Interacting charged particles in an electric field and the Unruh effect

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We compute the transition amplitudes between charged particles of mass M and m accelerated by a constant electric field and interacting by the exchange of quanta of a third field. We work in second quantization in order to take into account both recoil effects induced by transitions and the vacuum instability of the charged fields. In spite of both effects, when the exchanged particle is neutral, the equilibrium ratio of the populations is simply $\exp[\pi(M^2-m^2)/eE]$. Thus, in the limit $(M-m)/M\rightarrow 0$, one recovers Unruh's result characterized by the temperature $a/2\pi$ where a is the acceleration. When the exchanged particle is charged, its vacuum instability prevents a simple description of the equilibrium state. However, in the limit wherein the charge of the exchanged particle tends to zero, the equilibrium distribution is once more Boltzmannian, but characterized not only by a temperature but also by the electric potential felt by the exchanged particle. This work therefore confirms that thermodynamics in the presence of horizons does not rely on a semiclassical treatment. The relationship with horizon thermodynamics and the role of the horizon area as an entropy are stressed. [S0556-2821(98)03210-X]

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I. INTRODUCTION

Shortly after Hawking's seminal discovery of black hole radiation [1], Unruh [2] showed that it possesses a flat space analogue, namely, that a uniformly accelerated detector perceives the Minkowski vacuum to be thermally populated at a temperature $T_U=a/2\pi$. In Unruh's original work, only the detector's internal states were treated quantum mechanically. Its position was treated classically and thus was insensitive to the transitions occurring between its internal states. This is an approximation that violates momentum conservation: the transitions are accompanied by the emission of a radiation quantum, but the energy and momentum transfer due to this emission is neglected since the detector's trajectory is fixed once and for all. In order to enforce momentum conservation, one must quantize the detector's position.

This enlargement of the quantum dynamics allows one to answer questions concerning the origin of the energy emitted during transitions and the consequences of the recoil effects [3,4]. Moreover, it provides new insight into the Unruh process and connects it with the Schwinger process [5,6] and horizon thermodynamics [7,8]. Thus, it may serve as a guide for other problems dealing with particle creation in the presence of horizons. Indeed, in all cases, background field approximation schemes have been used and should be abandoned in other to address the question of the quantum back reaction.

In Refs. [5,3,6], the enlargement of the dynamics has been carried out by modeling the detector by a ''two-level ion'' propagating in a constant electric field E. The ion has charge Q and its two levels have rest mass M and m. It therefore uniformly accelerates with acceleration $a_M=QE/M$ or $a_m=QE/m$ according to its mass. The ion can make transitions between its two levels by emitting or absorbing a quantum of a massless chargeless field Φ. Thus it behaves like an accelerated particle detector with mass gap $\Delta M=M-m$. Moreover, the transitions now satisfy Feynman rules, as in QED; see [9,10]. The main new insights concerning the Unruh effect which have been obtained in this way are the following.

(1) The detector can be described by a delocalized wave function, whereupon the classical geometric notion of a horizon no longer exists. (Of course, one may approximately recover the concept of a horizon by building well-localized wave packets.) Nevertheless, thermal rates for transitions of the detector still obtain, thereby confirming that thermodynamical relations still govern the physics when one goes beyond the semiclassical treatment.

(2) Each time the detector makes a transition, it recoils both in momentum and in energy in such a way that the total instantaneous Minkowski momentum and energy are conserved. From a space time point of view, i.e., if one builds wave packets, the transition induces a kink in the detector's trajectory, which accounts for its change in momentum and

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kinetic energy. This is explained in more detail in [3], pp. 245–246.[1]

(3) These recoils give rise to a decoherence of the detector-radiation system. This in turn implies that the detector emits a steady flux of radiation, contrary to the situation where the detector's position is treated classically [11,12,13,14].

(4) One of the most interesting consequences of this approach is that it relates the Unruh effect to the Schwinger process [15,16] of pair creation in an electric field. Indeed the two processes are ''in equilibrium'' [5].

(5) Finally both the Unruh effect and the Schwinger process are deeply related to horizon thermodynamics, since the area of the acceleration horizon [7] plays the role of an entropy in delivering the equilibrium population ratios [8].

The aim of the present work is to extend these results by taking the field Φ with which the detector interacts to be massive and charged. Its mass will be denoted μ and its charge α. The fields of the detector of mass M and m then have charge Q and q with $Q=q+\alpha$ to ensure charge conservation. We shall show that taking Φ to be massive does not modify any of the above points, but taking it to be charged, and hence accelerated, does modify the equilibrium properties.

In order to present these new properties, we first explain in more detail the physical content of points (4) and (5). When describing the charged fields by operators, i.e., by working in a second quantization, the electric field leads to a vacuum instability through pair creation of ions and anti-ions. The mean numbers of created pairs are

$$N_M=\frac{QEVT}{2\pi}\, e^{-\pi M^2/QE}, \quad N_m=\frac{qEVT}{2\pi}\, e^{-\pi m^2/qE}, \quad (1)$$

and similarly for the Φ field. The prefactor $Q(q)EVT/2\pi$ accounts for the number of quantum cells, i.e., orthogonal states, that are subject to pair creation when the electric field is turned on during a lapse T in a box of length L. These expressions are valid for scalar fields in $1+1$ dimensions when T and L $(=V)$ are much bigger than $1/a_M$; see [17] for more details. This prefactor will play no role in what follows since equilibrium distributions are governed by ratios which are dominated by the exponentials. We therefore introduce the probabilities per unit time and unit length of creating a particle of mass M:

$$P_M=\frac{N_M}{VT}=\frac{QE}{2\pi}\, e^{-\pi M^2/QE}, \quad (2)$$

and similarly for the other fields.

A priori independently of these creation processes, an ion will make transitions from its excited to its ground state at a rate $R_{M\to m}$ or from its ground to its excited state at a rate $R_{m\to M}$. But as suggested in [5], these two processes are intimately related. Indeed, when Φ is neutral $(Q=q,\ \alpha=0)$, the ratio of their rates is given by [6,18]

$$\frac{R_{m\to M}}{R_{M\to m}}=\frac{P_M}{P_m}=e^{-\pi(M^2-m^2)/QE}. \quad (3)$$

This motivates our saying that the two processes are in equilibrium since they determine the same distribution P_M/P_m of particles of mass M and m. We emphasize that Eq. (3) is exact in the sense that it takes into account all effects due to the finite mass of the detector, i.e., recoil effects, and the finite probability to create pairs of detectors.

Upon taking the limit $M,m\to\infty$, with $\bar{a}=2QE/(M+m)$ and ΔM constant, both recoil effects and pair creation amplitudes vanish. Therefore, one expects to recover Unruh's result which gives the equilibrium probabilities of an accelerated detector of given acceleration $\bar{a}$. Indeed, upon taking the above limit, Eq. (3) becomes

$$\frac{R_{m\to M}}{R_{M\to m}}=\frac{P_M}{P_m}=e^{-\pi\Delta M(M+m)/QE}=e^{-2\pi\Delta M/\bar{a}}. \quad (4)$$

In this paper, we shall derive the modified equations that replace Eqs. (3),(4) when the exchanged Φ field is charged. The new transition rates of the detector satisfy

$$\frac{R_{m\to M}}{R_{M\to m}}=e^{-\pi(M^2/QE-m^2/qE)}+O(e^{-\pi\mu^2/\alpha E}). \quad (5)$$

Using Eq. (2), we can reexpress this as

$$\frac{R_{m\to M}}{R_{M\to m}}=\frac{q}{Q}\,\frac{P_M}{P_m}+O(e^{-\pi\mu^2/\alpha E}). \quad (6)$$

Thus, in the limit wherein we can neglect the spontaneous creation of Φ quanta (i.e., $e^{-\pi\mu^2/\alpha E}\to 0$), we recover the equilibrium between the Schwinger and Unruh effects, up to the prefactor Q/q. This is the main result of our paper. We now display what we can learn from it.

The semiclassical limit is obtained by generalizing what lead to Eq. (4), i.e., by taking $M,m\to\infty$, $(Q-q)/Q=\alpha/Q\to 0$, $\mu^2/E\alpha\to\infty$, with ΔM and $\bar{a}=(Q+q)E/(M+m)$ constant. In this limit Eq. (5) becomes

$$\frac{R_{m\to M}}{R_{M\to m}}=e^{(-2\pi/\bar{a})(\Delta M-\alpha E/2\bar{a})+O(\alpha^2/Q^2)}+O(e^{-\pi\mu^2/\alpha E}). \quad (7)$$

This is again a thermodynamical relation. It shows that, in addition to the Unruh temperature $T_U=\bar{a}/2\pi$, there is now an electric potential $(=E/2\bar{a})$ which modifies the equilibrium.

This is strictly analogous to the ratio of the rates for charged particles to be emitted or absorbed by a charged

[1]Note that in this model, by treating the electric field classically, one also makes an approximation that neglects certain recoil effects. However, these effects are governed by the rest mass of the condenser plates that generate the electric field and not by the rest mass of the Unruh detector. It is therefore a much more legitimate approximation to neglect condenser recoils only. Treating the electric field classically also neglects self-interactions of the particles. These effects are proportional to Q^2, the square of the charge of the fields. On the other hand, the coupling to the external electric field is proportional to $QQ_{condenser}$ where $Q_{condenser}$ is the charge on the condenser plates. Hence the self-interactions can be neglected if $Q/Q_{condenser}\ll 1$.

black hole. In Hawking's derivation, this ratio is expressed in terms of the Bogoljubov coefficients $\gamma_\omega,\beta_\omega$ characterizing the mixing of *in* and *out* modes of the Φ field. However, to make clearer the contact with Eq. (7), we can express Hawking's result in terms of the rate $\mathcal{R}_{M\to M-\omega}$ to jump from a black hole of mass M and charge Q to a black hole characterized by $M-\omega$ and $Q-\alpha$ and the rate of the inverse process $\mathcal{R}_{M-\omega\to M}$. Then, Hawking's result can be expressed as

$$\left|\frac{\beta_\omega}{\gamma_\omega}\right|^2=e^{-\beta_H(\omega-\alpha\phi)}=\frac{\mathcal{R}_{M\to M-\omega}}{\mathcal{R}_{M-\omega\to M}}, \tag{8}$$

where β_H is the inverse Hawking temperature, ω is the energy of the quantum measured at spatial infinity, and ϕ is the difference of electric potential between the horizon and infinity. In Eq. (7), the equivalent of ϕ is $E/2\bar{a}$, the difference of the electric potential between the horizon and the accelerated trajectory where the charged quantum is emitted (absorbed).

