PHYSICAL REVIEW D VOLUME 57, NUMBER 10 15 MAY 1998

Massless Gupta-Bleuler vacuum on the (1+1)-dimensional de Sitter space-time

S. De Bièvre*
UFR de Mathématiques et URA GAT, Université des Sciences et Technologies de Lille, 59655 Villeneuve d'Ascq Cedex, France

J. Renaud†
LPTMC, Tour Centrale 3^{me} étage, Université Paris VII, 75251 Paris Cedex 05, France

(Received 7 July 1997; revised manuscript received 6 January 1998; published 1 May 1998)

We construct a causal, de Sitter, and conformally covariant massless free quantum field on the (1+1)-dimensional de Sitter space-time admitting a de Sitter invariant vacuum in an indefinite inner product space. The field is defined rigorously as an operator-valued distribution and is covariant in the usual strong sense: $\underline{V}_g^{-1}\varphi(x)\underline{V}_g=\varphi(g\cdot x)$ for any g in the de Sitter group, where $\underline{V}$ is a unitary representation of the de Sitter group on the space of states. We use the formalism of Gupta-Bleuler triplets which also allows for an explicit description of the gauge degree of freedom. As a consequence the model does not suffer from infrared divergences, contrary to what happened in previous treatments of this problem. The causality and the covariance of the theory are assured thanks to a suitable choice of the space of solutions of the classical field equation. We show that, although the field itself is not observable (it is gauge dependent), the stress tensor and the energy-momentum vector are. The energy operator P_0 is positive in all physical states, and vanishes in the vacuum. In addition, the field is conformally covariant and the model does not exhibit a conformal anomaly in the trace of the energy-momentum tensor. [S0556-2821(98)00212-4]

PACS number(s): 04.62.+v, 03.70.+k

I. INTRODUCTION

In this paper we construct a new massless free quantum field on the (1+1)-dimensional de Sitter space-time, transforming correctly under isometries, conformal transformations, and gauge transformations, which is free of conformal anomalies, and which admits a de Sitter invariant vacuum.

The massless free field on the (1+1)-dimensional de Sitter space-time is of interest partially because the difficulties encountered in its construction are similar to those that one has when considering the (1+3)-dimensional minimally coupled field [1–4] and the spin 2 field. It is well-known that a straightforward quantization of the classical massless free field in two-dimensional space-time leads to an infrared divergence in the case of the Minkowski space-time. It is then sometimes claimed that to avoid the divergence, the Lorentz symmetry must be broken [5]. A similar problem is known to occur in (3+1)-dimensional de Sitter space-time, where Allen has proven the nonexistence of a de Sitter covariant Fock vacuum for the massless minimally coupled field [1], which is also infrared divergent. Allen's result easily holds in 1+1 dimensions as well. To circumvent this problem, the covariance condition is often weakened one way or another: some authors studied vacua invariant under a subgroup of the de Sitter group only (spontaneous symmetry breaking), others choose to restrict the field to a subset of the de Sitter space-time, or consider invariance under the Lie algebra of the de Sitter group rather than under the full group action (see [2,6,7] and references therein). Kirsten and Garriga [3] have proposed an alternative vacuum, in which certain two-point functions are de Sitter covariant, and grow linearly in the proper time between the points. They exploit the observation that the free field equation admits a zero-frequency mode. We show below that this vacuum is not invariant in the usual strong sense.

Remarking that the classical free massless field in 1+1 dimensions is in addition to de Sitter, also gauge covariant, we show here that a rather straightforward application of the Gupta-Bleuler formalism, known to be well adapted to treat models with gauge symmetries, permits one to avoid the symmetry breaking altogether: the field we construct transforms correctly under de Sitter, conformal, and gauge transformations and acts on a state space containing a vacuum invariant under all of them. It is free of infrared divergence. In addition, a consequence of our construction is an automatic renormalization of the stress tensor which makes the so-called conformal anomaly disappear from the trace of the energy-momentum tensor.

Our construction is of the Gupta-Bleuler type, and the field acts on a space of states having the structure of a Fock space but containing both positive and negative norm vectors. To assure a reasonable interpretation of the theory, we therefore need to select the subspace of physical states. To do this, we recall that since de Sitter space-time is not stationary, there is *a priori* no natural time coordinate on it and hence no natural notion of ''positive frequency.'' We nevertheless select the physical states by demanding that they be positive frequencies with respect to the conformal time on de Sitter space-time. This choice, while *ad hoc*, is justified by the fact that the resulting theory has all the properties one might require from a free field on a space-time with high symmetry, as we now further explain.

First of all, it turns out that all physical states have positive norms, as required for a reasonable quantum mechanical interpretation of the model (even though it is not true that all positive norm states are physical).

*Email address: debievre@gat.univ-lille1.fr
†Email address: renaud@ccr.jussieu.fr

Second, the field φ we construct here is causal and it is convariant in the usual strong sense: there is a unitary representation $\underline{V}$ of the de Sitter group on the space of states and $\underline{V}_g^{-1}\varphi(x)\underline{V}_g=\varphi(g\cdot x)$ for any g in the de Sitter group (Sec. V). This implies in particular that the field is defined on the whole space-time.

Third, there is a one-parameter group of unitaries $U(\lambda)$ acting on the state space, that implement the gauge transformations:

$$U(-\lambda)\varphi(x)U(\lambda)=\varphi(x)+\lambda. \tag{1}$$

Fourth, the gauge degree of freedom entails a notion of physical equivalence between states: basically, two states differing by a state containing ''gauge states'' are physically equivalent. The space of physical states contains a vector invariant under the action of the de Sitter group, which is unique up to physical equivalence. We call it the Gupta-Bleuler vacuum. The vacuum is gauge-invariant in the sense that a gauge transformation transforms the vacuum into a physically equivalent state. Observables are those self-adjoint operators having expectation values that are gauge independent (Sec. VI). With these definitions, it turns out that the field itself is not an observable: this is as expected and can be seen by calculating the mean value of Eq. (1) in the vacuum. The components of the energy-momentum tensor on the other hand are observables. Furthermore, in spite of the fact that the operator $T_{00}(x)$ is not positively definite as an operator on the full space of states, we show that the expected value of $T_{00}(x)$ between excited physical states of the form $|k_1^{n_1}\dots k_j^{n_j}\rangle$ is given by

$$\langle k_1^{n_1}\dots k_j^{n_j}|T_{00}(x)|k_1^{n_1}\dots k_j^{n_j}\rangle=\frac{1}{2\pi}\sum_{i=1}^{j}n_i|k_i|,$$

which is clearly positive. This assures a reasonable physical interpretation of the model.

