PHYSICAL REVIEW D VOLUME 57, NUMBER 10 15 MAY 1998

Dilatonic black holes in higher curvature string gravity. II. Linear stability

P. Kanti,[1] N. E. Mavromatos,[2] J. Rizos,[1] K. Tamvakis,[1] and E. Winstanley[2]
[1]*Division of Theoretical Physics, Physics Department, University of Ioannina, Ioannina GR 451 10, Greece*
[2]*University of Oxford, Department of Physics (Theoretical Physics), 1 Keble Road, Oxford, OX1 3NP United Kingdom*
(Received 27 March 1997; published 28 April 1998)

We demonstrate the linear stability of dilatonic black holes appearing in a string-inspired higher-derivative gravity theory with a Gauss-Bonnet curvature-squared term. The proof is accomplished by mapping the system to a one-dimensional Schrödinger problem which admits no bound states. This result is important in that it constitutes a linearly stable example of a black hole that bypasses the ''no-hair conjecture.'' However, dilaton hair is *secondary* in the sense that it is not accompanied by any new quantum number for the black hole solution. [S0556-2821(98)03310-4]

PACS number(s): 04.70.Bw, 04.20.Jb, 04.50.+h, 11.25.Mj

I. INTRODUCTION

In Ref. [1] we presented analytic arguments in favor of, and demonstrated numerically, the existence of dilatonic black hole solutions with nontrivial scalar hair, in a string-inspired higher-derivative gravity theory with a Gauss Bonnet (GB) curvature-squared term. The numerical solutions clearly demonstrated the existence of a regular event horizon and asymptotic flatness of the four-dimensional spherically symmetric space-time configurations considered in the analysis. There is a nontrivial dilaton (global) charge, which, however, is related to the Arnowitt-Deser-Misner (ADM) mass of the black hole, and hence the hair is of secondary type [2]: the gravitational field acts as a source for the scalar hair and one does not obtain a new independent set of quantum numbers characterizing the black hole.[1] Subsequent to the work of Ref. [1], other researchers confirmed these results by discussing the internal structure of the solutions behind the horizon, and demonstrating numerically the existence of curvature singularities [5]. Also an extension of the analysis of Ref. [1] to incorporate gauge fields became possible [6,7].

The key feature for the existence of such hairy black holes is the bypass of the no-scalar-hair theorems [8] due to the fact that, as a result of the GB term, the scalar field stress tensor becomes *negative* near the horizon [1], thereby violating one of the main assumptions in the proof of the no-hair theorem [8]. An additional element was the fact that the higher-derivative GB term provides a sort of ''repulsion'' that balances the gravitational attraction of the standard Einstein terms, and a black hole is formed. In this respect the GB term plays a similar role to the *non-Abelian* gauge field kinetic terms in Einstein-Yang-Mills-Higgs theories [4], which are also notable exceptions of the no-scalar-(Higgs-) hair theorem.

An important question which arises concerns the stability of the dilatonic black holes. In the Yang-Mills case, the structures are similar to the sphaleron solutions of flat-space Yang-Mills theories, and thus unstable [4]. This may be easily understood by the fact that the black hole solutions owe their existence to a delicate balance between the gravitational attraction and the Yang-Mills repulsive forces. On the other hand, the dilatonic black hole solutions are entirely due to the existence of a *single* force, that of gravity. This already prompts one to think that such structures might be stable.

It is the purpose of this article to argue that the dilatonic black holes are indeed stable under linear time-dependent perturbations of the classical solutions. To this end, we shall map the system of gravitational-dilaton equations for spherically symmetric solutions into a one-dimensional Schrödinger problem, where the instabilities are equivalent to bound states. We shall prove that our dilaton-graviton system admits *no bound states*. This result is important, since it constitutes an example of a hairy black hole structure that appears to be, at least linearly, stable. Its importance is also related to the fact that such higher curvature gravity theories are effective theories obtained from superstrings, which may imply that there is plenty of room in the gravitational sector of string theory to allow for physically sensible situations that are not covered by the no-hair theorem as stated [8]. Unfortunately, at present nonlinear stability of the dilaton-graviton-GB system cannot be checked analytically, and is left for future investigations.

II. RELEVANT FORMALISM

We start by considering the action of the Einstein-Dilaton-Gauss-Bonnet (EDGB) theory:

$$S=\int d^4x\ \sqrt{-g}\left(\frac{R}{2}+\frac{1}{4}\partial_\mu\phi\partial^\mu\phi+\frac{\alpha' e^\phi}{8g^2}\mathcal{R}^2_{GB}\right), \qquad (1)$$

where

$$\mathcal{R}^2_{GB}=R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}-4R_{\mu\nu}R^{\mu\nu}+R^2. \qquad (2)$$

[1]A similar situation occurs in the Higgs hair of the Einstein-Yang-Mills-Higgs systems ([3] and [4]), where the Yang-Mills field acts as a source for the nontrivial configurations of the Higgs field outside the horizon.

The spherically symmetric ansatz for the metric takes the form

$$ds^2 = e^{\Gamma(r,t)}dt^2 - e^{\Lambda(r,t)}dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\varphi^2). \quad (3)$$

The equations of motion derived from Eq. (1) are:

$$\phi'' + \phi'\left(\frac{\Gamma'}{2} - \frac{\Lambda'}{2} + \frac{2}{r}\right) - e^{\Lambda-\Gamma}\left[\ddot\phi + \frac{\dot\phi}{2}(\dot\Lambda - \dot\Gamma)\right]$$
$$= \frac{\alpha' e^{\phi}}{g^2 r^2}\left\{\Gamma'\Lambda' e^{-\Lambda} - \dot\Lambda^2 e^{-\Gamma} + (1-e^{-\Lambda})\right.$$
$$\times\left[\Gamma'' + \frac{\Gamma'}{2}(\Gamma' - \Lambda')\right] - (1-e^{-\Lambda})\, e^{\Lambda-\Gamma}$$
$$\left.\times\left[\ddot\Lambda + \frac{\dot\Lambda}{2}(\dot\Lambda - \dot\Gamma)\right]\right\}, \quad (4)$$

$$\Lambda'\left[1 + \frac{\alpha' e^{\phi}\phi'}{2g^2 r}(1-3e^{-\Lambda})\right]$$
$$= \frac{r}{4}(\phi'^2 + e^{\Lambda-\Gamma}\dot\phi^2) + \frac{1-e^{\Lambda}}{r} + \frac{\alpha' e^{\phi}}{g^2 r}(1-e^{-\Lambda})$$
$$\times\left[\phi'' + \phi'^2 - \frac{\dot\phi\dot\Lambda}{2}e^{\Lambda-\Gamma}\right], \quad (5)$$

$$\Gamma'\left[1 + \frac{\alpha' e^{\phi}\phi'}{2g^2 r}(1-3e^{-\Lambda})\right]$$
$$= \frac{r}{4}(\phi'^2 + e^{\Lambda-\Gamma}\dot\phi^2) + \frac{e^{\Lambda}-1}{r} + \frac{\alpha' e^{\phi}}{g^2 r}(1-e^{-\Lambda})\, e^{\Lambda-\Gamma}$$
$$\times\left[\ddot\phi + \dot\phi^2 - \frac{\dot\phi\dot\Gamma}{2}\right], \quad (6)$$