The occurrence of this electromagnetic potential can be derived in a more direct way by quantizing the field Φ in Rindler coordinates and calculating the Bogoljubov transformation that relates the Rindler modes to the Unruh modes. This semiclassical approach is identical to the one used by Unruh [2] and Hawking in their seminal calculations. We hope to report on it in a future publication [19].

The main point of this discussion is that in contradistinction to the semiclassical treatment which yields directly and only to Eqs. (4),(7), we are now able to show how these semiclassical equilibrium ratios arise from Eqs. (3),(5) by taking, *a posteriori*, variations limited to first order in α and ΔM. Therefore, we can analyze the *finite* differences and not only the first order changes delivering the above canonical concepts through differentiation. This will play a crucial role in what follows.

The interest of relating the Unruh process to the Schwinger process as in Eq. (3) is further enhanced when one recalls that the rate of pair creation due to the Schwinger process can be expressed in terms of the change of the area of the acceleration horizon [7,8]:

$$P_M \simeq e^{-\Delta A_H(M,Q)/4}. \tag{9}$$

The quantity that appears in the exponential $\Delta A_H(M,Q) = A_H^0 - A_H(M,Q)$ is the *finite difference* between the infinite area of the acceleration horizon in flat space, A_H^0, and the infinite area if a pair of particles of mass M and charge Q is emitted, $A_H(M,Q)$. This difference is calculated by enclosing the system in a fictitious box, whereupon the areas are finite, and then taking the limit as the size of the box tends to infinity; see [7].

The simplest way to derive Eq. (9) is to note that the Schwinger process possesses a Euclidean instanton, which is a circle in the Euclidean continuation of Rindler space time. The probability of pair creation is given by the action to go once round the circle:

$$P_M \simeq e^{-S_{inst}(M,Q)}, \quad S_{inst}(M,Q)=\pi M^2/QE. \tag{10}$$

One can also include the gravitational field of the instanton to obtain a self-consistent Euclidean solution of the Einstein matter equations. Using the Hamiltonian decomposition of the Einstein-Hilbert action, one finds that

$$S_{inst}(M,Q)=\Delta A_H/4. \tag{11}$$

This instanton approach only gives the leading exponent of the Schwinger process. In this case Eq. (3) can be rewritten as

$$\frac{R_{m\to M}}{R_{M\to m}}=e^{-\{\Delta A_H(M,Q)-\Delta A_H(m,Q)\}/4}=e^{-\Delta_H(\Delta M)/4}, \tag{12}$$

where in the last equality we have written $\Delta A_H(\Delta M) = A_H(m,Q) - A_H(M,Q)$ as the difference of the horizon area between the initial and final states.

When Φ is charged, if we neglect the second term in Eq. (6), the transition rates can also be related to changes of the accelerating area horizon:

$$\frac{R_{m\to M}}{R_{M\to m}}\simeq e^{-\{\Delta A_H(M,Q)-\Delta A_H(m,q)\}/4}=e^{-\Delta A_H(\Delta M,\alpha)/4}. \tag{13}$$

In the case of black hole evaporation, a similar rewriting of emission rates in terms of changes of the area of the horizon is a reexpression of the first law of black hole thermodynamics. In the present case, since Eq. (7) deals with the change in the accelerated horizon,

$$\frac{\Delta A_H(\Delta M,\alpha)}{4}\underset{lin.}{=}\frac{2\pi}{\bar{a}}\left(\Delta M-\alpha\frac{E}{2\bar{a}}\right) \tag{14}$$

should be considered as the (linearized) first law of accelerated-horizon thermodynamics.

However, contrary to the black hole case, we have obtained an expression for the rates containing finite differences [see Eq. (5)], and not only linearized (canonical) expressions. Upon still neglecting the vacuum instability of the exchanged Φ field, we find that the logarithm of the ratio of the transition rates is given by the following *finite* difference:

$$\frac{\Delta A_H}{4}=\pi\left(\frac{M^2}{QE}-\frac{m^2}{qE}\right). \tag{15}$$

This shows that a quarter of the area of the horizon not only delivers canonical distributions and thermodynamics but truly determines quantum processes as in statistical mechanics. One-quarter of the Rindler horizon area is therefore completely analogous to an entropy as far as as the Unruh process is concerned.

To conclude the Introduction, let us emphasize the generic character of the agreement of Eqs. (3),(5) with the background field expressions, Eqs. (4),(6) when a first order expansion in the light changes is performed. Equations (3),(5) are derived in an enlarged quantum setting wherein the trajectory of the heavy ion is quantum mechanically treated whereas Eqs. (4),(6) are based on a background field approximation in which the trajectory is classically determined. The agreement of the transition rates evaluated from both treatments arises when the following procedure is applied to transition amplitudes evaluated in the more quantum framework. Upon working with WKB waves and performing first order expansions in $\Delta M/M$ and in the momentum transfer, these amplitudes coincide with the corresponding amplitudes evaluated at the background field approximation; see [20] for details. What guarantees this agreement is that first order expansions of the WKB phases are controlled by Hamilton-Jacobi equations. The same relation will therefore hold when one considers gravity. (This has been explicitly verified in mini-superspace in [21].) To first order in the matter energy change, transition amplitudes computed in quantum gravity with WKB waves are equal to the corresponding amplitudes evaluated from quantum field theory in a given classical geometry. Only second order changes, i.e., the nonlinear response of gravity, involve the Planck mass.

The present article is organized as follows. Section I is this Introduction. Section II is devoted to recalling the quantization of a charged field in an electric field and the Schwinger process. In Sec. III we introduce our detector model, and consider the case when the detector interacts with a massive, but neutral, field. The techniques developed in this section are then used in Sec. IV to analyze the more complicated case of interactions with a charged field. Appendix A is devoted to the analytical evaluation of the transition amplitudes and Appendix B deals with the limit $\alpha\mapsto 0$ of the transition amplitudes.

II. CHARGED PARTICLES IN AN ELECTRIC FIELD

The aim of this section is to review the quantization of a massive charged scalar field in an external electric field E. For the reader interested in a more complete treatment we refer to [17] and references therein. Classically, the equation of motion of a relativistic charged particle of mass M and charge Q in an electric field is

$$M\,\frac{d^2x^\mu}{d\tau^2}=-QE\varepsilon^{\mu\nu}\,\frac{dx_\nu}{d\tau}. \tag{16}$$

Its trajectory is a hyperbola with parametric equations given by

$$t+\frac{k}{QE}=\frac{1}{A}\sinh A(s-s_0), \tag{17}$$

$$x-x_0=\frac{1}{A}\cosh A(s-s_0). \tag{18}$$

Here $A=QE/M$ is the classical acceleration of the particle. The time coordinate of the turning point of the hyperbola is $t_0=-k/QE$ where $k=(M\dot{x}+A_x)$ is the conserved momentum canonically conjugate to the variable x in the gauge $A_t=0$ and $A_x=-Et$.

The corresponding Klein-Gordon equation for the field is

$$[\Box+M^2]\psi_M(t,x)\equiv[(-i\partial_x+QEt)^2+\partial_t^2+M^2]\psi_M(t,x)=0. \tag{19}$$

The general solution of this equation can be written as a superposition of modes,

$$\psi_{M,k}(t,x)=\frac{e^{ikx}}{\sqrt{2\pi}}\,\chi_{M,k}(t), \tag{20}$$

and their complex conjugates. The functions $\chi_{M,k}(t)$ obey the equation

$$[\partial_t^2+M^2+(k+QEt)^2]\chi_{M,k}(t)=0. \tag{21}$$

Because of the t dependence of this potential, there will be some backscattering. To interpret this in the context of the Klein-Gordon equation, we note that the current operator is $J_0=-iQ\overleftrightarrow{\partial}_t$. Thus the backscattering corresponds to a mixing of positive and negative charges. In a second quantized context this corresponds to pair creation [5,17]. To describe them we introduce a set of solutions with only positive or negative charge for $t\to-\infty$ (*in* modes),

$$\begin{aligned}\psi^{p\,in}_{M,k}(t,x)&=\frac{e^{ikx}}{\sqrt{2\pi}}\,\frac{e^{(-\pi/4)\epsilon_M}}{(2QE)^{1/4}}\\&\quad\times D_{i\epsilon_M-1/2}\left[e^{3i\pi/4}\left(t+\frac{k}{QE}\right)\sqrt{2QE}\right]\\&=\frac{e^{ikx}}{\sqrt{2\pi}}\,\mathcal{D}_{\epsilon_M}[\lambda],\end{aligned} \tag{22}$$

$$\psi^{a\,in}_{M,-k}(t,x)=\frac{e^{-ikx}}{\sqrt{2\pi}}\,\mathcal{D}_{\epsilon_M}[\lambda], \tag{23}$$

and a set of solutions with only positive or negative charge for $t\to+\infty$ (*out* modes),

$$\begin{aligned}\psi^{p\,out}_{M,k}(t,x)&=[\psi^{p\,in}_{M,-k}(-t,x)]^*\\&=\frac{e^{ikx}}{\sqrt{2\pi}}\,\mathcal{D}^*_{\epsilon_M}[-\lambda],\end{aligned} \tag{24}$$

$$\begin{aligned}\psi^{a\,out}_{M,-k}(t,x)&=[\psi^{a\,in}_{M,-k}(-t,x)]^*\\&=\frac{e^{-ikx}}{\sqrt{2\pi}}\,\mathcal{D}^*_{\epsilon_M}[-\lambda],\end{aligned} \tag{25}$$

where $\epsilon_M=M^2/(2QE)$ and where the superscripts p and a refer, respectively, to particle and antiparticle wave functions. We have introduced a synthetic notation for the parabolic cylinder functions and their argument:

$$\begin{aligned}\mathcal{D}_{\epsilon_M}[\lambda]&=\frac{e^{(-\pi/4)\epsilon_M}}{(2QE)^{1/4}}\,D_{i\epsilon_M-1/2}\left[e^{3i\pi/4}\left(t+\frac{k}{QE}\right)\sqrt{2QE}\right],\\ \lambda&=\left(t+\frac{k}{QE}\right).\end{aligned} \tag{26}$$

We also note that the parabolic cylinder function has the following integral representation [17]:

$$\begin{aligned}\mathcal{D}_{\epsilon_M}[\lambda]&=\frac{e^{(-\pi/2)\epsilon_M}e^{-i\pi/8}}{(2QE)^{1/4}\Gamma\left(\frac{1}{2}-i\epsilon_M\right)}\,e^{(+i/2)\lambda^2QE}\\&\quad\times\int_0^\infty dv\,e^{-i\lambda\sqrt{2QE}v+iv^2/2}v^{-i\epsilon_M-1/2},\end{aligned} \tag{27}$$

where the integration parameter v is classically related to t and its conjugate momentum p_t by [22]

$$v=\sqrt{QE/2}\left[\frac{p_t}{QE}+\left(t+\frac{k_0}{QE}\right)\right]=\sqrt{\frac{m}{2A}}\,e^{As}. \tag{28}$$