The results of this paper show that Allen's result does not imply that a fully invariant vacuum for the free massless field does not exist on (1+1)-dimensional de Sitter space-time. There indeed does not exist an invariant Fock vacuum state in the usual sense, i.e., a vacuum belonging to a Fock space constructed over a *Hilbert* space. We do however construct an invariant vacuum in a Fock space constructed over an indefinite inner product space that we referred to as a Gupta-Bleuler vacuum above. In view of the obvious gauge invariance of the field equation, this is really not too surprising: the Gupta-Bleuler formalism was invented to avoid Lorentz symmetry breaking through gauge fixing. As we shall show, our construction is quite easy to implement: it is a matter of adapting the Gupta-Bleuler quantization of the free electromagnetic field [8].

Let us recall that in electrodynamics the Gupta-Bleuler triplet $V_g\subset V\subset V'$ is defined as follows [9,10]. The space V_g is the space of scalar photon states or ''gauge states,'' the space V is the space of positive frequency solutions of the field equation verifying the Lorentz condition, and V' is the space of all positive frequency solutions of the field equation, containing nonphysical states. The Klein-Gordon inner product defines an indefinite inner product on V' that is Poincaré and locally and conformally invariant. All three of these spaces carry representations of the Poincaré group but V_g and V are not covariantly complemented. The quotient space V/V_g of states up to a gauge transformation is the space of physical one-photon states. The quantized field acts on the Fock space built on V', which is not a Hilbert space, but an indefinite inner product space.

The same scheme will appear in our construction. The Lagrangian

$$\mathcal{L}=\sqrt{|g|}g^{\mu\nu}\partial_\mu\phi\partial_\nu\overline{\phi}$$

of the free massless field is invariant when adding to ϕ a constant function. As a consequence, in the ''one-particle sector'' of the field the space of gauge states is simply the space of constant functions written $\mathcal{N}$ in the following: The analogue of the space V above, written $\mathcal{K}$ and called the physical one-particle space in the following, is a space of positive frequency solutions of the field equation equipped with the degenerate (but positive) Klein-Gordon inner product. It was studied in [11] and will be described in detail below. It is the principal ingredient of the construction. The role of the space V' is played by a larger space of solutions, described below as well, and for which we write $\mathcal{H}$: $\mathcal{H}$ is called the total space and it is a Krein space [12] when equipped with the Klein-Gordon inner product. Let us recall that a Krein space is the orthogonal sum of a Hilbert space and an anti-Hilbert space (with a negative definite inner product). Hence, the Klein-Gordon inner product on $\mathcal{H}$ is nondegenerate, but not positively definite. As usual in a Gupta-Bleuler model, the quantum field is written on a Fock space built on $\mathcal{H}$. We do insist on the fact that all of these spaces carry representations of the de Sitter group and that the construction of the field is completely covariant as a result of this and of the nondegeneracy of the inner product on $\mathcal{H}$. Again, this is not in contradiction with the result of Allen [1] because $\mathcal{H}$ is not a Hilbert space.

All of these properties are not restricted to the (1+1)-dimensional massless field. In a future work [4] it will be proved that a similar construction yields similar properties for the minimally coupled massless field on the (1+3)-dimensional de Sitter space-time.

We would like to add one more comment. One might object that there is no reason to insist on the correct transformation properties of the field, since it is unobservable anyway and since, at any rate, most space-times do not have much symmetry to begin with. We feel this is not justified for two reasons.

First, one could conceivably construct a field that does not transform correctly but gives rise to observables that do transform correctly: this does not seem so easy to implement and it is certainly not what one does for the electromagnetic vector potential and field.

Second, the (1+1)-dimensional equation is conformally invariant, and this invariance is important since it survives for the free massless field on all two-dimensional space-times.

In our construction, the first two spaces ($\mathcal{N}$,$\mathcal{K}$) are in addition invariant under the conformal group and we obtain the conformal invariance of the field from this. Davies and Fulling [13] have studied a vacuum for the massless free quantum field on arbitrary (1+1)-dimensional space-times. As they point out, their treatment is incomplete on the de Sitter space-time, since they neglect the zero-frequency mode, but they suggest that once quantized, this mode does not contribute substantially to the energy-momentum tensor. We show in Sec. III that the neglect of the zero-frequency mode is the source of two other problems: the field constructed by Davies and Fulling is neither causal, nor de Sitter, or conformally covariant. This is not too surprising, in view of Allen's result [1]. On the contrary, our construction is conformally covariant, and the expected value of all components of the stress-energy tensor vanish in the Gupta-Bleuler vacuum, and hence there is no conformal anomaly in the trace of the energy-momentum tensor.

One might actually argue that the conformal anomaly appearing in the trace of the energy momentum tensor in the field of Davies and Fulling and others is a result of the fact that their constructions break the conformal invariance from the outset.

The rest of this paper is organized as follows. In Sec. II we recall some elementary facts about the (1+1)-dimensional de Sitter space-time and fix the notations. In Sec. III we present the standard approach to free field quantization on curved space-times in order to show how it breaks down for the free massless field on the (1+1)-dimensional de Sitter space-time. In Sec. IV we build the quantum field and we discuss its invariance in Sec. V. In Sec. VI we compute the stress tensor. Some elementary facts about indefinite inner product Fock spaces are recalled in the Appendix.

II. THE de SITTER UNIVERSE AND THE MASSLESS FIELD EQUATION

The (1+1)-dimensional de Sitter universe can be realized as the submanifold M of $\mathbb{R}^3$ defined by

$$(y^1)^2-(y^2)^2-(y^3)^2=-1,$$

with the metric $ds^2=(dy^1)^2-(dy^2)^2-(dy^3)^2$. The following ''conformal'' coordinates will be useful for our purposes:

$$\begin{cases} y^1=\tan\rho, \\ y^2=\dfrac{\sin\alpha}{\cos\rho}, \quad \alpha\in\mathbb{R}/2\pi\mathbb{Z},\quad \rho\in\,]-\pi/2,\pi/2[, \\ y^3=\dfrac{\cos\alpha}{\cos\rho}. \end{cases}$$

In these coordinates the metric reads $ds^2=\cos^{-2}\rho(-d\rho^2+d\alpha^2)$. Note that the coordinate ρ is timelike and α is spacelike. Let G be the connected component of the isometry group containing the identity: this is the so-called de Sitter group, $G=SO_0(1,2)$, of the de Sitter universe which is generated by the three infinitesimal generators:

$$X_{2,3}=\frac{\partial}{\partial\alpha},\quad X_{1,3}=-\sin\alpha\sin\rho\frac{\partial}{\partial\alpha}+\cos\alpha\cos\rho\frac{\partial}{\partial\rho},$$

$$X_{2,1}=-\cos\alpha\sin\rho\frac{\partial}{\partial\alpha}-\sin\alpha\cos\rho\frac{\partial}{\partial\rho}.$$

The G-invariant measure reads $d\mu=d\rho d\alpha/\cos^2\rho$. The Laplace-Beltrami operator is

$$\Box=-\frac{1}{\sqrt{g}}(\partial_\nu\sqrt{g}g^{\nu\mu}\partial_\mu)=\cos^2\rho\left(\frac{\partial^2}{\partial\rho^2}-\frac{\partial^2}{\partial\alpha^2}\right).$$

Recall that the de Sitter universe is not stationary since it does not allow for a timelike Killing vector field. Some authors nevertheless refer to the generator $\hat{X}=iX_{1,3}$ [14] as the Hamiltonian and one can see that it contracts to the usual Hamiltonian of the Minkowski plane when the curvature tends to 0. Since this vector field is not everywhere timelike it can, however, not be used to define a separation into positive and negative frequency solutions of the field equation.