$$\dot\Lambda\left[1 + \frac{\alpha' e^{\phi}\phi'}{2g^2 r}(1-3e^{-\Lambda})\right] = \frac{r\dot\phi\phi'}{2} + \frac{\alpha' e^{\phi}}{g^2 r}(1-e^{-\Lambda})$$
$$\times\left(\dot\phi\phi' + \dot\phi' - \frac{\dot\phi\Gamma'}{2}\right), \quad (7)$$

$$\Gamma'' + \frac{\Gamma'}{2}(\Gamma' - \Lambda') + \frac{(\Gamma' - \Lambda')}{r} - e^{\Lambda-\Gamma}\left[\ddot\Lambda + \frac{\dot\Lambda}{2}(\dot\Lambda - \dot\Gamma)\right]$$
$$= +\frac{1}{2}(e^{\Lambda-\Gamma}\dot\phi^2 - \phi'^2) + \frac{8\alpha'}{r}e^{-\Lambda}\left\{f'\Gamma'' + f''\Gamma'\right.$$
$$+ \frac{f'\Gamma'}{2}(\Gamma' - 3\Lambda') + e^{\Lambda-\Gamma}\left[\Lambda'\ddot f - 2\dot f'\dot\Lambda\right.$$
$$\left.\left. + \frac{\dot f\dot\Lambda}{2}(\Gamma' + \Lambda') - f'\ddot\Lambda + (f'\dot\Lambda - \Lambda'\dot f)\,\frac{(\dot\Gamma + \dot\Lambda)}{2}\right]\right\}, \quad (8)$$

where $f = e^{\phi}/8g^2$ and the prime and overdot denote differentiation with respect to r and t, respectively.

For later use we note that for the derivatives of the dilaton field at the horizon in the static case [1] one has the following behavior:

$$\phi_h' = \frac{g^2}{\alpha'} r_h e^{-\phi_h}\left(-1 \pm \sqrt{1 - \frac{6(\alpha')^2 e^{2\phi_h}}{g^4 r_h^4}}\right), \quad (9)$$

which implies that black hole solutions exists only if

$$e^{\phi_h} < \frac{g^2 r_h^2}{\alpha'\sqrt{6}}. \quad (10)$$

We now notice that (1) only one of the two branches of solutions in Eq. (9), the one with the + sign, leads to asymptotic flatness of the fields, and this is the branch we shall consider here. The equation for ϕ'' near r_h is

$$\phi'' = -\frac{1}{2}\,\frac{\left[\left(\frac{\alpha'}{g^2}\right)e^{\phi}\phi' + 2r\right]\left[6\left(\frac{\alpha'}{g^2}\right)e^{\phi} + \left(\frac{\alpha'}{g^2}\right)e^{\phi}\phi'^2 r^2 + 2\phi' r^3\right]}{-6\left(\frac{\alpha'^2}{g^4}\right)e^{2\phi} + \left(\frac{\alpha'}{g^2}\right)e^{\phi}\phi' r^3 + 2r^4}\,\Gamma' + O(1), \quad (11)$$

which is finite $O(1)$, as a result of Eq. (9). The asymptotic form of the dilaton field and the metric components near the even horizon $r \simeq r_h$ are

$$e^{-\Lambda(r)} = \lambda_1(r-r_h) + \lambda_2(r-r_h)^2 + \cdots,$$
$$e^{\Gamma(r)} = \gamma_1(r-r_h) + \gamma_2(r-r_h)^2 + \cdots, \quad (12)$$

$$\phi(r) = \phi_h + \phi_h'(r-r_h) + \phi_h''(r-r_h)^2 + \cdots,$$

where

$$\lambda_1 = 2/(\alpha' e^{\phi_h}\phi_h'/g^2 + 2r_h), \quad (13)$$

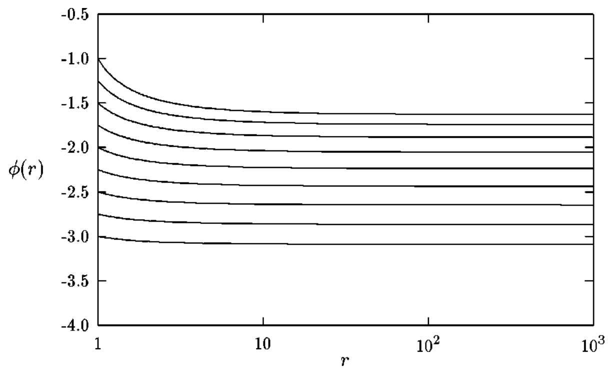

FIG. 1. Dilaton configurations for a family of black hole solutions, corresponding to fixed $r_h=1$, for various values of ϕ_h. Notice the monotonic behavior of the solutions and the fact that the curves do not intersect.

and γ_1 is an arbitrary finite *positive* integration constant, which cannot be fixed by the equations of motion, since the latter involve only $\Gamma'(r)$ and not $\Gamma(r)$. This constant is fixed by the asymptotic limit of the solutions at infinity. At infinity, one uses the following asymptotic behavior:

$$e^{\Lambda(r)}=1+\frac{2M}{r}+\frac{16M^2-D^2}{4r^2}+O\left(\frac{1}{r^3}\right),$$

$$e^{\Gamma(r)}=1-\frac{2M}{r}+O\left(\frac{1}{r^3}\right), \tag{14}$$

$$\phi(r)=\phi_\infty+\frac{D}{r}+\frac{MD}{r^2}+O\left(\frac{1}{r^3}\right),$$

which guarantees asymptotic flatness of the space time. Above, M denotes the ADM mass of the black hole and D the dilaton charge. The numerical analysis of Ref. [1] has shown that M and D are not independent quantities, thereby leading to the secondary nature of the dilaton hair [1,2].

The black hole solutions of Ref. [1] are characterized uniquely by two parameters (ϕ_h, r_h). Note, however, that the equations of motion remain invariant under a shift $\phi\to\phi+\phi_0$ as long as it is accompanied by a radial rescaling $r\to re^{\phi_0/2}$. Because of above invariance, it is sufficient to vary only one of r_h and ϕ_h. In the present analysis we choose to keep r_h fixed $(r_h=1)$, and to vary ϕ_h. We also set $\alpha'/g^2=1$ for convenience. A typical family of solutions has dilaton configurations (outside the horizon) of the form depicted in Fig. 1. The solutions are characterized by negative ϕ_h and *monotonic, nonintersecting* behavior from r_h until infinity. These are essential features of the solutions, which we shall make use of in our linear stability analysis.

