Non-Abelian solitons in $N=4$ gauged supergravity and leading order string theory

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We study static, spherically symmetric, and purely magnetic solutions of the $N=4$ gauged supergravity in four dimensions. A systematic analysis of the supersymmetry conditions reveals solutions which preserve 1/4 of the supersymmetries and are characterized by a BPS-monopole-type gauge field and a globally hyperbolic, everywhere regular geometry. We show that the theory in which these solutions arise can be obtained via compactification of ten-dimensional supergravity on the group manifold. This result is then used to lift the solutions to ten dimensions. [S0556-2821(98)06410-8]

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I. INTRODUCTION

In the past few years there has been considerable interest in supersymmetric solitons originating from effective field theories of superstrings and heterotic strings (see [1] for review). These solutions play an important role in the study of the non-perturbative sector of string theory and in understanding string dualities. A characteristic feature of such solutions is that supersymmetry is only partially broken, and associated with each of the unbroken supersymmetries there is a Killing spinor fulfilling a set of linear differential constraints. The corresponding integrability conditions can be formulated as a set of non-linear Bogomol'nyi equations for the solitonic background, which can often be solved analytically.

The analysis of the supersymmetry conditions has proven to be an efficient way of studying the non-perturbative dynamics. So far, however, the investigations have mainly been restricted to the Abelian theory and little is known about the structure of the non-Abelian sector, which presumably is due to the complexity of the problem. At the same time, the gauge group arising in the context of string theory is fairly general. It includes the U(1) group as a subgroup, but otherwise it is clear that the restriction to the Abelian sector truncates most of the degrees of freedom.

In view of this it seems reasonable to focus on studying supergravity solitons with non-Abelian gauge fields. It turns out that all known solutions of this type can be classified according to two different methods applied to obtain them. The first of these methods is employed in the heterotic five-brane construction [2]. Specifically, the geometry of the four-dimensional space transverse to the brane is supposed to be conformally flat. This allows one to choose for the Yang-Mills field living in this space any known solution of the self-duality equations in flat Euclidean space. Choosing all possible self-dual configurations, one can obtain in this way a large variety of different five-branes [3,4]. The ten dimensional solutions then further modify upon reducing to four dimensions, displaying, nevertheless, a number of common features due to the common origin. In this connection it is also worth mentioning the work in Ref. [5], where the equations of a supergravity model with non-Abelian vector fields were directly attacked. It was shown later [6] that, for one special value of the dilaton coupling constant, the solutions obtained exactly correspond to the reduction of the five-brane-type solution described in [3].

Another way to construct non-Abelian solutions is to embed the gravitational connection into the gauge group; see [7] and references therein. In this approach one starts from a solution of leading order string theory, which is sometimes obtained by uplifting a four-dimensional Abelian solution. Its spin-connections are then identified with the gauge field potential. As a result one obtains a solution of the theory with string corrections, which sometimes can be exact in all orders of string expansion.

No other non-Abelian supergravity solitons are known than those obtained by applying the described two methods. All known solutions are thus either essentially Abelian, or flat-space non-Abelian. In this sense, they can be regarded as too special, since none of them really reflect the interplay between gravity and the non-Abelian gauge field. At the same time, the famous example of the (non-supersymmetric) Bartnik-McKinnon particles [8] shows that such an interplay can result in an unusually rich variety of properties of the solutions.

Motivated by the arguments above, we study solitons in a four-dimensional supergravity model with non-Abelian Yang-Mills multiplets. The model we consider is the $N=4$ gauged SU(2)×SU(2) supergravity [9], which can be regarded as $N=1$, $D=10$ supergravity compactified on the group manifold $S^3\times S^3$. Note that all previously known non-Abelian supergravity solitons have the Yang-Mills field already in ten dimensions, and their compactification gives the different matter content in $D=4$. Our choice of the model therefore ensures that we do not reproduce any known solutions. Note also that the non-gauged version of the same model, corresponding to the toroidal compactification of ten-dimensional supergravity, has been extensively studied in the past [10,11]. The Abelian solutions in the gauged version of the model have been studied in [12].

In order to obtain the solutions we carry out the component analysis of the supersymmetry constraints, which gives us a set of the first integrals for the field equations. We investigate static, spherically symmetric, purely magnetic field configurations choosing for the gauge group either $SU(2)\times SU(2)$ or $SU(2)\times[U(1)]^3$. It turns out that in the first case there are no supersymmetric solutions. The second choice, however, leads to the obtaining of the variety of nontrivial solutions with 1/4 of supersymmetries preserved, all of which can be described analytically [13]. Among them we discover a one-parameter family of globally regular solutions with quite unusual properties. First, the solutions are characterized by a regular-Bogomol'nyi-Prasad-Sommerfield-(BPS)-monopole-type gauge field with non-vanishing magnetic charge. This is very surprising, since a Higgs field is not present in the problem, in which case it would be reasonable to expect the regular solutions to be neutral. Secondly, the geometry of the solutions turns out to be globally hyperbolic. This is also quite remarkable, because the standard gauge supergravity ground states usually lack of global hyperbolicity.

Having obtained the solutions we lift them to ten dimensions. For this we first show how to obtain the $N=4$ gauged supergravity via compactification of the $N=1$, $D=10$ supergravity on the group manifold, which is either $S^3\times S^3$ or $S^3\times T^3$. It turns out that the corresponding procedure has not been described in the literature. Applying to the four dimensional solutions the procedure inverse to the dimensional reduction, we thus obtain the solutions of the leading order equations of motion of the string effective action in ten dimensions.

The rest of the paper is organized as follows. In Sec. II we describe the action and supersymmetry transformations of the $N=4$ gauged supergravity, derive the field equations and present their first integrals following from the dilatational symmetry. Our procedure to handle the supersymmetry constraints, that is, the equations for the Killing spinors, is described in Sec. III. The supersymmetry conditions, which are the consistency conditions for the supersymmetry constraints, are derived in Sec. IV in the form of the first order Bogomol'nyi equations for the bosonic background. This section contains also the solutions for the Killing spinors. Solutions of the Bogomol'nyi equations are presented in Sec. V. Section VI describes the compactification of $N=1$, $D=10$ supergravity on the group manifold. The results obtained in that section then used to lift the four-dimensional solutions to ten dimensions. The lifted solutions and some of their properties are described in Sec. VII. The last section contains concluding remarks.

Our notation is as follows: Greek, Latin, and capital Latin letters stand for the four-dimensional, internal six-dimensional, and general ten-dimensional indices, respectively. The early letters refer to the tangent space whereas the late ones denote the base space indices. The six-dimensional space, whose indices are a,b,c, . . . and $m,n,p,\ldots$, splits further into two three-dimensional spaces. The three-dimensional indices are a,b,c, which stand also for the group indices, and i,j,k. The spacetime metric is denoted by $\mathbf{g}$, whereas g stands for the gauge coupling constant(s).

II. THE MODEL

The action of the $N=4$ gauged $SU(2)\times SU(2)$ supergravity includes a vierbein e^{α}_{μ}, four Majorana spin-3/2 fields $\psi_\mu\equiv\psi^{\mathrm{I}}_\mu$ ($\mathrm{I}=1,\ldots 4$), vector and pseudovector non-Abelian gauge fields $A^{(1)\,a}_\mu$ and $A^{(2)\,a}_\mu$ with independent gauge coupling constants g_1 and g_2, respectively, four Majorana spin-1/2 fields $\chi\equiv\chi^{\mathrm{I}}$, the axion $\mathbf{a}$ and the dilaton ϕ [9]. The bosonic part of the action reads

$$S=\int\Bigg(-\frac{1}{4}R+\frac{1}{2}\,\partial_\mu\phi\,\partial^\mu\phi+\frac{1}{2}\,e^{-4\phi}\partial_\mu\mathbf{a}\,\partial^\mu\mathbf{a}\\ -\frac{1}{4}\,e^{2\phi}\sum_{s=1}^{2}F^{(s)\,a}_{\mu\nu}F^{(s)\,a\mu\nu}\\ -\frac{1}{2}\,\mathbf{a}\sum_{s=1}^{2}F^{(s)\,a}_{\mu\nu}{*F}^{(s)\,a\mu\nu}+\frac{g^2}{8}\,e^{-2\phi}\Bigg)\sqrt{-\mathbf{g}}\,d^4x. \tag{2.1}$$

Here $g^2=g_1^2+g_2^2$, the gauge field tensor $F^{(s)\,a}_{\mu\nu}=\partial_\mu A^{(s)\,a}_\nu-\partial_\nu A^{(s)\,a}_\mu+g_s\,\varepsilon_{abc}\,A^{(s)\,b}_\mu A^{(s)\,c}_\nu$ (there is no summation over $s=1,2$), and ${*F}^{(s)\,a}_{\mu\nu}$ is the dual tensor. The dilaton potential can be viewed as an effective negative, position-dependent cosmological term $\Lambda(\phi)=-\frac{1}{4}\,g^2\,e^{-2\phi}$. The ungauged version of the theory corresponds to the case where $g_1=g_2=0$.