To deal with the problems surrounding conformal invariance it is helpful to recall that the de Sitter universe can be realized as a dense open subset of the torus $S^1\times S^1$, the compactified space-time, which is a homogeneous space for the conformal group $SO_0(2,2)/\{\pm Id\}\equiv G_c$. The point $(u^+,u^-)\in S^1\times S^1$ is identified with the point (ρ,α) by the formulas $u^\varepsilon=\rho-\varepsilon\alpha$. Conversely $\rho\equiv(u^++u^-)/2$ with $\rho\in\,]-\pi/2,\pi/2[$ and $\alpha=\rho-u^+$. The generators of the conformal group action are obtained by adjoining the following ones to $X_{2,3}$, $X_{1,3}$, $X_{2,1}$:

$$X_{2,0}=-\cos\alpha\cos\rho\frac{\partial}{\partial\alpha}+\sin\alpha\sin\rho\frac{\partial}{\partial\rho},\quad X_{1,0}=-\frac{\partial}{\partial\rho},\quad X_{3,0}=\sin\alpha\cos\rho\frac{\partial}{\partial\alpha}+\cos\alpha\sin\rho\frac{\partial}{\partial\rho}.$$

The conformally coupled field equation reads

$$\Box\phi=0, \tag{2}$$

which becomes in u^ε coordinates: $\partial^2\phi/\partial u^+\partial u^-=0$, the solutions of which are the functions $f_+(u^+)+f_-(u^-)$. Note that the conformal Killing vector field $X_{1,0}$ is clearly timelike everywhere, and we will use it below to select positive frequency solutions. Note also that the lightlike coordinates $(u^+,u^-)\in S^1\times S^1$ allow us to see the commutative decomposition $G_c=SO_0(2,2)/\{\pm Id\}=SO_0(1,2)\times SO_0(1,2)$ by defining, for $\varepsilon=\pm$;

$$Y_0^\varepsilon=\tfrac{1}{2}(X_{3,2}+\varepsilon X_{0,1}),\quad Y_1^\varepsilon=\tfrac{1}{2}(X_{1,2}+\varepsilon X_{0,3}),$$
$$Y_2^\varepsilon=\tfrac{1}{2}(X_{1,3}+\varepsilon X_{2,0}). \tag{3}$$

Straightforward computation gives

$$Y_0^\varepsilon=\varepsilon\partial_{u^\varepsilon},\quad Y_1^\varepsilon=-\varepsilon\sin u^\varepsilon\partial_{u^\varepsilon},\quad Y_2^\varepsilon=\cos u^\varepsilon\partial_{u^\varepsilon}.$$

Hence the group is decomposed in the commutative direct product of two copies of $SO_0(1,2)$, each of them acting on only one coordinate. Note that the de Sitter group $SO_0(1,2)=G$ is *not* one of these factors but is generated by the set of diagonal elements of the product, as can be easily seen from Eq. (3) because $X_{3,2}=Y_0^++Y_0^-$, etc.

This space-time is globally hyperbolic, hence the so-called commutator $\tilde{G}=G^{\mathrm{adv}}-G^{\mathrm{ret}}$ is uniquely defined [15]. Let us recall that these propagators are defined by $\Box_xG^{\mathrm{adv}}(x,y)=\Box_xG^{\mathrm{ret}}(x,y)=-\delta(x,y)$ and for fixed y the support in x of $G^{\rm adv}$ (resp. $G^{\rm ret}$) lies in the past (resp. future) cone of y. This commutator $\widetilde{G}(x,y)$ is equal to $+1/2$ when x is in the future cone of y, $-1/2$ when x is in the past cone of y, and 0 elsewhere. The Klein-Gordon inner product between solutions of Eq. (2) is defined by

$$\langle \phi,\psi\rangle = i\int_{\rho=0}\overline{\phi(\alpha,\rho)}\overleftrightarrow{\partial}_\rho\psi(\alpha,\rho)d\alpha \tag{4}$$

when it makes sense. This inner product is invariant: $\langle V_g\phi,V_g\psi\rangle=\langle\phi,\psi\rangle$ where V is the regular representation of the de Sitter group: $V_g\psi(x)=\psi(g^{-1}x)$ for any $g\in G$ and $x\in M$. This is linked to the commutator by

$$\phi(x)=i\int_{\rho=0}-i\widetilde{G}(x,(\alpha,\rho))\overleftrightarrow{\partial}_\rho\phi(\alpha,\rho)d\alpha$$

for (at least) any smooth solution of the field equation with compact support.

We shall finally also use the invariant inner product on $L^2(M)$ denoted by parentheses:

$$(f,g)=\int\overline{f(x)}g(x)d\mu(x), \tag{5}$$

where $d\mu(x)=d\rho d\alpha/\cos^2\rho$ is the invariant measure.

III. DESCRIPTION OF THE PROBLEM

In this section we explain how the difficulty in quantizing the massless field on M arise.

We first recall the canonical quantization of the bosonic scalar Klein-Gordon field $\varphi(x)$,

$$(\Box+m^2)\varphi(x)=0, \tag{6}$$

on a globally hyperbolic space-time M. To construct the quantum field one looks for a set of modes ϕ_k, solutions to Eq. (6) satisfying the following properties. Firstly,

$$\langle\phi_k,\phi_l\rangle=\delta_{k,l}, \quad \langle\phi_k,\overline{\phi_l}\rangle=0, \tag{7}$$

where $\langle\cdot,\cdot\rangle$ denotes the Klein-Gordon inner product. Secondly, the ϕ_k and the $\overline{\phi_k}$ are requested to span the space of smooth solutions to Eq. (6). Given such ϕ_k, one considers the Hilbert space S they span and the corresponding bosonic Fock space $\underline{\mathcal{S}}$. The field is then defined by

$$\varphi(x)=\sum_k \phi_k(x)a_k+\sum_k\overline{\phi_k(x)}a_k^\dagger, \tag{8}$$