III. LINEAR STABILITY ANALYSIS

We now consider perturbing Eqs. (4)–(8) by time-dependent *linear* perturbations of the form

$$\Gamma(r,t)=\Gamma(r)+\delta\Gamma(r,t)=\Gamma(r)+\delta\Gamma(r)e^{i\sigma t},$$

$$\Lambda(r,t)=\Lambda(r)+\delta\Lambda(r,t)=\Lambda(r)+\delta\Lambda(r)e^{i\sigma t}, \tag{15}$$

$$\phi(r,t)=\phi(r)+\delta\phi(r,t)=\phi(r)+\delta\phi(r)e^{i\sigma t},$$

where the variations $\delta\Gamma$, $\delta\Lambda$, and $\delta\phi$ are assumed small (bounded), and the quantities without a δ prefactor denote classical time-independent solutions of Eqs. (4)–(8). The above *harmonic* time dependence is sufficient for a *linear stability* analysis, since by assumption the linear variations are characterized by a well-defined Fourier expansion in time t [4,9,10]. The linear stability analysis proceeds by mapping the algebraic system of variations of the equations of motion under consideration to a stationary one-dimensional Schrödinger problem, in an appropriate potential well, in which the ''squared frequencies'' σ^2 will constitute the energy eigenvalues. In the present problem, the ''wave function'' turns out to be the dilaton linear variation $\delta\phi(r)e^{i\sigma t}$. Instabilities, then, correspond to negative energy eigenstates (''bound states''), i.e., imaginary frequencies σ. As we shall show in this article, for the system of variations corresponding to Eqs. (4)–(8), (15) the corresponding Schrödinger problem admits *no bound states*, thereby proving the linear stability of the EDGB hairy black holes.

As we shall discuss below, some technical complications arise, as usual [4], in the above process due to the fact that the ''naive'' stationary Schrödinger equation with respect to the original coordinate r is not well defined at some points of the domain of $r\in[r_h,\infty)$. This necessitates a change of coordinates in such a way that the resulting Schrödinger problem is well defined. A convenient choice is provided by the so-called ''tortoise'' coordinate r^* [4,10], which is defined in such a way so that the domain $[r_h,\infty)$ is extended over the entire real axis $r\to r^*\in(-\infty,\infty)$. In our specific problem, we shall define the ''tortoise'' coordinate r^* as [4]

$$\frac{dr^*}{dr}=e^{-(\Gamma-\Lambda)/2}. \tag{16}$$

As we shall show, then, the associated stationary Schrödinger equation, pertaining to the dilaton variation in Eqs. (15), will be of the form

$$p_*^2u+\mathcal{V}(r^*)u(r^*)=-\sigma^2u(r^*);\quad p_*\equiv\frac{d}{dr^*}, \tag{17}$$

where $u(r^*)$ is related to the dilaton variation $\delta\phi(r)$ in Eqs. (15), and the potential $\mathcal{V}$ is well defined over the domain of validity of r^*.

Let us now proceed with our analysis. The perturbed equations (4)–(8), under the variations (15), read

$$\delta\phi''+\delta\phi'\left(\frac{\Gamma'}{2}-\frac{\Lambda'}{2}+\frac{2}{r}\right)-\delta\phi\left[\phi''+\phi'\left(\frac{\Gamma'}{2}-\frac{\Lambda'}{2}+\frac{2}{r}\right)\right]-e^{\Lambda-\Gamma}\delta\ddot{\phi}+\delta\Gamma'\left\{\frac{\phi'}{2}-\frac{\alpha' e^{\phi}}{g^2r^2}\right.$$
$$\times\left.\left[\Lambda' e^{-\Lambda}+(1-e^{-\Lambda})\left(\Gamma'-\frac{\Lambda'}{2}\right)\right]\right\}+\delta\Lambda\,\frac{\alpha' e^{\phi}}{g^2r^2}\left\{\Gamma'\Lambda' e^{-\Lambda}-e^{-\Lambda}\left[\Gamma''+\frac{\Gamma'}{2}(\Gamma'-\Lambda')\right]\right\}$$
$$-\delta\Lambda'\left\{\frac{\phi'}{2}+\frac{\alpha' e^{\phi}}{g^2r^2}\left[\Gamma' e^{-\Lambda}-(1-e^{-\Lambda})\frac{\Gamma'}{2}\right]\right\}+\frac{\alpha' e^{\phi}}{g^2r^2}(1-e^{-\Lambda})\,e^{\Lambda-\Gamma}\delta\ddot{\Lambda}$$
$$-\frac{\alpha' e^{\phi}}{g^2r^2}(1-e^{-\Lambda})\,\delta\Gamma''=0, \tag{18}$$

$$\delta\Lambda'\left[1+\frac{\alpha' e^{\phi}\phi'}{2g^2r}(1-3e^{-\Lambda})\right]+\delta\phi'\left[-\frac{r\phi'}{2}+\frac{\alpha' e^{\phi}}{2g^2r}\Lambda'(1-3e^{-\Lambda})\right]+\delta\Lambda\left\{\frac{e^{\Lambda}}{r}+\frac{\alpha' e^{\phi-\Lambda}}{g^2r}\left[\frac{3\phi'\Lambda'}{2}-(\phi''+\phi'^2)\right]\right\}$$
$$+\delta\phi\frac{\alpha' e^{\phi}}{2g^2r}\phi'\Lambda'(1-3e^{-\Lambda})-\frac{\alpha' e^{\phi}}{g^2r}(1-e^{-\Lambda})\left[\delta\phi''+2\phi'\,\delta\phi'+\delta\phi(\phi''+\phi'^2)\right]=0, \tag{19}$$

$$\delta\Gamma'\left[1+\frac{\alpha' e^{\phi}\phi'}{2g^2r}(1-3e^{-\Lambda})\right]+\delta\Lambda\left[-\frac{e^{\Lambda}}{r}+\frac{3\alpha' e^{\phi}}{2g^2r}\phi'\Gamma' e^{-\Lambda}\right]+\delta\phi\frac{\alpha' e^{\phi}}{2g^2r}\phi'\Gamma'(1-3e^{-\Lambda})$$
$$+\delta\phi'\left[-\frac{r\phi'}{2}+\frac{\alpha' e^{\phi}}{2g^2r}\Gamma'(1-3e^{-\Lambda})\right]-\frac{\alpha' e^{\phi}}{g^2r}(1-e^{-\Lambda})\,e^{\Lambda-\Gamma}\,\delta\ddot{\phi}=0, \tag{20}$$

$$\delta\dot{\Lambda}\left[1+\frac{\alpha' e^{\phi}\phi'}{2g^2r}(1-3e^{-\Lambda})\right]=\frac{r\phi'}{2}\,\delta\dot{\phi}+\frac{\alpha' e^{\phi}}{g^2r}(1-e^{-\Lambda})\left(\delta\dot{\phi}\,\phi'+\delta\dot{\phi}'-\delta\dot{\phi}\,\frac{\Gamma'}{2}\right), \tag{21}$$

$$\delta\Gamma''\left(1-\frac{\alpha' e^{\phi-\Lambda}}{g^2r}\phi'\right)+\frac{\delta\Gamma'}{2}(\Gamma'-\Lambda')+(\delta\Gamma'-\delta\Lambda')\left(\frac{\Gamma'}{2}+\frac{1}{r}\right)-e^{\Lambda-\Gamma}\delta\ddot{\Lambda}-\frac{\alpha' e^{\phi-\Lambda}}{g^2r}\left\{\delta\phi'\,\Gamma''+(\delta\phi''+2\phi'\,\delta\phi')\Gamma'\right.$$
$$\left.+\delta\Gamma'(\phi''+\phi'^2)+\frac{\delta\phi'\Gamma'}{2}(\Gamma'-3\Lambda')+\frac{\phi'\delta\Gamma'}{2}(\Gamma'-3\Lambda')+\frac{\phi'\Gamma'}{2}(\delta\Gamma'-3\delta\Lambda')+e^{\Lambda-\Gamma}(\Lambda'\delta\ddot{\phi}-\phi'\delta\ddot{\Lambda})\right\}$$
$$+\phi'\,\delta\phi'-\frac{\alpha' e^{\phi-\Lambda}}{g^2r}\left[\phi'\Gamma''+\Gamma'(\phi''+\phi'^2)+\frac{\phi'\Gamma'}{2}(\Gamma'-3\Lambda')\right](\delta\phi-\delta\Lambda)=0. \tag{22}$$