Since the parabolic cylinder functions have the property

$$\mathcal{D}^*_{\epsilon_M}[\lambda]=\alpha_M\mathcal{D}_{\epsilon_M}[-\lambda]+\beta_M\mathcal{D}^*_{\epsilon_M}[-\lambda], \tag{29}$$

the *in* and *out* modes are related by the linear transformation

$$\psi^{p\ out}_{M,k}=\alpha_M\psi^{p\ in}_{M,k}+\beta_M(\psi^{a\ in}_{M,-k})^*,$$
$$\psi^{p\ in}_{M,k}=\alpha^*_M\psi^{p\ out}_{M,k}-\beta_M(\psi^{a\ out}_{M,-k})^*, \tag{30}$$

where

$$\alpha_M=\frac{\sqrt{2\pi}e^{-i\pi/4}e^{(-\pi/2)\epsilon_M}}{\Gamma\left(\frac{1}{2}+i\epsilon_M\right)}\quad\text{and}\quad\beta_M=ie^{-\pi\epsilon_M}. \tag{31}$$

These Bogoljubov coefficients are k independent, but mass and charge dependent. The second quantized field Ψ_M should be decomposed in either the *in* or the *out* bases,

$$\Psi_M(t,x)=\int dk[\psi^{p\{^{in}_{out}}_{M,k}(t,x)a^{\{^{in}_{out}}_M(k)$$
$$+(\psi^{a\{^{in}_{out}}_{M,k}(t,x))^*b^{\dagger\{^{in}_{out}}_M(k)], \tag{32}$$

to define the *in* and *out* operators. From Eq. (30) we obtain

$$a^{in}_k=\alpha_Ma^{out}_k-\beta_Mb^{out\dagger}_{-k},$$
$$b^{in}_k=\alpha_Mb^{out}_k-\beta_Ma^{out\dagger}_{-k}. \tag{33}$$

The Heisenberg state $|0,in\rangle$ contains no particles at early times; i.e., it is annihilated by the *in* destruction operators. At late times it contains pairs of particles, as expressed by the relation

$$|0,in\rangle_M=\mathcal{N}^{-1}_M\exp\left(\frac{\beta_M}{\alpha_M}\int a^{out\dagger}_kb^{out\dagger}_{-k}dk\right)|0,out\rangle_M, \tag{34}$$

where

$$\mathcal{N}^{-1}_M={}_M\langle 0,out|0,in\rangle_M \tag{35}$$

is a normalization factor. The mean number of created particles [per quantum cell; see Eq. (1)] is

$$\mathcal{P}_M={}_M\langle 0,in|b^{out\dagger}_{-k}b^{out}_{-k}|0,in\rangle_M=|\beta_M|^2=e^{-2\pi M^2/QE}. \tag{36}$$

III. PARTICLE INTERACTIONS: EMISSION OF A NEUTRAL PARTICLE

In this section we consider a uniformly accelerated detector interacting with a neutral scalar field Φ_μ of mass μ. We shall show that Eq. (3), i.e., the relation between Schwinger and radiative processes which was obtained in [6] for a massless field, still holds when the exchanged quanta are massive. Moreover, we shall see that all amplitudes linear in the coupling constant can be expressed in terms of a single amplitude describing the creation of a pair of charged quanta. The techniques developed in this section will be generalized in the next section where the field Φ_μ is both massive and charged.

The detector is described by two charged scalar fields Ψ_M and Ψ_m, with mass M and m and the same charge Q, propagating in the electric field E. As in [5,3,6], the interaction between the fields is supposed to be given by

$$H^{int}=g\int dx(\Psi^\dagger_M\Psi_m+\Psi_M\Psi^\dagger_m)\Phi_\mu. \tag{37}$$

A first amplitude of interest is the amplitude $\mathcal{A}$ of transition from an *in* M particle of momentum k into an *out* m particle of momentum k' and a μ particle of momentum k'', which we schematically write as $M(k)\to m(k')+\mu(k'')$. It corresponds to spontaneous deexcitation of the detector (since $M>m$). In the interaction representation, to first order in g, it is given by

$$\mathcal{A}(k|k',k'')=-ig\int dtdx\langle 0,out|a_\mu(k'')a^{out}_m(k')\Psi_M\Psi^\dagger_m\Phi_\mu a^{in\dagger}_M(k)|0,in\rangle. \tag{38}$$

The factorization of the vacuum states as a tensor product of three vacua,

$$|0,\{^{in}_{out}\rangle=|0,\{^{in}_{out}\rangle_M\otimes|0,\{^{in}_{out}\rangle_m\otimes|0\rangle_\mu, \tag{39}$$

leads to the expression (see [9,10])

$$\mathcal{A}(k|k',k'')=\frac{1}{\alpha_M\mathcal{N}_M}\frac{1}{\alpha_m\mathcal{N}_m}(-i)g\times\int dt\ dx\ \psi^{p\ out}_{Mk}\psi^{p\ in*}_{mk'}\phi^*_{\mu k''}, \tag{40}$$

where $\mathcal{N}_M$, $\mathcal{N}_m$ are the overlaps [see Eqs. (34),(35)].

At this point it is interesting to note that the quantity $\mathcal{A}(k|k',k'')$ describes several different processes in addition to Eq. (38). Indeed it is not difficult to verify using Eqs. (22)–(25) that the four transitions

$$M(k)\rightarrow m(k')+\mu(k''),$$
$$\overline{M}(-k)\rightarrow \overline{m}(-k')+\mu(-k''),$$
$$\overline{m}(k')+\mu(k'')\rightarrow \overline{M}(k),$$
$$m(-k')+\mu(-k'')\rightarrow M(-k), \tag{41}$$

where $\overline{M}$ and $\overline{m}$ denote antiparticles, all have the same amplitude $\mathcal{A}(k|k',k'')$. These different amplitudes are related by combinations of substituting particles for antiparticles, changing the sign of the momentum $k\rightarrow -k$, and permuting incoming and outgoing quanta. Their origin will be further explained in the next section. Similar properties will obtain for all the amplitudes we shall introduce. Thus the notation we use for $\mathcal{A}(k|k',k'')$, and which is also used for all the other amplitudes in this paper, is that the vertical bar separates incoming from outgoing quanta, and that $\pm k$ is the momentum of the Ψ_M quanta, $\pm k'$ of the Ψ_m quanta, and $\pm k''$ of the Φ_μ quanta.

Similarly, the amplitude $\mathcal{B}$ of spontaneous excitation of an m particle into an M particle accompanied by the emission of a μ particle is given by

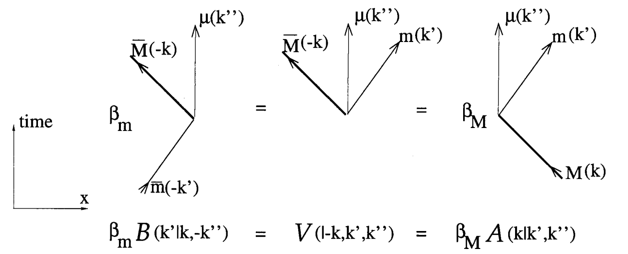

FIG. 1. A graphic representation of the relation between amplitudes $\mathcal{B},\mathcal{V},\mathcal{A}$ of Sec. III. Particles are denoted by $M(k), m(k'), \mu(k'')$ according to their mass and momentum, and antiparticles by $\overline{M},\overline{m}$ (there is no antiparticle for the Φ_μ field which is real). The momenta in each diagram are always conserved, $k=k'+k''$. The coefficients β_M,β_m which weight the different amplitudes are the amplitudes to produce a pair of $M,\overline{M}$ or $m,\overline{m}$ particles from vacuum. The picture is oriented both in time and space: particles (antiparticles) are incoming from, and outgoing to, the right (left) according to their acceleration in the electric field. The chargeless particles μ are represented as vertical lines because they are not accelerated.

$$\mathcal{B}(k'|k,-k'')=-ig\int dt dx\langle 0,out|a_\mu(-k'')a_M^{out}(k)\Psi_M^\dagger\Psi_m\Phi_\mu a_m^{in\dagger}(k')|0,in\rangle$$
$$=\frac{1}{\alpha_M\mathcal{N}_M}\frac{1}{\alpha_m\mathcal{N}_m}(-ig)\int dt\; dx\,\psi^{p\ in*}_{M,k}\,\psi^{p\ out}_{m,k'}\,\phi^*_{\mu,-k''}. \tag{42}$$

Because of the uniform acceleration, this spontaneous excitation amplitude is nonvanishing. As in the Unruh treatment, the ratio of the rates of spontaneous excitation to spontaneous deexcitation is simply given by

$$\frac{R_{m\rightarrow M}}{R_{M\rightarrow m}}=\frac{|\mathcal{B}|^2}{|\mathcal{A}|^2}, \tag{43}$$

since the norm of the amplitudes is independent of the momenta k and k'.

Thus, when the detector reaches equilibrium, the ratio of the probabilities to find the detector in its excited or ground state is

$$\frac{P_{M\ equil}}{P_{m\ equil}}=\frac{R_{m\rightarrow M}}{R_{M\rightarrow m}}=\frac{|\mathcal{B}|^2}{|\mathcal{A}|^2}. \tag{44}$$

Our task is to calculate this ratio and to confirm the relation with the Schwinger process, Eq. (36). To this end we introduce a third amplitude $\mathcal{V}$ corresponding to the creation from a vacuum of an *out* M antiparticle, an *out* m particle, and a μ particle. To first order in g, it is given by

$$\mathcal{V}(|-k,k',k'')=-ig\int dt dx\langle 0,out|a_\mu(k'')a_m^{out}(k')b_M^{out}(-k)\Psi_M\Psi_m\Phi_\mu^\dagger|0,in\rangle$$
$$=\frac{1}{\alpha_M\mathcal{N}_M}\frac{1}{\alpha_m\mathcal{N}_m}(-ig)\int dt dx\,\psi^{a\ in*}_{M,-k}\,\psi^{p\ in*}_{m,k'}\,\phi^*_{\mu,k''}. \tag{45}$$

Using the Bogoljubov transformation for the M-particle wave function, Eq. (30), we can reexpress $\mathcal{V}$ in terms of the $\mathcal{A}$ amplitude:

$$\mathcal{V}(|-k,k',k'')=\frac{-ig}{\alpha_M\mathcal{N}_M}\frac{1}{\alpha_m\mathcal{N}_m}\int dtdx[\alpha_M\psi^{a\ out*}_{M,-k}+\beta_M\psi^{p\ out}_{M,k}]\psi^{p\ in*}_{mk'}\phi^*_{\mu k''}=\beta_M\mathcal{A}(k|k',k'')+\alpha_M\mathcal{I}(k,k',k''). \quad (46)$$