where the a_k and the $a_k^\dagger$ are the usual annihilation and creation operators of the mode ϕ_k. Note that this construction depends crucially on the choice made for the ϕ_k or, more precisely, on the space $\mathcal{S}$ they span. To make sure this yields a physically acceptable theory, one normally requires the following additional properties of $\varphi(x)$. First, φ needs to be causal: actually, $[\varphi(x),\varphi(x')]$ is required to equal the commutator function $-i\widetilde{G}(x,x')$ on M to ensure that the field satisfies the correct equal time commutation relations with its conjugate momentum. Next, if M is stationary, one imposes a positive frequency requirement, which restricts the possible choices for the ϕ_k. In addition, one wishes all the symmetries of the classical equation (6) to survive in the quantized theory. This means that one expects the Fock space $\underline{\mathcal{S}}$ to carry a unitary representation $\underline{V}$ of the isometry group of M (and of all other symmetries of the theory), and that one requires the field to transform correctly [meaning $\underline{V}_g^{-1}\varphi(x)\underline{V}_g=\varphi(g\cdot x)$], and the vacuum to be invariant. For what follows it is of importance to recall that, *in the above setting*, it is sufficient to require the invariance of the solution space $\mathcal{S}$ under the natural representation of the isometry group (which extends in the obvious way to the full Fock space) to obtain the correct transformation properties of the field.

To see this, it is convenient to smear the field with a real test function $f\in\mathcal{C}_0^\infty(M)$:

$$\begin{aligned}\varphi(f)&=\int f(x)\varphi(x)d\mu(x)\\ &=\sum_k\int\phi_k(x)f(x)d\mu(x)a_k\\ &\quad+\sum_k\int\overline{\phi_k(x)}f(x)d\mu(x)a_k^\dagger\\ &=\sum_k(\overline{\phi_k},f)a_k+\sum_k(\phi_k,f)a_k^\dagger,\end{aligned}$$

where the parentheses designate the L^2 inner product (5). As pointed out in the Appendix, $a_k=a(\phi_k)$ and $a_k^\dagger=a^\dagger(\phi_k)$ where a and $a^\dagger$ are, respectively, antilinear and linear in the argument ϕ_k. Hence we can rewrite the smeared field in the following manner:

$$\begin{aligned}\varphi(f)&=\sum_k(\overline{\phi_k},f)a(\phi_k)+\sum_k(\phi_k,f)a^\dagger(\phi_k)\\ &=a\left(\sum_k(\phi_k,f)\phi_k\right)+a^\dagger\left(\sum_k(\phi_k,f)\phi_k\right).\end{aligned}$$

Defining

$$p(f)=\sum_k(\phi_k,f)\phi_k\in\mathcal{S}, \tag{9}$$

we have

$$\varphi(f)=a(p(f))+a^\dagger(p(f)). \tag{10}$$

Let us now consider p, which is a vector valued distribution taking values in the space $\mathcal{S}$ generated by the modes. Its role is to associate to a test function f an element of the physical one-particle space, so that we can consider the associated annihilation and creation operators. Note that $p(f)$ in Eq. (9) is the *unique* vector in $\mathcal{S}$ for which

$$\langle p(f),\psi\rangle=(f,\psi), \quad \forall\psi\in\mathcal{S}, \tag{11}$$

where we refer to Eqs. (4) and (5) for the definitions of the two inner products, both invariant under isometries. One concludes immediately from Eq. (11) and the nondegeneracy condition (7) that, if $\mathcal{S}$ is invariant under the action of the isometry group, then $V_g p(f) V_{g^{-1}} = p(V_g f)$ and as a result, $\varphi(x)$ also transforms correctly (see also Sec. V for more details).

It is well known that the above picture tends to break down in the presence of gauge invariance, and in particular when $m=0$ in Eq. (6). Allen [1] has indeed proved that on the (3+1)-dimensional de Sitter space-time no de Sitter covariant quantum field of the above type can exist when $m=0$. In particular, the theory obtained by taking $m\to 0$ in a massive theory is infrared divergent and this phenomenon is interpreted as symmetry breaking. The same divergence appears in 1+1 dimensions [7]. What we show in this paper is that a causal, gauge, and conformally covariant quantum field of the Gupta-Bleuler type can nevertheless be constructed by dropping the positivity requirement in Eq. (7).

Before doing this, we start by following the above canonical approach in order to pin down precisely where it breaks down. Although M does not allow a global timelike Killing field, $\hat{X}_0 = -iX_{1,0}$ is a global timelike *conformal* Killing field and we choose the modes ϕ_k to be positive frequency for $\hat{X}_0$, i.e., simultaneous solutions to $\hat{X}_0\phi = \omega\phi$, $\omega \geqslant 0$, and Eq. (6) with $m=0$. They are easily seen to be given by

$$\phi_0(\alpha,\rho) = \frac{1}{\sqrt{4\pi}},$$

$$\phi_k(\alpha,\rho) = \frac{e^{-i|k|\rho + ik\alpha}}{\sqrt{4\pi|k|}} \quad \text{for any} \quad k\in \mathbb{Z}\backslash\{0\}, \tag{12}$$

with $\omega = |k|$. The sign of k distinguishes the right-moving modes $(k>0)$ and the left-moving modes $(k<0)$. We have, for $k,l \in \mathbb{Z}\backslash\{0\}$,

$$\langle \phi_k, \phi_l\rangle = \delta_{k,l}, \quad \langle \phi_k, \overline{\phi_l}\rangle = 0, \tag{13}$$

but, for all $l\in\mathbb{Z}$, including $l=0$,

$$\langle \phi_0, \phi_l\rangle = 0 = \langle \phi_0, \overline{\phi_l}\rangle, \tag{14}$$

which means the inner product is now degenerate, in contrast to Eq. (7). This suggests dropping ϕ_0 and constructing the field as in Eq. (8) with only the ϕ_k, $k\neq 0$. This is precisely the field considered in [13] and leads to two problems at least. A simple computation shows that this field is not causal and does not transform correctly under the de Sitter group $SO_0(1,2) = G$. This is due to the fact that the space spanned by the ϕ_k for $k\neq 0$ is not invariant under the de Sitter group. For instance, a direct computation gives $(-X_{2,1} + iX_{1,3})\phi_1 = \phi_0$. Let us recall that the infrared problem on (1+1)-dimensional Minkowski space-time is due to the existence of solutions of arbitrary small frequencies. One could think that this problem, and the symmetry breaking associated with it, will disappear on the de Sitter space-time because the frequencies are now discretized since the space-time is spatially compact. But this is not quite true in the sense that, as we have just seen, the covariance of the theory forces one to include the null frequency solution itself in the normal mode decomposition of the field. This is of course in perfect agreement with Allen's result cited above.