For a purely bosonic configuration, the supersymmetry transformation laws are [9]

$$\delta\bar{\chi}=\frac{i}{\sqrt{2}}\,\bar{\epsilon}\,(-\partial_\mu\phi+i\gamma_5\,e^{-2\phi}\,\partial_\mu\mathbf{a})\gamma^\mu-\frac{1}{2}e^{\phi}\,\bar{\epsilon}\,\mathcal{F}_{\mu\nu}\,\sigma^{\mu\nu}\\ +\frac{1}{4}\,e^{-\phi}\,\bar{\epsilon}\,(g_1+i\gamma_5\,g_2),$$

$$\delta\bar{\psi}_\rho=\bar{\epsilon}\left(\overleftarrow{D}_\rho-\frac{i}{2}\,e^{-2\phi}\,\partial_\rho\mathbf{a}\,\gamma_5\right)-\frac{i}{2\sqrt{2}}\,e^{\phi}\,\bar{\epsilon}\,\mathcal{F}_{\mu\nu}\,\gamma_\rho\,\sigma^{\mu\nu}\\ +\frac{i}{4\sqrt{2}}\,e^{-\phi}\,\bar{\epsilon}\,(g_1+i\gamma_5 g_2)\,\gamma_\rho, \tag{2.2}$$

whereas the variations of the bosonic fields vanish. Here

$$\bar{\epsilon}\overleftarrow{D}_\rho\equiv\bar{\epsilon}\left(\overleftarrow{\partial}_\rho-\frac{1}{2}\,\omega_\rho^{\ \alpha\beta}\,\sigma_{\alpha\beta}+\frac{1}{2}\sum_{s=1}^{2}g_s\,\mathbf{T}_{(s)\,a}\,A^{(s)\,a}_\rho\right),$$

$$\mathcal{F}_{\mu\nu}\equiv\mathbf{T}_{(1)\,a}\,F^{(1)\,a}_{\mu\nu}+i\gamma_5\mathbf{T}_{(2)\,a}\,F^{(2)\,a}_{\mu\nu}. \tag{2.3}$$

In these formulas, $\epsilon\equiv\epsilon^{\mathrm{I}}$ are four Majorana spinor supersymmetry parameters, $\omega_\rho^{\ \alpha\beta}$ is the spin-connection, $\sigma_{\alpha\beta}=\frac{1}{4}[\gamma_\alpha\gamma_\beta]$, and $\mathbf{T}_{(s)\,a}\equiv\mathbf{T}_{(s)\,a\,\mathrm{IJ}}$ are the $SU(2)\times SU(2)$ gauge group generators, whose explicit form will be given below.

Throughout this paper we shall specialize to the static, purely magnetic fields. In this case the axion decouples and one can consistently put $\mathbf{a}=0$. Choosing the metric in the form

$$ds^2=e^{2V}dt^2-e^{-2V}h_{ik}\,dx^i dx^k, \tag{2.4}$$

the action becomes

$$S=\int dt\int\Bigg(-\frac{1}{4}\,{}^{(3)}R-\frac{1}{2}\,\partial_i V\,\partial^i V-\frac{1}{2}\,\partial_i\phi\,\partial^i\phi$$
$$-\frac{1}{4}\,e^{2\phi+2V}\sum_{s=1}^{2}F_{ik}^{(s)\,a}F^{(s)\,aik}$$
$$+\frac{g^2}{8}\,e^{-2\phi-2V}\Bigg)\sqrt{h}\,d^3x. \tag{2.5}$$

This admits a global symmetry

$$V\to V+\epsilon,\quad \phi\to\phi-\epsilon. \tag{2.6}$$

As a consequence, there exists a conserved Noether current $\Theta^i=\sqrt{h}(\partial^i V-\partial^i\phi)$. The corresponding conservation law is

$$\tilde{\nabla}_i\tilde{\nabla}^i(V-\phi)=0, \tag{2.7}$$

where $\tilde{\nabla}_i$ is the covariant derivative with respect to h_{ik}. As a result, the following condition

$$V=\phi-\phi_0 \tag{2.8}$$

with constant ϕ_0 can be imposed.

Let us now further specialize to the case of spherical symmetry. For this we choose the spacetime metric and the gauge fields as

$$ds^2=N\sigma^2dt^2-\frac{dr^2}{N}-r^2(d\theta^2+\sin^2\theta\,d\varphi^2),$$

$$\mathbf{T}_{(s)\,a}\,A_\mu^{(s)\,a}dx^\mu=\frac{1}{g_s}(w_s\,\{-\mathbf{T}_{(s)\,2}\,d\theta+\mathbf{T}_{(s)\,1}\sin\theta\,d\varphi\}$$
$$+\mathbf{T}_{(s)\,3}\cos\theta\,d\varphi), \tag{2.9}$$

where there is no summation over s. We assume that the functions N, σ, w_s and the dilaton ϕ depend only on the radial coordinate $r\in[0,\infty)$. Substituting Eqs. (2.9) into Eq. (2.1) and omitting the surface term, the action becomes

$$S=-4\pi\int dt\int_0^\infty dr\;\sigma\,\Bigg\{\frac{r}{2}\,(1-N)\,\frac{\sigma'}{\sigma}+\frac{r^2}{2}N\phi'^2$$
$$+\frac{1}{2}\,(NW+U)-\frac{g^2}{8}\,r^2\,e^{-2\phi}\Bigg\}, \tag{2.10}$$

where

$$W\equiv W_1+W_2=2e^{2\phi}\sum_{s=1}^{2}\frac{w_s'^2}{g_s^2},$$

$$U\equiv U_1+U_2=e^{2\phi}\sum_{s=1}^{2}\frac{(w_s^2-1)^2}{g_s^2\,r^2}. \tag{2.11}$$

This action admits a symmetry

$$r\to\epsilon\,r,\quad \sigma\to\frac{1}{\epsilon}\,\sigma,\quad \phi\to\phi+\ln\epsilon,\quad w_s\to w_s,\quad N\to N, \tag{2.12}$$

which is the analog of that in Eq. (2.6). The corresponding Noether current is

$$\Xi=\sum_j\frac{\partial L}{\partial u_j{}'}\left(r u_j'-\frac{\partial u_j}{\partial\epsilon}\right)\Bigg|_{\epsilon=1}-rL\equiv\text{const}, \tag{2.13}$$

where $L=L(r,u_j,u_j')$ is the Lagrangian density corresponding to the action (2.10).

The field equations following from the action read

$$(rN)'+r^2N\phi'^2+NW+U+r^2\Lambda(\phi)=1,$$

$$\sigma'/\sigma=r\phi'^2+W/r,$$

$$(\sigma N r^2\phi')'=\sigma\,\{NW+U-r^2\Lambda(\phi)\},$$

$$(N\sigma e^{2\phi}\,w_s')'=\sigma e^{2\phi}\,w_s(w_s^2-1)/r^2. \tag{2.14}$$

Now, there are two first integrals for these equations which provide the solution for the metric variables N and σ. First, the condition (2.8) ensures that

$$\sigma^2N=e^{2(\phi-\phi_0)}. \tag{2.15}$$

In addition, putting $\Xi=0$ in Eq. (2.13) yields

$$N=\frac{1-U+g^2r^2\,e^{-2\phi}/4}{1+2r\phi'-r^2\phi'^2-W}. \tag{2.16}$$

These two first integrals arise as a result of the dilatational symmetry of the action. They provide the most general solutions for the metric variables in the case where the metric is regular at the origin. In addition, as we shall see, these conditions are precisely what is required by supersymmetry. One may wonder why the same symmetry, being expressed in the two different forms (2.6) and (2.12), leads to the two apparently different expressions (2.15) and (2.16). It turns out that, although Eqs. (2.15) and (2.16) are indeed independent, they are equivalent up to an equation of motion. Specifically, Eq. (2.16) can be obtained by inserting Eq. (2.15) into the $G_r^r=2\,T_r^r$ Einstein equation.

Our goal is to solve the remaining equations in the system (2.14). For this we are turning now to the analysis of the supersymmetry constraints, which will give us the additional first integrals.

III. SUPERSYMMETRY CONSTRAINTS

The field configuration (2.9) is supersymmetric provided that there are non-trivial supersymmetry Killing spinors ϵ for which the variations of the fermion fields defined by Eqs. (2.2) vanish. Putting in Eqs. (2.2) $\delta\overline{\chi}=\delta\overline{\psi}_\mu=0$, we arrive at the supersymmetry constraints given in the form of a system of equations for the spinor supersymmetry parameter ϵ:

$$2\sqrt{2}\,e^{\phi}\,\overline{\epsilon}\,\gamma^\mu\,\partial_\mu\phi-2i\,e^{2\phi}\,\overline{\epsilon}\,\mathcal{F}_{\mu\nu}\,\sigma^{\mu\nu}+\overline{\epsilon}\,(ig_1-\gamma_5\,g_2)$$
$$=0, \tag{3.1}$$

$$4\sqrt{2}\ e^{\phi}\ \bar{\epsilon}\overleftarrow{D}_{\rho}-2i\ e^{2\phi}\ \bar{\epsilon}\ \mathcal{F}_{\mu\nu}\ \gamma_{\rho}\ \sigma^{\mu\nu}+\bar{\epsilon}\ (ig_1-\gamma_5 g_2)\ \gamma_{\rho}=0. \tag{3.2}$$

Here D_{ρ} and $\mathcal{F}_{\mu\nu}$ are defined by Eqs. (2.3) and the background fields are specified by Eqs. (2.9). This system consists of 80 linear equations for the 16 independent real components of ϵ. At most, in the maximally supersymmetric case, there could be 16 independent non-trivial solutions. It is clear, however, that generically the system has no non-trivial solutions at all. To find out under what conditions the non-trivial solutions are possible, our strategy is to analyze the equations in components.

First, we choose the vierbein $e_{\alpha}^{\ \mu}$ to be a ''half-null'' complex tetrad:

$$e_0=\frac{1}{\sigma\sqrt{N}}\frac{\partial}{\partial t},\quad e_1=\sqrt{N}\ \frac{\partial}{\partial r},$$

$$e_2=\frac{1}{\sqrt{2}r}\left(\frac{\partial}{\partial\vartheta}+\frac{i}{\sin\ \vartheta}\frac{\partial}{\partial\varphi}\right),\quad e_3=e_2^{*}. \tag{3.3}$$

The non-zero components of the tetrad metric $\eta_{\alpha\beta}=(e_{\alpha},e_{\beta})$ are $\eta_{00}=-\eta_{11}=-\eta_{23}=1$. The dual tetrad e^{α} determines the spin-connection coefficients $\omega^{\alpha\beta}=\omega_{\rho}^{\ \alpha\beta}dx^{\rho}$ via the structure equation, $de^{\alpha}+\omega^{\alpha}_{\ \beta}\wedge e^{\beta}=0$.