We can integrate Eq. (21) to obtain

$$\delta\Lambda\left[1+\frac{\alpha' e^{\phi}\phi'}{2g^2r}(1-3e^{-\Lambda})\right]$$
$$=\frac{\alpha' e^{\phi}}{g^2r}(1-e^{-\Lambda})\left(\delta\phi\,\phi'+\delta\phi'-\delta\phi\,\frac{\Gamma'}{2}\right)+\frac{r\phi'}{2}\,\delta\phi$$
$$+\mu(r), \tag{23}$$

where $\mu(r)$ is an arbitrary function of r which can be set equal to zero if we require that $\delta\Lambda=0$ when $\delta\phi=0$. An independent check of this assertion can be obtained as follows: first we differentiate Eq. (23) with respect to r and then use Eq. (19), as well as the time-independent equations of motion. Thus, we obtain the following differential equation for $\mu(r)$:

$$\mu'(r)+\mu(r)\left(\frac{\Gamma'}{2}-\frac{\Lambda'}{2}+\frac{1}{r}\right)=0. \tag{24}$$

This can be integrated to give $\mu(r)\sim e^{(\Lambda-\Gamma)/2}/r$. When $r\to r_h$, $e^{(\Lambda-\Gamma)/2}\to\infty$ and $\mu\to\infty$, which is incompatible with the assumption of a small $\delta\Lambda(r)$ required for the linear stability analysis. Rejecting this solution, we are left with the trivial one $\mu=0$. For calculational convenience we also set $\alpha'/g^2=1$, from now on, which is achieved by an appropriate rescaling of the dilaton field.

Rearranging the above equations, one obtains, after a tedious but straightforward procedure, an equation for $\delta\phi$ which has the following structure:

$$A\ \delta\phi''+2B\ \delta\phi'+C\ \delta\phi+\sigma^2\ E\ \delta\phi=0, \tag{25}$$

where A, B, C, and E are rather complicated functions of ϕ, ϕ', ϕ'', Λ, Λ', Γ, Γ', and Γ''. In the limit $r\to r_h$ these coefficients take the form

$$A=\frac{2\sqrt{1-\left(\frac{6e^{2\phi_h}}{r_h^4}\right)}}{1+\sqrt{1-\left(\frac{6e^{2\phi_h}}{r_h^4}\right)}}+O\ (r-r_h), \tag{26}$$

$$B=\frac{\sqrt{1-\left(\frac{6e^{2\phi_h}}{r_h^4}\right)}}{1+\sqrt{1-\left(\frac{6e^{2\phi_h}}{r_h^4}\right)}}\ \frac{1}{(r-r_h)}+O\ (1), \tag{27}$$

$$C=\frac{2\ e^{2\phi_h}}{r_h^4\left[1+\sqrt{1-\frac{6e^{2\phi_h}}{r_h^4}}\right]}\quad\frac{1}{(r-r_h)^2}+O\left(\frac{1}{r-r_h}\right), \tag{28}$$

$$E=\frac{r_h\sqrt{1-\frac{6e^{2\phi_h}}{r_h^4}}}{\gamma_1}\ \frac{1}{(r-r_h)^2}+O\left(\frac{1}{r-r_h}\right), \tag{29}$$

where we have used the asymptotic behavior (12) near the event horizon.

On the other hand, when $r\to\infty$ we obtain

$$A=1+\frac{e^{\phi_\infty}DM}{r^4}+O\left(\frac{1}{r^5}\right), \tag{30}$$

$$B=\frac{1}{r}+\frac{M}{r^2}+O\left(\frac{1}{r^5}\right), \tag{31}$$

$$C=\frac{D^2}{2r^4}+O\left(\frac{1}{r^5}\right), \tag{32}$$

$$E=1+\frac{4M}{r}+\frac{4M^2}{r^2}+\frac{e^{\phi_\infty}DM}{r^4}+O\left(\frac{1}{r^5}\right), \tag{33}$$

where we have used the asymptotic behavior (14) near infinity.

As we can see from Eqs. (26)–(29) the coefficients of the Schrödinger equation (25) are not finite at the boundary $r=r_h$, where the variation $\delta\phi$ is bounded. As mentioned previously, to arrive at a well-defined Schrödinger problem, one can use the ''tortoise'' coordinate (16). Then, the perturbed equation for the dilaton field takes the form

$$A\ \frac{d^2\delta\phi}{dr^{*2}}+\left[\ 2B\ e^{(\Gamma-\Lambda)/2}-\frac{A}{2}\ \frac{d(\Gamma-\Lambda)}{dr^*}\ \right]\frac{d\delta\phi}{dr^*}$$
$$+e^{\Gamma-\Lambda}(C+\sigma^2E)\ \delta\phi=0 \tag{34}$$

or

$$\mathcal{A}\ \frac{d^2\delta\phi}{dr^{*2}}+2\mathcal{B}\frac{d\delta\phi}{dr^*}+(\mathcal{C}+\sigma^2\mathcal{E})\ \delta\phi=0, \tag{35}$$

where

$$\mathcal{A}=A,\quad \mathcal{B}=B\ e^{(\Gamma-\Lambda)/2}-\frac{A}{4}\ \frac{d(\Gamma-\Lambda)}{dr^*}, \tag{36}$$

$$\mathcal{C}=e^{\Gamma-\Lambda}C,\quad \mathcal{E}=e^{\Gamma-\Lambda}E. \tag{37}$$

Note that, near the horizon,

$$e^{(\Gamma-\Lambda)/2}=\sqrt{\gamma_1\lambda_1}\ (r-r_h)+O(r-r_h)^2, \tag{38}$$

$$\frac{d(\Gamma-\Lambda)}{dr^*}=2\sqrt{\gamma_1\lambda_1}+O(r-r_h), \tag{39}$$

where λ_1 is given by Eq. (13). As a result, all the coefficients in Eq. (34) are now well behaved near the horizon r_h. In order to eliminate the term proportional to $\delta\phi'$, we first divide Eq. (34) by $\mathcal{A}$ and then we use the function

$$F=\exp\left(\int_{-\infty}^{r^*}\frac{\mathcal{B}}{\mathcal{A}}dr^{*\prime}\right). \tag{40}$$

Then, the equation for $\delta\phi$ takes the form

$$p_*^2u+\left[\ \frac{\mathcal{C}}{\mathcal{A}}+\sigma^2\frac{\mathcal{E}}{\mathcal{A}}-\frac{\mathcal{B}^2}{\mathcal{A}^2}-p_*\left(\frac{\mathcal{B}}{\mathcal{A}}\right)\right]u=0, \tag{41}$$

where

$$p_*\equiv\frac{d}{dr^*} \tag{42}$$

and we have set $u=F\,\delta\phi$.