At this point, wishing to recover an invariant theory, one is naturally lead to reinclude ϕ_0, to consider the positive semidefinite inner product space $\mathcal{K}$ spanned by the ϕ_k, and to construct the field φ as in Eq. (8) on the Fock space $\underline{\mathcal{K}}$. This still makes sense, in spite of the degeneracy of the inner product. Since, as is shown in Sec. IV, $\mathcal{K}$ is invariant under both $SO_0(1,2)$ and the conformal group, one would expect the resulting field to be $SO_0(1,2)$ and conformally covariant, i.e., $\underline{V}_g^{-1}\varphi(x)\underline{V}_g = \varphi(g\cdot x)$ for any g in the de Sitter group, where $\underline{V}$ is the natural representation of this group on the Fock space. *This turns out to be wrong: in spite of the fact that the space $\mathcal{K}$ generated by the modes in Eq.* (12) *is closed under the action of the group, the field is not covariant as a consequence of the degeneracy of the inner product on $\mathcal{K}$.* The reason is that the basis (12) is not orthonormal and as a result the corresponding distribution p, as defined in Eq. (9), does not verify Eq. (11) anymore, so the reasoning leading to the invariance of the field no longer holds up. It turns out that the distribution p is indeed *not covariant*: p does not intertwine the (regular) representations of the group on $\mathcal{C}_0^\infty(M)$ and $\mathcal{K}$. Direct computation shows for example that $\hat{X}_{1,3}p(f) \neq p(\hat{X}_{1,3}f)$ where $\hat{X}_{1,3} = -iX_{1,3}$. Note that, if the inner product is nondegenerate, one can define p equally well through Eqs. (9) and (11), the two being equivalent. In contrast, since the Klein-Gordon inner product is degenerate on $\mathcal{K}$, there no longer exists a $p(f)$ satisfying Eq. (11), as is easily seen. We note in passing that the degeneracy of the inner product is directly related to the indecomposability of the representation of $SO_0(2,1)$ on $\mathcal{K}$ (see Sec. IV). *To summarize, in order to construct a covariant quantum field theory via Eq. (8), it seems one must have a nondegenerate inner product; it is in particular not enough to have a representation of the de Sitter group on the Fock space where the field operators act to guarantee the covariance of the field.*

Incidentally, one could try to ignore this invariance problem and carry on with the above field. But one then notices that the commutator built with this field is still not causal, so the theory remains at any rate unacceptable.

At this point, one should notice that the modes ϕ_k do not form a complete set in the sense specified above; since the field equation admits a so-called zero-frequency mode,

$$\psi_p = \frac{1}{\sqrt{4\pi}}(1 - i\rho),$$

verifying $\langle \psi_p, \psi_p\rangle = 1$. One might think our troubles come from ignoring this mode and it seems natural to consider the complete set of modes ϕ_k, ψ_p, with $k\neq 0$ [which satisfy Eq. (7)] and to construct the corresponding field that we denote by φ_p for reasons explained in Sec. V:

$$\varphi_p(x) = \sum_{k\neq 0} \phi_k(x)a_k + \sum_{k\neq 0} \overline{\phi_k}(x)a_k^\dagger + a_p\psi_p(x) + a_p^\dagger\overline{\psi_p}(x).$$

Introducing $Q_p = (a_p + a_p^\dagger)/\sqrt{2}$ and $P_p = -i(a_p - a_p^\dagger)/\sqrt{2}$, one sees $[Q_p, P_p] = i$, and the field can be rewritten

$$\varphi_p(x)=\sum_{k\neq 0}\phi_k(x)a_k+\sum_{k\neq 0}\overline{\phi_k}(x)a_k^\dagger+\frac{1}{\sqrt{2\pi}}(Q_p+P_p\rho). \tag{15}$$

This field acts on the Fock space over $\mathcal{H}_p=\mathcal{K}_p\oplus\mathbb{C}\psi_p$, where $\mathbb{C}\psi_p$ is the one-dimensional space spanned by ψ_p and $\mathcal{K}_p$ is the Hilbert space generated by the ϕ_k's for $k\neq 0$. It is causal, but not covariant, since it is easy to see (Sec. IV) that $\mathcal{H}_p$ is not an invariant space of solutions to the field equation.

This is exactly the field considered in [5,3], except that in these papers it is made to act on the direct product of the Fock space over $\mathcal{K}'$ with the Fock space over $\mathbb{C}\psi_p$. It cannot be covariant for the same reason. Nevertheless, treating the (3+1)-dimensional case, Kirsten and Garrida have found a state for it, which we will denote by $|0_p\rangle$ in the following, distinct from the Fock vacuum, and which is covariant in a weak sense. It is characterized by $a_k|0_p\rangle=0=P_p|0_p\rangle$. As an eigenvector of the ''momentum'' operator P_p it is of course not normalizable, but it is chosen to minimize the energy operator which contains, in addition to the usual $a_k^\dagger a_k$ terms, a term proportional to ${P_p}^2$. Kirsten and Garriga compute $\langle 0_p|[\varphi(x)-\varphi(y)]^2|0_p\rangle$ and find, after point splitting renormalization, that it depends only on the proper distance between x and y. In this sense, they say their state is ''invariant.'' Our analysis in the next section will show that in the 1+1 dimension it cannot be invariant in the stronger (and usual) sense recalled above. It will be proved in [4] that the same feature appears in the 1+3 dimension. As we show in the rest of this paper, the problems of the above fields can be solved at once by enlarging the space of states into an indefinite but nondegenerate inner product space $\mathcal{H}$ (called the total space) containing nonphysical states as well as $\mathcal{K}$ and $\mathcal{H}_p$.

IV. THE QUANTUM FIELD ON (1+1)-DIMENSIONAL de SITTER SPACE-TIME

We begin with the construction of the space $\mathcal{K}$ of physical one-particle states. First, recall from Sec. II that the de Sitter space-time M can be realized as an open dense subset of the torus $S^1\times S^1$ and that the conformally coupled (=minimally coupled) equation is

$$4\cos^2\left(\frac{u^++u^-}{2}\right)\partial_u+\partial_u-\psi=0,\quad u^\varepsilon\in S^1.$$

Any *smooth* solution of this equation on the torus reads $f(u^+,u^-)=f_+(u^+)+f_-(u^-)$, the decomposition is not unique because of the presence of the constant functions. We define

$$\mathcal{K}=\left\{\sum_{k\in\mathbb{Z}}c_k\phi_k,\quad\text{with}\ \sum_{k\in\mathbb{Z}}|c_k|^2<\infty\right\},$$

where

$$\phi_0(u^+,u^-)=\frac{1}{\sqrt{4\pi}},\quad \phi_k(u^+,u^-)=\frac{e^{-iku^+}}{\sqrt{4\pi k}}\quad\text{when}\ k>0,$$

$$\phi_k(u^+,u^-)=\frac{e^{-i|k|u^-}}{\sqrt{4\pi|k|}}\quad\text{when}\ k<0,$$

which is the same as Eq. (12). Then we have

$$\langle\phi_0,\phi_k\rangle=0\,\forall k\in\mathbb{Z},\quad\langle\phi_k,\phi_l\rangle=\delta_{kl}\,\forall k,l\in\mathbb{Z}\backslash\{0\}.$$