The gamma matrices $\gamma^{\alpha}\gamma^{\beta}+\gamma^{\beta}\gamma^{\alpha}=2\eta^{\alpha\beta}$ are chosen to be

$$\gamma^0=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad \gamma^1=\begin{pmatrix}0 & -\sigma^3\\ \sigma^3 & 0\end{pmatrix},$$

$$\gamma^2=\frac{1}{\sqrt{2}}\begin{pmatrix}0 & -\sigma^-\\ \sigma^- & 0\end{pmatrix},\quad \gamma^3=\frac{1}{\sqrt{2}}\begin{pmatrix}0 & -\sigma^+\\ \sigma^+ & 0\end{pmatrix},$$

$$\gamma^5=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix},\quad C=\begin{pmatrix}i\sigma^2 & 0\\ 0 & -i\sigma^2\end{pmatrix}, \tag{3.4}$$

where $\gamma^5=\gamma_5=-(i/4!)\sqrt{-\eta}\ \varepsilon_{\alpha\beta\gamma\delta}\ \gamma^{\alpha}\gamma^{\beta}\gamma^{\gamma}\gamma^{\delta}$ with $\varepsilon_{0123}=-1$ (note that $\sqrt{-\eta}=i$ since $\det(\eta_{\alpha\beta})=1$); and the charge conjugation matrix $C\gamma^{\alpha}C^{-1}=-(\gamma^{\alpha})^{\mathrm{T}}$. The Pauli matrices are

$$\sigma^1=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix},\quad \sigma^2=\begin{pmatrix}0 & -i\\ i & 0\end{pmatrix},\quad \sigma^3=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},$$

and $\sigma^{\pm}=\sigma^1\pm i\sigma^2$.

The SU(2)×SU(2) group generators $\mathbf{T}_{(s)\ a}$, which are subject to the conditions $[\mathbf{T}_{(1)\ a},\mathbf{T}_{(2)\ b}]=0$ and $\mathbf{T}_{(s)\ a}\mathbf{T}_{(s)\ b}=-\epsilon_{abc}\mathbf{T}_{(s)\ c}-\delta_{ab}$, are chosen to be

$$\mathbf{T}_{(1)\ 1}=\begin{pmatrix}0 & -\sigma^2\\ \sigma^2 & 0\end{pmatrix},\quad \mathbf{T}_{(1)\ 2}=\begin{pmatrix}0 & -\sigma^1\\ \sigma^1 & 0\end{pmatrix},$$

$$\mathbf{T}_{(1)\ 3}=\begin{pmatrix}-i\sigma^3 & 0\\ 0 & -i\sigma^3\end{pmatrix},\quad \mathbf{T}_{(2)\ 1}=\begin{pmatrix}0 & i\sigma^3\\ i\sigma^3 & 0\end{pmatrix},$$

$$\mathbf{T}_{(2)\ 2}=\begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix},\quad \mathbf{T}_{(2)\ 3}=\begin{pmatrix}-i\sigma^3 & 0\\ 0 & i\sigma^3\end{pmatrix}. \tag{3.5}$$

Note that this representation of the group generators differs from that in [9] by a unitary transformation.

The Majorana condition for ϵ requires that its Dirac conjugate is equal to the Majorana conjugate [14]:

$$(\epsilon^{\mathrm{I}})^{*\mathrm{T}}\ \gamma^0=\Omega^{\mathrm{I}}_{\ \mathrm{J}}\ (\epsilon^{\mathrm{J}})^{\mathrm{T}}C. \tag{3.6}$$

Here $\Omega^{\mathrm{I}}_{\ \mathrm{J}}$ is defined by the requirement that the condition (3.6) is invariant with respect to the gauge transformations, which demands that

$$\Omega\ \mathbf{T}_{(s)\ a}+(\mathbf{T}_{(s)\ a})^{\mathrm{T}}\ \Omega=0,\quad \Omega\ \Omega^{*}=1. \tag{3.7}$$

The solution of these equations, in the representation (3.5), is given by

$$\Omega=\begin{pmatrix}\sigma^1 & 0\\ 0 & \sigma^1\end{pmatrix}. \tag{3.8}$$

As a result, denoting the components of $\bar{\epsilon}^{\mathrm{I}}$ by ψ_q^{I}, the Majorana condition can be expressed as a set of the following relations between ψ_q^{I}'s:

$$\psi_1^2=-(\psi_4^1)^*,\quad \psi_2^2=(\psi_3^1)^*,\quad \psi_3^2=(\psi_2^1)^*,\quad \psi_4^2=-(\psi_1^1)^*,$$

$$\psi_1^4=-(\psi_4^3)^*,\quad \psi_2^4=(\psi_3^3)^*,\quad \psi_3^4=(\psi_2^3)^*,\quad \psi_4^4=-(\psi_1^3)^*. \tag{3.9}$$

Now we can proceed to solving Eqs. (3.1) and (3.2). First, we choose ϵ to be time independent. At this stage one can obtain the first supersymmetry condition. Specifically, let us multiply the $\rho=0$ equation in (3.2) by γ^0 from the right and subtract the result from Eqs. (3.1). Using the fact that the electric part of $\mathcal{F}_{\mu\nu}$ vanishes, and also that γ^0 commutes with σ^{ik}, the result is

$$\bar{\epsilon}\gamma^{\mu}\partial_{\mu}\phi-2\bar{\epsilon}\overleftarrow{D}_0\gamma^0=0. \tag{3.10}$$

Computing $\bar{\epsilon}\overleftarrow{D}_0=-(1/2)\ \bar{\epsilon}\ \omega_0^{\ \alpha\beta}\ \sigma_{\alpha\beta}$ this condition is equivalent to

$$\bar{\epsilon}\gamma^1(\ln(\sigma^2Ne^{-2\phi}))'=0, \tag{3.11}$$

which finally requires that

$$\sigma^2N=e^{2(\phi-\phi_0)}, \tag{3.12}$$

thus reproducing Eq. (2.15). As a result, we can omit Eq. (3.1) and concentrate on the four gravitino supersymmetry constraints (3.2).

Our procedure is straightforward: by inserting the above definitions into Eqs. (3.2) and projecting the equations onto the tetrad, we work out the result in components (we do not present here the expressions explicitly in view of their complexity). The next step is to separate the angular variables, and for this we take advantage of the special properties of the spinor representation chosen. Specifically, it turns out that the spherical variables enter the resulting equations only in such a way that they form certain differential operators. The structure of these operators coincides with the one for the raising and lowering operators in the well-known recurrence relations for the spin-weighted spherical harmonics ${}_{\kappa}Y_{jm}$ [15]:

$$\left(\frac{\partial}{\partial\vartheta}\mp\frac{i}{\sin\,\vartheta}\,\frac{\partial}{\partial\varphi}\pm\kappa\cot\vartheta\right)\,{}_{\kappa}Y_{jm}=\pm\sqrt{(j\pm\kappa)(j\mp\kappa+1)}\;{}_{\kappa\mp1}Y_{jm}. \tag{3.13}$$

This suggests choosing the spinor components $\psi_q^{\rm I}$ in the following form:

$$\psi_q^{\rm I}\,(r,\vartheta,\varphi)=R_q^{\rm I}(r)\;{}_{\kappa}Y_{jm}(\vartheta,\varphi). \tag{3.14}$$

The spin weights of the amplitudes, $\kappa=\kappa_q^{\rm I}$, are determined by the direct inspection of the equations:

$$\kappa_1^1=\kappa_3^1=-\kappa_2^2=-\kappa_4^2=\frac{1-\nu_1-\nu_2}{2},$$

$$\kappa_2^1=\kappa_4^1=-\kappa_1^2=-\kappa_3^2=-\frac{1+\nu_1+\nu_2}{2},$$

$$\kappa_1^3=\kappa_3^3=-\kappa_2^4=-\kappa_4^4=\frac{1-\nu_1+\nu_2}{2},$$

$$\kappa_2^3=\kappa_4^3=-\kappa_1^4=-\kappa_3^4=-\frac{1+\nu_1-\nu_2}{2}. \tag{3.15}$$

Here $\nu_s=1$ if $g_s\neq0$ and $\nu_s=0$ otherwise.

The quantum number j, which is the same for all amplitudes, has the meaning of the total angular momentum including orbital angular momentum, spin and isospin. Its values are restricted by the condition $j\geqslant|\kappa|$, since ${}_{\kappa}Y_{jm}$ vanishes otherwise. We fix the value of j by requiring that

$$j=\;\min\,|\kappa_q^{\rm I}|, \tag{3.16}$$

where $\kappa_q^{\rm I}$'s are given by Eq. (3.15). This can be regarded as a consistent truncation of the system, since all amplitudes with $|\kappa_q^{\rm I}|$ exceeding the minimal value vanish. The values of the azimuthal quantum number m are restricted by the condition $-j\leqslant m\leqslant j$. Since m does not enter the equations, its entire effect is to increase the degeneracy of the solutions.

At this stage, the complete separation of the angular variables is achieved in the equations. The supersymmetry constraints reduce to a set of algebraic and ordinary differential equations for the radial amplitudes $R_q^{\rm I}(r)$. Note that the spin weights in Eq. (3.15) and, correspondingly, the structure of the resulting equations essentially depend on whether some of the coupling constants g_s vanish or not. As a result, there arise three basically different cases to consider : (1) None of g_s's vanish, which corresponds to the full SU(2)×SU(2) gauge symmetry. (2) Either g_1 or g_2 vanishes—the gauge symmetry is truncated to SU(2)×[U(1)]^3. (3) $g_1=g_2=0$–the gauge group is [U(1)]^6. It turns out that in the first case there are no solutions to the supersymmetry constraints (apart from the trivial one). If both coupling constants vanish, the non-trivial Killing spinors exist and the underlying supersymmetric backgrounds are the well-known Abelian dilaton black holes [10,11]. Our main thrust will be on the second case, where the gauge symmetry is truncated to SU(2)×[U(1)]^3.

IV. THE SUPERSYMMETRY CONSISTENCY CONDITIONS

If one of the coupling constants is zero, we assume that the corresponding Abelian gauge field vanishes too. At the same time, the other coupling constant can be set to unity via the appropriate rescaling of the fields in the action. As a result, one has either $g_1=1$, $g_2=0$ or $g_1=0$, $g_2=1$. It turns out that in both of these cases there is the same number of non-trivial solutions of the supersymmetry constraints. The corresponding consistency conditions are identical up to the replacement $w_1\leftrightarrow w_2$. We shall therefore consider explicitly only the case where $g_1=0$, $g_2=1$, since the equations contain then only real coefficients.