It is straightforward to see that

$$\frac{\mathcal{B}}{\mathcal{A}}\to 0\ \text{ for }\ r\to r_h\ \text{ and }\ r\to\infty, \tag{43}$$

independently of r_h, ϕ_h. In addition,

$$\frac{\mathcal{B}}{\mathcal{A}}\frac{dr^*}{dr}=\text{finite}\quad\text{for}\quad r\to r_h. \tag{44}$$

Moreover, as shown in Fig. 2, the function $\mathcal{B}/\mathcal{A}$ is well behaved over the entire domain outside the horizon, implying the integrability of the function F. Also, the quantity $\mathcal{E}/\mathcal{A}$ is finite and always *positive* outside the horizon of the numeri-

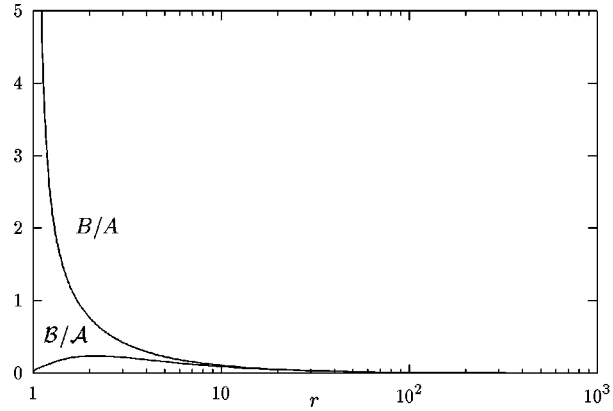

FIG. 2. The graph depicts the coefficients B/A and $\mathcal{B}/\mathcal{A}$, for a typical member of the family of the black hole solutions of Fig. 1 corresponding to $\phi_h=-1$, $r_h=1$. It is clear that the coefficient $\mathcal{B}/\mathcal{A}$, incorporating the tortoise coordinate, is finite in the entire domain outside the horizon, thereby implying that the quantity F is well defined and integrable.

cal black hole solutions of Ref. [1] (see Fig. 3). It is also immediately seen from Eq. (15) that the eigenfunction u_0, corresponding to the eigenvalue $\sigma^2=0$, can be constructed out of the difference of any two time-independent solutions of Eqs. (4)–(8), i.e., out of the difference of any two curves in Fig. 1. From the *monotonic* and *nonintersecting* nature of the various members of the family of the numerical solutions of Fig. 1, then, one can conclude that u_0 has *no nodes* in the domain $r^*\in(-\infty,\infty)$, and that $p_*^2u_0/u_0p_*^2u_0/u_0 = e^{\Gamma-\Lambda}[\frac{1}{2}(\Gamma' p_*^2u_0/u_0 = e^{\Gamma-\Lambda}[\frac{1}{2}(\Gamma'-\Lambda')u_0'+u_0'']/u_0$ is *finite*. This, together with the finite and smooth form of $\mathcal{B}/\mathcal{A}$ (Fig. 2), implies, on account of Eq. (41), the finiteness of the coefficient $\mathcal{C}/\mathcal{A}$ outside the horizon, without the need for an explicit numerical computation. Thus, Eq. (41) assumes the form of an ordinary Schrödinger with *regular* coefficients over the entire domain of r^*.

If we solve Eq. (41) near $r^*=-\infty$, that is, near the horizon, with $\sigma^2=0$, we find an oscillatory behavior for u_0 which means that u_0 remains always bounded near the horizon and it can take on constant finite values. The same equation, near $r^*\to\infty$, allows u_0 and $\delta\phi$ to take on constant values as well. Since u_0 does not vanish at the boundaries, *it is not a physical perturbation* but *a shift of solution*. Nevertheless, it still remains an acceptable solution of the perturbed equations, a mathematical tool, which we can exploit to show that there are no physical perturbations, that is, no solutions with $\sigma^2<0$.

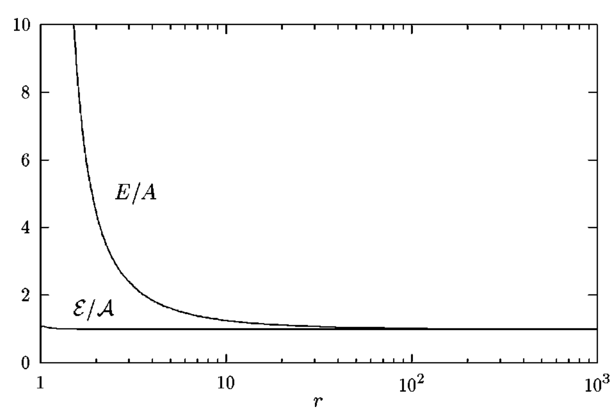

FIG. 3. This diagram depicts the coefficients E/A and $\mathcal{E}/\mathcal{A}$ for a typical member of the family of the black hole solutions depicted in Fig. 1 ($\phi_h=-1$, $r_h=1$); the coefficient E/A diverges at the horizon as $1/(r-r_h)^2$. On the other hand, $\mathcal{E}/\mathcal{A}=e^{\Gamma-\Lambda}E/A$, appearing in Eq. (40), is finite at the horizon. The positive definiteness of both coefficients is clear.

From Eqs. (40) and (43) one obtains, for the eigenfunction u_0 on the boundaries,

$$p_*u_0|_{r^*=\pm\infty} = \frac{\mathcal{B}}{\mathcal{A}}u_0|_{r^*=\pm\infty} + Fp_*\delta\phi|_{r^*=\pm\infty}. \tag{45}$$

The $\delta\phi$ remains bounded at $r^*=\pm\infty$. From the asymptotic behavior (14) it becomes clear that, at the $r=\infty$ boundary, $\mathcal{B}/\mathcal{A}\to B/A\sim 1/r$, independently of ϕ_h. Thus, $F(r^*\to\infty) = \exp[\int_{-\infty}^{r}(B/A)\,dr]\sim r$ for $r\to\infty$. Hence, the first term in Eq. (45) becomes $\delta\phi_\infty+\mathcal{O}(1/r)$, while the second term vanishes as $1/r$, for $r\to\infty$ [see Eqs. (14)]. Hence, at the $r=\infty$ boundary the boundary values of p_*u_0 are proportional to those of u_0:

$$p_*u_0|_{r^*=\infty} = \frac{u_0}{r}\bigg|_{r^*=\infty}. \tag{46}$$

On the other hand, from Eqs. (40) and (44) it becomes clear that at the boundary $r^*=-\infty$ (horizon), $F(r^*=-\infty)=1$. Moreover, $p_*u_0|_{r^*=-\infty}$ is given by

$$\begin{aligned} p_*u_0|_{r^*=-\infty} &= (r-r_h)(\phi^{(2)'}-\phi^{(1)'})|_{r=r_h} \\ &= \left((r-r_h)\frac{\partial\phi_h'}{\partial\phi_h}u_0\right)\bigg|_{r=r_h}, \end{aligned} \tag{47}$$

since, in our construction, each family of solutions is uniquely characterized [1] by the value ϕ_h for fixed r_h [we remind the reader that here we chose to keep $r_h=1$ (fixed) and vary ϕ_h]. From Eq. (9) one easily observes that for linear variations $\delta\phi\equiv\phi^{(2)}-\phi^{(1)}$, the difference $\phi_h^{(2)'}-\phi_h^{(1)'}$ is *finite*. Hence,

$$p_*u_0|_{r^*=-\infty}\to 0. \tag{48}$$

We now discuss the $\sigma^2<0$ unstable modes u_σ. It can be easily seen from Eqs. (30)–(33) that the asymptotic form of Eq. (41) at the $r^*=\infty$ boundary reads

$$p_*^2u_\sigma = -\sigma^2u_\sigma. \tag{49}$$

Hence, the bound-state solution u_σ behaves as[2]

