# Lattice black holes

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We study the Hawking process on lattices falling into static black holes. The motivation is to understand how the outgoing modes and Hawking radiation can arise in a setting with a strict short distance cutoff in the free-fall frame. We employ two-dimensional free scalar field theory. For a falling lattice with a discrete time-translation symmetry we use analytical methods to establish that, for Killing frequency $\omega$ and surface gravity $\kappa$ satisfying $\kappa \ll \omega^{1/3} \ll 1$ in lattice units, the continuum Hawking spectrum is recovered. The low frequency outgoing modes arise from exotic ingoing modes with large proper wave vectors that ''refract'' off the horizon. In this model with time translation symmetry the proper lattice spacing goes to zero at spatial infinity. We also consider instead falling lattices whose proper lattice spacing is constant at infinity and therefore grows with time at any finite radius. This violation of time translation symmetry is visible only at wavelengths comparable to the lattice spacing, and it is responsible for transmuting ingoing high Killing frequency modes into low frequency outgoing modes.
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## I. INTRODUCTION

As field modes emerge from the vicinity of the horizon they are infinitely redshifted. In ordinary field theory there is an infinite density of states at the horizon to supply the outgoing modes. How do these outgoing modes arise if the short distance physics supports no infinite density of states? And how does the short distance physics affect the Hawking radiation in these modes? By insisting on a fully sensible resolution of the apparent conflict between black holes and short distance finiteness we hope that some deep lessons can be learned. That is the underlying motivation for the present work.

One way to avoid an infinite density of states is to have some physical cutoff at short distances, related to a fundamental graininess of spacetime.[1] Analogies with condensed matter systems such as fluids, crystals, Fermi liquids, etc. suggest that in this case the long wavelength collective modes which are described by field theory will propagate with a dispersion relation that deviates from the linear, Lorentz-invariant form at high frequencies. A number of (two dimensional) linear field theory models with such behavior have now been studied [1,2,3,4]. It turns out that the Hawking radiation is extremely insensitive to the short distance physics, as long as neither the black hole temperature nor the frequency at which the spectrum is examined is too close to the scale of the new physics.[2] What is striking is that this is so even though the behavior of the field modes is rather bizarre: the outgoing modes that carry the Hawking radiation arise from exotic ingoing modes that bounce off the horizon.[3]

Although these models do provide a mechanism for generating the outgoing modes without an infinite density of states at the horizon, they still behave unphysically: wave packets cannot be propagated backwards in time all the way out to ''infinity'' (i.e. the asymptotic region far from the black hole). For example, using Unruh's dispersion relation [1], which has a group velocity that drops monotonically to zero at infinite wave vector, the wave vector diverges as the wave packet goes (backwards in time) farther from the black hole. So, in this case, the infinite redshift is just moved to a new location. Evidently the theory is pushed into the trans-Planckian regime after all. To make sense of—or at least to sensibly model—the true origin of the outgoing modes, it therefore seems necessary to work with a theory that has a well-behaved physical cutoff. A simple way to implement such a cutoff is to discretize space, and work with a lattice theory preserving continuity in time. This is roughly similar to what is happening in a condensed matter system, but we can preserve strict linearity for the lattice field theory and still model the key effect of the cutoff.

In this paper we study two lattice models of this nature obtained by discretizing the spatial coordinate in a freely

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[1]String theory provides a different way, in which the states are less localizable than in ordinary field theory.

[2]The ultra low frequency part of the spectrum, which has not yet been computed, might turn out to be non-thermal.

[3]This is what happens in the case where the group velocity is subluminal at high frequencies. In the superluminal case the outgoing modes arise from superluminal modes that emerge from behind the horizon. Ultimately these modes come from the singularity (for a neutral black hole), so it is not so clear one can make sense of this case. However, Unruh [5] and Corley [6] have recently shown that if one simply imposes a vacuum boundary condition on these modes behind the horizon the usual Hawking radiation outside the black hole is recovered. A black hole with an inner horizon (such as a charged one) behaves very differently in the superluminal case however. The ergoregion inside the black hole is unstable to self-amplifying Hawking radiation [7].

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falling coordinate system. (If instead a static coordinate is discretized then the lattice points have diverging acceleration as the horizon is approached, and their worldlines are spacelike inside the horizon. This leads to pathological behavior of the field.) If there really is a cutoff in some preferred frame, then that frame should presumably fall from the ''cosmic'' rest frame at infinity in towards a black hole. This would be like Unruh's sonic analog of a black hole [8,1,9] or the helium-3 texture analog [10], where the short distance cutoff is provided by the atomic structure of the fluid which is freely flowing across the phonon or other quasiparticle horizon.

One particular choice of discretization has the feature that a discrete remnant of time translation invariance survives. This makes the model easier to study analytically, and we exploit this to show here in Secs. II–V that in a leading approximation the black hole radiates thermally at the Hawking temperature. The same result was found previously by Unruh [11] using numerical evolution of the lattice field equation. Unfortunately, however, this particular lattice model is still not satisfactory as a model of physics with a fundamental cutoff, because the proper lattice spacing goes to zero at infinity.

It is easy to avoid the vanishing lattice spacing by discretizing instead a spatial coordinate which measures proper length on some initial spacelike surface all the way out to infinity. However, since we also want the lattice points to fall freely into the black hole, this results in a lattice spacing that grows in time, as shown in Sec. VI. The growth of the lattice spacing suggests that we have still not found a satisfactory model with a short distance cutoff. (An alternative which avoids this problem will be discussed at the end of this paper.) However, it is rather instructive to understand the physics of this model with the growing lattice spacing. In this model time translation symmetry is violated for short wavelength modes but not for long wavelengths. In fact, the Killing energy of an outgoing mode can be much lower than that of the ingoing mode that gave rise to it. This is essential to producing the outgoing long wavelength modes in this model since a long wavelength ingoing mode will of course sail across the horizon into the black hole rather than converting to an outgoing mode. This mechanism is studied in Sec. VI with the help of the eikonal approximation.

We adopt units in which $\hbar=c=\delta=1$, where $\delta$ is the coordinate lattice spacing, and we use the ''timelike'' metric signature.

## II. FALLING LATTICE MODELS

Our goal is now to ''latticize'' the theory of a scalar field propagating in a static black hole spacetime. For each spherical harmonic, the physics reduces to a two-dimensional problem in the time-radius subspace. The short distance phenomena we wish to study have nothing to do with the scattering of modes off of the angular momentum barrier, so nothing essential is lost in dropping the angular dependence and studying instead the physics in a two dimensional black hole spacetime.