Similarly, using the Bogoljubov transformation for the m particle we can reexpress $\mathcal{V}$ in terms of the $\mathcal{B}$ amplitude:

$$\mathcal{V}(|k,k',k'')=\beta_m\mathcal{B}(k'|k,-k'')+\alpha_m\mathcal{I}'(k,k',k''). \quad (47)$$

Using the identities (which constitute the main mathematical result of this section, and are proved hereafter)

$$\mathcal{I}=\mathcal{I}'=0, \quad (48)$$

one obtains

$$\beta_m\mathcal{B}(k'|k,-k'')=\mathcal{V}(|-k,k',k'')=\beta_M\mathcal{A}(k|k',k''), \quad (49)$$

relations that we have depicted schematically in Fig. 1. Inserting Eq. (49) into Eq. (43) yields the link between radiative and Schwinger processes discussed in the Introduction, Eq. (3):

$$\frac{P_{M\ equil}}{P_{m\ equil}}=\frac{N_M}{N_m}=e^{-\pi[(M^2-m^2)/QE]}. \quad (50)$$

It remains to establish Eq. (48). To this end, we write

$$\mathcal{I}(k,k',k'')=Cst\int dxdte^{ikx}e^{-ik'x}e^{-ik''x}\times\mathcal{D}_{\epsilon_M}[-\lambda]\mathcal{D}^*_{\epsilon_m}[\lambda']e^{i\omega''t}=2\pi\delta(k-k'-k'')\int dt\mathcal{D}_{\epsilon_M}[-\lambda]\times\mathcal{D}^*_{\epsilon_m}[\lambda']e^{i\omega''t}, \quad (51)$$

where we have introduced $\lambda=(t+k/QE)$, $\lambda'=(t+k'/QE)$. To perform the t integration we replace the Whittaker's functions $\mathcal{D}$ by their integral representations, Eq. (27), to obtain

$$\mathcal{I}(k,k',k'')=Cst'\,\delta(k-k'-k'')\int dte^{(i/2)QE[t+(k/Q)E]^2}e^{-(i/2)QE[t+(k'/Q)E]^2}e^{i\omega''t}\times\int_0^\infty dudv\,e^{+iu\sqrt{2QE}[t+(k/Q)E]+iu^2/2}u^{-i\epsilon_M-1/2}e^{+i[t+(k'/Q)E]\sqrt{2QE}v-iv^2/2}v^{i\epsilon_m-1/2}. \quad (52)$$

The quadratic phases in t cancel, and the t integral yields $\delta(u+v+(\omega''+k'')/\sqrt{2QE})$. The argument of the delta function never vanishes on the domain of integration of u and v since $\omega''+k''\neq 0$ for $\mu\neq 0$. Hence $\mathcal{I}=0$. A similar reasoning shows that $\mathcal{I}'=0$.

The interested reader will now find the calculation of the amplitudes themselves. By virtue of Eq. (49), we only need to calculate the amplitude $\mathcal{V}$. We start from Eq. (45) in which we reexpress $(\Psi^{a\ in}_{M,-k})^*$ as $[(\Psi^{a\ out}_{M,-k})^*+\beta_M(\Psi^{p\ in}_{M,k})]/\alpha^*_M$. The first term does not contribute since it is equal to $\mathcal{I}(k,k',k'')/|\alpha|^2$. The second term gives

$$\begin{aligned}\mathcal{V}(|-k,k',k'')&=Cst\int dxdte^{ikx}e^{-ik'x}e^{-ik''x}\mathcal{D}_{\epsilon_M}[\lambda]\mathcal{D}^*_{\epsilon_m}[\lambda']e^{i\omega''t}\\&=Cst'\,\delta(k-k'-k'')\int dte^{(i/2)QE[t+(k/Q)E]^2}e^{(-i/2)QE[t+(k'/Q)E]^2}e^{i\omega''t}\\&\quad\times\int_0^\infty dudv\,e^{-iu\sqrt{2QE}[t+(k/Q)E]+iu^2/2}u^{-i\epsilon_M-1/2}e^{i[t+(k'/Q)E]\sqrt{2QE}v-iv^2/2}v^{i\epsilon_m-1/2}\\&=Cst''\,\delta(k-k'-k'')\int_0^\infty dudv\,\delta\left(-u+v+\frac{\omega''+k''}{\sqrt{2QE}}\right)\\&\quad\times e^{i[(k'/Q)E]\sqrt{2QE}v-iv^2/2}v^{i\epsilon_m-1/2}u^{-i\epsilon_M-1/2}e^{-iu\sqrt{2QE}[(k/Q)E]+iu^2/2}\\&=Cst'''\,\delta(k-k'-k'')\int_0^\infty dv\,e^{iv(\omega-k'')/\sqrt{2QE}}v^{i\epsilon_m-1/2}\left(v+\frac{\omega+k''}{\sqrt{2E}}\right)^{-i\epsilon_M-1/2}.\end{aligned} \quad (53)$$

The last integral gives an integral representation of a Whittaker's function (see [23], formula 3.383.4). Reinstating the value of Cst''' yields

$$\mathcal{V}(|-k,k',k'')=g\,\frac{1}{\alpha_M\mathcal{N}_M\alpha_m\mathcal{N}_m}\,\delta(k-k'-k'')\,\frac{e^{(-3\pi/4)\epsilon_m}e^{(-3\pi/4)\epsilon_M}}{2QE\sqrt{2\omega}}\,\frac{1}{\mu}\,e^{(-i/2QE)(k+k')\omega}\times\left(\frac{\omega-k''}{\omega+k''}\right)^{(i/2)(\epsilon_M-\epsilon_m)}W_{(-i/2)(\epsilon_m+\epsilon_M),(i/2)(\epsilon_m-\epsilon_M)}\left[-\frac{i\mu^2}{2QE}\right].\tag{54}$$

It is interesting to note that the delta function $\delta(-u+v+(\omega''+k'')/\sqrt{2QE})$ has the interpretation of ensuring local energy conservation. Indeed, replacing u and v by their classical relation to t and p_t, Eq. (28), yields simply $-p_t+p_t'+\omega=0$ where p_t is the energy of the M particle, p_t' of the m particle, and ω of the μ particle.

We conclude this section by calculating the $\mu\to 0$ limit of the above amplitude. This will allow us to make contact with the expressions of [6], which were obtained using a different method. The evaluation of this limit needs some precautions. Indeed, for $k''<0$, the support of the delta function given by the integral over t in Eq. (52) rejoins the boundary of the domain of integration. In order to avoid ambiguities, we have to substitute in Eq. (45), according to the sign of k'', the decomposition of $(\Psi^{a\ in}_{M,-k})^*$ or $(\Psi^{p\ in}_{m,k'})^*$ in terms of *in* and *out* fields and take into account the infinitesimal imaginary part that the squared masses of the fields share. For small value of μ one finds

$$\mathcal{V}(|-k,k',k'')\underset{\mu\to 0}{=}g\,\frac{1}{\alpha_M\mathcal{N}_M\alpha_m\mathcal{N}_m}\,\delta(k-k'-k'')\,\frac{e^{(-\pi/2)(\epsilon_m-\epsilon_M)}}{2QE\sqrt{2\omega}}\,\frac{e^{3i\pi/4}}{\sqrt{2QE}}\,e^{(-i/2QE)(k+k')\omega}\times\Bigg\{\theta(k'')\left(\frac{2k''}{QE}\right)^{(i/2)(\epsilon_m-\epsilon_M)}\frac{1}{\Gamma\left(\frac{1}{2}+i\epsilon_M\right)\Gamma(1+i\epsilon_m-i\epsilon_M)}-\theta(-k'')e^{(\pi/2)(\epsilon_m-\epsilon_M)}\times\left(\frac{2k''}{QE}\right)^{(i/2)(\epsilon_M-\epsilon_m)}\frac{1}{\Gamma\left(\frac{1}{2}+i\epsilon_m\right)\Gamma(1-i\epsilon_m+i\epsilon_M)}\Bigg\}.\tag{55}$$

Note the surprising fact that in this limit, the amplitudes of decay of the same process but with the opposite momenta differ. However, this is peculiar to two dimensions. Indeed in higher dimensions, the mass μ contains the squared transversal momentum and so, even in the massless limit, vanishes only on a domain of zero measure in phase space.

IV. PARTICLE INTERACTIONS: EMISSION OF A CHARGED PARTICLE

In this section we shall suppose that the Φ_μ field is also charged. Thus there are three fields Ψ_M,Ψ_m,Φ_μ, with masses M,m,μ and charges Q,q,α, respectively. These three fields interact through the Hamiltonian

$$H^{int}=g\int dx(\Psi_M^\dagger\Psi_m\Phi_\mu+\Psi_M\Psi_m^\dagger\Phi_\mu^\dagger).\tag{56}$$

Charge conservation requires

$$Q=q+\alpha.\tag{57}$$

The amplitude $\mathcal{A}$ of transition from an *in* M particle into an *out* m particle and a *out* μ particle is given at first order in perturbation theory by

$$\begin{aligned}\mathcal{A}(k|k',k'')&=\left\langle 0,out\left|a_\mu^{out}(k'')a_m^{out}(k')(-i)\int dtH^{int}a_M^{in\dagger}(k)\right|0,in\right\rangle\\&=\frac{1}{\alpha_M\mathcal{N}_M}\,\frac{1}{\alpha_m\mathcal{N}_m}\,\frac{1}{\alpha_\mu\mathcal{N}_\mu}\,(-i)g\\&\quad\times\int dtdx\,\psi^{p\ out}_{Mk}\,\psi^{p\ in*}_{mk'}\,\phi^{p\ in*}_{\mu k''}\\&=C\,\delta(k-k'-k'')\mathbf{A}(k|k',k''),\end{aligned}\tag{58}$$

where

$$C=\frac{-ig}{\sqrt{2\pi}}\,\frac{1}{\alpha_M\mathcal{N}_M\alpha_m\mathcal{N}_m\alpha_\mu\mathcal{N}_\mu}\tag{59}$$

and we have introduced the reduced amplitude

$$\mathbf{A}(k|k',k'')=\int_{-\infty}^{+\infty}dt\mathcal{D}^*_{\epsilon_M}[-\lambda]\mathcal{D}^*_{\epsilon_m}[\lambda']\mathcal{D}^*_{\epsilon_\mu}[\lambda'']\tag{60}$$

and used the abbreviated notation for the parabolic cylinder functions introduced in Eq. (26). Note also the appearance of the factor $1/\alpha_\mu\mathcal{N}_\mu$ which arises due to the vacuum instability of the μ field.