Putting $A_\mu^{(1)\,a}=0$, the field equations are obtained from Eqs. (2.14)–(2.16) by omitting the terms W_1 and U_1 in Eq. (2.11). The gauge field $A_\mu^{(2)\,a}$ is given by Eq. (2.9), where w_2 will be denoted by w. Equations (3.15) imply that $\min|\kappa_q^{\rm I}|=0$, and so we put in (3.14) $j=0$. Note that this can be regarded as a manifestation of the spin-isospin coupling: since both spin and isospin are half-integer, the total angular momentum is integer and hence its lowest value is zero. For $j=0$ all spin-weighted harmonics with $\kappa>0$ vanish, while ${}_0Y_{00}=$const. As a result, the non-vanishing spinor components are

$$\overline{\epsilon}^{\,1}=(R_1^1(r),0,R_3^1(r),0),\quad \overline{\epsilon}^{\,3}=(0,R_2^3(r),0,R_4^3(r)), \tag{4.1}$$

and

$$\overline{\epsilon}^{\,2}=(0,R_2^2(r),0,R_4^2(r)),\quad \overline{\epsilon}^{\,4}=(R_1^4(r),0,R_3^4(r),0). \tag{4.2}$$

Among these components those in Eq. (4.1) can be chosen to be independent, whereas

$$R_2^2=(R_3^1)^*,\quad R_4^2=-(R_1^1)^*,\quad R_1^4=-(R_4^3)^*,\quad R_3^4=(R_2^3)^*, \tag{4.3}$$

in view of the Majorana conjugation (3.9). The equations for R_q^2 and R_q^4 also can be obtained from those for R_q^1 and R_q^3 by applying the conjugation rule (4.3). We shall therefore concentrate only on the independent variables R_q^1 and R_q^3.

Making the linear combinations

$$\Psi^1=R_1^1+R_3^1,\quad \Psi^2=R_2^3+R_4^3,$$

$$\Psi^3=R_1^1-R_3^1,\quad \Psi^4=R_2^3-R_4^3, \tag{4.4}$$

the supersymmetry constraints can be represented as follows: The temporal component $(\rho=0)$ of Eqs. (3.2) gives the relations

$$A^{+}\Psi^{1}+C\Psi^{2}=0,\quad C\Psi^{1}-A^{-}\Psi^{2}=0,$$
$$A^{-}\Psi^{3}-C\Psi^{4}=0,\quad C\Psi^{3}+A^{+}\Psi^{4}=0,\qquad(4.5)$$

whereas the angular components of the equations ($\rho=\vartheta,\varphi$) together require that

$$b^{-}\Psi^{1}-w\beta\,\Psi^{2}=0,\quad -w\beta\,\Psi^{1}+b^{+}\Psi^{2}=0,$$
$$b^{+}\Psi^{3}-w\beta\,\Psi^{4}=0,\quad -w\beta\,\Psi^{3}+b^{-}\Psi^{4}=0.\qquad(4.6)$$

Finally, the radial component yields

$$\gamma\,(\Psi^{1})'+(B+1)\Psi^{1}-C\Psi^{2}=0,$$
$$\gamma\,(\Psi^{2})'-(B+1)\Psi^{2}+C\Psi^{1}=0,$$
$$\gamma\,(\Psi^{3})'-(B+1)\Psi^{3}+C\Psi^{4}=0,$$
$$\gamma\,(\Psi^{4})'+(B+1)\Psi^{4}-C\Psi^{3}=0.\qquad(4.7)$$

The coefficients in these equations are given by

$$B=\frac{2}{r^{2}}\,e^{2\phi}(w^{2}-1),\quad C=\frac{4}{r}\,e^{\phi}\sqrt{N}w',$$
$$\beta=\frac{4}{r}\,e^{\phi},\quad \gamma=4\sqrt{2N}\,e^{\phi},$$
$$A^{\pm}=2\sqrt{2N}\,e^{\phi}\phi'\pm(B+1),$$
$$b^{\pm}=\beta\sqrt{N}\pm\sqrt{2}(B-1).\qquad(4.8)$$

The algebraic equations (4.5) and (4.6) have non-trivial solutions if only the corresponding determinants vanish:

$$A^{+}A^{-}+C^{2}=0,\quad b^{+}b^{-}-w^{2}\beta^{2}=0,\qquad(4.9)$$

under which conditions the solutions are

$$\Psi^{1}=\frac{A^{-}}{C}\,\Psi^{2},\quad \Psi^{4}=\frac{A^{-}}{C}\,\Psi^{3},\qquad(4.10)$$

for Eqs. (4.5), and

$$\Psi^{1}=\frac{w\beta}{b^{-}}\,\Psi^{2},\quad \Psi^{4}=\frac{w\beta}{b^{-}}\,\Psi^{3},\qquad(4.11)$$

for Eqs. (4.6), respectively. It is clear that these solutions agree if only

$$A^{-}b^{-}=w\beta\,C.\qquad(4.12)$$

We thus arrive at the three consistency conditions given by Eqs. (4.9) and (4.12), under which the solution of the algebraic equations (4.5) and (4.6) is expressed by Eqs. (4.10) and (4.11) in terms of two independent functions Ψ^2 and Ψ^3. Next, inserting this solution into Eq. (4.7) gives an additional consistency condition

$$\gamma C\left(\frac{A^{-}}{C}\right)'+2(B+1)A^{-}-A^{-\,2}-C^{2}=0,\qquad(4.13)$$

and a pair of differential equations for Ψ^2 and Ψ^3

$$\gamma(\Psi^{2})'+(A^{-}-B-1)\Psi^{2}=0,$$
$$\gamma(\Psi^{3})'+(A^{-}-B-1)\Psi^{3}=0.\qquad(4.14)$$

Remarkably, it can be verified that the condition in Eq. (4.13) is a differential consequence of the algebraic conditions (4.9) and (4.12). The latter therefore provide the full set of the consistency conditions, under which the solution of the supersymmetry constraints is given by Eq. (4.10) [or Eq. (4.11)] and Eq. (4.14).

Taking into account the definitions in Eq. (4.8), the consistency conditions (4.9) and (4.12) can be explicitly expressed as follows:

$$N=1+\frac{r^{2}}{8}e^{-2\phi}\left(1+2e^{2\phi}\frac{w^{2}-1}{r^{2}}\right)^{2},\qquad(4.15)$$

$$r\phi'=\frac{r^{2}}{8N}e^{-2\phi}\left(1-4e^{4\phi}\,\frac{(w^{2}-1)^{2}}{r^{4}}\right),\qquad(4.16)$$

$$rw'=-2w\frac{r^{2}}{8N}e^{-2\phi}\left(1+2e^{2\phi}\frac{w^{2}-1}{r^{2}}\right).\qquad(4.17)$$

Together with

$$N\sigma^{2}=e^{2(\phi-\phi_{0})}\qquad(4.18)$$

these equations provide the full set of the consistency conditions under which the supersymmetry constraints have nontrivial solutions. It can be verified that these conditions are compatible with the field equations (2.14). One can check with the help of Eqs. (4.16) and (4.17) that the expression for N given by Eq. (4.15) is equivalent to that in Eq. (2.16).

The supersymmetry Killing spinors are given by Eqs. (4.1)–(4.3) with

$$R_{1}^{1}=\varepsilon_{1}F_{1}+\varepsilon_{2}F_{2},\quad R_{3}^{1}=\varepsilon_{1}F_{1}-\varepsilon_{2}F_{2},$$
$$R_{2}^{3}=\varepsilon_{1}F_{2}+\varepsilon_{2}F_{1},\quad R_{4}^{3}=\varepsilon_{1}F_{2}-\varepsilon_{2}F_{1},\qquad(4.19)$$

where

$$F_{2}=\exp\left\{-\frac{\phi}{2}-\int_{0}^{r}\frac{\sqrt{N-1}}{r\sqrt{N}}dr\right\},$$
$$F_{1}=\frac{F_{2}}{w}\left\{e^{\phi}(\sqrt{N}-\sqrt{N-1})-\frac{r}{\sqrt{2}}\right\},\qquad(4.20)$$

and ε_1, ε_2 are two complex integration constants. One can see that there are altogether four independent Killing spinors.

The same supersymmetry conditions arise in the case where $g_2=A_{\mu}^{(2)\,a}=0$, whereas $A_{\mu}^{(1)\,a}\neq 0$, $g_1=1$. Then there are also four independent Killing spinors. We therefore conclude that the Bogomol'nyi equations (4.15)–(4.18) specify the $N=1$ supersymmetric BPS states in the $N=4$ gauged supergravity with the gauge group SU(2)$\times$[U(1)]^3.

Let us describe briefly what happens in the two other cases, where the gauge symmetry is either Abelian or totally non-Abelian. If $g_1=g_2=0$, we make the gauge fields in Eq. (2.9) Abelian by setting $w_1=w_2=0$:

$$\mathbf{T}_{(s)\ a}\ A_{\mu}^{(s)\ a}dx^{\mu}=\mathbf{T}_{(s)\ 3}\cos\theta\ d\varphi, \tag{4.21}$$

which corresponds to the Dirac monopole type fields. The supersymmetry constraints split then into four independent groups, one group for each of the four spinors $\overline{\epsilon^{\mathrm{I}}}$. The spinors $\overline{\epsilon^1}$ and $\overline{\epsilon^3}$ can be chosen to be independent, $\overline{\epsilon^2}$ and $\overline{\epsilon^4}$ being their Majorana conjugates. The separation of the angular variables is achieved by choosing

$$\overline{\epsilon^{1}}=[R_1^1(r)_{-1/2}Y_{(1/2)\,m}\,,\quad R_2^1(r)_{-1/2}Y_{(1/2)\,m}\,,$$
$$R_3^1(r)_{1/2}Y_{(1/2)\,m}\,,\quad R_4^1(r)_{1/2}Y_{(1/2)\,m}], \tag{4.22}$$

and similarly for $\overline{\epsilon^{3}}$. It turns out then that if one of the two gauge fields in Eq. (4.21) vanishes, no matter which, the supersymmetry constraints admit two independent solutions for the radial amplitudes R_q^1, and similarly for R_q^3, provided that the following consistency conditions hold:

$$N\sigma^2=e^{2(\phi-\phi_0)},\quad 2Nr^2\phi'^2=\frac{e^{2\phi}}{r^2},\quad N(1+r\phi')^2=1. \tag{4.23}$$

In addition, the fact that the azimuthal quantum number m in Eq. (4.22) assumes two values, $m=\pm 1/2$, doubles the number of solutions, which finally corresponds to eight supersymmetries. The solutions to Eqs. (4.23) describe well-known magnetic dilaton black holes [10], the fact that they have $N=2$ supersymmetry was established in [11].

