z}_q(A)}{\mathcal{Z}(A)}\right|^2 \tag{13}$$

whose physical meaning in this case is that only a very few number of q's shall be accessible with a finite free energy.

We shall now derive a more convenient formula for Eq. (10) in terms of the dimensionless parameters of the theory. We define the dimensionless quantity $b\equiv(M/|M|)(e\Phi/2\pi)$, which essentially measures the magnetic flux in units of the elementary flux quantum $(e\Phi/2\pi)$. We then note that after some elementary algebra, Eq. (10) may be written as follows:

$$\frac{\mathcal{Z}_q(A)}{\mathcal{Z}(A)}=\int_{-\pi}^{\pi}\frac{d\theta}{2\pi}e^{-i\theta(q-b/2)}\left(\frac{1+e^{-2\beta|M|}e^{-i\theta}}{e^{-2\beta|M|}+e^{-i\theta}}\right)^{b/2}. \tag{14}$$

The change of integration variable $z=e^{-i\theta}$ maps the integration path to a unit circle in the complex plane:

$$\frac{\mathcal{Z}_q(A)}{\mathcal{Z}(A)}=\frac{i}{2\pi}\oint_C\frac{dz}{z}z^{q-b/2}\left(\frac{1+e^{2\beta|M|}z}{e^{2\beta|M|}+z}\right)^{b/2} \tag{15}$$

which, if b is even, say $b=2k$ for an integer k, can be evaluated as the sum of the residues over the two poles inside the unit circle. The result of this procedure may be put as

$$\frac{\mathcal{Z}_q(A)}{\mathcal{Z}(A)}=\frac{\Theta(q\leqslant k)}{(k-q)!}\lim_{z->0}\frac{d^{k-q}}{dz^{k-q}}\left[\frac{1+e^{2\beta|M|}z}{e^{2\beta|M|}+z}\right]^k +\frac{\Theta(k<0)}{(k-1)!}\lim_{z\rightarrow -e^{-2\beta|M|}}\frac{d^{k-1}}{dz^{k-1}}\times[z^{q-k-1}(1+e^{-2\beta|M|}z)^{-k}], \tag{16}$$

where the symbol $\Theta(\textit{inequality})$ is defined to be one if the inequality is true, and zero otherwise. This is not a closed form but may be exactly evaluated for any set of values for q, k, and βM. Note that when the sign of the magnetic flux is the same as the one the mass, k becomes positive, and so the second term in Eq. (16) vanishes:

$$\left[\frac{\mathcal{Z}_q(A)}{\mathcal{Z}(A)}\right]_{k>0}=\frac{\Theta(q\leqslant k)}{(k-q)!}\lim_{z->0}\frac{d^{k-q}}{dz^{k-q}}\left[\frac{1+e^{2\beta|M|}z}{e^{2\beta|M|}+z}\right]^k. \tag{17}$$

From a numerical evaluation of this expression, we see that finite temperature effects strongly affect the properties of the free energy. In particular, for $\beta|M|$ of order 1, the maximum of the ratio $\mathcal{Z}_q(A)/\mathcal{Z}(A)$ is reached when q is equal to k. This means that, when the system is heated, the Chern-Simons term makes states with total charge proportional to the total flux more convenient energetically. The situation is qualitatively similar for an odd number of fluxes, though we could only check that numerically.

We shall now discuss the issue of the meaning of the fixed-charge ensembles in the ''anomalous'' case, namely, when the effective action is not invariant under large gauge transformations. Invariance under large gauge transformations is, in our case, tantamount to periodicity in θ. Coming back to the definition of the fixed-charge partition function (2), we may say that the effect of the induced Chern-Simons term, in the anomalous case, is equivalent to having states of *fractional* charge. And indeed, a trivial way of recovering a fixed-charge ensemble for this case also would be to fix the total charge to a fractional value. An equivalent way of saying this is that, if the parity anomaly term is lacking, the effective action is no longer 2π periodic, but has a period of 4π, what can be attached to a redefinition of the charge operator.

It is important to realize that, had we used the perturbative result [9–19] for the induced Chern-Simons term, no structure such as the ones we are seeing here would arise. Indeed, the very problem of defining the fixed charge ensemble would be ill defined, since for the perturbative Chern-Simons term the periodicity in $\tilde{A}_3$ is lost, and cannot be rescued by a simple interpretation in terms of a fractional charge. If the perturbative result is used there is no periodicity whatsoever.