---

[2]Here, and in the following, we insist on bounded or—at most—linearly divergent u_σ, at the boundaries $r^*=\pm\infty$. This is due to the fact that, since $u=F\delta\phi$, and F is independent of σ and at most linearly divergent at $r^*=\infty$, then it is only for such a behavior that the variation $\delta\phi$ remains bounded, as required by the linear stability analysis. An exponentially divergent $u_\sigma\sim e^r$ at the boundaries, would imply $\delta\phi\sim e^r/F$, and, hence, is not acceptable.

$$u_\sigma(r^*=\infty)=e^{-|\sigma|r^*}\to 0. \tag{50}$$

On the other hand, from Eqs. (26)–(29) it is obvious that, on the horizon, Eq. (41) assumes the form (for the case $r_h=1$)

$$p_*^2 u_\sigma + k^2 u_\sigma = 0, \tag{51}$$

$$k^2 \equiv \frac{2\gamma_1 e^{2\phi_h}}{(1+\sqrt{1-6e^{2\phi_h}})\sqrt{1-6e^{2\phi_h}}} + \sigma^2 = k_0^2 + \sigma^2. \tag{52}$$

From Eq. (52) one can see two possibilities near the horizon of the black hole

(i) The ''total energy'' is such that $0>\sigma^2>-2\gamma_1 e^{2\phi_h}/[(1+\sqrt{1-6e^{2\phi_h}})\sqrt{1-6e^{2\phi_h}}]$. Taking into account the asymptotic form at $r^*=\infty$, Eq. (49), one also observes that in this case the spectrum of the respective Schrödinger equation is *continuous* and *nondegenerate*. The general solution of the perturbation u_σ *near the horizon* is, therefore, oscillatory (unbound state):

$$u_\sigma^\pm \sim e^{\pm ikr^*}, \quad r^* \sim -\infty. \tag{53}$$

Such a continuum of states *cannot exist* in our case by continuity. Indeed, because of the nondegenerate nature of the eigenvalue problem, the limiting case $\sigma\to 0$ should yield the solution u_0. However, in the limit $\sigma^2\to 0$, and in terms of r, one obtains

$$u_0^\pm \sim \cos\left(\frac{e^{\phi_h}}{(1-6e^{2\phi_h})^{1/4}}\ln(r-r_h)+\varphi_0\right), \tag{54}$$

where φ_0 is a constant phase shift, and we have taken u_0 to be the real part of Eq. (53). Then,

$$p_* u_0 \sim -k_0 \sin\left(\frac{e^{\phi_h}}{(1-6e^{2\phi_h})^{1/4}}\ln(r-r_h)+\varphi_0\right). \tag{55}$$

The above result is not in agreement with Eq. (48), thereby contradicting the nondegenerate nature of these solutions, which, in turn, implies the absence of such solutions in the problem (41).

(ii) This leaves one with the second possibility of a *discrete spectrum* of *bound states*, which would occur for

$$\sigma^2 < -\frac{2\gamma_1 e^{2\phi_h}}{(1+\sqrt{1-6e^{2\phi_h}})\sqrt{1-6e^{2\phi_h}}} < 0. \tag{56}$$

As we shall show below this is also *not realized* due to the special form of u_0.

To this end, one first observes that such bound states would *vanish* exponentially at the $r^*=-\infty$ boundary:

$$u_\sigma(r^*=-\infty)\sim e^{|k|\,r^*}\to 0. \tag{57}$$

Note that Eqs. (50) and (57) force the eigenfunction u_σ to vanish at both boundaries $r^*=\pm\infty$. In this way, u_σ not only remains bounded but is also a physical perturbation [9]. Thus, on account of Eqs. (46), (48), (50), (57) the Wronskian of any two solutions u_1, u_2 of Eq. (41) with $\sigma^2\leq 0$ vanishes at the boundaries:

$$W=(u_1 p_* u_2 - u_2 p_* u_1)|_{r^*=\pm\infty}=0. \tag{58}$$

To count the unstable gravitational modes of the original problem, one needs to count the nodes of the wave function u of the one-dimensional Schrödinger problem (41). Fortunately, this can be done without detailed knowledge of the solutions. As we shall discuss below, all one needs to observe is the monotonic and nonintersecting nature of the dilaton curves in Fig. 1. To this end, one first observes that a standard ''node rule'' for the discrete spectrum of Eq. (41) applies, which is a direct consequence of Fubini's theorem of ordinary differential equations [11]. This theorem can be stated as follows: consider two differential equations

$$u''+2p_1u'+q_1u=0, \tag{59}$$

$$u''+2p_2u'+q_2u=0. \tag{60}$$

If

$$p_2'+p_2^2-q_2 \leq p_1'+p_1^2-q_1 \tag{61}$$

throughout the interval $[a,b]$, then, between *any two consecutive zeroes of a solution of* Eq. (59), in the interval $[a,b]$, *there is at least one zero of a solution of* Eq. (60).

In our case, we can apply this theorem for two different eigenfunctions u_1, u_2, corresponding to eigenvalues σ_1^2 and σ_2^2 of Eq. (41). The interval $[a,b]$ is the entire domain of validity of the solutions of Eq. (41) $(-\infty,\infty)$, including the boundaries at infinity. In this case,

$$p_i=0, \quad q_i=\frac{\mathcal{C}}{\mathcal{A}}+\sigma_i^2\frac{\mathcal{E}}{\mathcal{A}}-\frac{\mathcal{B}^2}{\mathcal{A}^2}-p_*\left(\frac{\mathcal{B}}{\mathcal{A}}\right), \quad i=1,2. \tag{62}$$

Then, the (sufficient) condition for a ''node rule'' (61) reads simply

$$\frac{\mathcal{E}}{\mathcal{A}}\sigma_2^2 \geq \frac{\mathcal{E}}{\mathcal{A}}\sigma_1^2. \tag{63}$$

The positivity of $\mathcal{E}/\mathcal{A}$ (Fig. 3) implies that the condition (63) becomes simply

$$\sigma_2^2 \geq \sigma_1^2. \tag{64}$$

This special version of the theorem is known as Sturm's theorem [12]. As a corollary of the Fubini-Sturm theorem one obtains the standard ''node rule'' for the number of zeros of the eigenfunctions in the discrete spectrum of bound states, according to which if the eigenfunctions are ranked in order of increasing energy, then the nth eigenfunction has $n-1$ nodes (excluding the boundary zeros) [12] (''node rule'').