As in the previous section, the processes

$$M(k)\to m(k')+\mu(k''),$$

$$\overline{M}(-k)\to\overline{m}(-k')+\overline{\mu}(-k''),$$

$$\overline{m}(k')+\overline{\mu}(k'')\to\overline{M}(k),$$

$$m(-k')+\mu(-k'')\to M(-k), \tag{61}$$

all have the same amplitude $\mathcal{A}(k|k',k'')$. Note that these differ slightly from Eq. (41) because one has to distinguish between μ and $\overline{\mu}$ since the field Φ_μ is charged.

Similarly the amplitude $\mathcal{B}$ for an m antiparticle to spontaneously excite into an M antiparticle and a μ particle is

$$\mathcal{B}(k'|k,-k'')=-i\int dtdx\langle 0,out|a_\mu^{out}(k'')b_M^{out}(-k)\Psi_M\Psi_m^\dagger\Phi_\mu^\dagger b_m^{in\dagger}(-k')|0,in\rangle$$
$$=C\delta(k-k'-k'')\mathbf{B}(k'|k,-k''), \tag{62}$$

where

$$\mathbf{B}(k'|k,-k'')=\int_{-\infty}^{+\infty}dt\mathcal{D}^*_{\epsilon_M}[\lambda]\mathcal{D}^*_{\epsilon_m}[-\lambda']\mathcal{D}^*_{\epsilon_\mu}[\lambda'']. \tag{63}$$

Another amplitude that we need to introduce is the amplitude $\mathcal{C}$ for a μ antiparticle to spontaneously excite into an M antiparticle and an m particle,

$$\mathcal{C}(k''|-k',k)=-i\int dtdx\langle 0,out|b_m^{out}(-k')a_M^{out}(k)\Psi_M^\dagger\Psi_m\Phi_\mu a_\mu^{in\dagger}(k'')|0,in\rangle$$
$$=C\delta(k-k'-k'')\mathbf{C}(k''|k,-k'), \tag{64}$$

where

$$\mathbf{C}(k''|k,-k')=\int_{-\infty}^{+\infty}dt\mathcal{D}^*_{\epsilon_M}[\lambda]\mathcal{D}^*_{\epsilon_m}[\lambda']\mathcal{D}^*_{\epsilon_\mu}[-\lambda''], \tag{65}$$

and the amplitude $\mathcal{V}$ for spontaneous creation from a vacuum of an M antiparticle, an m particle, and a μ particle:

$$\mathcal{V}(|-k,k',k'')=-i\int dtdx\langle 0,out|a_m^{out}(k')b_M^{out}(-k)a_\mu^{out}(k'')\Psi_M^\dagger\Psi_m\Phi_\mu|0,in\rangle$$
$$=C\delta(k-k'-k'')\mathbf{V}(|-k,k',k''), \tag{66}$$

where

$$\mathbf{V}(|-k,k',k'')=\int_{-\infty}^{+\infty}dt\mathcal{D}^*_{\epsilon_M}[\lambda]\mathcal{D}^*_{\epsilon_m}[\lambda']\mathcal{D}^*_{\epsilon_\mu}[\lambda'']. \tag{67}$$

The rules governing the product of $\mathcal{D}$ functions in the reduced amplitudes **A**, **B**, **C**, **V** can be summarized as follows.

The sum of the momenta is conserved. We always assume that $k=k'+k''$.

Equations (22) and (23) tell us that a function $\mathcal{D}^*_{\epsilon_M}[-\lambda]$ is associated with an incoming particle (of mass M, charge $Q>0$, and momentum k) or an incoming antiparticle (of mass M, charge $-Q$, and momentum $-k$).

Equations (24) and (25) tell us that a function $\mathcal{D}^*_{\epsilon_M}[\lambda]$ is associated with an outgoing particle (of mass M, charge $Q>0$, and momentum k) or an outgoing antiparticle (of mass M, charge $-Q$, and momentum $-k$).

The change of variable $t\to -t$ in the integrals accompanied by a change in sign of momentum $k\to -k$ replaces incoming quanta by outgoing quanta and *vice versa*, without changing the value of the integral.

These rules give directly the relation (61) and similar ones for **B**, **C**, and **V**. Therefore the 64 possible amplitudes describing first order interactions between Ψ_M, Ψ_m, and Φ_μ quanta can all be equal to either **A**, **B**, **C**, or **V**.[2]

There is a further identity which will play a crucial role, namely,

$$\mathcal{I}=\int_{-\infty}^{+\infty}dt\mathcal{D}_{\epsilon_M}[-\lambda]\mathcal{D}^*_{\epsilon_m}[\lambda']\mathcal{D}^*_{\epsilon_\mu}[\lambda'']=0. \tag{68}$$

[2]Note that these amplitudes are all integrals of products of three $\mathcal{D}^*$ functions. Mathematically one could also consider integrals of products of three $\mathcal{D}^*$ and $\mathcal{D}$ functions. The Bogoljubov transformation (30) ensures that they can be reduced to a combination of **A**, **B**, **C**, **V**.

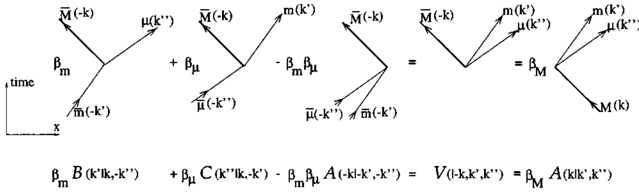

FIG. 2. A graphic representation of the relation between the amplitudes **A**, **B**, **C**, **V** when the field Φ_μ is charged. The relation between the amplitudes is more complicated than in Fig. 1 because of the nonvanishing amplitude β_μ to produce pairs of $\mu,\bar{\mu}$ quanta. The conventions are the same as in Fig. 1, except that μ and $\bar{\mu}$ denote a particle and antiparticle of the Φ_μ field, and are represented by slanted lines since they are also accelerated by the electric field. Note that this picture is not the unique representation of the amplitudes **A**, **B**, **C**, **V** since as indicated in the text each amplitude corresponds to four different processes. We have used this to represent the amplitude $\mathbf{A}(-k|-k',-k'')$ as the amplitude $\bar{m}(k')+\bar{\mu}(k'')\to\bar{M}(k)$, but it could equally well be represented by $M(-k)\to m(-k')+\mu(-k'')$.

It generalizes Eq. (48) and it is proved also by using the integral representation of the $\mathcal{D}$ functions given in Eq. (27). As in Eq. (52), one verifies that $\mathcal{I}$ still vanishes simply because due to charge conservation the quadratic phases in t cancel each other and the remaining t integral yields to a delta function whose argument is strictly positive on the domain of integration. This identity implies relations between the amplitudes $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$, $\mathcal{V}$ which generalize Eq. (49). Indeed from Eqs. (29) and (68) we obtain immediately

$$\mathbf{V}(|-k,k',k'')=\beta_M\mathbf{A}(k|k',k''). \tag{69}$$

On the other hand, starting from Eq. (67) and using the identity (29) to split into sums, successively, the functions $\mathcal{D}^*_{\epsilon_m}[\lambda']$, $\mathcal{D}^*_{\epsilon_\mu}[\lambda'']$, and $\alpha_m\mathcal{D}^*_{\epsilon_m}[-\lambda']$ and finally using the complex conjugate version of relation (68), we obtain

$$\mathbf{V}(|-k,k',k'')=\beta_m\mathbf{B}(k'|k,-k'')+\beta_\mu\mathbf{C}(k''|k,-k') - \beta_\mu\beta_m\mathbf{A}(-k|-k',-k''). \tag{70}$$

These relations are illustrated in Fig. 2. Using Eq. (69) to eliminate $\mathbf{V}(|-k,k',k'')$ yields

$$\frac{\mathbf{B}(k'|k,-k'')}{\mathbf{A}(k|k',k'')}=\frac{\beta_M}{\beta_m}-\frac{\beta_\mu}{\beta_m}\frac{\mathbf{C}(k''|k,-k')}{\mathbf{A}(k|k',k'')} +\beta_\mu\frac{\mathbf{A}(-k|-k',-k'')}{\mathbf{A}(k|k',k'')}. \tag{71}$$

This shows that when the charge of the exchanged particle tends to zero, i.e., $\beta_\mu=e^{-\pi\mu^2/2\alpha}\to 0$, the ratio of the amplitudes $\mathbf{A}(k|k',k'')$ and $\mathbf{B}(k'|k,-k'')$ is still directly related to the amplitudes for the Schwinger process:

$$\frac{\mathbf{B}(k'|k,-k'')}{\mathbf{A}(k|k',k'')}=\frac{\beta_M}{\beta_m}+O(\beta_\mu). \tag{72}$$

This estimate is proven in Appendix B where the $\alpha\to 0$ limit is studied in detail. It is shown that while each amplitude involving a charged μ particle involves polynomial corrections in α/μ, the ratio of the population of m and M species is in thermodynamical equilibrium up only to nonperturbative corrections, as expressed in Eq. (72).

Thus the equilibrium distribution of the detector states is given by

$$\frac{P_{M\ equil}}{P_{m\ equil}}=\frac{|\beta_M|^2}{|\beta_m|^2}+(\beta_\mu) = e^{-\pi M^2/QE+\pi m^2/qE}+O(\beta_\mu). \tag{73}$$

Therefore this equilibrium distribution is governed by the finite change of the horizon area; cf. the Introduction from Eq. (9) to Eq. (15).

Upon taking the double limit $\delta M/M\to 0$, $\alpha/Q\to 0$ with the mean acceleration

$$\bar{a}=\frac{\left(\frac{Q+q}{2}\right)E}{\left(\frac{M+m}{2}\right)} \tag{74}$$

fixed, one gets the linearized expression

$$\frac{P_{M\ equil}}{P_{m\ equil}}=\exp\left[-\frac{2\pi}{\bar{a}}\left(\Delta m-\alpha\frac{E}{2\bar{a}}\right)+O(\alpha^2)\right]+O(\beta_\mu) \tag{75}$$

governed by a temperature $\bar{a}/2\pi$ and an electric potential $(=E/2\bar{a})$. Indeed, in Rindler coordinates τ, ρ, the (static) potential is

$$A_\tau(\rho)=E\bar{a}\rho^2/2. \tag{76}$$

Evaluated at $\rho=1/\bar{a}$, it gives $E/2\bar{a}$.

Thus when the charge of the emitted particle is small enough with respect to its mass square (so as to neglect its vacuum instability), its effect, at the linearized level, is to modify Unruh equilibrium by the addition of an electric potential, thereby enlarging the relation to thermodynamics in a nontrivial way; cf. the Introduction for a comparison with charged black hole thermodynamics.