We shall here extend the previous discussion to the non-Abelian case. It seems that we should now deal with a large number of fixed charges. However, one should remember [7] that in statistical mechanics not all the charges can be fixed but only a subset of them that commutes with the Hamiltonian and with all the other charges.[1] Thus, when considering the partition function for fermions in a non-Abelian magnetic background, θ will have to be a matrix commuting with the spatial components A_j of the non-Abelian gauge field. By identifying again θ with A_3, this is precisely the kind of configuration that has been considered in [5]. Obviously, the number of different integrations will depend on the group. For example, for SU(2) there will only be one such θ, and we have the analogous of Eq. (10), the only change being a different expression for $\Gamma_{\rm odd}$. In the general case, $\vec{\theta}\equiv(\theta^a)$ will have a number f of components in internal space corresponding to the ''directions'' of the fixed charges. Obviously the maximum allowed value for f shall depend on the group, for example, $f=1$ for the group SU(2). Denoting by $\vec{q}$ the values of such charges, the corresponding partition function is, in the same approximation we used for the Abelian case,

$$\frac{\mathcal{Z}_{\vec{q}}(A)}{\mathcal{Z}(A)}\simeq\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\frac{d\vec{\theta}}{(2\pi)^f}e^{-i\vec{\theta}\cdot\vec{q}-\Gamma_{\rm odd}(\vec{\theta},A)}, \tag{18}$$

where

[1]This also happens in the grand canonical ensemble, where only such a subset of charges may carry chemical potentials.

$$\Gamma_{\rm odd}=\frac{ig}{4\pi}\,{\rm tr}\left\{\arctan\left[\tanh\left(\frac{\beta M}{2}\right)\tan(\theta)\right]\int d^2x\,\varepsilon_{ij}F_{ij}\right\}, \tag{19}$$

where $\theta\equiv\theta^a\tau^a$, and we are using the same conventions as in [5].

We conclude by saying that the use of the nonperturbative parity-breaking term in the effective action is essential for the definition of fixed-charge ensembles in $2+1$ dimensions. Even if one is going to assume that the fermions are coupled to a dynamical gauge field, the constant θ will appear coupled to the fermionic current together with the third component of the gauge field, and again θ cannot be assumed to be small since periodicity (and the interpretation as a fixed-charge ensemble) would be lost.

The author acknowledges G. L. Rossini and F. A. Schaposnik for reading this manuscript.

[1] E. Fradkin, *Field Theories of Condensed Matter Systems*, Frontiers in Physics Vol. 82 (Addison-Wesley, New York, 1991).
[2] S. Deser, R. Jackiw, and S. Templeton, Phys. Rev. Lett. **48**, 975 (1982); Ann. Phys. (N.Y.) **140**, 372 (1982).
[3] S. Deser, L. Griguolo, and D. Seminara, Phys. Rev. Lett. **79**, 1976 (1997).
[4] C. D. Fosco, G. L. Rossini, and F. A. Schaposnik, Phys. Rev. Lett. **79**, 1980 (1997); **79**, 4296(E) (1997).
[5] C. D. Fosco, G. L. Rossini, and F. A. Schaposnik, Phys. Rev. D **56**, 6547 (1997).
[6] I. J. R. Aitchison and C. D. Fosco, Phys. Rev. D **57**, 1171 (1998).
[7] J. Kapusta, *Finite-Temperature Field Theory* (Cambridge University Press, Cambridge, England, 1993), Appendix A.3, and references therein.
[8] K. Redlich and L. Turko, Z. Phys. C **5**, 201 (1980).
[9] J. Kapusta, Nucl. Phys. **B196**, 1 (1982).
[10] A. J. Niemi and G. W. Semenoff, Phys. Rev. Lett. **51**, 2077 (1983).
[11] A. J. Niemi, Nucl. Phys. **B251**, 55 (1985).
[12] A. J. Niemi and G. W. Semenoff, Phys. Rep. **135**, 99 (1986).
[13] K. Babu, A. Das, and P. Panigrahi, Phys. Rev. D **36**, 3725 (1987).
[14] A. Das and S. Panda, J. Phys. A **25**, L245 (1992).
[15] I. J. R. Aitchinson, C. D. Fosco, and J. Zuk, Phys. Rev. D **48**, 5895 (1993).
[16] E. R. Poppitz, Phys. Lett. B **252**, 417 (1990).
[17] M. Burgess, Phys. Rev. D **44**, 2552 (1991).
[18] W. T. Kim, Y. J. Park, K. Y. Kim, and Y. Kim, Phys. Rev. D **46**, 3674 (1992).
[19] K. Ishikawa and T. Matsuyama, Nucl. Phys. **B280**, 523 (1987).