We begin with a generic static two dimensional spacetime, and choose coordinates so that the line element takes the form

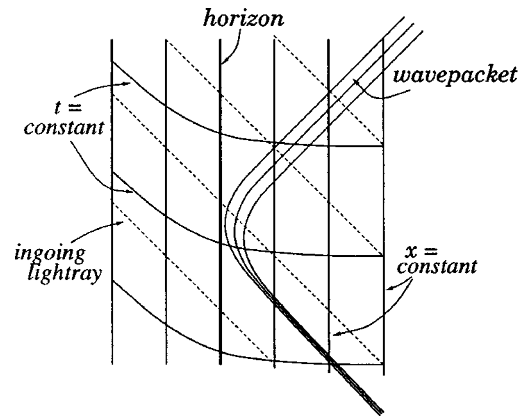


FIG. 1. Painlevé-Gullstrand coordinates and ingoing light rays. The trajectory of a wave packet that is outgoing with low wave vector at late times is sketched.

$$ds^2=dt^2-(dx-v(x)dt)^2. \tag{1}$$

The curves of constant $x$ are orbits of the Killing field $\chi=\partial_t$. The curves with $dx=v(x)dt$ are geodesics which are at rest with respect to the Killing field where $v(x)=0$, and the proper time along these geodesics is $t$. The constant $t$ time-slices are orthogonal to these geodesics, and the proper distance along these time-slices is $x$. In Appendix A we explain why such a coordinate system can always be chosen.

To represent a black hole spacetime with an asymptotically flat region at $x\to\infty$, we choose $v(x)$ to be a negative, monotonically increasing, function with $v(\infty)=0$. The event horizon is located where the Killing vector becomes lightlike, i.e. where $v(x)=-1$. For the Schwarzschild black hole this coordinate system corresponds to the Painlevé-Gullstrand coordinates [12,13], with $x\equiv r$ and $v(x)=-\sqrt{2GM/x}$. A sketch of the relation between these coordinates and the ingoing Eddington-Finkelstein null coordinate $v$ is given in Fig. 1. (The wave packet trajectory is discussed in Sec. IV.)

A new coordinate $y$ that is constant on the free-fall worldlines $dx=v(x)dt$ is defined by

$$y:=t-\int \frac{dx}{v(x)}. \tag{2}$$

This yields the line element

$$ds^2=dt^2-v^2(x)dy^2 \tag{3}$$

where $x$ is now a function of $t-y$ obtained by solving (2) for $x$. In these coordinates the Killing vector is given by

$$\chi=\partial_t+\partial_y. \tag{4}$$

The action for a real scalar field in these coordinates is

$$S=\frac{1}{2}\int dt dy \sqrt{-g} g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi \tag{5}$$

$$=\frac{1}{2}\int dt dy\left(\left|v(x)\right|(\partial_t\phi)^2-\frac{1}{|v(x)|}(\partial_y\phi)^2\right) \tag{6}$$

and the equation of motion is

$$-\partial_t^2\phi+\frac{1}{v^2(x)}\partial_y^2\phi-v'(x)\partial_t\phi+\left(\frac{v'(x)}{v^2(x)}\right)\partial_y\phi=0. \quad (7)$$

We could now alter the theory to include high frequency dispersion by replacing $\partial_y$ by $F(\partial_y)=\partial_y+a\partial_y^3+\cdots$ in the action (6). This is similar to what was done in the models already studied [1–3], the only difference being that there it was $\partial_x$ that was replaced by $F(\partial_x)$. Since $\partial_y=-v(x)\partial_x$, these two modifications are essentially the same near the horizon where $v(x)=-1$, and in fact they are quite similar in all regions where $v(x)$ is of order unity. It is only asymptotically, where $v(x)$ goes to zero, that their behavior should differ substantially. We previously preferred to modify $\partial_x$ since it is the derivative with respect to *proper* distance on a constant $t$ surface everywhere. Now however we want to *discretize* the spatial coordinate and, as explained in the Introduction, we do not want to discretize $x$ because it is infinitely accelerated at the horizon. Instead, we discretize the free-fall coordinate $y$.

One possible spatial discretization of the action is[4]

$$S=\frac{1}{2}\sum_m\int dt\left(|v_m(t)|(\partial_t\phi_m(t))^2-\frac{(D\phi_m(t))^2}{|v_{m+1}(t)+v_m(t)|/2}\right) \quad (8)$$

where $D$ is the forward differencing operator $D\phi_m(t) := (\phi_{m+1}(t)-\phi_m(t))/\delta$, $\delta$ is the lattice spacing in the $y$ coordinate, and $v_m(t):=v(x(t-m\delta))$. *In the remainder of this paper we shall work in units of the lattice coordinate spacing*, so that $\delta=1$. Varying the action (8) gives the equation of motion for $\phi_m(t)$:

$$\partial_t(v_m(t)\partial_t\phi_m(t))-D\left(\frac{D\phi_{m-1}(t)}{(v_m(t)+v_{m-1}(t))/2}\right)=0. \quad (9)$$

This lattice action has a discrete symmetry

$$(t,m)\to(t+1,m+1) \quad (10)$$

which is the remnant of the Killing symmetry generated by (4). The meaning of this is that shifting forwards in time by one unit at fixed static coordinate $x$ is just enough time for the next lattice point to fall from $x(t,y+1)$ to $x(t,y)$. This symmetry will be heavily exploited in the following analysis.[5]

Note that the $y$ coordinate is infinitely bunched up as $v\to 0$ [see (3)], which occurs at infinity for a black hole type metric. Therefore the uniform discretization $y_m=m$ yields a *proper* lattice spacing that goes to zero at infinity. This is undesirable from a physical point of view, but it is a convenient choice mathematically, since unlike other discretizations it preserves the symmetry (10). Also, as long as we do not try to evolve the scalar field modes all the way to infinity, the decreasing proper lattice spacing is benign and has no effect on the physics of the Hawking process. However, since our goal is to understand how the outgoing modes can be accounted for in a theory that has a ''reasonable'' short distance cutoff, we shall return to this issue in Sec. VI.

The lattice model defined by (8) was studied numerically by Unruh [11]. He found by propagating wave packets backward in time that the outgoing modes come from exotic ingoing modes and, if these ingoing modes are in their ground states, then the outgoing modes are thermally occupied at the Hawking temperature. In the next three sections we use analytic methods to understand the propagation of these wave packets and the computation of the flux of radiation from the black hole. Our results are in agreement with Unruh's numerical results.

## III. LATTICE DISPERSION RELATION

Due to the symmetry (10) of the lattice action (8) there exist mode solutions of the form

$$\phi_m(t)=e^{-i\omega t}f(m-t). \quad (11)$$

Under the discrete symmetry (10) the mode (11) changes by a phase factor as $\phi_m(t)\to e^{-i\omega}\phi_m(t)$. This identifies $\omega$ as the *Killing frequency* which is defined modulo $2\pi n$ and is conserved.