Finally, in the totally non-Abelian case the supersymmetry constraints are given by the most general expressions described above. Similarly to the Abelian case, the minimal value of the angular momentum required by the condition (3.16) is 1/2. This is due to the presence of the two independent isospins, which ensures that the total angular momentum is half-integer. However, the equations in this case do not allow for any non-trivial solutions.

Summarizing, the gauged SU(2)×SU(2) $N=4$ supergravity admits no supersymmetric solutions at all—in the static, spherically symmetric, purely magnetic sector. The ''half-gauged'' SU(2)×$[\mathrm{U}(1)]^3$ model has solutions with $N=1$ supersymmetry that will be presented below. The non-gauged theory admits solutions with $N=2$ supersymmetry described in [10,11].

V. SOLUTIONS OF THE BOGOMOL'NYI EQUATIONS

In order to find the general solution of the Bogomol'nyi equations (4.15)–(4.18) we start from the case where $w(r)$ is constant. The only possibilities are $w(r)=\pm 1$ or $w(r)=0$.

For $w(r)=\pm 1$ the Yang-Mills field is a pure gauge. Equation (4.17) requires then that $\exp(-2\phi)=0$, which means that $\phi(r)=\phi_0\to\infty$, implying that the metric is flat.

The $w(r)=0$ choice corresponds to the Dirac monopole gauge field. The only non-trivial equation, Eq. (4.16), then reads

$$r\phi'=\frac{r^2-2e^{2\phi}}{r^2+2e^{2\phi}}, \tag{5.1}$$

whose general solution is given by

$$\phi+\ln\frac{r}{r_0}=\frac{r^2}{4}e^{-2\phi}, \tag{5.2}$$

with constant r_0. The corresponding metric turns out to be singular both at the origin and at infinity.

Suppose now that $w(r)$ is not a constant. Introducing the new variables $x=w^2$ and $R^2=\frac{1}{2}r^2e^{-2\phi}$, Eqs. (4.15)–(4.17) become equivalent to one differential equation

$$2xR\ (R^2+x-1)\ \frac{dR}{dx}+(x+1)\ R^2+(x-1)^2=0. \tag{5.3}$$

If $R(x)$ is known, the radial dependence of the functions, $x(r)$ and $R(r)$, can be determined from (4.16) or (4.17). Equation (5.3) is solved by the following substitution:

$$x=\rho^2\ e^{\xi(\rho)},\quad R^2=-\rho\frac{d\xi(\rho)}{d\rho}-\rho^2\ e^{\xi(\rho)}-1, \tag{5.4}$$

where $\xi(\rho)$ is a solution of

$$\frac{d^2\xi(\rho)}{d\rho^2}=2\ e^{\xi(\rho)}. \tag{5.5}$$

The most general (up to reparametrizations) solution of this equation which ensures that $R^2>0$ is $\xi(\rho)=-2\ln\sinh(\rho-\rho_0)$. This gives us the general solution of Eqs. (4.15)–(4.18). The metric is non-singular at the origin if only $\rho_0=0$, in which case

$$R^2(\rho)=2\rho\coth\rho-\frac{\rho^2}{\sinh^2\rho}-1\,. \tag{5.6}$$

One has $R^2(\rho)=\rho^2+O(\rho^4)$ as $\rho\to 0$, and $R^2(\rho)=2\rho+O(1)$ as $\rho\to\infty$. The last step is to obtain $r(s)$ from Eq. (4.17), which finally gives us a family of completely regular solutions of the Bogomol'nyi equations:

$$ds^2=2\,e^{2\phi}\,\{dt^2-d\rho^2-R^2(\rho)(d\vartheta^2+\ \sin^2\ \vartheta d\varphi^2)\}, \tag{5.7}$$

$$w=\pm\frac{\rho}{\sinh\rho},\quad e^{2\phi}=a^2\ \frac{\sinh\rho}{2\,R(\rho)}, \tag{5.8}$$

where $0\leqslant\rho<\infty$, and we have chosen in Eq. (4.18) $2\phi_0=-\ln 2$. The appearance of the free parameter a in the solutions reflects the scaling symmetry of Eqs. (4.15)–(4.18): $r\to ar$, $\phi\to\phi+\ln a$. The geometry described by the line element (5.7) is everywhere regular, the coordinates covering the whole space whose topology is R^4. It is instructive to express the solutions in Schwarzschild coordinates, where the metric functions $N(r)$ and $\sigma(r)$ are given parametrically by

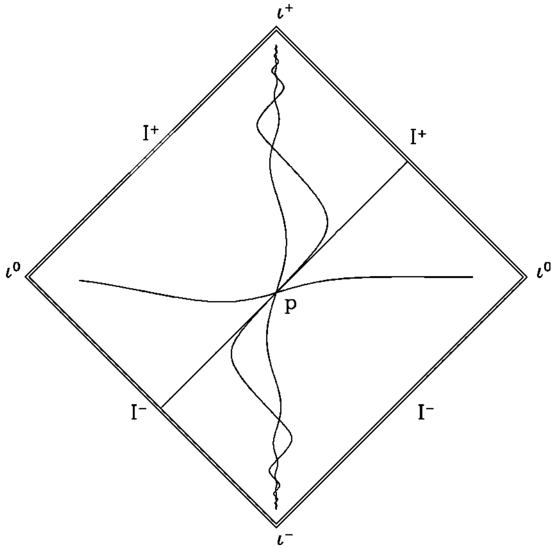

FIG. 1. The conformal diagram for the spacetime described by the line element (5.7).

$$r=a\sqrt{R(\rho)}\,\sinh\rho,\quad N=\frac{\rho^2}{R^2(\rho)},\quad \sigma=\frac{r}{\rho}. \tag{5.9}$$

At the origin, $r\to 0$, one has

$$N=1+\frac{r^2}{9a^2}+O(r^4),\quad N\sigma^2=2e^{2\phi}=a^2+\frac{2r^2}{9}+O(r^4),$$

$$w=1-\frac{r^2}{6a^2}+O(r^4), \tag{5.10}$$

whereas in the asymptotic region, $r\to\infty$,

$$N\propto\ln r,\quad N\sigma^2=2e^{2\phi}\propto\frac{r^2}{4\,\ln r},\quad w\propto\frac{4\,\ln r}{r^2}. \tag{5.11}$$

The geometry is flat at the origin, but asymptotically it is not flat. Specifically, all curvature invariants vanish in the asymptotic region, however, not fast enough. For example, the non-vanishing Weyl tensor invariant $\Psi_2\propto -1/6r^2$ as $r\to\infty$.

The global structure of the solutions is well illustrated by the conformal diagram. Inspecting the t-ρ part of the metric, it is not difficult to see that the conformal diagram in this case is actually identical to the one for Minkowski space, even though the geometry is not asymptotically flat (see Fig. 1). The spacetime is therefore geodesically complete and globally hyperbolic. The latter property is quite remarkable, since global hyperbolicity is usually lacking for the known supersymmetry backgrounds in gauged supergravity models. The geodesics through a spacetime point p are shown in the diagram, each geodesic approaching infinity for large absolute values of the affine parameter. Although the global behavior of geodesics is similar to that for Minkowski space, they locally behave differently. For $\rho<\infty$ the cosmological term $\Lambda(\phi)$ is non-zero and negative, thus having the focusing effect on timelike geodesics, which makes them oscillate around the origin. Unlike the situation in the anti–de Sitter case, each geodesic has its own period of oscillations, such that the geodesics from a point p never refocus again.

The shape of the amplitude $w(\rho)$ in Eq. (5.8) corresponds to the gauge field of the regular magnetic monopole with unit magnetic charge. In fact, assuming for a moment that ρ is the standard radial coordinate, the amplitude exactly coincides with that for the flat space BPS solution. This result is quite surprising, since the model has no Higgs field, in which case it would be natural to expect the existence of only neutral solutions [8]. A manifestation of this is the fact that, without a Higgs field, the magnetic charge has no gauge invariant meaning and can only be defined for a certain class of gauges. In addition, since all fields in the problem are massless, it is clear that w cannot in fact exhibit exactly the same behavior as the one for the flat space BPS monopole amplitude. Indeed, passing to the physical radial coordinate r, the amplitude w for $r\to\infty$ decays polynomially, and not exponentially; see Eq. (5.11).

In conclusion, Eqs. (5.7), (5.8) describe globally regular, supersymmetric backgrounds of a new type. The existence of unbroken supersymmetries suggests that the configurations should be stable, and we expect that the stability proof can be given along the same lines as in [16]. Being solutions of $N=4$ quantum supergravity in four dimensions, they presumably receive no quantum corrections. On the other hand, they can be considered in the framework of the string theory, and then the issue of string corrections can be addressed. In order to study this problem, we first of all need to lift the solutions to ten dimensions.

VI. COMPACTIFICATION OF $D=10$ SUPERGRAVITY ON THE GROUP MANIFOLD

Our aim now is to promote the solutions of the four-dimensional supergravity model obtained above to the solutions of $N=1$ supergravity in ten dimensions. This would make it possible to link the solutions to string theory. It is a well-known fact that ungauged $N=4$ supergravity in four dimensions can be obtained via toriodal compactification of ten-dimensional supergravity [17]. Similarly, the gauged supergravity can be obtained by compactification on the group manifold. This fact is, however, less known, although one could have conjectured this by studying the compactification of eleven-dimensional supergravity on the seven sphere [18]. Because this is not covered in the literature we shall outline below the compactification procedure in some detail. We shall restrict ourselves to the purely bosonic sector and describe the reduction of the action and the fermionic supersymmetry transformations. The corresponding procedure for the full theory, including fermion interactions, can be derived similarly but will not be given here.