Note added. After this work was completed, we learned of a similar investigation of the interaction of three charged fields propagating in an external electric field by Nikishov and Ritus [24]. The main point of that work is that when the charges and masses of the particles are such that they propagate semiclassically in a substantially different way than in Unruh's model, then the rates of transitions differ from those obtained by Unruh, since they are severely modified by recoil effects or the spontaneous creation of particles in the electric field. In this limit, the analogy with thermodynamics is of course no longer valid (recall that temperature emerges from microcanonical ensembles only in the reservoir limit). On the other hand, the present work emphasizes the recovery of an extended thermodynamics in the limit of a small charge α of the emitted particle.

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APPENDIX A: AMPLITUDES OF DECAY DUE TO THE EXCHANGE OF CHARGED PARTICLES

For completeness we give here closed forms for the amplitudes **A**, **B**, **C**, **V**. We first discuss the $\mathcal{A}(k|k',k'')$ amplitude. We have to evaluate the integral (60), which, in terms of Whittaker's functions, reads as

$$\mathcal{I}_1=\int dt D_{-i\epsilon_M-1/2}\left[-e^{-3i\pi/4}\left(t+\frac{k}{MA}\right)\sqrt{2MA}\right]D_{-i\epsilon_m-1/2}\left[e^{-3i\pi/4}\left(t+\frac{k'}{q}E\right)\sqrt{2qE}\right]\times D_{-i\epsilon_\mu-1/2}\left[e^{-3i\pi/4}\left(t+\frac{k''}{\mu\alpha}\right)\sqrt{2\mu\alpha}\right]. \quad \text{(A1)}$$

The evaluation of this integral is similar to the pattern followed to obtain Eq. (45). First we split the function $D_{-i\epsilon_M-1/2}[e^{i\pi/4}(t+k/QE)\sqrt{2QE}]$ into two others functions thanks to the relation (29) and reexpress all the parabolic cylinder functions involved in terms of their integral representation (27). This yields

$$\mathcal{I}_1=\frac{e^{-\pi\epsilon_m/4}e^{-\pi\epsilon_\mu/4}}{\Gamma(i\epsilon_m+1/2)\Gamma(i\epsilon_\mu+1/2)}\frac{e^{i\pi/4}}{\sqrt{2\pi}}\int dt e^{(-i/2)qE[t+(k'/qE)]^2}e^{-(i/2)\alpha E[t+(k''/\alpha E)]^2}e^{(i/2)QE[t+(k/QE)]^2}\times\int_0^\infty du dv dw e^{iv\sqrt{2qE}[t+(k'/qE)]-iv^2/2}e^{iw\sqrt{2\alpha E}[t+(k''/\alpha E)]-iw^2/2}v^{i\epsilon_m-1/2}w^{i\epsilon_\mu-1/2}u^{-i\epsilon_M-1/2}\times[e^{(\pi/4)\epsilon_M}e^{-3i\pi/8}e^{-iu\sqrt{2QE}[t+(k/QE)]+iu^2/2}+e^{(-3\pi/4)\epsilon_M}e^{i\pi/8}e^{iu\sqrt{2QE}[t+(k/QE)]+iu^2/2}], \quad \text{(A2)}$$

with $k=k'+k''$. Charge conservation $Q=q+\alpha$ eliminates the quadratic terms in t and the t integration leads to two delta functions $\delta(\sqrt{qE}v+\sqrt{\alpha E}w\mp\sqrt{QE}u)$. The positivity of the u, v, and w variables makes only the first one contributing to the amplitude. After some elementary algebra, we obtain for $\mathcal{I}_1$ the expression

$$\mathcal{I}_1=\frac{e^{(\pi/4)(\epsilon_M-\epsilon_m-\epsilon_\mu)}e^{-i\pi/8}}{\Gamma(i\epsilon_m+1/2)\Gamma(i\epsilon_\mu+1/2)}\frac{\sqrt{2\pi}}{\sqrt{2QE}}\exp\left[\frac{i}{2}\left(\frac{k^2}{QE}-\frac{k'^2}{qE}-\frac{k''^2}{\alpha E}\right)\right]\mathcal{I}_2, \quad \text{(A3)}$$

where

$$\mathcal{I}_2=\int_0^\infty dv dw \exp\left[-\frac{i}{2QE}(\sqrt{\alpha E}v-\sqrt{qE}w)^2+i\sqrt{2}\frac{QEk'-qEk}{\sqrt{QEqE\alpha E}}\left(\frac{\sqrt{\alpha E}}{\sqrt{QE}}v-\frac{\sqrt{qE}}{\sqrt{QE}}w\right)\right]\times\left(\frac{\sqrt{\alpha E}}{\sqrt{QE}}w+\frac{\sqrt{qE}}{\sqrt{QE}}v\right)^{-i\epsilon_M-1/2}v^{i\epsilon_m-1/2}w^{i\epsilon_\mu-1/2}$$
$$=\left(\frac{\sqrt{QE}}{\sqrt{\alpha E}}\right)^{i\epsilon_m+1/2}\left(\frac{\sqrt{QE}}{\sqrt{qE}}\right)^{i\epsilon_\mu+1/2}\left(\frac{\sqrt{\alpha E}}{\sqrt{qE}}\right)^{i\epsilon_M+1/2}\Gamma\left(i\mathcal{E}+\frac{1}{2}\right)e^{i\Omega^2/2}e^{\pi\mathcal{E}/4}e^{-i\pi/8}\mathcal{I}_3, \quad \text{(A4)}$$

with

$$\Omega=\frac{qEk-QEk'}{\sqrt{QEqE\alpha E}}, \quad \text{(A5)}$$

$$\mathcal{E}=\epsilon_m+\epsilon_\mu-\epsilon_M, \quad \text{(A6)}$$

and

$$\mathcal{I}_3=\left\{D_{-i\mathcal{E}-1/2}[+\sqrt{2}e^{i\pi/4}\Omega]B\left(i\epsilon_\mu+\frac{1}{2},-i\mathcal{E}+\frac{1}{2}\right){}_2F_1\left(i\epsilon_M+\frac{1}{2},i\epsilon_\mu+\frac{1}{2},1+i\epsilon_M-i\epsilon_m;-\frac{\alpha E}{qE}\right)+D_{-i\mathcal{E}-1/2}[\sqrt{2}e^{-i\pi/4}\Omega]B\left(i\epsilon_m+\frac{1}{2},-i\mathcal{E}+\frac{1}{2}\right){}_2F_1\left(i\epsilon_M+\frac{1}{2},i\epsilon_m+\frac{1}{2},1+i\epsilon_M-i\epsilon_\mu;-\frac{qE}{\alpha E}\right)\right\}. \quad \text{(A7)}$$

The way to obtain this last result and others similar is postponed to the end of this appendix. Collecting all these results, the amplitude $\mathbf{A}(k|k',k'')$ reads

$$\mathbf{A}(k|k',k'')=\frac{\sqrt{2\pi}e^{-i\pi/4}e^{(-\pi/2)(\epsilon_m+\epsilon_\mu)}e^{\pi\mathcal{E}/8}\exp\left[\frac{i}{2}\left(\frac{k^2}{QE}-\frac{k'^2}{qE}-\frac{k''^2}{\alpha E}\right)\right]e^{i\Omega^2/2}}{(2QE)^{1/4}(2qE)^{3/4}(2\alpha E)^{1/4}\Gamma(i\epsilon_m+1/2)\Gamma(i\epsilon_\mu+1/2)}\left(\frac{Q}{q}\right)^{(i/2)\epsilon_\mu}\left(\frac{\alpha}{q}\right)^{(i/2)\epsilon_M}\left(\frac{Q}{\alpha}\right)^{(i/2)\epsilon_m}\Gamma\left(i\mathcal{E}+\frac{1}{2}\right)\mathcal{I}_3. \tag{A8}$$

As expected this result is symmetric with respect to the exchange $(m,a,k')\leftrightarrow(\mu,\alpha,k'')$.

The computation of $\mathbf{B}(k'|k,-k'')$ is somewhat different. Starting from the general expression (63), we have to evaluate the integral

$$\mathcal{J}_1=\int dt D_{-i\epsilon_M-1/2}\left[e^{-3i\pi/4}\left(t+\frac{k}{QE}\right)\sqrt{2QE}\right]D_{-i\epsilon_m-1/2}\left[-e^{-3i\pi/4}\left(t+\frac{k'}{qE}\right)\sqrt{2qE}\right]D_{-i\epsilon_\mu-1/2}\left[e^{-3i\pi/4}\left(t+\frac{k''}{\alpha E}\right)\sqrt{2\alpha E}\right], \tag{A9}$$

with $k=k'+k''$. After the same transformations as those performed to evaluate $\mathcal{I}_1$, we obtain

$$\mathcal{J}_1=\frac{e^{(-\pi/4)(\epsilon_m+\epsilon_\mu)}}{\Gamma(i\epsilon_m+1/2)}\frac{1}{\Gamma(i\epsilon_\mu+1/2)}\frac{e^{i\pi/4}}{\sqrt{2\pi}}\int dt e^{(-i/2)qE(t+k'/qE)^2}e^{(-i/2)\alpha E(t+k''/\alpha E)^2}e^{(i/2)QE(t+k/QE)^2}$$
$$\times\int_0^\infty du dv dw e^{-iv\sqrt{2qE}(t+k'/qE)-iv^2/2}e^{iw\sqrt{2\alpha E}(t+k''/\alpha E)-iw^2/2}v^{i\epsilon_m-1/2}w^{i\epsilon_\mu-1/2}u^{-i\epsilon_M-1/2}$$
$$\times[e^{(\pi/4)\epsilon_M}e^{-3i\pi/8}e^{iu\sqrt{2QE}(t+k/QE)+iu^2/2}+e^{(-3\pi/4)\epsilon_M}e^{i\pi/8}e^{-iu\sqrt{2QE}(t+k/QE)+iu^2/2}]. \tag{A10}$$

