Inhomogeneous cosmological models in $D=6$, $N=2$ Kaluza-Klein supergravity

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(Received 31 October 1997; published 16 April 1998)

We obtain a cosmological solution in an $R^1\times R^3\times S^2$ spacetime for an inhomogeneous distribution of matter obeying an equation of state, $p=-\rho\neq p_1$, where p and p_1 are the isotropic pressures in the 3-space and extra space, respectively. Our model admits exponential expansion of the three-dimensional (3D) space, while the extra space is amenable to dimensional reduction. Interestingly, aside from the well known singularity at the big bang our inhomogeneous solutions are spatially regular everywhere, including the center of symmetry $r=0$. Moreover, our model seems to suggest an alternative mechanism pointing to a smooth transition from a primorial multidimensional, inhomogeneous phase to a 4D homogeneous one. [S0556-2821(98)02210-3]

PACS number(s): 98.80.Hw, 04.50.+h

Field theories in more than four spacetime dimensions are considered as models for the unification of all interactions. In these $(4+D)$-dimensional models D spacelike dimensions are spontaneously compactified. The symmetries of this space appear as gauge symmetries of the effective four-dimensional (4D) theory. Internal spaces admitting the gauge group SU(3)×U(2)×U(1) or grand unified groups have been proposed [1,2]. Even though a fully realistic model still remains to be found, the idea is elegant enough to warrant serious investigation of some of its implications.

In usual Kaluza-Klein supergravity theories [3] we encounter a large 4D cosmological constant Λ_4 as a result of compactification of the extra dimensions. However, this large value of the cosmological constant does not at all fit in with our present universe because the observational upper limit on Λ_4 is less than 10^{-120} m^2 pl. On the other hand to circumvent this problem we have to take recourse to an elaborate fine-tuning of Λ_D to get rid of Λ_4. This fine-tuning is too artificial and unnatural as we have no fundamental principle to choose the specific value of Λ_D.

In this context, the $D=6$, $N=2$ Kaluza-Klein (KK) supergravity theory [4–6], where it is possible that six dimensions compactify into a 4D Minkowski spacetime and a 2D sphere, $S(2)$ as a product space, deserves serious attention. It is encouraging to point out that in the supergravity theory referred to in [4–6] one is not constrained to fine-tune the physical parameters like a coupling constant to obtain the Minkowski spacetime. This theory is, therefore, a good candidate for a realistic KK cosmology.

In an earlier work Maeda and Nishino [7] obtained two families of cosmological solutions in this $N=2$, $D=6$ supergravity theory in a homogeneous background which includes both the vacuum and radiation dominated cases and showed that in the second case the solutions approach the Friedmann universe asymptotically. In this Brief Report we have taken the same form of the line-element in 6D such that it compactifies into a four-dimensional conformally flat spacetime X and a two dimensional sphere S^2. The line element is given by

$$ds^2=dt^2-B^2(dr^2+r^2dq^2+r^2\sin^2q\,df^2)-C^2(d\psi^2+\sin^2\psi\,d\zeta^2). \tag{1}$$

However, our metric differs essentially from the Maeda-Nishino case in that although $B=B(t)$ here the metric component C depends both on time and the radial component r. So here we assume that the physical 3-space is flat and homogeneous while the inhomogeneity is introduced through the extra space. This, however, makes the total 6D spacetime an inhomogeneous one. Furthermore, note that since the extra space depends on a radial coordinate also it is evident that we are not dealing here with a simple product space; the shape of the internal space is different at different points of the 4D world. We shall see subsequently that the fact that inhomogeneity is being introduced via the extra space has far reaching implications in the cosmological evolution of our model.

Homogeneous KK extension of the Friedmann-Robertson-Walker (FRW) model has been fairly adequately discussed in the literature by a host of authors [8,9]. Starting from a topology of $R^1\times R^3\times S^d$ it is shown that both the ''standard'' and extra space expand initially after which curvature effects become significant and the extra compact space collapses to a singularity. It is conjectured that some sort of quantum gravity effect stabilizes the compact space at the Planckian length and thereafter the visible universe expands in the usual FRW way.