1. The action in D=10. The starting point is the bosonic part of the action of $N=1$ supergravity in ten dimensions:

$$S_{10}=\int\left(-\frac{\hat{e}}{4}\hat{R}+\frac{\hat{e}}{2}\,\partial_M\hat{\phi}\,\partial^M\hat{\phi}+\frac{\hat{e}}{12}\,e^{-2\hat{\phi}}\hat{H}_{MNP}\,\hat{H}^{MNP}\right)$$
$$\times d^4x\;d^6z\equiv S_{\hat{G}}+S_{\hat{\phi}}+S_{\hat{H}}. \tag{6.1}$$

The notation is as follows: the hatted symbols are used for the 10-dimensional quantities. Late capital Latin letters stand for the base space indices $(M,N,P, \ldots)$ and the early letters refer to the tangent space indices $(A,B,C, \ldots)$. For space-time indices taking 4 values, late and early Greek letters denote base space and tangent space indices, respectively. Similarly, the internal base space and tangent space indices are denoted by late and early Latin letters, respectively:

$$\{M\}=\{\mu=0,\ldots,3; \quad m=1,\ldots,6\},$$

$$\{A\}=\{\alpha=0,\ldots,3; \quad \mathrm{a}=1,\ldots,6\}. \tag{6.2}$$

The general coordinates $\hat{x}^M$ consist of spacetime coordinates x^μ and internal coordinates z^m. The flat Lorentz metric of the tangent space is chosen to be $(+,-,\ldots,-)$ with the internal dimensions all spacelike. One has $\hat{e}=|\hat{e}^A_{\ M}|$, the metric is related to the vielbein by $\hat{\mathbf{g}}_{MN}=\hat{\eta}_{AB}\hat{e}^A_{\ M}\hat{e}^B_{\ N}=\eta_{\alpha\beta}\hat{e}^\alpha_{\ M}\hat{e}^\beta_{\ N}-\delta_{\mathrm{ab}}\hat{e}^{\mathrm{a}}_{\ M}\hat{e}^{\mathrm{b}}_{\ N}$, and the antisymmetric tensor field strength is

$$\hat{H}_{MNP}=\partial_M\hat{B}_{NP}+\partial_N\hat{B}_{PM}+\partial_P\hat{B}_{MN}. \tag{6.3}$$

The internal space spanned by z^m is assumed to form a compact group space. This means that there are functions $U^{\mathrm{a}}_{\ m}(z)$ subject to the condition

$$(U^{-1})_{\mathrm{b}}^{\ m}(U^{-1})_{\mathrm{c}}^{\ n}(\partial_m U^{\mathrm{a}}_{\ n}-\partial_n U^{\mathrm{a}}_{\ m})=\frac{f_{\mathrm{abc}}}{\sqrt{2}}, \tag{6.4}$$

where f_{abc} are the group structure constants. The volume of the space is

$$\Omega=\int |U^{\mathrm{a}}_{\ m}|d^6z. \tag{6.5}$$

In particular, we shall be considering the case where the internal space is the product manifold SU(2)×SU(2). It is convenient to parametrize then the 6 internal coordinates by a pair of indices: $\{m\}=\{(s),i\}$, where $s=1,2$ and $i=1,2,3$; similarly for the tangent space coordinates:$\{\mathrm{a}\}=\{(s),a\}$, $a=1,2,3$. Each of the two S^3's admits invariant 1-forms $\theta^{(s)\,a}=\theta^{(s)\ a}_{\ i}dz^{(s)\,i}$:

$$d\theta^{(s)\,a}+\frac{1}{2}\,\epsilon_{abc}\,\theta^{(s)\,b}\wedge\theta^{(s)\,c}=0. \tag{6.6}$$

If we choose

$$U^{\mathrm{a}}_{\ m}\equiv U^{(s)\ a}_{\ i}=-\frac{\sqrt{2}}{g_s}\,\theta^{(s)\ a}_{\ i}, \tag{6.7}$$

where g_s are the two gauge coupling constants, then the structure constants determined by Eq. (6.4) will be

$$f_{\mathrm{abc}}\equiv f^{(s)}_{abc}=g_s\,\epsilon_{abc}. \tag{6.8}$$

Similarly, if one of the gauge coupling constants vanishes, say $g_2=0$, the internal space is SU(2)×[U(1)]^3. Choosing in this case $g_1=1$,

$$U^{(1)\ a}_{\ i}=-\sqrt{2}\,\theta^{(1)\ a}_{\ i}, \quad U^{(2)\ a}_{\ i}=\delta^a_i \Rightarrow f^{(1)}_{abc}=\epsilon_{abc},$$

$$f^{(2)}_{abc}=0. \tag{6.9}$$

2. The metric and the dilaton. Let us now return to the general parametrization of the internal space. The dimensional reduction of the action (6.1) starts by choosing the vielbein and the dilaton in the following form:

$$\hat{e}^\alpha_{\ \mu}=e^{-(3/4)\,\phi}\,e^\alpha_{\ \mu}, \quad \hat{e}^{\mathrm{a}}_{\ \mu}=\sqrt{2}\,e^{(1/4)\,\phi}\,A^{\mathrm{a}}_{\ \mu},$$

$$\hat{e}^\alpha_{\ m}=0, \quad \hat{e}^{\mathrm{a}}_{\ m}=e^{(1/4)\,\phi}\,U^{\mathrm{a}}_{\ m}, \quad \hat{\phi}=-\frac{\phi}{2}, \tag{6.10}$$

where all quantities on the right, apart from $U^{\mathrm{a}}_{\ m}$, depend only on x^μ. One has $\hat{e}=e^{-3\phi/2}|U^{\mathrm{a}}_{\ m}|\,e$. The dual basis is given by

$$\hat{e}_\alpha^{\ \mu}=e^{(3/4)\,\phi}\,e_\alpha^{\ \mu}, \quad \hat{e}_{\mathrm{a}}^{\ \mu}=0,$$

$$\hat{e}_\alpha^{\ m}=-\sqrt{2}\,e^{(3/4)\,\phi}\,e_\alpha^{\ \mu}\,A^{\mathrm{a}}_\mu\,(U^{-1})_{\mathrm{a}}^{\ m},$$

$$\hat{e}_{\mathrm{a}}^{\ m}=e^{-(1/4)\,\phi}\,(U^{-1})_{\mathrm{a}}^{\ m}. \tag{6.11}$$

The metric components are obtained from Eq. (6.10):

$$\hat{\mathbf{g}}_{\mu\nu}=e^{-(3/2)\,\phi}\,\mathbf{g}_{\mu\nu}-2\,e^{(1/2)\,\phi}\,A^{\mathrm{a}}_\mu A^{\mathrm{a}}_\nu,$$

$$\hat{\mathbf{g}}_{\mu m}=\sqrt{2}\,e^{(1/2)\,\phi}\,A^{\mathrm{a}}_\mu U^{\mathrm{a}}_{\ m},$$

$$\hat{\mathbf{g}}_{mn}=-e^{(1/2)\,\phi}\,U^{\mathrm{a}}_{\ m}U^{\mathrm{a}}_{\ n}; \tag{6.12}$$

similarly for $\hat{\mathbf{g}}^{\mu\nu}$. Using these expressions, the application of the standard formulas [19] gives for the gravitational and dilaton terms in the action (6.1)

$$S_{\hat{G}}+S_{\hat{\phi}}=\Omega\int e\left(-\frac{1}{4}R+\frac{1}{2}\,\partial_\mu\phi\,\partial^\mu\phi-\frac{1}{8}\,e^{2\phi}\,F^{\mathrm{a}}_{\mu\nu}F^{\mathrm{a}\mu\nu}+\frac{1}{32}\,e^{-2\phi}\,f^2_{\mathrm{abc}}\right)d^4x, \tag{6.13}$$

where

$$F^{\mathrm{a}}_{\mu\nu}=\partial_\mu A^{\mathrm{a}}_\nu-\partial_\nu A^{\mathrm{a}}_\mu+f_{\mathrm{abc}}A^{\mathrm{b}}_\mu A^{\mathrm{c}}_\nu. \tag{6.14}$$

3. The two-form. Now, the important role is played by the antisymmetric tensor field. The corresponding ansatz is

$$\hat{B}_{\mu\nu}=B_{\mu\nu}, \quad \hat{B}_{\mu m}=-\frac{1}{\sqrt{2}}A^{\mathrm{a}}_\mu\,U^{\mathrm{a}}_{\ m}, \quad \hat{B}_{mn}=\widetilde{B}_{mn}, \tag{6.15}$$

where $B_{\mu\nu}=B_{\mu\nu}(x)$, while $\widetilde{B}_{mn}$ depend only on z. Computation of the field strength according to the rule (6.3) gives

$$\hat{H}_{\mu\nu\rho}=H_{\mu\nu\rho}\equiv\partial_\mu B_{\nu\rho}+\partial_\nu B_{\rho\mu}+\partial_\rho B_{\mu\nu},$$

$$\hat{H}_{\mu\nu m}=-\frac{1}{\sqrt{2}}\,(\partial_\mu A^{\mathrm{a}}_\nu-\partial_\nu A^{\mathrm{a}}_\mu)\ U^{\mathrm{a}}{}_m,$$

$$\hat{H}_{\mu mn}=\frac{1}{2}\,f_{\mathrm{abc}}\,A^{\mathrm{a}}_\mu\ U^{\mathrm{b}}{}_m\ U^{\mathrm{c}}{}_n,$$

$$\hat{H}_{mnp}=\partial_m\widetilde{B}_{np}+\partial_n\widetilde{B}_{pm}+\partial_p\widetilde{B}_{mn}. \tag{6.16}$$

We require that

$$\hat{H}_{mnp}=\frac{1}{2\sqrt{2}}\,f_{\mathrm{abc}}\,U^{\mathrm{a}}{}_m\ U^{\mathrm{b}}{}_n\ U^{\mathrm{c}}{}_p. \tag{6.17}$$

This relation should be regarded as a system of equations for $\widetilde{B}_{mn}$. One can see that the solution exists in the cases that we are interested in. Indeed, if the internal space is $S^3\times S^3$ Eq. (6.17) assures that the 3-form $\hat{H}_{mnp}$ is proportional to the volume form on $S^3\times S^3$. Since this form is closed, the integrability conditions for the system are locally satisfied. On the other hand, since the volume form is not exact, the solution exists only locally. However, the gauge invariance

$$\hat{B}_{mn}\rightarrow\hat{B}_{mn}+\partial_m\Lambda_n-\partial_n\Lambda_m \tag{6.18}$$

allows one to globally extend the local solutions by choosing the non-trivial transition functions in the overlapping regions. A similar argument applies when one of the manifolds is T^3.