$\mathcal{K}$ is a space of (conformal) positive frequency solutions of the wave equation on $S^1\times S^1$. It contains all smooth solutions on $S^1\times S^1$. Crucial for what follows is the fact, proved in [11], that $\mathcal{K}$ is invariant under the left regular representation of the conformal group $G_c=\mathrm{SO}_0(2,2)/\{\pm Id\}$, which also preserves the above positive semidefinite Klein-Gordon inner product on $\mathcal{K}$. This representation was studied in detail in [11] and we recall its essential features here. Defining $\mathcal{N}=\mathcal{K}\cap\mathcal{K}^\perp=\mathbb{C}\phi_0$, one easily sees that $\mathcal{N}$ is an invariant subspace of $\mathcal{K}$, which is not orthogonally and not invariantly complemented: the representation is therefore indecomposable. The Hilbert space $\mathcal{B}=:\mathcal{K}/\mathcal{N}$ carries a unitary representation of $\mathrm{SO}_0(2,2)/\{\pm Id\}=\mathrm{SO}_0(1,2)\times\mathrm{SO}_0(1,2)$ which is the direct sum of two irreducible unitary representations $\mathrm{SO}_0(1,2)$. It is proved in [11] that each of these representations of $\mathrm{SO}_0(1,2)$ is the first term of the discrete series of representations of this group.

Let us finally recall that the generator of (conformal) time translations is $X_{1,0}=-\partial_\rho=-\partial_{u^+}-\partial_{u^-}$, so that $iX_{1,0}\phi_k=|k|\phi_k$; hence the operator $iX_{1,0}$ induces an operator on $\mathcal{B}$ with a positive spectrum. Thinking of $iX_{1,0}$ as a (conformal) Hamiltonian, this is the spectral condition satisfied by the one-particle physical states, a direct consequence of our choice of positive frequency solutions in the construction of $\mathcal{K}$.

Now we have to define $\mathcal{H}$, the third term of the Gupta-Bleuler triplet, which will contain nonphysical modes. We first recall from the previous section that the ϕ_k do not form a complete set of modes on M. Indeed, the field equation has an obvious solution ψ_s on de Sitter space-time, defined by

$$\psi_s(\alpha,\rho)=-\frac{2i\rho}{\sqrt{4\pi}},$$

which is of norm zero: $\langle\psi_s,\psi_s\rangle=0$: ψ_s is a zero-frequency mode [13,5,3]. This solution does not extend to a smooth solution on $S^1\times S^1$, though, and, as a matter of fact, ψ_s does not belong to $\mathcal{K}+\overline{\mathcal{K}}$, where $\overline{\mathcal{K}}$ is the set of the complex conjugates of the elements of $\mathcal{K}$: ψ_s is in this sense not a superposition of the modes ϕ_k. To see this, one can, for example, remark that $\langle\psi_s,\phi_0\rangle=1\neq 0$, which is impossible for elements of $\mathcal{K}+\overline{\mathcal{K}}$. We will show how the Gupta-Bleuler formalism allows a very natural quantization of this mode, together with the others, different from the one of [13,5,3] that we reviewed in the previous section.

We now define the total space $\mathcal{H}$ by

$$\mathcal{H}=(\mathcal{K}+\overline{\mathcal{K}})\oplus\mathbb{C}\psi_s,$$

where $\overline{\mathcal{K}}$ is the set of the complex conjugates of the elements of $\mathcal{K}$.

Note that the first sum is not a direct sum because $\phi_0 \in \mathcal{K} \cap \overline{\mathcal{K}}$. Recalling that $\mathcal{N} = \mathbb{C}\phi_0$ we obtain our Gupta-Bleuler triplet $\mathcal{N} \subset \mathcal{K} \subset \mathcal{H}$. Note that $\mathcal{H}$ contains negative norm states (for example, the elements of $\overline{\mathcal{K}}$), and it cannot be the space of physical states that we have just identified as $\mathcal{K}$. Note nevertheless that the negative norm states are indispensable as intermediate objects to assure the covariance and the gauge covariance of the theory. As shown in Sec. VI below, these states also play a role in the renormalization procedure but their use does not imply the appearance of negative energy: the mean value of P_0 on physical states is always positive.

We shall now prove that $\mathcal{H}$ is a Krein space containing $\mathcal{K}$ as a closed subset. To that end, consider $\psi_p = \phi_0 + (\psi_s/2)$ and $\psi_n = \phi_0 - (\psi_s/2) = \overline{\psi_p}$, and $\mathcal{K}_p = \{\Sigma_{k\neq 0} c_k \phi_k ; \Sigma_{k\neq 0} |c_k|^2 < \infty\}$ and $\mathcal{H}_p = \mathbb{C}\psi_p \oplus \mathcal{K}_p$, and also $\mathcal{H}_n = \mathbb{C}\psi_n \oplus \overline{\mathcal{K}_p}$. The equality

$$\mathcal{H} = \mathcal{H}_p \oplus \mathcal{H}_n , \tag{16}$$

realizes $\mathcal{H}$ as a direct sum of a Hilbert space and an anti-Hilbert space which indeed proves that $\mathcal{H}$ is a Krein space. Note that neither $\mathcal{H}_p$ nor $\mathcal{H}_n$ carries a representation of the de Sitter group, so the previous decomposition is not covariant. This can be seen from the fact that the action of $SO_0(1,2)$ on ψ_p generates $\mathcal{K}$ as well as $\overline{\mathcal{K}}$, where $\overline{\mathcal{K}}$ is the set of complex conjugates of $\mathcal{K}$ which is a space of negative frequency solutions [for instance $(X_{1,3} + iX_{2,1})\psi_p = -i(\overline{\phi_1} + \phi_{-1})/2$]. Moreover, one can prove that $\mathcal{K}$ is a closed subspace of $\mathcal{H}$ by remarking that $\mathcal{K} = (\mathcal{K})^{\perp}$. Note that ψ_p is not a physical state ($\psi_p \notin \mathcal{K}$) in spite of the fact that $\langle \psi_p , \psi_p \rangle > 0$: the condition of positivity of the inner product is not a sufficient condition for selecting physical states.