However, inhomogeneous cosmological models in higher dimensions have not, so far, attracted the attention they deserve. The recent analysis made by de Lapparent *et al.* [10] of the CFA redshift survey and also the observations by Saunders *et al.* [11] of the Infrared Astronomy Satellite (RAS) survey indicate that the large scale structure of the universe does not show itself as a smooth and homogeneous distribution of matter as was thought earlier. At the same time the failure of the theoretical considerations such as statistical fluctuations in the FRW models to explain the large scale structure suggests that the inhomogeneity factor in any

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cosmological model can no longer be avoided. Motivated by these considerations some of us have attempted to study inhomogeneous models and their implications in a series of papers in 5D spacetime [12–15]. The present work is a natural generalization of some of these ideas in 6D spacetime where we have obtained an exponential inflation of 3D scale with corresponding dimensional reduction of the extra space, assuming a specific form of the equation of state.

Let us now work out the energy-momentum tensor in our model. Unlike the Maeda-Nishino (MN) case [7] where an additional scalar field is introduced to trigger the compactification, we have here only classical matter field. Three possibilities present themselves: (i) a vacuum state, (ii) a low temperature state, i.e., $T<1/C$ where evidently $1/C$ gives the curvature scale of S^2, and (iii) a high temperature state, i.e., $T>1/C$. In our present work we take the case (iii) where the energy of the particles is higher than the excitation energy of the internal space such that the higher excitations in the internal space can no longer be neglected. Therefore, the higher dimensional stress $p_1 \neq 0$. Furthermore, the isometries of our metric (1) tell us that the stress tensor $T_{\alpha\beta}$ in comoving coordinates should be of the form

$$T^0_0=\rho, \quad T^1_1=T^2_2=T^3_3=-p, \quad T^4_4=T^5_5=-p_1. \tag{2}$$

From Einstein's field equations $G_{\mu\nu}=-T_{\mu\nu}$ we get the following set of independent equations for the line element (1):

$$G^0_0=\frac{2C''}{B^2C}+\frac{C'^2}{B^2C^2}+\frac{4C'}{B^2Cr}-\frac{3\dot{B}^2}{B^2}-\frac{6\dot{B}\dot{C}}{BC}-\frac{\dot{C}^2}{C^2}-\frac{1}{C^2}=-\rho, \tag{3}$$

$$G^1_1=\frac{4C'}{B^2Cr}+\frac{C'^2}{B^2C^2}-\frac{1}{C^2}-\frac{2\ddot{B}}{B}-\frac{\dot{B}^2}{B^2}-\frac{4\dot{B}\dot{C}}{BC}-\frac{2\ddot{C}}{C}-\frac{\dot{C}^2}{C^2}=p, \tag{4}$$

$$G^2_2=G^3_3=\frac{2C''}{B^2C}+\frac{2C'}{B^2Cr}+\frac{C'^2}{B^2C^2}-\frac{1}{C^2}-\frac{2\ddot{B}}{B}-\frac{\dot{B}^2}{B^2}-\frac{4\dot{B}\dot{C}}{BC}-\frac{2\ddot{C}}{C}-\frac{\dot{C}^2}{C^2}=p, \tag{5}$$

$$G^4_4=G^5_5=-3\,\frac{\ddot{B}}{B}-3\,\frac{\dot{B}^2}{B^2}-3\,\frac{\dot{B}\dot{C}}{BC}-\frac{\ddot{C}}{C}+\frac{2C'}{B^2Cr}+\frac{C''}{B^2C}=p_1, \tag{6}$$

$$G_{10}=2\,\frac{\dot{C}'}{C}-2\,\frac{\dot{B}C'}{BC}=0, \tag{7}$$

where a prime denotes the derivative with respect to "r" and a dot with respect to time "t."

For economy of space we shall skip the details of the mathematical steps and give the final results only. Utilizing the property of isotropy of 3D pressure we get, from Eqs. (4) and (5),

$$C=A(t)r^2/2+\alpha(t), \tag{8}$$

where A and α are arbitrary functions of integration. This result, when compared with Eq. (7), finally yields

$$C=KB(t)r^2/2+\alpha(t). \tag{9}$$

At this stage, let us assume an equation of state $\rho=-p$ which, when used in Eqs. (3) and (4), gives both

$$B=e^{at}, \tag{10}$$

where a is an integration constant and also an equation involving α as

$$B\ddot{\alpha}-\dot{B}\dot{\alpha}-\frac{\dot{B}^2}{B}\,\alpha+\ddot{B}\alpha+K=0. \tag{11}$$

Solving for α we finally get, via Eq. (9),

$$C=e^{at}(\gamma-Kr^2)+\frac{K}{a^2}\,e^{-at}+\delta. \tag{12}$$

Of the two arbitrary constants γ and δ, the latter is set equal to zero without any loss of generality.