The next step is to compute the vielbein projections of the expressions in (6.16), (6.17). The result is

$$\hat{H}_{\alpha\beta\gamma}=e^{(9/4)\ \phi}\ (H_{\alpha\beta\gamma}-\omega_{\alpha\beta\gamma}),$$

$$\hat{H}_{\alpha\beta\mathrm{a}}=-\frac{1}{\sqrt{2}}\,e^{(5/4)\ \phi}\ F^{\mathrm{a}}_{\alpha\beta},$$

$$\hat{H}_{\alpha\mathrm{ab}}=0,\quad \hat{H}_{\mathrm{abc}}=\frac{1}{2\sqrt{2}}\,e^{-(3/4)\ \phi}\quad f_{\mathrm{abc}}, \tag{6.19}$$

where $F^{\mathrm{a}}_{\alpha\beta}=e_\alpha{}^\mu e_\beta{}^\nu F^{\mathrm{a}}_{\mu\nu}$ are the tetrad projections of the gauge field tensor, and $\omega_{\alpha\beta\gamma}$ are the tetrad projections of the gauge field Chern-Simons 3-form

$$\omega_{\mu\nu\rho}=-6\left(A^{\mathrm{a}}_{[\mu}\partial_\nu A^{\mathrm{a}}_{\rho]}+\frac{1}{3}\,f_{\mathrm{abc}}\,A^{\mathrm{a}}_\mu\ A^{\mathrm{b}}_\nu\ A^{\mathrm{c}}_\rho\right). \tag{6.20}$$

Using Eq. (6.19) it is now straightforward to compute the last term in the action (6.1):

$$S_{\hat{F}}=\Omega\int e\ \left(-\frac{1}{8}\,e^{2\phi}\ F^{\mathrm{a}}_{\mu\nu}F^{\mathrm{a}\mu\nu}-\frac{1}{96}\,e^{-2\phi}\ f^2_{\mathrm{abc}}\right.$$
$$\left.+\frac{1}{12}\,e^{4\phi}\ H'_{\mu\nu\rho}H'^{\mu\nu\rho}\right)\ d^4x, \tag{6.21}$$

where

$$H'_{\mu\nu\rho}=H_{\mu\nu\rho}-\omega_{\mu\nu\rho}.$$

Now, taking advantage of the identity

$$\varepsilon^{\sigma\mu\nu\rho}\ \partial_\sigma H_{\mu\nu\rho}=0 \tag{6.22}$$

it is easy to see that the expression

$$-\Omega\int\left(\frac{1}{6}\,\varepsilon^{\sigma\mu\nu\rho}\ \partial_\sigma\mathbf{a}\ H_{\mu\nu\rho}\right)d^4x \tag{6.23}$$

vanishes up to a surface term; here **a** is a Lagrange multiplier. Adding this to the action (6.21) it is possible to go to a first order formalism where both $H_{\mu\nu\rho}$ and **a** are treated as independent fields. The equation of motion of **a** implies that $H_{\mu\nu\rho}$ is a closed form and can be expressed locally as the curl of $B_{\mu\nu}$ thus giving the action (6.21). Alternatively we can integrate the field $H_{\mu\nu\rho}$ from the action as it appears quadratically. This is equivalent to varying $H_{\mu\nu\rho}$ in the action with the result

$$H_{\mu\nu\rho}=\omega_{\mu\nu\rho}+e^{-4\phi}\varepsilon_{\sigma\mu\nu\rho}\ \partial^\sigma\mathbf{a}, \tag{6.24}$$

and then eliminating $H_{\mu\nu\rho}$ from the action in favor of **a**. Adding Eqs. (6.13) and (6.21), the result is

$$S_{10}=\Omega\int e\ \left(-\frac{1}{4}\,R+\frac{1}{2}\,\partial_\mu\phi\ \partial^\mu\phi\right.$$
$$+\frac{1}{2}\,e^{-4\phi}\ \partial_\mu\mathbf{a}\ \partial^\mu\mathbf{a}-\frac{1}{4}\,e^{2\phi}\ F^{\mathrm{a}}_{\mu\nu}F^{\mathrm{a}\mu\nu}$$
$$\left.-\frac{1}{2}\,\mathbf{a}\ F^{\mathrm{a}}_{\mu\nu}{*F}^{\mathrm{a}\mu\nu}+\frac{1}{48}\,e^{-2\phi}\ f^2_{\mathrm{abc}}\right)\ d^4x. \tag{6.25}$$

Finally, choosing $U^{\mathrm{a}}{}_m$ and f_{abc} in accordance with Eqs. (6.7) and Eqs. (6.8), respectively, gives $(f_{\mathrm{abc}})^2=6\,(g_1^2+g_2^2)$, and thus the dimensionally reduced action (6.25) exactly reproduces the bosonic part of the action of the $N=4$ supergravity in Eq. (2.1)—up to an overall factor. Similarly, the choice (6.9) leads to the truncated model considered above.

4. The fermions. Consider the supersymmetry transformations for the spinor fields in ten dimensions (for a purely bosonic background):

$$\delta\hat{\psi}_P=\hat{D}_P\ \hat{\epsilon}+\frac{1}{48}\,e^{-\hat{\phi}}\ (\hat{\Gamma}^{MNQ}{}_P+9\ \delta^M_P\ \hat{\Gamma}^{NQ})\hat{\epsilon}\ \hat{H}_{MNQ},$$

$$\delta\hat{\chi}=\frac{i}{\sqrt{2}}\,(\partial_Q\hat{\phi})\ \hat{\Gamma}^Q\ \hat{\epsilon}+\frac{i}{12\sqrt{2}}\,e^{-\hat{\phi}}\ \hat{\Gamma}^{MNQ}\ \hat{\epsilon}\ \hat{H}_{MNQ}. \tag{6.26}$$

Here the $D=10$ Dirac matrices satisfy $\hat{\Gamma}_M\hat{\Gamma}_N+\hat{\Gamma}_N\hat{\Gamma}_M=2\,\hat{\mathbf{g}}_{MN}$, one has $\hat{\Gamma}_{M\ldots Q}=\hat{\Gamma}_{[M}\ldots\hat{\Gamma}_{Q]}$. In order to descend to four dimensions, we first notice that for the bosinic background defined by Eqs. (6.10) and (6.19) the vector fields coming from the vielbein and those from the two-form are identified, while the 36 scalar fields are truncated. For this to be consistent with supersymmetry, the fermionic fields which are in the same supermultiplets should also be truncated simultaneously. In complete analogy with the case of toroidal compactification one must set:

$$\hat{\psi}_{\mathrm{a}}-\frac{i}{2\sqrt{2}}\hat{\Gamma}_{\mathrm{a}}\hat{\chi}\equiv 0. \tag{6.27}$$

In order to be consistent with the reduction procedure, the variation of the above should remain zero. This implies that a Killing spinor η exists such that

$$D_m\eta-\frac{g_{\mathrm{s}}}{4\sqrt{2}}\Gamma_m\eta=0, \tag{6.28}$$

where $g_{\mathrm{s}}=g_1$ for $m=1,2,3$, and $g_{\mathrm{s}}=g_2$ for $m=4,5,6$. The dependence of the spinors on internal coordinates z is factorized through the η dependence:

$$\epsilon\,(x,z)=\epsilon\,(x)\;\eta(z). \tag{6.29}$$

The next step is to represent the $D=10$ 32-component Majorana-Weyl spinors in the form

$$\hat{\psi}_\alpha=\begin{pmatrix}\psi_\alpha\\ -i\gamma_5\,\psi_\alpha\end{pmatrix},\quad \hat{\chi}=\begin{pmatrix}\chi\\ i\gamma_5\,\chi\end{pmatrix},\quad \hat{\epsilon}=\begin{pmatrix}\epsilon\\ -i\gamma_5\,\epsilon\end{pmatrix}, \tag{6.30}$$

where $\epsilon\equiv\epsilon^{\mathrm{I}}$ with $\mathrm{I}=1,2,3,4$ and ϵ^{I}'s are four-component spinors; similarly for ψ_α and χ. The Dirac matrices are chosen to be

$$\hat{\Gamma}^m=\gamma^m\otimes 1,\quad \hat{\Gamma}^{1\,a}=\gamma_5\otimes\begin{pmatrix}0 & \mathbf{T}_{(1)\,a}\\ \mathbf{T}_{(1)\,a} & 0\end{pmatrix},$$

$$\hat{\Gamma}^{2\,a}=\gamma_5\otimes\begin{pmatrix}-\mathbf{T}_{(2)\,a} & 0\\ 0 & \mathbf{T}_{(2)\,a}\end{pmatrix}, \tag{6.31}$$

where $\mathbf{T}_{(s)\,a}$ are defined by Eq. (3.5). Finally, let us introduce the following linear combinations:

$$\psi_\mu=e^{-\frac{3}{4}\phi}\left(e^\alpha_{\ \mu}\,\psi_\alpha-\frac{3i}{2\sqrt{2}}\,\gamma_\mu\,\chi\right), \tag{6.32}$$

and rescale

$$\chi\rightarrow -2\,e^{-\frac{3}{4}}\,\chi. \tag{6.33}$$

The straightforward application of all the above definitions allows one to verify that the relation between the variations $\delta\psi_\mu$ and $\delta\chi$ of the spinors defined by Eqs. (6.30), (6.32), (6.33) and ϵ in Eq. (6.30) coincides with the $D=4$ supersymmetry transformation rules in Eq. (2.2) up to the Dirac conjugation. This completes the compactification procedure.

VII. LIFTING THE SOLUTIONS TO TEN DIMENSIONS

The results of the previous section imply that any solution of the gauged supergravity model in four dimensions given in terms of the metric $\mathbf{g}_{\mu\nu}$, gauge fields $A^{(s)\,a}_\mu$, the axion $\mathbf{a}$ and the dilaton ϕ, can be lifted to ten dimensions as a solution of the $N=1$ supergravity. The ten-dimensional metric, the vielbein and the dilaton $\hat{\phi}$ are then given by Eqs. (6.10)–(6.12), where the functions $U^{\mathrm{a}}_{\ m}$ are defined by either Eq. (6.7) for the SU(2)×SU(2) gauge group or by Eq. (6.9) when the symmetry is SU(2)×$[\mathrm{U}(1)]^3$. If the gauge group is $[\mathrm{U}(1)]^6$ one has $U^{\mathrm{a}}_{\ m}=\delta^{\mathrm{a}}_m$. The vielbein projections of the three-form are given by Eqs. (6.19), from where the two-form components can be obtained.