As previously charge conservation implies that the quadratic terms in t cancel and the t integration yields a delta functions $\delta(\sqrt{\alpha E}w-\sqrt{qE}v\pm\sqrt{QE}u)$. However, this time, both terms will contribute to $\mathcal{J}_1$ and we are left with the following expression:

$$\mathcal{J}_1=\frac{e^{(-\pi/4)(\epsilon_m+\epsilon_\mu)}e^{i\pi/4}}{\Gamma(i\epsilon_m+1/2)\Gamma(i\epsilon_\mu+1/2)}\sqrt{2\pi}\exp\left(\frac{i}{2}\left\{\frac{k^2}{QE}-\frac{k'^2}{qE}-\frac{(k'')^2}{\alpha E}\right\}\right)\left[\frac{1}{\sqrt{2qE}}e^{(\pi/4)\epsilon_M}e^{-3i\pi/8}\mathcal{J}_2+\frac{1}{\sqrt{2\alpha E}}e^{(-3\pi/4)\epsilon_M}e^{i\pi/8}\mathcal{J}_3\right], \tag{A11}$$

where

$$\mathcal{J}_2=\int_0^\infty du dw \exp\left[-\frac{i}{2qE}(\sqrt{\alpha E}u+\sqrt{QE}w)^2+i\sqrt{2}\frac{qEk-QEk'}{\sqrt{QEqE\alpha E}}\left(\frac{\sqrt{\alpha E}}{\sqrt{qE}}u+\frac{\sqrt{QE}}{\sqrt{qE}}w\right)\right]$$
$$\times\left(\frac{\sqrt{\alpha E}}{\sqrt{qE}}w+\frac{\sqrt{QE}}{\sqrt{qE}}u\right)^{i\epsilon_m-1/2}u^{-i\epsilon_M-1/2}w^{i\epsilon_\mu-1/2} \tag{A12}$$

and

$$\mathcal{J}_3=\int_0^\infty du dv \exp\left[-\frac{i}{2\alpha E}(\sqrt{QE}v+\sqrt{qE}u)^2+i\sqrt{2}\frac{qEk-QEk'}{\sqrt{QEqE\alpha E}}\left(\frac{\sqrt{QE}}{\sqrt{\alpha E}}v+\frac{\sqrt{qE}}{\sqrt{\alpha E}}u\right)\right]$$
$$\times\left(\frac{\sqrt{QE}}{\sqrt{\alpha E}}u+\frac{\sqrt{qE}}{\sqrt{\alpha E}}v\right)^{i\epsilon_\mu-1/2}v^{i\epsilon_m-1/2}u^{-i\epsilon_M-1/2}. \tag{A13}$$

Note that the last integral $\mathcal{J}_3$ can be obtained from the first one $\mathcal{J}_2$ by exchanging (m,a) with (μ,α) and (k,k',k'') with $(-k,-k'',-k')$. In particular, note the invariance $qEk-QEk'\mapsto-\alpha Ek+QEk''=qEk-QEk'$. We display here the result for $\mathcal{J}_2$:

$$\mathcal{J}_2=\left(\frac{\sqrt{qE}}{\sqrt{\alpha E}}\right)^{-i\epsilon_M+1/2}\left(\frac{\sqrt{qE}}{\sqrt{QE}}\right)^{i\epsilon_\mu+1/2}\Gamma\left(i\mathcal{E}+\frac{1}{2}\right)e^{i\Omega^2/2}e^{\pi\mathcal{E}/4}e^{-i\pi/8}D_{-i\mathcal{E}-1/2}[-\sqrt{2}e^{i\pi/4}\Omega]\left(\frac{\sqrt{QE}}{\sqrt{\alpha E}}\right)^{2i\epsilon_M-i\epsilon_m+1/2}$$
$$\times B\left(i\epsilon_\mu+\frac{1}{2},-i\epsilon_M+\frac{1}{2}\right){}_2F_1\left(i\mathcal{E}+\frac{1}{2},-i\epsilon_M+\frac{1}{2},1+i(\epsilon_\mu-\epsilon_M);\frac{qE}{QE}\right). \tag{A14}$$

Therefore, the $\mathbf{B}(k'|k,-k'')$ amplitude reads

$$\mathbf{B}(k'|k,-k'')=\left(\frac{q}{Q}\right)^{(i/2)(\epsilon_\mu-i\epsilon_M)}\left(\frac{\alpha}{Q}\right)^{(i/2)(\epsilon_m-i\epsilon_M)}\frac{e^{(-\pi/2)(\epsilon_m+\epsilon_\mu)}e^{i\pi/4}\sqrt{2\pi}\,\exp\left[-\frac{i}{2}\left(\frac{k^2}{QE}-\frac{k'^2}{qE}+\frac{k''^2}{\alpha E}\right)\right]}{(2QE)^{3/4}(2qE)^{1/4}(2\alpha E)^{1/4}\Gamma(i\epsilon_m+1/2)\Gamma(i\epsilon_\mu+1/2)}$$
$$\times\Gamma\left(i\mathcal{E}+\frac{1}{2}\right)e^{i\Omega^2/2}e^{\pi\mathcal{E}/8}e^{-i\pi/8}D_{-i\mathcal{E}-1/2}[\sqrt{2}e^{i5\pi/4}\Omega]\mathcal{J}_4, \tag{A15}$$

with

$$\mathcal{J}_4=\left\{e^{-3i\pi/8}B\left(i\epsilon_\mu+\frac{1}{2},-i\epsilon_M+\frac{1}{2}\right){}_2F_1\left(i\mathcal{E}+\frac{1}{2},\frac{1}{2}-i\epsilon_M,1+i\epsilon_\mu-i\epsilon_M;\frac{qE}{QE}\right)\right.$$
$$\left.+e^{-\pi\epsilon_M}e^{i\pi/8}B\left(i\epsilon_m+\frac{1}{2},-i\epsilon_M+\frac{1}{2}\right){}_2F_1\left(i\mathcal{E}+\frac{1}{2},\frac{1}{2}-i\epsilon_M,1+i\epsilon_m-i\epsilon_M;\frac{\alpha E}{QE}\right)\right\}. \tag{A16}$$

Finally, notice that the $\mathbf{C}(k''|k,-k')$ amplitude can be easily obtained from $\mathbf{B}(k'|k,-k'')$ by the substitution $m,q,k'\mapsto\mu,\alpha,k''$, and hence $\Omega\mapsto-\Omega$. To complete the calculation we have to evaluate the integrals $\mathcal{I}_2$, $\mathcal{J}_2$, and $\mathcal{J}_3$. As an example we consider the integral $\mathcal{I}_2$; the others follow a similar pattern:

$$\mathcal{I}_2=\int_0^\infty dv\,dw\,\exp\left[-\frac{i}{2QE}(\sqrt{\alpha E}v-\sqrt{qE}w)^2+i\sqrt{2}\,\frac{-qEk+QEk'}{\sqrt{QEqE\alpha E}}\left(\frac{\sqrt{\alpha E}}{\sqrt{QE}}v-\frac{\sqrt{qE}}{\sqrt{QE}}w\right)\right]$$
$$\times\left(\frac{\sqrt{\alpha E}}{\sqrt{QE}}w+\frac{\sqrt{qE}}{\sqrt{QE}}v\right)^{-i\epsilon_M-1/2}v^{i\epsilon_m-1/2}w^{i\epsilon_\mu-1/2}. \tag{A17}$$

By using the reduced variables

$$X=\sqrt{\frac{\alpha E}{QE}}\,v,\quad Y=\sqrt{\frac{qE}{QE}}\,w,\quad \Omega=\frac{kqE-QEk'}{\sqrt{QEqE\alpha E}},\quad c=e^\gamma=\sqrt{\frac{qE}{\alpha E}}, \tag{A18}$$

this integral becomes

$$\left(\sqrt{\frac{QE}{qE}}\right)^{i\epsilon_\mu+1/2}\left(\sqrt{\frac{QE}{\alpha E}}\right)^{i\epsilon_m+1/2}\mathcal{I}, \tag{A19}$$

where

$$\mathcal{I}=\int_0^\infty dX\,dY\,e^{(-i/2)(X-Y)^2-i\sqrt{2}\Omega(X-Y)}(cX+c^{-1}Y)^{-i\epsilon_M-1/2}X^{i\epsilon_m-1/2}Y^{i\epsilon_\mu-1/2}. \tag{A20}$$

The change of variables

$$X=\frac{\rho e^\theta}{2},\quad Y=\frac{\rho e^{-\theta}}{2},\quad 0<\rho<\infty,\quad -\infty<\theta<\infty, \tag{A21}$$

and $z=\rho\sinh\theta$ factorizes the double integral:

$$\mathcal{I}=2^{-i\mathcal{E}+1/2}\int_0^\infty dz\,d\theta\,e^{(-i/2)z^2}z^{i\mathcal{E}-1/2}(\sinh\theta)^{-i\mathcal{E}-1/2}\{e^{i\theta(\epsilon_m-\epsilon_\mu)}(e^{\theta+\gamma}+e^{-\theta+\gamma})^{-i\epsilon_M-1/2}e^{-i\sqrt{2}\Omega z}$$
$$+e^{-i\theta(\epsilon_m-\epsilon_\mu)}(e^{\gamma-\theta}+e^{-\theta+\gamma})^{-i\epsilon_M-1/2}e^{i\sqrt{2}\Omega z}\}, \tag{A22}$$

where the two terms come from the separation of the positive and negative values of the integration variable θ. The z integrals are the representation (27) of the parabolic cylinder function, and so

$$\mathcal{I}=2\Gamma\left(i\frac{\mathcal{E}}{2}+\frac{1}{2}\right)e^{i\Omega^2/2}e^{\pi\mathcal{E}/4}e^{-i\pi/8}\int_0^\infty d\theta(2\sinh\theta)^{-i\mathcal{E}-1/2}\{D_{-i\mathcal{E}-1/2}(\sqrt{2}e^{i\pi/4}\Omega)e^{i\theta(\epsilon_m-\epsilon_\mu)}(e^{\theta+\gamma}+e^{-\theta+\gamma})^{-i\epsilon_M-1/2}$$
$$+D_{-i\mathcal{E}-1/2}(\sqrt{2}e^{i\pi/4}\Omega)e^{-i\theta(\epsilon_m-\epsilon_\mu)}(e^{\gamma-\theta}+e^{\theta-\gamma})^{-i\epsilon_M-1/2}\}. \tag{A23}$$

A final change of variable $s=e^{2\theta}$ reduces the last integrals to products of Euler and hypergeometric functions ([23], formula 3.197.2). The final result for $\mathcal{I}_2$ reads

$$\mathcal{I}_2=\left(\sqrt{\frac{Qe}{qE}}\right)^{i\epsilon_\mu+1/2}\left(\sqrt{\frac{QE}{\alpha E}}\right)^{i\epsilon_m+1/2}\Gamma\left(\frac{i\mathcal{E}}{2}+\frac{1}{2}\right)e^{i\Omega^2/2}e^{\pi\mathcal{E}/4}e^{-i\pi/8}\Bigg\{D_{-i\mathcal{E}-1/2}[\sqrt{2}e^{i\pi/4}\Omega]$$
$$\times B\left(i\epsilon_\mu+\frac{1}{2},-i\mathcal{E}+\frac{1}{2}\right)\,{}_2F_1\left(\frac{1}{2}+i\epsilon_M,i\epsilon_\mu+\frac{1}{2},1-i\epsilon_m+i\epsilon_M;-\frac{\alpha E}{qE}\right)+D_{-i\mathcal{E}-1/2}[-\sqrt{2}e^{i\pi/4}\Omega]$$
$$\times B\left(i\epsilon_m+\frac{1}{2},-i\mathcal{E}+\frac{1}{2}\right)\,{}_2F_1\left(\frac{1}{2}+i\epsilon_M,i\epsilon_m+\frac{1}{2},1-i\epsilon_\mu+i\epsilon_M;-\frac{qE}{\alpha E}\right)\Bigg\}. \tag{A24}$$

Similarly, for the amplitude $\mathbf{B}(k'|k,-k'')$ the last integration can be carried out by using ([23], formula 3.197.1).