To derive the dispersion relation we plug the ansatz

$$\phi_m(t)=e^{-i\omega t}e^{ik(m-t)}=e^{-i(\omega+k)t}e^{ikm} \quad (12)$$

into the equation of motion (9) and treat $v_m(t)$ as a constant. The result is

$$|v|(\omega+k)=\pm 2\sin(k/2). \quad (13)$$

The *free-fall frequency*, i.e. the frequency measured along the free-fall lines of constant $y$, is defined by $\partial_t\phi=-i\omega_{\rm ff}\phi$. The form of the modes (12) then shows that

$$\omega_{\rm ff}=\omega+k. \quad (14)$$

To understand what range of $\omega$ and $k$ are considered distinct, note that the modes defined by (12) are invariant under the simultaneous shifts

$$k\to k+2n\pi \quad (15)$$

$$\omega\to\omega-2n\pi \quad (16)$$

for any integer $n$. Thus we can transform any $(\omega,k)$ pair into an equivalent pair $(\omega',k')$ where $k'$ lies within a fixed range of length $2\pi$ (the standard choice being $-\pi<k'<\pi$). The value of $\omega'$ is unconstrained with this range of $k'$. One choice of fundamental domain of $(\omega,k)$ pairs is therefore given by

$$-\pi<k<\pi, \quad -\infty<\omega<\infty. \quad (17)$$

Conversely, we could just as well use the above transformation to force $\omega'$ to lie within a fixed range of length $2\pi$ leaving $k'$ arbitrary.

[4]For later convenience in the WKB approximation we take the average of $v_{m+1}$ and $v_m$ in the second term in the action.

[5]The existence of this discrete remnant of the Killing symmetry was pointed out to us by W. G. Unruh. In Secs. VI and VII we study a similar model in which a reparametrization of $y$ is discretized and no discrete symmetry survives.

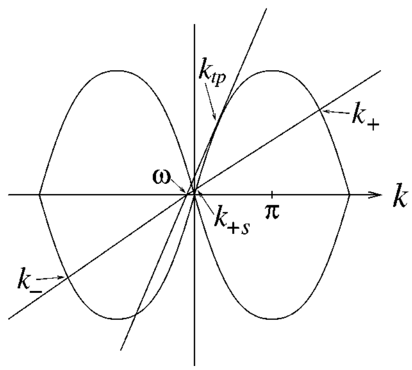


FIG. 2. Graphical representation of the dispersion relation (13).

The dispersion relation (13) has a useful graphical representation (see Fig. 2): On a graph with abscissa $k$, the straight line with slope $|v|$ and $k$-intercept $-\omega$ intersects the curve $\pm 2\sin(k/2)$ at a $k$ that is a solution or ''root'' of the dispersion relation. A wave packet constructed from modes of the form (11) with Killing frequency near $\omega$ will propagate through the lattice spacetime with conserved Killing frequency. This propagation can be represented graphically in the WKB approximation by following a point on the dispersion curve. Since the Killing frequency is conserved, the $k$-intercept of the straight line is fixed, while the slope $|v(x)|$ of the straight line changes according to where the wave packet is located. The direction of motion with respect to the static position coordinate $\xi := y - t = m - t$ is determined by the group velocity $d\xi/dt$ which is given by

$$v_g = d\omega/dk = \pm\frac{\cos(k/2)}{|v|} - 1. \tag{18}$$

Therefore the *sign* of the group velocity is the sign of the difference between the slope of the $\pm\sin$ curve at the intersection point and the slope of the straight line. The group velocity in terms of $y$ is $dy/dt = \pm\cos(k/2)/|v|$, which is always less than the speed of light according to the line element (3).

## IV. ORIGIN OF THE OUTGOING MODES

In this section we argue using the dispersion relation that outgoing low wave vector wave packets indeed originate as ingoing high wave vector wave packets which ''bounce'' off of the horizon. A spacetime diagram of the process is sketched in Fig. 1.

To see where the outgoing modes come from, consider a late-time, positive Killing frequency, outgoing packet centered on a small positive wave vector $k_{+s}$. This wave packet is represented on the dispersion curve in Fig. 2 as the point labeled $k_{+s}$. Following this back in time using the graphical method described in the previous section we find that it moves up the dispersion curve until it reaches the tangency point $k_{tp}$ at which the group velocity (18) vanishes. This is the turning point, where the WKB approximation fails. If $\omega \ll 1$, the straight line is extremely close to the sine curve for many $k$ values. This means that when the wave packet is close to the horizon it is really a superposition of many $k$ values, including negative ones. The amplitude of the negative wave vector piece, which determines the Hawking radiation, is of order $\exp(-\pi\omega/\kappa)$ where $\kappa$ is the surface gravity of the horizon. The positive and negative wave vector pieces both propagate back away from the horizon, evolving into the modes $k_+$ and $k_-$ respectively.[6] Thus we see that the outgoing positive Killing frequency modes come from *ingoing* large wave vector modes which ''bounce'' off the horizon. This continuous evolution from one type of modes to another is called *mode conversion*. The same phenomenon occurs in the continuum models in which the high frequency dispersion is put into the theory by adding higher spatial derivative terms to the action.

Now let us compute the values of the wave vectors $k_\pm$ and $k_{+s}$ corresponding to a fixed frequency $\omega$ as $|v|\to 0$ at infinity. From the dispersion relation (13) or Fig. 2, one sees that all three wave vectors $k_{+s}, k_-$, and $k_+$ converge to zero modulo $2\pi n$ independent of the value of $\omega$. This rather strange result follows because the continuum metric has the form $ds^2 = dt^2 - v^2 dy^2$, and so the $y$-lattice spacing goes to zero as $v$ goes to zero. Therefore any mode of finite proper wavelength will have infinite coordinate wavelength and zero coordinate wave vector. To resolve these modes we can look at their proper wave vectors $k_p = k/|v|$ instead of the coordinate wave vectors. For the $k_{+s}$ wave vector, as $v\to 0$ we may approximate $2\sin(k/2) \approx k$ in the dispersion relation (13) (with the plus sign), which yields $k \simeq |v|\omega$, so the proper wave vector goes to just $\omega$. For the $k_\pm$ wave vectors, we first use the symmetry relation (15),(16) to shift the coordinate wave vectors (and therefore also the frequency $\omega$) so that they converge to zero as $v\to 0$, and then use the small $k$ approximation in the dispersion relation to obtain the proper wave vectors $k_{p,\pm} = -(\omega \pm 2\pi)$. Therefore the late time, long wavelength, outgoing Hawking particle arises from a pair of short but finite proper wavelength ingoing modes. It follows from the discussion below (18) that, at spatial infinity, the group velocity for these wave vectors is equal to the speed of light.

In the next section we compute the amplitudes of the $k_+$ and $k_-$ pieces of the ingoing wave packet. Crucial to the validity of the approximation used in this calculation is the maximum value of the wave vectors in the wave packet solution near the horizon. We can estimate this maximum by a simple calculation using the dispersion relation. The classical turning point is located where the straight line of Fig. 2 is tangent to the sine curve, labeled $k_{tp}$ in the figure. Although the wave packet tunnels beyond the classical turning point, it is not propagating there, so its shortest wavelength near the horizon should be roughly given by the wavelength at the classical turning point. The wave vector at this point satisfies

[6]Other modes get excited as well, but only slightly. From ''reflecting'' off the background curvature a small negative wave vector piece will arise. This will have extremely small amplitude however, for the following reason. There is no scattering at all for a massless scalar field in the continuum due to conformal invariance of the action. On the lattice this symmetry will remain approximately for wavelengths much longer than the lattice spacing, and short wavelength modes will not see the curvature. As $v$ becomes smaller, there are also more wave vector roots to the dispersion relation with $|k| > 2\pi$ which are also presumably excited slightly by scattering.

the dispersion relation (13) (with the plus sign) and the relation

$$|v| = \cos(k/2) \tag{19}$$

expressing equality of the slopes of the two curves. If $\omega \ll 1$ (which is the case of interest when the surface gravity $\kappa \ll 1$), then Fig. 2 shows clearly that $k \ll 1$ as well. Using small $k$ approximations in (13) and (19) respectively and solving for $k$ yields

$$k_{\mathrm{tp}} \approx (12\omega)^{1/3} \ll 1, \tag{20}$$

consistent with our approximations.