Using these results we further get

$$\rho=-p=10a^2+\frac{1+3K^2r^2-4Km}{C^2}, \tag{13}$$

$$p_1=\frac{6Ke^{-at}}{C}-10a^2. \tag{14}$$

It has not also escaped our notice that here ρ and p vary with t and r. This is strikingly different from the analogous 4D case where both p and ρ are separately constant in the inflationary era. The fact that ρ is not a constant also follows from the time component of the Bianchi identity

$$\dot{\rho}+\frac{3\dot{B}}{B}\,(\rho+p)+\frac{2\dot{C}}{C}\,(\rho+p_1)=0, \tag{15}$$

when the equation of state $\rho+p=0$ and $p_1\neq 0$ is taken.

We can, at this stage, calculate the Kretschmann scalar, $R_{ijkl}R^{ijkl}$ for our metric. Explicit calculations (also checked and verified with the help of a computer) show that the only surviving components of the Riemannian tensors are R_{0441}, R_{0551}, and R_{5454} such that

$$R_{ijk}R^{ijkl}=R_{0441}R^{0441}+R_{0551}R^{0551}+R_{5454}R^{5454}=\frac{2}{B^2C^2}\,(C^{11})^2+\frac{1}{C^4B^4}\,[B^2+B^2(\dot{C}')^2(C')^2]. \tag{16}$$

It is encouraging to point out that when we use the value of B and C for our metric (1) we find that the scalar is regular everywhere including the point $r=0$, which may be called the center of symmetry as in many inhomogeneous distributions.

Furthermore, if we choose K to be negative, then as time evolves, C approaches zero at some finite time $t=t_0$, where

$$t_0=\frac{1}{2a}\,\ln\!\left(\frac{1}{a^2}\,\frac{|K|}{\gamma+|K|r^2}\right). \tag{17}$$

It must be emphasized that this dimensional reduction is *local* contrary to what is observed in homogeneous models because here t_0 is a function of r also. However, this particular feature is generic to all inhomogeneous models. When $K>0$, the extra space starts from and evolves to infinity after bouncing off a minimum (not equal to zero). In both the cases the usual 3-space expands exponentially giving the well known inflationary scenario, analogous to that due to vacuum energy in the 4D case (see Shafi and Wetterich [16], Dereli and Tucker [17], and also [14] for the homogeneous model).

It may not be out of place to point out an essential difference from the analogous models in 5D referred to earlier [13]. One can calculate the 5-space curvature of the t-constant hypersurface for the line-element (1) from the expression [13]

$$R_i^i = R^{*(5)} + \dot{\theta} + \theta^2 - 2\omega^2 + u^i;i,$$

where θ is the expansion scalar and the last two terms give vorticity and acceleration and $i=1-5$. After a long but straightforward calculation we get

$$R^{*(5)} = \frac{2}{C^2}\,(1-K^2r^2) + \frac{24K}{BC}. \tag{18}$$

When $K=0$, our model becomes homogeneous but the 5D curvature does not vanish. This is in contrast with our earlier works on 5D cosmology where homogeneity ($K=0$) necessarily implies zero spatial curvature.

Before concluding a final remark may be in order. As mentioned earlier, it is conjectured that during dimensional reduction the extra dimensions finally stabilize at a very small length and then lose their dynamical character. Thereafter the cosmology enters the 4D phase without having any reference to the extra dimensions. For our model this transition has far reaching implications because the very existence of the extra space, so to speak, seems to induce inhomogeneity in our case. So not only do we enter a 4D era, we also envisage a smooth transition from a multidimensional, inhomogeneous model to a 4D homogeneous one. Interestingly this desirable transition takes place without forcing us to choose very special initial conditions as is the practice in conventional four-dimensional models. This, in our opinion, is a very important feature of our model. So, in short, we here describe an inhomogeneous scenario where the 3D space expands exponentially while the extra space shrinks with time indefinitely. Furthermore, the promordial inhomogeneity dies down in a natural way as we enter the 4D world. Both these results are new and sufficiently interesting to warrant further investigations in this direction.

T.G. wishes to thank the DST, India for financial support and S.C. the University of Zululand, South Africa for local hospitality where part of the work was done. We also thank Professor R. Maharaj, University of Westville, South Africa for the computer-aided calculations for the field equations and relevant scalars.

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