Let us now apply these formulas to the family of solutions obtained in Sec. V. Choosing $A^{(2)\,a}_\mu=g_2=0$ and $g_1=1$, the lifted solutions can be represented as follows. The metric and the dilaton are

$$\hat{\mathbf{g}}_{MN}=2e^{-\hat{\phi}}\,\tilde{\mathbf{g}}_{MN},\quad \hat{\phi}=-\frac{\phi(\rho)}{2}, \tag{7.1}$$

where the metric in the string frame, $\tilde{\mathbf{g}}_{MN}$, is specified by the line element

$$d\tilde{s}^2=dt^2-d\rho^2-R^2(\rho)\,d\Omega_2^2-\Theta^a\Theta^a-(dz^4)^2$$
$$-(dz^5)^2-(dz^6)^2. \tag{7.2}$$

Here $d\Omega_2^2$ is the standard metric on unit 2-sphere,

$$\Theta^a\equiv A^a-\theta^a=A^a_\mu\,dx^\mu-\theta^a_i\,dz^i, \tag{7.3}$$

where θ^a are the Maurer-Cartan forms on S^3 parametrized by $\{z^i\}=\{z^1,z^2,z^3\}$:

$$d\theta^a+\frac{1}{2}\,\epsilon_{abc}\,\theta^b\wedge\theta^c=0. \tag{7.4}$$

If $\mathbf{T}_a$ are the SU(2) group generators, $[\mathbf{T}_a,\mathbf{T}_b]=i\epsilon_{abc}\mathbf{T}_c$, then the gauge field is given by

$$A\equiv\mathbf{T}_aA^a\equiv\mathbf{T}_aA^a_\mu dx^\mu$$
$$=w(\rho)\,\{-\mathbf{T}_2\,d\theta+\mathbf{T}_1\sin\theta\,d\varphi\}+\mathbf{T}_3\cos\theta\,d\varphi. \tag{7.5}$$

The non-vanishing vielbein projections of the antisymmetric tensor field are

$$\hat{H}_{\alpha\beta a}=-\frac{1}{2\sqrt{2}}\,e^{-\frac{3}{4}\phi}\,F^a_{\alpha\beta},\quad \hat{H}_{abc}=\frac{1}{2\sqrt{2}}\,e^{-\frac{3}{4}\phi}\,\epsilon_{abc}, \tag{7.6}$$

where $F^a_{\alpha\beta}$ are the tetrad projections of the gauge field tensor corresponding to the gauge field (7.5) for the tetrad e_α specified by the four-dimensional part of the string metric (7.2). These can be read off from

$$\frac{1}{2}\,\mathbf{T}_aF^a_{\alpha\beta}\,e^\alpha\wedge e^\beta=-\mathbf{T}_2\,\frac{w'}{R}\,e^1\wedge e^2+\mathbf{T}_1\,\frac{w'}{R}\,e^1\wedge e^3$$
$$+\mathbf{T}_3\,\frac{w^2-1}{R^2}\,e^2\wedge e^3. \tag{7.7}$$

Finally, for the sake of completeness, we write down the functions $R(\rho)$, $w(\rho)$ and $\phi(\rho)$ in Eqs. (7.1)–(7.7):

$$R^2=2\,\rho\,\coth\rho-\frac{\rho^2}{\sinh^2\rho}-1,\quad w=\pm\frac{\rho}{\sinh\rho},$$

$$e^{2(\phi-\phi_0)}=\frac{\sinh\rho}{2\,R(\rho)},\tag{7.8}$$

where ϕ_0 is a free parameter.

One can verify that the lifted solutions given by Eqs. (7.1)–(7.8) indeed fulfill the equations of motion of ten-dimensional supergravity:

$$\hat{\nabla}_M\hat{\nabla}^M\,\hat{\phi}=-\frac{1}{6}\,e^{-2\hat{\phi}}\,\hat{H}_{MNP}\hat{H}^{MNP},\tag{7.9}$$

$$\hat{\nabla}_M(e^{-2\hat{\phi}}\,\hat{H}^{MNP})=0,\tag{7.10}$$

$$\hat{R}_{MN}=2\,\partial_M\hat{\phi}\,\partial_N\hat{\phi}+e^{-2\hat{\phi}}\,\hat{H}_{MPQ}\hat{H}_N{}^{PQ}$$
$$-\frac{1}{12}\,e^{-2\hat{\phi}}\,\hat{\mathbf{g}}_{MN}\,\hat{H}_{PQS}\hat{H}^{PQS}.\tag{7.11}$$

The direct verification is, however, rather difficult. Although the dilaton equation can be checked straightforwardly, already for the antisymmetric tensor field the procedure is much more involved. The equations then split into three groups depending on values of the indices N and P in (7.10). Equations of the first group are satisfied by virtue of the geometrical properties of the invariant forms θ^a, whereas equations of the second and the third groups eventually reduce to the Yang-Mills equations in $D=4$. Finally, we have had computer check the Einstein equations (7.11).

Note that the gauge potential A in Eq. (7.5) can be arbitrarily gauge transformed, since any gauge transformation can now be viewed as a diffeomorphism in ten dimensions. It is instructive to see how it works at the linearized level. Consider an infinitesimal gauge transformation

$$A\rightarrow A+d\xi+i[\xi,A],\tag{7.12}$$

where $\xi=\mathbf{T}_a\xi^a(x)$. Consider at the same time a diffeomorphism

$$z^i\rightarrow z^i+\theta_a^i(z)\;\xi^a(x),\tag{7.13}$$

where $\theta^a_i\theta_b^i=\delta^a_b$, and the remaining seven coordinates are intact. This causes a change in the Maurer-Cartan form $\theta\equiv\mathbf{T}_a\theta^a$:

$$\theta\rightarrow\theta+d\xi+i[\xi,\theta].\tag{7.14}$$

As a result one has

$$\Theta^a\Theta^a=2\,\mathrm{tr}\,(A-\theta)^2\rightarrow 2\,\mathrm{tr}\,(A-\theta+i[\xi,A-\theta])^2$$
$$=\Theta^a\Theta^a+O(\alpha^2).\tag{7.15}$$

The $D=10$ metric therefore remains invariant, and the same can be shown to be true for the antisymmetric tensor field. This shows that the effect of gauge transformations can be compensated by that of the diffeomorphisms.

Finally, let us describe some properties of the solutions in Eqs. (7.1)–(7.8). They preserve 1/4 of the supersymmetries and differ essentially from all other known solutions of leading order string theory [1] in that the gauge field, which now appears as off-diagonal components of the metric, is non-Abelian. For this reason we call the solutions non-Abelian. Specifically, the gauge field in the metric combines with the non-Abelian isometries of the internal space. At first glance, the solutions exhibit some similarities with p-branes in $D=10$. Here $p=3$ because the expressions do not depend on three spatial coordinates z^4, z^5, z^6. However, the analogy is incomplete, since there is no 5-form to couple to the 3-brane. In addition, the six-dimensional transverse space is not asymptotically flat and topologically is $\mathrm{R}^3\times S^3$, which spoils the resemblance with an extended object moving through the ten-dimensional spacetime. Moreover, we cannot introduce the notion of mass of the brane per unit 3-volume.

One can regard the solutions as describing interpolating solitons [20]. The reason for this is the observation that for small ρ one can choose the gauge where the gauge field vanishes in the limit $\rho\rightarrow 0$, and the geometry in string frame is described by the standard metric on $\mathcal{M}^7\times S^3$, where $\mathcal{M}^7$ is seven-dimensional Minkowski spacetime. In the opposite limit, $\rho\rightarrow\infty$, introducing the radial coordinate $\tilde{r}=\sqrt{2\rho}$, the geometry is given by the metric on $\mathcal{M}^4\times\mathcal{V}^6$. Here $\mathcal{V}^6$ is a manifold whose metric is a ''warped'' product of the standard metric on S^3 and that on the three-dimensional paraboloid:

$$ds^2=\tilde{r}^2(d\tilde{r}^2+d\vartheta^2+\sin^2\vartheta\,d\varphi^2)$$
$$+\delta_{ab}\,(\theta^a-\delta^a_3\cos\vartheta\,d\varphi)\,(\theta^b-\delta^b_3\cos\vartheta\,d\varphi).\tag{7.16}$$

Note that this does not correspond to any known supergravity vacuum.

Although we have not studied the issue of α' corrections for our solutions, we expect them to get corrected. These corrections could probably be balanced by adding the ten-dimensional Yang-Mills field [7], however, the definite conclusion cannot be reached without special analysis. This issue is currently under investigation. Another interesting problem to analyze is the study of dual partners to the solutions found here.

VIII. SUMMARY

In this paper we have studied non-Abelian BPS solutions in $N=4$ gauged supergravity and leading order string theory. Our main motivation for this was to develop a systematic procedure for handling non-Abelian gauge fields in the context of supergravity models, a problem not well covered in the literature. The procedure we have employed is the straightforward component analysis of the equations for Killing spinors. Although the procedure is rather involved (we had to resort to computer calculations) it has given us a set of the first integrals (4.15)–(4.18) for the field equations (2.14) in the static, spherically symmetric, purely magnetic case with the gauge group $\mathrm{SU}(2)\times[\mathrm{U}(1)]^3$. These first order Bogomol'nyi equations are considerably easier to solve than the second order field equations, with the solutions given by Eqs. (5.7), (5.8).

Having obtained the solutions, we show that the $N=4$

gauged supergravity in four dimensions can be obtained via compactification of $N=1$, $D=10$ supergravity on the group manifold. This fact, although quite plausible, has not been covered in the literature before. Applying a procedure inverse to dimensional reduction, we have lifted the $D=4$ solutions to ten dimensions, where they can be regarded as solutions to the leading order equations of motion of the string effective action.

We expect our results to be applicable in the following ways. First, we are currently investigating the properties of the solutions obtained above by performing the stability analysis in four dimensions and studying the issue of string corrections and the duality transformations in $D=10$. Second and more important, we expect that our approach can be applied to obtain more general solutions, also in the context of other supergravity models. An interesting example would be $N=2$ supergravity with non-Abelian matter in four dimensions.

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