This very important result states that although the scale of the new physics is the lattice spacing $\delta(=1)$, the effects of the new physics occur long before that scale is ever reached. With the ordinary wave equation the maximum wave vector near the horizon is *infinite* due to the infinite blueshift (actually it is finite but trans-Planckian if the black hole is formed by collapse). One might have expected that on the lattice $k_{\mathrm{tp}}$ would be of order the inverse lattice spacing $\delta^{-1}$ but (20) shows that this is not the case (although $k \sim \delta^{-1}$ does occur far from the horizon—see for example the roots $k_+$ and $k_-$ in Fig. 2 and the accompanying discussion). This fact—which is also true in continuum models with high frequency dispersion—was not noticed in earlier work on dispersive models. As long as $\omega \ll 1$ (20) shows that the physics near the horizon that determines the Hawking flux depends only on the low order terms in $k$. This result is absolutely essential for the validity of the approximation used in the next section.

In Sec. VI we will discuss ways to avoid the problem of vanishing lattice spacing at infinity. This problem plays no role in the calculation of the rate of particle production however, so we will now explain how this rate can be obtained in a leading order approximation.

## V. HAWKING RADIATION

The lattice theory can be quantized in strict analogy with the quantization of linear field theory in curved spacetime so we will not spell it out here. A difference peculiar to the lattice theory (or dispersive continuum field theories) is that the local notion of the ground state (or vacuum) is not Lorentz invariant but refers to the preferred free-fall frame. In a region where the function $v(x)$ is constant—or is approximately constant on the scale of the relevant wavelengths—the line element (3) is flat and the action (8) is that of a chain of identical masses coupled by identical springs. The ground state of this system is just the usual ground state of the normal modes, i.e., it is annihilated by annihilation operators for complex solutions to the oscillator equation with time dependence of the form $\exp(-i\omega_{\mathrm{ff}}t)$ with *positive* $\omega_{\mathrm{ff}}$, that is, positive free-fall frequency (14). This is the *free-fall vacuum*.

Given this initial vacuum state we would like to compute the particle flux seen by an observer sitting at a fixed location (fixed $x$ coordinate) far outside the black hole. The natural notion of particle for such an observer coincides with that defined by Killing frequency, therefore we shall compute the number expectation value for an outgoing positive Killing frequency packet in a state which at some initial time is the free-fall vacuum. The standard method of computing this [14] is to propagate the outgoing packet backward in time to the hypersurface where the vacuum state is defined. The norm of the negative free-fall frequency part of this packet is then (minus) the number expectation value. The norm referred to here is given by

$$\|\phi\|^2 = i\sum_m |v_m(t)|(\phi_m^*(t)\partial_t\phi_m(t) - \phi_m(t)\partial_t\phi_m^*(t)), \tag{21}$$

and is the sum over a constant $t$ surface of the $t$-component of the current associated with phase invariance of the action (8) (generalized to complex fields).

Several methods can be used to compute the rate of Hawking radiation. One approach is to evolve a wave packet backwards in time by numerical solution of the lattice wave equation (9), as was done by Unruh [11]. Alternatively, since the problem has time translation symmetry, one can just work with modes of definite Killing frequency. This is the approach we take here. The outgoing wave packet is composed of wave vectors around $k_{+s}$ (and has positive Killing frequency) and arises from a pair of packets composed of wave vectors around $k_+$ and $k_-$ respectively (which have positive and negative free-fall frequency respectively). Using the arguments in [3], modified to the lattice model, it is straightforward to show that, for an outgoing packet narrowly peaked about the frequency $\omega$, the number expectation value is

$$N(\omega) = \frac{|(k_-(\omega)+\omega)v_g(k_-(\omega))c_-(\omega)^2|}{|(k_{+s}(\omega)+\omega)v_g(k_{+s}(\omega))c_{+s}(\omega)^2|} \tag{22}$$

where $c_-(c_{+s})$ is the constant coefficient of the $k_-(k_{+s})$ mode located far outside the black hole [where $v(x)$ is essentially constant]. We now turn to the computation of these coefficients.

### A. Mode equation

The mode solutions to the lattice wave equation (9) are of the form (11), (11)

$$\phi_m(t) = e^{-i\omega t} f(m-t), \tag{23}$$

where $\omega$ is the conserved Killing frequency. Plugging this into the equation of motion (9) produces a delay-differential equation (DDE)

$$v(\xi)(f''(\xi) + i2\omega f'(\xi) - \omega^2 f(\xi)) + v'(\xi)(f'(\xi) + i\omega f(\xi)) - \frac{2(f(\xi-1) - f(\xi))}{(v(\xi-1)+v(\xi))} + \frac{2(f(\xi) - f(\xi+1))}{(v(\xi)+v(\xi+1))} = 0, \tag{24}$$

where we have defined the new variable $\xi := (m-t)$, and $v(\xi) := v(x(\xi))$. A wave packet that is outgoing at late times is composed of mode solutions that decay inside the horizon (see [3] for a discussion of the analogous boundary condition in a dispersive continuum model). We therefore need to solve (24) subject to this boundary condition.

The DDE (24) can be solved numerically, however it is more instructive, and sufficient for our purposes, to find an approximate analytic solution. We use the same analytical techniques as used in [6]. We first find an approximate solution (satisfying the above boundary condition) in a neighborhood of the horizon by the method of Laplace transforms, and then extend this solution far outside the black hole by matching to the WKB approximation. The mode coefficients $c_i$ can then be read off directly.

### B. Near horizon approximation

To solve the mode equation (24) near the horizon we first approximate $v(\xi)$ as

$$v(\xi)\approx -1+\kappa\xi, \tag{25}$$

where $\kappa$ is the surface gravity of the black hole, and neglect all terms of order $(\kappa\xi)^2$. This requires that we stay close enough to the horizon that $\kappa\xi \ll 1$.