APPENDIX B: SMALL CHARGE LIMIT OF THE AMPLITUDES

In this appendix we compute the small charge limit of the amplitudes $\mathbf{A}(k|k',k'')$ and $\mathbf{B}(k'|k,-k'')$. We also prove Eq. (72), i.e., that the ratio of the amplitudes $\mathbf{A}(k|k',k'')$ and $\mathbf{B}(k'|k,-k'')$ is given by the ratios of the Schwinger amplitudes, up to terms proportional to $e^{-\pi\epsilon_\mu}$. Moreover, we shall evaluate $\mathbf{A}(k|k',k'')$ in the limit $\alpha\to 0$. As emphasized in the main text, a consequence of this calculation is that while the amplitudes $\mathbf{A}(k|k',k'')$ and $\mathbf{B}(k'|k,-k'')$ differ polynomially in the variable α/μ from their chargeless limits, their ratio is nevertheless given by Eq. (72), which differs from the chargeless limit only by nonperturbative corrections. When α is small, the two hypergeometric functions giving $\mathcal{J}_4$ in Eq. (A17) can be combined into one thanks to the formulas (9.131.2) and (9.131.1) of Ref. [23]:

$$\mathcal{J}_4=\frac{e^{i\pi/8}e^{-\pi\epsilon_M}\sqrt{\alpha E}}{QE}\,(\sqrt{\alpha E})^{-\epsilon_M-i\epsilon_m}$$
$$\times(\sqrt{QE})^{-i\epsilon_m-3i\epsilon_\mu-3i\epsilon_m}B\left(\frac{1}{2}-i\epsilon_M,\frac{1}{2}+i\epsilon_m\right)$$
$$\times\left\{\frac{\Gamma(1+i\epsilon_m-i\epsilon_M)\Gamma(i\epsilon_\mu-i\epsilon_M)}{\Gamma\left(\frac{1}{2}+i\mathcal{E}\right)\Gamma\left(\frac{1}{2}-i\epsilon_M\right)}\,{}_2F_1\left(\frac{1}{2}+i\epsilon_m,\right.\right.$$
$$\left.\left.\frac{1}{2}+i\epsilon_M,1-i\epsilon_\mu+i\epsilon_M,-\frac{qE}{\alpha E}\right)+e^{-\pi\epsilon_\mu}O(\alpha/\mu)\right\}. \tag{B1}$$

Note that (and this constitutes a check of the exactness of the calculation) in the limit $\alpha/\mu\to 0$, there appears diverging phases in the first factor of this expression which cancel each other. If we omit the small corrections proportional to the Schwinger factor $e^{-\pi\epsilon_\mu}$, most of the prefactors are common between the amplitudes $\mathcal{A}(k|k'k'')$ and $\mathcal{B}(k'|k-k'')$ and thus can be omitted in the ratio $\mathcal{A}/\mathcal{B}$ which reduces to

$$\frac{\mathcal{A}(k|k'k'')}{\mathcal{B}(k'|k-k'')}=e^{-\pi/2}\,\frac{\Gamma\left(\frac{1}{2}-i\mathcal{E}\right)\Gamma\left(\frac{1}{2}+i\mathcal{E}\right)e^{\pi\epsilon_M}}{\Gamma(i\epsilon_\mu-i\epsilon_M)\Gamma(1-i\epsilon_\mu+i\epsilon_M)}$$
$$+e^{-\pi\epsilon_\mu}O(\alpha/\mu)$$
$$=e^{-\pi(\epsilon_m-\epsilon_M)}+e^{-\pi\epsilon_\mu}O(\alpha/\mu), \tag{B2}$$

which is the sought for result. Now we discuss in more detail the limit of the amplitude $\mathbf{A}(k|k',k'')$ when $\alpha\to 0$, and check that we recover the $\alpha=0$ result in the limit. The computation is done in three steps. Equations (A9),(A8) express $\mathbf{A}(k|k',k'')$ as products of phases and Γ functions with a sum of products of Eulerian (B) functions, Whittaker's ($\mathcal{D}$) functions, and hypergeometric functions.

First, we obtain, by a saddle point evaluation, appropriate approximations of Whittaker's functions. Then we estimate the limits of the hypergeometric functions as confluent hypergeometric functions. Finally, applying several times the Stirling and reflection formulae on the Γ and B Euler functions we obtain the result.

The integrals

$$\Gamma\left(\frac{i\mathcal{E}}{2}+\frac{1}{2}\right)e^{i\Omega^2/2}e^{\pi\mathcal{E}/8}e^{-i\pi/8}D_{-i\mathcal{E}/2-1/2}[\pm\sqrt{2}e^{i\pi/4}\Omega]$$
$$=\int_0^\infty dv\;e^{\pm i\sqrt{2}\Omega v-iv^2/2}v^{i\mathcal{E}-1/2} \tag{B3}$$

have saddle points located respectively at

$$v_s=\frac{\pm\sqrt{2}\Omega+\sqrt{2}(\Omega^2+\mathcal{E})^{1/2}}{2}. \tag{B4}$$

They are approximated by

$$\frac{\sqrt{2\pi}}{\sqrt{2\omega}}\,e^{-i\pi/4}(2\alpha E)^{1/4}e^{i\varphi_s^{\pm}}, \tag{B5}$$

with

$$\varphi_s^{\pm}=\frac{3}{4}\,\frac{(k+k')^2}{\alpha E}-\frac{\omega^2}{4\alpha E}\mp\frac{(k+k')\omega}{2\alpha E}+\frac{\mathcal{E}}{2}\ln\frac{\omega\mp(k+k')}{\sqrt{2\alpha E}}$$
$$-\frac{(k+k')(k-k')}{2QE}\pm\frac{(k-k')\omega}{2QE}. \tag{B6}$$

Let us emphasize that both integrals (B3) have their modulus of the same order of magnitude in α/μ.

To obtain the limits of the hypergeometric functions appearing in Eq. (52) as confluent hypergeometric functions when $\alpha\to 0$ is straightforward. For small value of α we obtain

$$ {}_2F_1\left(i\epsilon_M+\frac{1}{2},i\epsilon_\mu+\frac{1}{2},1+i\epsilon_M-i\epsilon_m;-\frac{\alpha E}{qE}\right) = M\left(i\epsilon_M+\frac{1}{2},1+i\epsilon_M-i\epsilon_m,-\frac{i\mu^2}{2qE}\right)+O\left(\frac{\alpha}{\mu}\right) \tag{B7} $$

and

$$ {}_2F_1\left(i\epsilon_M+\frac{1}{2},i\epsilon_m+\frac{1}{2},1+i\epsilon_M-i\epsilon_\mu;-\frac{qE}{\alpha E}\right) = e^{-i\mu^2/4qE}\left(\frac{\mu^2}{2qE}\right)^{i(\epsilon_M+\epsilon_m)/2} e^{(\pi/4)(\epsilon_M+\epsilon_m)} \times W_{(-1/2)(\epsilon_m+\epsilon_M),(1/2)(\epsilon_m-\epsilon_M)}\left[-\frac{i\mu^2}{4qE}\right]+O\left(\frac{\alpha}{\mu}\right). \tag{B8} $$

Here also these two functions are of the same order of magnitude in α/μ but the prefactors multiplying them in the amplitude $\mathbf{A}(k|k',k'')$ are quite different. The first one is multiplied by

$$ B\left(i\epsilon_\mu+\frac{1}{2},-i\epsilon_\mu+i(\epsilon_M-\epsilon_m)+\frac{1}{2}\right) = \frac{2\pi e^{i(\epsilon_M-\epsilon_m)}e^{\pi\epsilon_M/2}e^{-\pi\epsilon_m/2}e^{-\pi\epsilon_\mu}}{\Gamma(1+i(\epsilon_M-\epsilon_m))}\left[1+O\left(\frac{\alpha}{\mu}\right)\right], \tag{B9} $$

whereas the second one is pondered by

$$ B\left(i\epsilon_m+\frac{1}{2},-i\epsilon_\mu+i(\epsilon_M-\epsilon_m)+\frac{1}{2}\right) = \Gamma\left(i\epsilon_m+\frac{1}{2}\right)\epsilon_\mu^{-i\epsilon_m-1/2}e^{-\pi\epsilon_m/2}e^{i\pi/4}\left[1+O\left(\frac{\alpha}{\mu}\right)\right]. \tag{B10} $$

We see that the exponential factor $e^{-\pi\epsilon_\mu}$ appearing in Eq. (B9) makes the first term of $\mathcal{I}_3$ in Eq. (A7) negligible with respect to the second one in the limit $\alpha\to 0$ at a fixed nonvanishing value of μ. Collecting all the results (B7), (B9), and (B1), we obtain the limit we are discussing. At zero order in α/μ, once more (as expected) all the diverging phases cancel in the first term and the remaining factors group together to give the expression (55) with $qE=QE$. So we obtain, at the end

$$ \mathcal{A}(k|k'k'')=[\mathcal{A}_0(k'|kk'')+O(\alpha/\mu)]+e^{-\pi\epsilon_\mu}O(\alpha,\mu), \tag{B11} $$

where $\mathcal{A}_0(k|k'k'')$ is the transition amplitude in the neutral case (55). Using Eq. (A17), one can similarly show that

$$ \mathcal{B}(k|k'k'')=[\mathcal{B}_0(k'|kk'')+O(\alpha/\mu)]+e^{-\pi\epsilon_\mu}O(\alpha/\mu), \tag{B12} $$

where $\mathcal{B}_0(k|k'k'')$ is the transition amplitude in the neutral case. From these relations (B11),(B12) one can only *a priori* deduce that their ratio behaves as

$$ \frac{\mathcal{A}(k|k'k'')}{\mathcal{B}(k'|k-k'')}=e^{-\pi(\epsilon_m-\epsilon_M)}\left[1+O\left(\frac{\alpha}{\mu}\right)\right]+e^{-\pi\epsilon_\mu}O(\alpha/\mu), \tag{B13} $$

while our previous computation, Eq. (B2), shows that actually all the polynomial corrections in α/μ to the Boltzmann factor cancel each other.

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