We are now ready to define the (unsmeared) massless quantum field as follows. Following Mincev [16] (see Appendix), one can build the Fock space $\underline{\mathcal{H}}$ over the Krein space $\mathcal{H}$. We then define the quantum field φ on $\underline{\mathcal{H}}$ by

$$\varphi(x) = \sum_{k\neq 0} \phi_k(x) a_k - \sum_{k\neq 0} \overline{\phi_k}(x) b_k + \phi_0(x) a_s + \psi_s(x) a_0 + \sum_{k\neq 0} \overline{\phi_k}(x) a_k^{\dagger} - \sum_{k\neq 0} \phi_k(x) b_k^{\dagger} + \phi_0(x) a_s^{\dagger} - \psi_s(x) a_0^{\dagger} , \tag{17}$$

where, $a_0 = a(\phi_0)$, $a_0^{\dagger} = a^{\dagger}(\phi_0)$, $a_s = a(\psi_s)$, $a_s^{\dagger} = a^{\dagger}(\psi_s)$, $b_k = a(\overline{\phi_k})$, and $b_k^{\dagger} = a^{\dagger}(\overline{\phi_k})$ (see Appendix). The nonvanishing commutation relations between these operators are for $k \neq 0$:

$$[a_k , a_k^{\dagger}] = 1, \quad [b_k , b_k^{\dagger}] = -1, \quad [a_s , a_0^{\dagger}] = 1, \quad [a_0 , a_s^{\dagger}] = 1. \tag{18}$$

Note the minus sign which follows from the formulas $[a(\phi), a^{\dagger}(\phi')] = \langle \phi, \phi' \rangle$ and $\langle \overline{\phi_k}, \overline{\phi_k} \rangle = -1$. Note also that this field is clearly real as the sum of an operator and its conjugate. For later use, we remark that it can be rewritten as follows:

$$\varphi(x) = \varphi_p(x) + \varphi_n(x), \tag{19}$$

where

$$\varphi_p(x) = \sum_{k\neq 0} \phi_k(x) a_k + \sum_{k\neq 0} \overline{\phi_k}(x) a_k^{\dagger} + \frac{1}{\sqrt{2\pi}} (Q_p + P_p \rho), \tag{20}$$

and

$$\varphi_n(x) = -\sum_{k\neq 0} \overline{\phi_k}(x) b_k - \sum_{k\neq 0} \phi_k(x) b_k^{\dagger} + \frac{1}{\sqrt{2\pi}} (Q_n + P_n \rho), \tag{21}$$

with

$$Q_n = -\frac{1}{\sqrt{2}} (a_n + a_n^{\dagger}), \quad Q_p = \frac{1}{\sqrt{2}} (a_p + a_p^{\dagger}),$$

$$P_n = -\frac{i}{\sqrt{2}} (a_n - a_n^{\dagger}), \quad P_p = -\frac{i}{\sqrt{2}} (a_p - a_p^{\dagger}).$$

We claim that this field is covariant in the strong sense, conformally covariant and causal. Moreover the stress tensor is an observable and the energy operator P_0 has positive expectation values in all physical states.

In order to prove these claims, we proceed as in Eqs. (10), (11) to introduce the smeared field, which is easier to work with. For any test function $f \in C_0^{\infty}(M)$, we define $p(f)$ as the only element of $\mathcal{H}$ verifying

$$\langle p(f), \psi \rangle = (f, \psi), \quad \forall \psi \in \mathcal{H}.$$

Unlike on $\mathcal{K}$, this definition makes sense on $\mathcal{H}$, since the Klein-Gordon inner product is nondegenerate on it. As is the case for any distribution, one can prefer the unsmeared form which reads $\langle p(x), \psi \rangle = \psi(x)$. The nondegeneracy of the inner product in $\mathcal{H}$ is of course crucial in this definition. Let us remark that the range of p is a dense subset of $\mathcal{H}$ because $(\mathrm{Ran}(p))^{\perp} = \{0\}$. It is actually a covariant vector-valued distribution verifying the field equation. It can be expanded in the basis, and in the unsmeared form, it reads

$$p(x) = \sum_{k\neq 0} \overline{\phi_k}(x) \phi_k - \sum_{k\neq 0} \phi_k(x) \overline{\phi_k} + \phi_0(x) \psi_s - \psi_s(x) \phi_0 , \tag{22}$$

$$= \sum_{k\neq 0} \overline{\phi_k}(x) \phi_k - \sum_{k\neq 0} \phi_k(x) \overline{\phi_k} + \overline{\psi_p}(x) \psi_p - \overline{\psi_n}(x) \psi_n . \tag{23}$$

Then, using Eqs. (17) and (22) one can readily verify that

$$\varphi(x) = a(p(x)) + a^{\dagger}(p(x)).$$

In smeared form this reads

$$\varphi(f)=a(p(f))+a^\dagger(p(f)),$$

which is well defined on a suitable domain $\underline{\mathcal{H}}_0$ of $\underline{\mathcal{H}}$ ($\underline{\mathcal{H}}_0$ is the set of finite length elements of $\underline{\mathcal{H}}$). We note in passing that the definition of the field does not depend on the modes but on the space $\mathcal{H}$ they generate, the modes being only a tool for computation.

To establish the causality of the field, we compute W, the kernel of p, defined formally by

$$p(f)(x')=\int W(x',x)f(x)d\mu(x),$$

where $d\mu$ is the invariant measure. As a consequence we have

$$\begin{aligned}\langle p(f_1),p(f_2)\rangle&=(f_1,p(f_2))\\&=\int\int \overline{f_1}(x')W(x',x)f_2(x)d\mu(x)d\mu(x'),\end{aligned}$$

that is to say in the unsmeared form

$$W(x,x')=\langle p(x),p(x')\rangle. \tag{24}$$

From Eqs. (22), (23), and (24) one obtains

$$\begin{aligned}W(x,x')&=\sum_{k\neq 0}\overline{\phi_k}(x)\phi_k(x')-\psi_s(x')+\psi_s(x)\\&\quad-\sum_{k\neq 0}\phi_k(x)\overline{\phi_k}(x')\\&=\sum_{k\neq 0}\overline{\phi_k}(x)\phi_k(x')+\overline{\psi_p}(x)\psi_p(x')\\&\quad-\overline{\psi_n}(x)\psi_n(x')-\sum_{k\neq 0}\phi_k(x)\overline{\phi_k}(x').\end{aligned} \tag{25}$$

Explicit computation gives

$$W(x,x')=-i\widetilde{G}(x,x'),$$

where we recover the commutator defined in Sec. II. The vector-valued distribution p is therefore just the kernel of the natural commutator $-i\widetilde{G}$.

The causality of this field now follows immediately from this definition and from the formula $[a(\phi),a^\dagger(\phi')]=\langle\phi,\phi'\rangle$:

$$\begin{aligned}[\varphi(x),\varphi(x')]&=2i\ \mathrm{Im}\langle p(x),p(x')\rangle\\&=2i\ \mathrm{Im}\ W(x,x')\\&=-2i\widetilde{G}(x,x').\end{aligned}$$

We conclude the field is causal because $\widetilde{G}$ vanishes when x and x' are spacelike separated.