Next we ''localize'' the DDE by first Taylor expanding $f(\xi-1), v(\xi-1)$, etc., and then truncating the expansions. Which terms to keep can be estimated as follows. The Taylor expansions produce the equation

$$0=v(\xi)(f''(\xi)+i2\omega f'(\xi)-\omega^2 f(\xi))+v'(\xi)(f'(\xi)+i\omega f(\xi))+\left(-\frac{f''(\xi)}{v(\xi)}+\frac{f'(\xi)v'(\xi)}{v^2(\xi)}\right)+\left(-\frac{f^{(iv)}(\xi)}{12v(\xi)}+\frac{f'''(\xi)v'(\xi)}{12v^2(\xi)}+\cdots\right)+\cdots \tag{26}$$

where we have grouped together terms in the expansion according to the total number of derivatives. The ellipses that appear inside parentheses denote other terms with a total of four derivatives and the other ellipses denote terms with six or more derivatives per term (only even numbers of derivatives occur in the expansion). Truncating the equation to second order in derivatives produces the ordinary wave equation. This is not sufficient for us because arbitrarily short wavelengths appear in the ordinary wave equation solution for the outgoing modes, so we must keep at least some of the higher derivative terms.

Let us define an effective local wave vector $k(\xi)$ by $f'(\xi)/f(\xi)=ik(\xi)$. Dropping the $f^{(vi)}(\xi)$ term compared to the $f^{(iv)}(\xi)$ term is accurate provided that $|k(\xi)|\ll 1$ in the near horizon region $|\xi|\ll 1/\kappa$. We can estimate $k(\xi)$ from the dispersion relation in the near horizon approximation just as we did in Sec. (IV). Outside the classical turning point (where $\xi_{tp}\sim\omega^{2/3}/\kappa$), but still in a region where $\xi\ll 1/\kappa$, all relevant wave vectors are real and the largest wave vector behaves as $k(\xi)\sim\sqrt{\kappa\xi}$, and therefore satisfies $|k(\xi)|\ll 1$. For $|\xi|<\xi_{tp}$, the relevant wave vector becomes complex and has a magnitude $|k(\xi)|\sim\omega^{1/3}$, therefore $|k(\xi)|\ll 1$ provided we only consider Killing frequencies satisfying $\omega^{1/3}\ll 1$. Even deeper inside the horizon where $-1/\kappa\ll\xi<-\xi_{tp}$, the wave vector is approximately imaginary with magnitude again given by $k(\xi)\sim\sqrt{\kappa|\xi|}$, and therefore $|k(\xi)|\ll 1$. Ignoring sixth and higher order derivatives in the equation (26) therefore requires that $\omega^{1/3}\ll 1$.

To further simplify the equation, note that the ratio of the $f^{(iv)}$ term to the $f'''$ term is

$$\left|\frac{f^{(iv)}(\xi)}{f'''(\xi)v'(\xi)}\right|\sim\frac{k(\xi)}{\kappa}. \tag{27}$$

From above we know that[7] $k(\xi)\gtrsim\omega^{1/3}$, so we will have $\kappa \ll |k(\xi)|$ provided that $\omega\gg\kappa^3$. As long as $\omega$ is not ultra small therefore we need only keep the fourth order derivative term[8] in the expansion (26). We therefore arrive at the ordinary differential equation (ODE)

$$\frac{1}{12}f^{(iv)}-2\kappa\xi f''-2(i\omega-\kappa)f'-i\omega(i\omega-\kappa)f\approx 0. \tag{28}$$

We show below by explicit calculation that the solution to (28) of interest to us is consistent with the approximations made above and therefore that this truncation of the mode equation is valid.

The ODE (28) is the same as that considered in [6] (except for the coefficient of the $f^{(iv)}$ term) where it was solved by the method of Laplace transforms with the same boundary conditions as discussed above. We therefore refer the reader to [6] for the details of this computation. Using the saddle point approximation to evaluate the Laplace transform for $\xi\gg 1$, we find that the solution satisfying the given boundary conditions can be expressed as

$$f(\xi)=f_+(\xi)+f_-(\xi)+f_{+s}(\xi) \tag{29}$$

where

$$f_+(\xi)\approx iNe^{3\pi\omega/(2\kappa)}\xi^{-3/4-i\omega/(2\kappa)}\exp\left(i\frac{2}{3}\sqrt{24\kappa}\xi^{3/2}\right) \tag{30}$$

$$f_-(\xi)\approx Ne^{\pi\omega/(2\kappa)}\xi^{-3/4-i\omega/(2\kappa)}\exp\left(-i\frac{2}{3}\sqrt{24\kappa}\xi^{3/2}\right) \tag{31}$$

$$f_{+s}(\xi)\approx 2e^{\pi\omega/\kappa}\sinh(\pi\omega/\kappa)\Gamma(-i\omega/\kappa)\xi^{i\omega/\kappa} \tag{32}$$

and

$$N:=e^{i\pi/4}\sqrt{2\pi}(6\kappa)^{1/4}(\sqrt{24\kappa})^{-1-i\omega/\kappa}. \tag{33}$$

To check the validity of our localization procedure, note for example that

$$\frac{f'_+(\xi)}{f_+(\xi)}=\left(\left(\frac{3}{4}+i\frac{\omega}{2\kappa}\right)\frac{1}{\xi}-i\sqrt{24\kappa\xi}\right). \tag{34}$$

The absolute values of the two terms on the right-hand-side of (34) are both much less than one provided we restrict $\xi$ to the range

$$1\ll\xi\ll\kappa^{-1} \tag{35}$$

which was already assumed in making the saddle point approximation (29) and the near horizon approximation (25). Expression (34) is also in agreement with our earlier esti-

[7]Actually the wave vector of the outgoing wave packet is smaller than this. For the outgoing packet though, all higher order derivative terms are negligible outside the classical turning point.

[8]We could in principle keep the third order derivative term as well and therefore enlarge the range of validity of our approximations in $\omega$, however for simplicity we work with the simpler equation.

mates of $f'(\xi)$ obtained by estimating the position dependent wave vector $k(\xi)$. Similar relations hold for the $f_-(\xi)$ and $f_{+s}(\xi)$ modes as well.

### C. Match to the far zone

The next step is to propagate the mode (29) away from the horizon to the constant $v(\xi)$ region. This is accomplished by computing approximate solutions to the non-local DDE (24) by the WKB method. [Since the wave vectors grow to order unity as $v(\xi)$ goes to zero, we must use the full non-local DDE at this stage.] Some details of this computation are given in Appendix B. The result is that there exist three different WKB solutions which, when evaluated near (but not too near) the horizon, take the same functional forms as the Laplace transform solutions given by (30), (31), (32). An appropriate linear combination of these WKB solutions can therefore be matched to the near horizon solution (29) yielding

$$f(\xi)=\sqrt{2\pi\kappa}(e^{3\pi\omega/(2\kappa)}f_+^{WKB}(\xi)+e^{\pi\omega/\kappa}f_-^{WKB}(\xi))$$
$$+2e^{\pi\omega/\kappa}\sinh\left(\frac{\pi\omega}{\kappa}\right)\Gamma\left(-i\frac{\omega}{\kappa}\right)f_{+s}^{WKB}(\xi). \qquad (36)$$

Since the WKB approximation holds far outside the horizon, we are free to evaluate the solution there, and thus read off the constant coefficients of the modes $\exp(ik\xi)$ with $k=k_{+s},k_+,k_-$ in the constant $v(x)$ region. These coefficients are simply given by the coefficients of the WKB solutions in (36) except that $f_\pm^{WKB}$ also contain the amplitude factors $(\omega\pm2\pi)^{-1/2}$ respectively (see Appendix B).