Consider, now, the case where σ_2 corresponds to the zero eigenvalue of Eq. (41), $\sigma_0=0$. As can be seen from the numerical solution of Fig. 1, the *monotonic* and *nonintersecting* character of the dilaton curves in the entire domain outside the horizon implies that the solution $u_2=u_0$, which, as we have mentioned earlier, can be constructed out of the difference of two such solutions, has *no nodes* in the domain $r^* \in (-\infty,\infty)$. Since any solution u_n from the *discrete set* of negative eigenvalues $\sigma_n^2 < 0$ (unstable modes) has at least two nodes at the boundaries, according to the Fubini-Sturm theorem, u_0 should have at least one in the domain $r^* \in (-\infty,\infty)$. This contradicts the fact that u_0 is nodeless. Thus, the only consistent situation is the one *without* such *negative energy* modes. This, in turn, implies *linear stability* for the dilaton-GB black holes of Ref. [1]. The reader might worry about the divergent boundary conditions of u_0 at $r^* = \infty$, which makes it *not an ordinary* eigenfunction of a Schrödinger problem. In the Appendix we argue that this is not an obstacle. In fact, we present an explicit proof of the absence of bound states in our case, following the same spirit used in the proof of the Fubini-Sturm theorem. The crucial element, which allows the standard proof to go through, is the special boundary condition for the Wronskian (58), which is valid for the entire spectrum of eigenfunctions of Eq. (41) with $\sigma^2 \leqslant 0$, including the nonstandard one u_0.

The above considerations can be extended straightforwardly to the case where $r_h \to 0$. All the coefficients of Eq. (41) are still well defined in this case, which implies that the stability in principle does not change. However, according to the analysis of Ref. [1], the case $r_h \to 0$ corresponds to a singular scalar curvature

$$R \sim \frac{2}{r_h^2} \quad r_h \to 0 \ . \tag{65}$$

Moreover, from the condition for the existence of black hole solutions (10), one observes that the only consistent value of ϕ_h for $r_h \to 0$ is [1] $\phi_h \to -\infty$. In this limit, the first derivative of the dilaton field (9) diverges. Both of the above results imply the absence of regular ''particlelike'' solutions in the EDGB system [1,13]. What we are left with is a stable ''pointlike'' spacetime singularity.

On the other hand, if we take the limit $\phi_h \to -\infty$ keeping r_h fixed at a nonvanishing value, Eq. (9) gives $\phi_h' \to 0$ which means that the dilaton remains constant. In addition, in this limit, the coupling between the dilaton and the Gauss-Bonnet term vanishes. This implies that in such a case the GB term in the action (1) becomes irrelevant, and one is left with the standard Einstein term, which admits only Schwarzschild black holes, known to be stable. This stability is confirmed by the smooth limit of the coefficients in Eq. (41) as $e^{\phi_h} \to 0$.

IV. REMARKS AND OUTLOOK

Above we have demonstrated the linear stability of the dilatonic black hole solutions in the EDGB system, found in Ref. [1]. This result is important, since it constitutes an example of a stable, albeit secondary, hair that bypasses the no-hair conjecture [4,8]. Nonlinear stability of the EDGB system, however, although expected, still remains an open issue.

Before closing we would like to compare our semianalytic results on linear stability with some remarks in favor of stability by virtue of a catastrophe theory approach made in Ref. [6]. As usual [4,14], catastrophe theory can only indicate *relative changes* of stability, and hence cannot constitute a ''proof'' of stability. In Ref. [6] a numerical solution was found for the (+) branch of solutions (9) which indicated the existence of a ''turning point'' (TP) in the r_h-M (or equivalently ϕ_h-M) graph. The TP occurs at the ''critical point'' for the existence of black hole solutions, which is the point where the black hole acquires a minimal mass, below which no solution is found. In the numerical solution of Ref. [6] a continuation beyond this critical point emerged, which ends at a point (''singular point'') where a singularity appears in the square of the Riemann tensor, as well as in ϕ_h''.

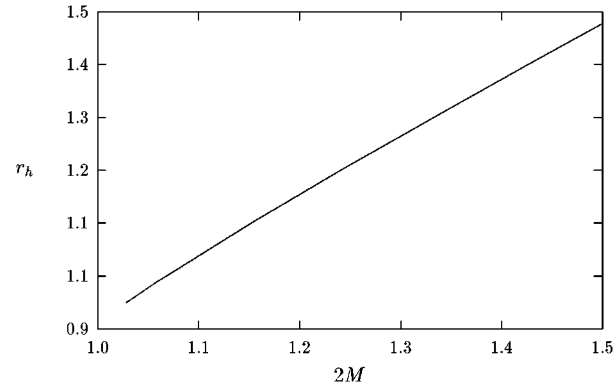

FIG. 4. The graph depicts r_h versus the ADM mass $2M$ of the black hole, for a fixed value of $\phi_h = -1$. The emergence of the asymptotic critical point, $r_h^4 \simeq 6\alpha'^2 e^{2\phi_h}/g^4$, below which there are no solutions, is apparent. At this point the mass becomes minimal.

The part of the solution from the critical point to the singular point was argued in Ref. [6] to be *relatively unstable*, as compared to the regular branch discussed here. Such a change in stability manifests itself as a cusp in an appropriate catastrophe theory diagram [4,14]. In Ref. [6] such a diagram has been chosen to be the diagram of the thermodynamic entropy [15] versus the mass of the black hole.

In our numerical solutions [1], used here, we found no evidence for such a TP. Our black hole solutions are uniquely specified by the pair (r_h, ϕ_h), which was essential in our linear stability analysis above. In this respect we are in agreement with the results of Ref. [5], where a branching of solutions was found only *inside* the event horizon. These authors have also given a graph of the entropy versus the mass of the black hole solution of Ref. [1] outside the horizon, and found, as expected, a smooth curve, with no cusps. In this article we have proved *analytically* the stability under linear perturbations of this (unique) branch of the black hole solutions, which in the r_h-M graphs of Figs. 4 and 5 appears to terminate at the minimum-mass critical point.

However, the apparent discrepancy between our results and that of Ref. [6], concerning the existence of a TP, is easily resolved if we notice that the corresponding graph of Ref. [6] was drawn with a different scaling, that is, for a fixed value of $\phi_\infty = 0$. Using the aforementioned invariance of the equations of motion, we reconstructed the r_h-$2M$ graph, keeping fixed, this time, the value of $\phi_\infty = 0$. Then, a cusp point and a continuation of our solutions beyond the critical point indeed emerged in accordance with Ref. [6]. Since we both study the same mode of the black hole solutions, that is, the radial mode, we reach the conclusion that the two different methods used by us and the authors of Ref.

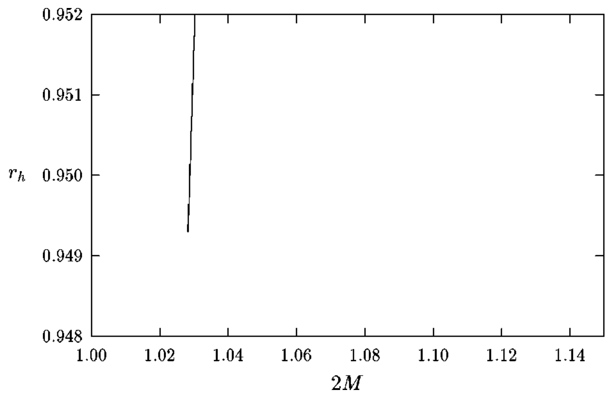

FIG. 5. The magnification around the critical point, depicted in the above figure, shows clearly the absence of a turning point.

[6] are equivalent and possibly complementary. However, in both cases one cannot draw any conclusions about the *non-linear* stability of the black hole solutions, which therefore remains an open issue.