### D. Kinematic factors

The only remaining ingredient in evaluating the number expectation value (22) is to compute the kinematic factors $[k(\omega)+\omega]$ and group velocity $v_g$ for each wave vector. From the dispersion relation (13) (with the plus sign for the roots $k_{+s},k_+,k_-$ corresponding to A, D and E respectively in Fig. 2) and the expression for the group velocity given by (18) it is straightforward to show that

$$(k(\omega)+\omega)v_g(\omega)=\frac{\cos(k/2)-|v|}{|v|^2/2}\sin(k/2). \qquad (37)$$

Plugging in the small $|v|$ expressions for the $k_-$ and $k_{+s}$ wave vectors computed in Sec. IV, we find that (37) reduces to $-(\omega-2\pi)/|v|$ for the $k_-$ root and $\omega/|v|$ for the $k_{+s}$ root. Putting all these results together we find, for the number expectation value (22),

$$N(\omega)=\frac{1}{e^{\omega/T_H}-1} \qquad (38)$$

where $T_H=\kappa/2\pi$ is the Hawking temperature. Therefore we see that to leading order in the lattice spacing the particle flux is thermal at the Hawking temperature in agreement with the ordinary wave equation.

This derivation is valid as long as (i) $\kappa\ll\omega^{1/3}\ll1$ and (ii) the WKB approximation can be used to connect the far zone with the zone $\kappa\xi\ll1$ near the horizon. This last condition should be satisfied as long as $\omega$ is not extremely small compared to $\kappa$, although we shall not attempt to write out the general conditions here (which are possibly more restrictive than the $\kappa\ll\omega^{1/3}$ condition already given).

## VI. MODELS WITH FINITE LATTICE SPACING AT SPATIAL INFINITY

One way to avoid the problem of vanishing lattice spacing at infinity is to simply not let $v(x)$ go to zero at infinity. It might seem that we have no freedom to make this choice, since the asymptotic form of the metric is determined by the black hole. However, we need not use a free-fall coordinate that is *at rest* at infinity. Instead, the coordinate lines can be chosen moving uniformly toward the black hole at infinity. In Appendix A it is shown that, in terms of the proper time $t'$ along the congruence of infalling geodesics of energy $E>1$ and the proper distance $x'$ along the spacelike slices orthogonal to these geodesics, the line element takes the form $dt'^2-(dx'-v_E(x')dt')^2$ for some function $v_E$. Note that this is the same form as (1), with a different function $v_E\neq v$ which, in particular, does not vanish at infinity: $v_E(\infty)=-(E^2-1)^{1/2}$. Proceeding as before one then arrives at the new line element (3), but with $v$ replaced by $v_E$. With this choice the preferred frame is not asymptotically at rest with respect to the black hole. Although this certainly solves the problem from a mathematical point of view, it is not physically satisfactory. Our ‘‘in’’ vacuum boundary condition depends on the choice of the preferred frame, and it just does not make much sense to rely on the assumption that the black hole is moving relative to the vacuum.

A more satisfactory resolution would be to choose the discretization such that the lattice spacing is a fixed proper distance on some initial slice. If we then let the lattice points fall into the black hole, the proper lattice spacing will not remain constant on the surfaces of equal proper time. Nevertheless, such a lattice will be perfectly well behaved at infinity, and the time dependence will be invisible to long wavelength modes that do not ‘‘see’’ the lattice at all. Although they are not ultimately satisfactory, we think it is instructive to understand the physics of such models with growing lattice spacing. We now describe a class of such models.

It is only necessary to reparametrize the $y$ coordinate (2) before discretizing. To this end, we define a new coordinate $z$ by

$$W(z)=y=t-\int_{x_h}^{x}dx'/v(x'), \qquad (39)$$

where $x_h$ is the value of $x$ at the event horizon, i.e., $v(x_h)=-1$. The original $x$ coordinate measures proper length on a constant $t$ surface in the metric (1), so we choose $z$ to agree with $x$ at $t=0$. This implies

$$W(z):=-\int_{x_h}^{z}dx'/v(x'). \qquad (40)$$

In terms of the function $W$, the defining relation for $z$ can be written as

$$W(z)=t+W(x), \tag{41}$$

which can be solved for $x(t,z)$ as

$$x=W^{-1}(W(z)-t). \tag{42}$$

In the coordinates $(t,z)$ the line element (1) becomes

$$ds^2=dt^2-\left(\frac{v(x)}{v(z)}\right)^2 dz^2, \tag{43}$$

where $x(t,z)$ is the function defined by (42). In these coordinates the Killing vector $\chi$ [which is $\partial_t$ in the $(t,x)$ coordinates and $\partial_t+\partial_y$ in the $(t,y)$ coordinates (4)] is given by

$$\chi=\partial_t-v(z)\partial_z. \tag{44}$$

When $\partial_z$ is modified in the action, either by higher derivatives or discretization, the presence of the factor $v(z)$ in (44) will prevent the survival of the symmetry generated by $\xi$. Not even a discrete remnant of the symmetry survives in the discrete case.

At any finite $t$, the spatial scale factor $v(x)/v(z)$ goes to unity as $z$ goes to infinity, as long as $v(x)$ goes to a constant (including zero) at infinity. Thus, the coordinate $z$ always measures proper distance sufficiently far from the black hole. Along a line of fixed $z$, $v(x)/v(z)$ grows as a function of $t$ as the horizon is approached, since $x$ is getting smaller and we are assuming $|v(x)|$ grows as $x$ decreases. That is, the proper spacing of the $z$ coordinate grows with $t$ because of the relative acceleration of the free-fall worldlines.

At the horizon $v(x_h)=-1$, $W(x_h)=0$, and therefore $z=W^{-1}(t)$. This yields the form of the line element evaluated at the horizon:

$$ds^2|_{\text{horizon}}=dt^2-[v(W^{-1}(t))]^{-2}dz^2. \tag{45}$$

Let us now consider two examples to see what this coordinate change yields. First, consider the Schwarzschild line element, for which $v(x)=-(2\kappa x)^{-1/2}$, where $\kappa$ is the surface gravity $1/4GM$. In this case the line element (43) becomes

$$ds^2=dt^2-\left(1-\frac{3t}{2(2\kappa z^3)^{1/2}}\right)^{-2/3}dz^2, \tag{46}$$

and at the horizon this reduces to

$$ds^2|_{\text{horizon}}=dt^2-(1+3\kappa t)^{2/3}dz^2. \tag{47}$$

For numerical calculation, it would be more convenient to have $v(x)$ go to zero more quickly than $x^{-1/2}$, so let us also consider the exponential velocity $v(x)=-\exp(-\kappa x)$. In this case the line element (43) becomes

$$ds^2=dt^2-(1-\kappa te^{-\kappa z})^{-2}dz^2, \tag{48}$$

and at the horizon this reduces to

$$ds^2|_{\text{horizon}}=dt^2-(1+\kappa t)^2dz^2. \tag{49}$$

Discretizing the $z$ coordinate will yield a new lattice theory in which the proper lattice spacing is constant at infinity, so it is possible to propagate wave packets in a sensible way all the way out to where $v(z)\approx 0$. Therefore the ingoing waves that produce the outgoing waves must originate at infinity as combinations of the standard flat space lattice modes. No exotic low frequency modes are available in this case. The low frequency ingoing waves behave like ordinary continuum ingoing waves which sail right across the horizon. They will *not* bounce off the horizon. So where can a low frequency outgoing mode come from?

The lack of even a discrete time translation symmetry seems to provide the answer. When a low frequency outgoing wave packet is propagated back close to the horizon, it gets blueshifted. Eventually its wave vector gets so large that it can sense the lack of time translation invariance in the lattice theory. At that point, there is no longer any reason for its Killing frequency to be conserved. Using an eikonal approximation we will show in the next section that the Killing frequency is indeed shifted so that, when the wave packet propagates backwards in time back out to infinity, it arrives with a large wave vector, on the order of the lattice spacing, and a correspondingly large Killing frequency. At this stage we have no solid proof that waves on the $z$-lattice will behave in the way indicated by the eikonal approximation. It should be possible to adapt Unruh's numerical computation on the $y$-lattice to see what in fact happens on the $z$-lattice.

## VII. ORIGIN OF THE OUTGOING MODES REVISITED

In deriving the eikonal approximation we forget that the space is discrete and just make the substitution $\partial_z\rightarrow\exp(\partial_z)-1$ in the continuum action in $(t,z)$ coordinates (in units of the lattice spacing):

$$S=\frac{1}{2}\int dtdz\left(\frac{1}{\sqrt{-g^{zz}}}(\partial_t\phi)^2-\sqrt{-g^{zz}}((e^{\partial_z}-1)\phi)^2\right). \tag{50}$$

This leads to an infinite order PDE to which the standard eikonal or geometrical optics approximation can be applied. One assumes that the wavelength and period of the wave are short compared with the length and time scales on which the background is varying and slowly changing on their own scales. This is reasonable for much of the trajectory of the wave packets we are interested in, but the latter condition fails at the turning point near the horizon. Nevertheless, the results obtained in this way seem reasonable and we would be surprised if a lattice calculation failed to confirm the general picture provided by this approximation.

Making this approximation, and assuming a wave of the form

$$\exp(-i\omega t)\exp(ikz), \tag{51}$$

we arrive at the dispersion relation

$$\omega^2=-g^{zz}(t,z)(F(k))^2, \tag{52}$$

where the function $F(k)$ is given by

$$F(k)=2\sin(k/2) \tag{53}$$

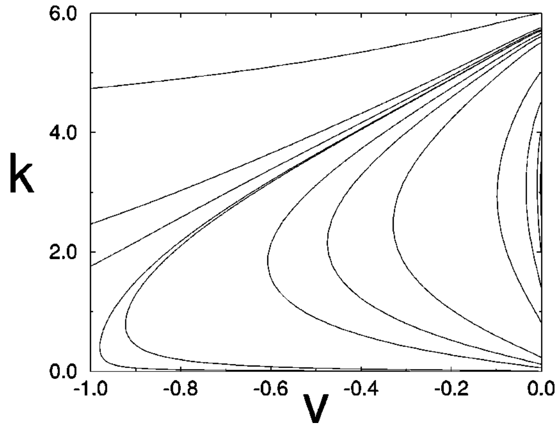


FIG. 3. Plot of the wave vector trajectories as a function of the background free-fall velocity function $v(x)$.

and, using (43),

$$-g^{zz}(t,z)=[v(z)/v(x(t,z))]^2. \tag{54}$$

Note that now $\omega$ (rather than $\omega_{\rm ff}$) stands for the free-fall frequency.

The eikonal approximation in this case amounts to Hamilton's equations for the phase space variables $(z,k)$ with the Hamiltonian

$$H=\pm\sqrt{-g^{zz}}F(k). \tag{55}$$

The free-fall frequency is just $\omega=H$, so the sign of this frequency is the sign of $\pm F(k)$. On the lattice, wave vectors differing by $2\pi n$ are identified, so a complete set of $k$ values is the interval $[0,2\pi)$. In this interval $F(k)$ in (53) is positive, so the sign of $\omega$ is the sign of the prefactor $\pm$. Instead of keeping this prefactor alternative, we can double the range of $k$ to $(-2\pi,2\pi)$ and always use the $+$ sign in the Hamiltonian (55), since $F(k)=-F(-k)$ is negative when $k\in(-2\pi,0)$.

Hamilton's equations are

$$dz/dt=\sqrt{-g^{zz}}\partial_k F \tag{56}$$

$$dk/dt=-\partial_z\sqrt{-g^{zz}}F. \tag{57}$$

We have solved these equations numerically for the case of the exponential velocity function $v(x)=-\exp(-\kappa x)$, for which (54) yields

$$\sqrt{-g^{zz}(t,z)}=1-\kappa t e^{-\kappa z}. \tag{58}$$

We used $\kappa=0.001$ and started the trajectories at the initial position $z(0)=10{,}000$ at $t=0$. The unit here is the lattice spacing in the $z$-coordinate. For each initial wave vector $k(0)$ we obtain a trajectory $[k(t),z(t)]$. To visualize the results, it is convenient to plot $k(t)$ versus $v(x(t))$ because the value of $v(x)$ indicates the static radial position whereas the $z$ coordinate lines are falling. [We could also have plotted versus $x(t)$ itself but it is helpful to be able to see the value of $v(x)$ on the same graph.] The results are given in Fig. 3. The equations of motion (56) and (57) are symmetric under $k\rightarrow -k$, so the solutions for negative $k$'s are obtained by changing the sign of $k$.

At spatial infinity, where $v(x)=0$, the right moving modes have $k\in(0,\pi)$ and the left moving modes have $k\in(\pi,2\pi)$. Thus we send in modes with $k$ in $(\pi,2\pi)$. The ones near $2\pi$ are equivalent to ordinary small negative $k$ modes and just cross the horizon. Since the group velocity (56) is always less than or equal to the speed of light $[-g_{zz}(dz/dt)^2=(\partial_k F)^2=\cos^2(k/2)\leq 1]$, these modes can never return to the outside once having crossed the horizon. Coming down from $2\pi$, at some critical value of $k$ there is a trajectory that asymptotes to the horizon and zero $k$. Below this critical $k$ are the exotic modes that bounce off the horizon and return to spatial infinity. The crucial thing to notice here is that an exotic ingoing mode can produce a nonexotic, very low wave vector outgoing mode. This is only possible because the lattice equations violate time translation symmetry at short wavelengths, so there is no conservation of Killing frequency to prevent this from happening.