As a final remark we would like to mention that the effects of gauge fields on the dilatonic black hole GB solutions have been considered in Refs. [6] and [7]. From the point of view of stability, one expects that, in the case of ''colored'' black holes, involving non-Abelian gauge fields, *instabilities* occur in *both* the gauge and gravitational sectors of the solutions. Instabilities in the gauge sector are of sphaleron type [4]. Those in the gravitational sector can be studied in a similar way as for colored black holes in Einstein-Yang-Mills theories [4]. One can go beyond linear stability analysis in such systems, by invoking catastrophe theory [4,14], which is capable of giving the relative stability of various branches of solutions for the colored EDGB black holes [6,7]. However, analytic methods can still be combined with the catastrophe theory approach [4] in order to *count* the *unstable modes* in both sectors, gauge and gravitational, by invoking appropriate maps of the system of perturbations into one-dimensional stationary Schrödinger problems [4,10]. We hope to return to a detailed analytic study of these issues in a future publication.

ACKNOWLEDGMENTS

Three of us (P.K., J.R., and K.T.) acknowledge traveling support from the EU-TMR Network ''Beyond the Standard Model.'' P.K. and K.T. wish to thank the Greek Ministry of Research and Technology for financial support (PENED 95). P.K. also thanks the Department of (Theoretical) Physics of Oxford University for the hospitality, and partial financial support, during the final stages of this work. The work of N.E.M. is supported by the P.P.A.R.C. (UK). E.W. thanks Oriel College, Oxford, for financial support.

APPENDIX: ABSENCE OF BOUND STATES

In this appendix we prove the absence of bound states in the problem (41) by following a Wronskian treatment for the entire set of eigenmodes with $\sigma^2 \leqslant 0$. This justifies the validity of the Fubini-Sturm theorem in our case. We stress that the crucial point in the proof is the special boundary conditions of the Wronskian (58). These allow a standard Wronskian treatment to go through for the u_0 solution of Eq. (41), despite its (linear) divergence at the $r^* = \infty$ boundary, which makes it not an ordinary eigenfunction of a Schrödinger problem.

To this end, we consider first the two solutions of Eq. (41), u_0, and u_b—the ground state, at the bottom of the discrete spectrum. Both of these have *no nodes* in the interior domain of r^*, excluding the boundaries (the node structure of u_b follows from the ''node rule''). We then employ properties of the Wronskian of the solutions as follows: first we multiply each equation by the other eigenfunction. Next, we subtract the resulting system of equations, and then integrate it over the entire domain of $r^* \in (-\infty,\infty)$. In this way one obtains, in a standard fashion [12],

$$\Delta W|_{r^*=\pm\infty} = (\sigma_b^2 - \sigma_0^2) \int_{-\infty}^{\infty} dr^* \frac{\mathcal{E}}{\mathcal{A}} u_b u_0, \qquad \text{(A1)}$$

where the left-hand side denotes the change in the Wronskian between the two boundaries. From Eq. (58) this *vanishes*. Moreover, as we have mentioned previously, $\mathcal{E}/\mathcal{A}$ is *positive definite* for the entire domain of $r^* \in (-\infty,\infty)$ (Fig. 3). Since u_b and u_0 have *no nodes* in the domain $(-\infty,\infty)$, excluding the boundaries, one obtains from Eq. (A1) that the *only consistent* case is the degenerate one $\sigma_b^2 = \sigma_0^2$. But $\sigma_0^2 = 0$, while $\sigma_b^2 < 0$ by assumption; this implies a contradiction, excluding σ_b from the spectrum.

One repeats the construction, using u_0 and any of the higher eigenfunctions of the discrete spectrum, u_n, corresponding to $\sigma_n^2 < 0$. The change in the Wronskian between $-\infty$ and the first encountered zero of u_n, at $r^* = z_0$, is then given by

$$\Delta W|_{r^*=-\infty}^{z_0} = -u_0 p_* u_n|_{z_0} = (\sigma_n^2 - \sigma_0^2) \int_{-\infty}^{z_0} dr^* \frac{\mathcal{E}}{\mathcal{A}} u_n u_0. \qquad \text{(A2)}$$

Without loss of generality, one may assume that $u_n > 0$ in the interval $(-\infty, z_0)$. Then $p_* u_n(z_0) < 0$. It is immediate to see that there is a contradiction in Eq. (A2). The middle part has the sign of u_0, while the right-hand side has the opposite sign of u_0. The case of a zero of u_n, at a point z_0, such that $p_* u(z_0) = 0$, is dealt with similarly. In that case the contradiction lies in the fact that the left-hand side vanishes, while the right-hand side is a negative number (for $u_n > 0$ in the interval). These results exclude the possibility of bound-state eigenfunctions with zeros in $(-\infty,\infty)$.

The above analysis, therefore, implies the *absence of negative energy modes* (bound states) in the problem (41), which, in turn, leads to *linear stability* for the Dilaton-Gauss-Bonnet black holes of Ref. [1]

[1] P. Kanti, N. E. Mavromatos, J. Rizos, K. Tamvakis, and E. Winstanley, Phys. Rev. D **54**, 5049 (1996).
[2] S. Coleman, J. Preskill, and F. Wilczek, Nucl. Phys. **B378**, 175 (1992).
[3] B. R. Greene, S. D. Mathur, and C. M. O'Neill, Phys. Rev. D **47**, 2242 (1993).
[4] N. E. Mavromatos and E. Winstanley, Phys. Rev. D **53**, 3190 (1996).
[5] S. O. Alexeyev and M. V. Pomazanov, Phys. Rev. D **55**, 2110 (1997).
[6] T. Torii, H. Yajima, and K. Maeda, Phys. Rev. D **55**, 739 (1996).
[7] P. Kanti and K. Tamvakis, Phys. Lett. B **392**, 30 (1996).
[8] J. Bekenstein, Phys. Rev. D **5**, 1239 (1972); **51**, 6608 (1995); A. Mayo and J. Bekenstein, *ibid*. **54**, 5059 (1996).
[9] N. Straumann and Z. H. Zhou, Phys. Lett. B **237**, 353 (1990); **243**, 33 (1990); M. Heusler, S. Droz, and N. Straumann, *ibid.* **285**, 21 (1992).
[10] P. Boschung, O. Brodbeck, F. Moser, N. Straumann, and M. S. Volkov, Phys. Rev. D **50**, 3842 (1994); M. S. Volkov and D. V. Gal'tsov, Phys. Lett. B **341**, 279 (1995); G. Lavrelashvili and D. Maison, *ibid*. **343**, 214 (1995); M. S. Volkov, O. Brodbeck, G. Lavrelashvili, and N. Straumann, *ibid*. **349**, 438 (1995); E. Winstanley and N. E. Mavromatos, *ibid*. **352**, 242 (1995); O. Brodbeck and N. Straumann, *ibid*. **324**, 309 (1994); J. Math. Phys. **37**, 1414 (1996).
[11] G. Birkhoff and G-C. Rota, *Ordinary Differential Equations* (Wiley, New York, 1989).
[12] See, for instance, A. Messiah, *Quantum Mechanics* (North-Holland, Amsterdam, 1970), Vol. I.
[13] E. E. Donets and D. V. Gal'tsov, Phys. Lett. B **352**, 261 (1995).
[14] T. Torii, K. Maeda, and T. Tachizawa, Phys. Rev. D **51**, 1510 (1995); **51**, 4054 (1995).
[15] G. 't Hooft, Nucl. Phys. **B256**, 727 (1985).