## VIII. DISCUSSION

It is intriguing that violation of time-translation invariance visible only at short wavelengths plays a crucial role in accounting for the outgoing modes. In our model this time-dependence is a consequence of the growing lattice spacing due to spreading of free-fall trajectories. At a more fundamental level, one expects the Killing symmetry of a black hole background to be violated by the gravitational back-reaction to the quantum fluctuations of the matter fields. A vague suggestion was made in [15] that the back-reaction might evade the conservation of Killing frequency and allow the outgoing modes to originate as ingoing modes from spatial infinity. Our simple model studied here seems to lend credence to this hypothesis, although the implementation is still in a background field approximation and has nothing obvious to do with the back-reaction.

It is scary to be violating time-translation invariance in the lattice theory. However, the characteristic time scale is long, $\kappa^{-1}$ according to either (47) or (49) for example, and even this time dependence is invisible to wavelengths long compared with the lattice spacing. It therefore seems that the low energy physics is immune from *direct* effects of this violation of time-translation symmetry, even though the outgoing modes owe their very existence to this violation.

We still do not have a satisfactory discretization of field theory in a black hole background. Either our lattice spacing goes to zero at infinity, or it grows as points fall in towards the horizon. For the Schwarzschild metric, the total amount of growth during the Hawking lifetime $M^3$ is, from (47), of order $M^{2/3}$. Thus if the lattice starts out with Planck spacing, it ends up with spacing of one angstrom after the evaporation of a solar mass black hole. But this is only the radial spacing. If the lattice points are falling on radial trajectories from radius $r_2$ to $r_1$ their transverse proper spacing *decreases* by the factor $r_1/r_2$.

It seems that to maintain a uniform lattice spacing in some preferred frame with a freely falling lattice of fixed topology is not possible. This suggests that one should be thinking about a lattice in which points can be created or annihilated in order to keep the spacing uniform.

An expanding cosmology provides a simpler setting than the black hole in which to contemplate the lattice question. As the universe expands, the lattice spacing will grow if the lattice points are at rest in the cosmic rest frame. Weiss [16] confronted this issue in trying to formulate lattice field theory in an expanding universe. He noted a very interesting point: if the couplings of an interacting field theory are fixed on the expanding lattice, then the renormalized parameters at a fixed proper scale will depend strongly on the cosmological epoch. One could of course adjust the lattice parameters as the scale factor evolves, but from a fundamental point of view that is artificial. Moreover, if the lattice spacing started out in the early universe at the Planck scale, it would quickly become too large to appear continuous at large scales. Both these problems would be eliminated if the lattice were itself dynamical, with points being added at the right rate to keep their density constant.

Allowing the lattice topology to be dynamical thus seems very natural. It would be interesting to see if field theory can be sensibly formulated on dynamical lattice models and, if so, to study the consequences for cosmology and black hole physics.

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## APPENDIX A: FREE-FALL COORDINATES

In this appendix we show that in a general static two-dimensional spacetime coordinates can always be chosen (at least locally) so the line element takes the form (1). Let $\chi^a$ be the time-translation Killing field, let $u^a$ be the unit tangent vector to a congruence of timelike geodesics all of the same energy $E$ and invariant under the symmetry, and let $s^a$ be the (unique up to sign) unit vector orthogonal to $u^a$. Then $u^a = E^{-1}\chi^a + v s^a$, where $v^2 = 1-(\chi^2/E^2)$. The assumed symmetry of $u^a$ implies $[\chi,s]=0$, so there exist coordinates $\tau$ and $x$ such that $E^{-1}\chi^a = (\partial_\tau)^a$ and $s^a = (\partial_x)^a$. In these coordinates the line element takes the form

$$ds^2 = (1-v^2)d\tau^2 + 2v\,d\tau dx - dx^2 \tag{A1}$$

$$= d\tau^2 - (dx - v\,d\tau)^2. \tag{A2}$$

Note that $\tau$ coincides with the proper time along the orbits of $u^a$, the lines of constant $\tau$ are orthogonal to these orbits, and $x$ measures the proper distance along these lines. Note also that, because of the symmetry, $v(t,x)=v(x)$ depends only on the coordinate $x$. If $\chi$ is normalized at infinity we have $v(\infty)=\pm(1-E^{-2})^{1/2}$.

## APPENDIX B: WKB SOLUTIONS TO THE DDE

In this appendix we discuss the application of the WKB approximation to finding approximate solutions to the DDE (24). We assume a solution of the form

$$f(\xi) = \exp\left(+i\int^{\xi} k(\xi)\right) \tag{B1}$$

and substitute into the DDE (24). This results in the equation

$$v(\xi)(+ik'(\xi) - k^2(\xi) - 2\omega k(\xi) - \omega^2) + iv'(\xi)(k(\xi)+\omega) \tag{B2}$$

$$-\frac{2\left[\exp\left(-i\int_{\xi}^{\xi-1}k(u)\right)-1\right]}{(v(\xi-1)+v(\xi))} + \frac{2\left[1-\exp\left(-i\int_{\xi}^{\xi+1}k(u)\right)\right]}{(v(\xi)+v(\xi+1))} = 0. \tag{B3}$$

We can rewrite the exponentials in a form more appropriate for the WKB approximation by Taylor expanding the integrand about $\xi$ and then evaluating the integrals, e.g.,

$$\int_{\xi}^{\xi+1} k(u)du = k(\xi) + \frac{1}{2}k'(\xi) + \cdots . \tag{B4}$$

For bookkeeping purposes, it is now convenient to make the substitution $\xi \to \alpha\xi$, which has the effect of scaling $n$th order derivatives in the equation by $1/\alpha^n$. Now expand $k(\xi)$ as

$$k(\xi) = k_0(\xi) + \frac{1}{\alpha}k_1(\xi) + \cdots , \tag{B5}$$

substitute into (B3), and demand that each coefficient of the separate powers of $1/\alpha$ vanish. The leading order equations are

$$v^2(\xi)(k_0(\xi)+\omega)^2 = [2\sin(k_0(\xi)/2)]^2 \tag{B6}$$

$$k_1 = +\frac{i}{2}\frac{d}{d\xi}\ln\left(v(\xi)(k_0(\xi)+\omega) - \frac{\sin(k_0(\xi))}{v(\xi)}\right). \tag{B7}$$

The first of these equations is of course the dispersion relation (13) that we derived in Sec. III, while the second produces the first order correction to the leading order root from the dispersion relation.

To solve the dispersion relation near the horizon (where $v \approx -1$) note that when $\omega \ll 1$ then $2\sin(k_0/2) \approx (k_0 - k_0^3/24)$. Using this approximation it is straightforward to show that the roots are

$$k_{0,\pm}(\xi) \approx \pm\sqrt{24(1-|v(\xi)|)} - \frac{\omega v^2(\xi)}{2(1-|v(\xi)|)} \tag{B8}$$

$$k_{0,+s} \approx \frac{\omega|v(\xi)|}{1-|v(\xi)|}. \tag{B9}$$

Substituting these into the expression for $k_1$ above gives the first order correction term.

To match the WKB solutions given here to the Laplace transform solutions given in Sec. IV we need only substitute the near horizon expansion for $v(\xi) \approx -1 + \kappa\xi$ into the expressions for $k_0$ and $k_1$ and evaluate the integrals given in (B1). Note that $k_1$ will yield in general a non-trivial amplitude factor.

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