Before studying the invariance of the field in the next section, we make an additional remark. Introducing the Bogoliubov transformation $A_k=a_k-b_k^\dagger$ and $A_p=a_0+a_s/2-a_0^\dagger+a_s^\dagger/2$. The above formula can be rearranged to read

$$\begin{aligned}\varphi(x)&=\sum_{k\neq 0}\phi_k(x)A_k+\sum_{k\neq 0}\overline{\phi_k}(x)A_k^\dagger+\psi_p(x)A_p\\&\quad+\overline{\psi_p}(x)A_p^\dagger,\end{aligned} \tag{26}$$

with $[A_k,A_k^\dagger]=[A_p,A_p^\dagger]=2$; note however that $A_k|0\rangle\neq 0$. This suggests that, after all, a Hilbert space Fock vacuum $|0'\rangle$ could be introduced. This vacuum verifying $A_k|0'\rangle=0$ would, via Eq. (26), lead to a field on a Hilbert space, but the invariance of the theory would be broken, as explained in Sec. III.

V. COVARIANCE

A. de Sitter covariance

The invariance of the total space $\mathcal{H}$ by the de Sitter group G can be proved in the following way. For X belonging to the above basis of the Lie algebra of G, one can calculate $\psi=\exp(\theta X)\psi_s-\psi_s$. It is enough to prove that $\psi\in\mathcal{K}+\overline{\mathcal{K}}$. One can do that by remarking that ψ can be extended in a $\mathcal{C}^\infty$ function on the whole torus. As a result its Fourier coefficients are rapidly decreasing and $\psi\in\mathcal{K}+\overline{\mathcal{K}}$.

We now remark that p intertwines the regular representation V of the de Sitter group (see p. 11) on $\mathcal{C}_0^\infty(M)$ and $\mathcal{H}$, actually for all $\psi\in\mathcal{H}$ we have

$$V_g p=pV_g, \tag{27}$$

for any g in the de Sitter group. Indeed we have

$$\begin{aligned}\langle V_g p(f),\psi\rangle&=\langle p(f),V_g^{-1}\psi\rangle\\&=(f,V_g^{-1}\psi)\\&=(V_g f,\psi)\\&=\langle p(V_g f),\psi\rangle.\end{aligned}$$

The representation V of the de Sitter group extends to a representation $\underline{V}$ of the same group on $\underline{\mathcal{H}}_0$ and

$$\begin{aligned}\underline{V}_g\varphi(f)\underline{V}_g^{-1}&=a(V_g p(f))+a^\dagger(V_g p(f))\\&=a(p(V_g f))+a^\dagger(p(V_g f))\\&=\varphi(V_g f).\end{aligned}$$

Remark: To the decomposition (16) corresponds to the decomposition

$$p=p_p+p_n \tag{28}$$

in a natural way. The expression (25) yields the corresponding decomposition of W: the kernel W decomposes into $W=W_p-W_n$ where these kernels are of a positive type. Explicitly, using the previous basis one obtains

$$4\pi W_p(x,x') = -\log[1 - 2\cos(\alpha-\alpha')e^{-i(\rho-\rho')} + e^{-2i(\rho-\rho')}] - i(\rho-\rho') + (1+\rho\rho')$$
$$= (1+\rho\rho') - \ln 2 - \ln|\cos(\alpha-\alpha') - \cos(\rho-\rho')| - 2i\pi\widetilde{G}(x,x'),$$

where log is the principal determination of the logarithm and ln is the Neperian logarithm. The decomposition (19) of the field is then obtained via $\varphi_p = a(p_p(x)) + a^\dagger(p_p(x))$ and $\varphi_n = a(p_n(x)) + a^\dagger(p_n(x))$. This field has been already discussed in Sec. III. One could be tempted to consider φ_p as the physical part of the field which has as Wightman function the function W_p defined above. But, once again, this object is not covariant because W_p is not. So all attempts to restore positivity of the inner product seem doomed to failure, in agreement of course with Allen's result.

B. Conformal covariance

Let us recall that the conformal group is $G_c = \mathrm{SO}_0(2,2)/\{\pm Id\}$. One would like to obtain a property which reads formally

$$\varphi(g^{-1}\cdot x) = \underline{V}_g \varphi(x) \underline{V}_g^{-1} \quad \text{for any } g \in G_c.$$

But several difficulties appear when dealing with such a formula. First, $\mathcal{K}$ is closed under the action of the conformal group but this is not the case for $\mathcal{H}$ and the formula can make sense only when taking expectation values between physical states. Second, the space of test functions is not invariant under the conformal group. Third, when smearing the distributions, one uses a measure which is not conformally invariant and one has to be careful when dealing with the action of the conformal group on distributions.

Let $f \in \mathcal{C}_0^\infty(M)$ and $g \in G_c$; we say that f and g are compatible if and only if there exists $X \in \mathrm{so}(2,2)$ such that $g = \exp X$ and $\exp \theta X \cdot x$ belongs to M for all $\theta \in [0,1]$ and for all x in the support of f. For f and g compatible we define a test function $\widetilde{V}_g f \in \mathcal{C}_0^\infty(M)$ by

$$\int \psi(x)(\widetilde{V}_g f)(x) d\mu(x) = \int \psi(g\cdot x) f(x) d\mu(x),$$

for any locally integrable function ψ. Note that, thanks to the compatibility condition, the right hand side makes sense. The (local) representation $\widetilde{V}$ is not the regular one because the measure is not conformally invariant. An explicit computation of the representation $\widetilde{V}$ allows one to identify it with the limit ($\sigma = 1$) of the complementary series of representations of $\mathrm{SO}_0(2,1)$ denoted $V^{0,1}$ in [17]. Let $\underline{\mathcal{K}}_0$ be the subspace of $\underline{\mathcal{H}}_0$ generated (as a tensor algebra) by $\mathcal{K}$. The representation V of the conformal group on $\mathcal{K}$ extends to a representation $\underline{V}$ of the same group on $\underline{\mathcal{K}}_0$. Then for any $f \in \mathcal{C}_0^\infty(M)$ and $g \in G_c$ compatible and for any $\psi_1, \psi_2 \in \underline{\mathcal{K}}_0$ one has

$$\langle \psi_1 | \varphi(\widetilde{V}_g f) | \psi_2 \rangle = \langle \underline{V}_g^{-1} \psi_1 | \varphi(f) | \underline{V}_g^{-1} \psi_2 \rangle.$$

More shortly, one can say that $\varphi(g^{-1}\cdot x) = \underline{V}_g \varphi(x) \underline{V}_g^{-1}$ on $\underline{\mathcal{K}}_0$.

Sketch of the proof: As for covariance, one can prove easily that $Xp = pX$ for any $X \in \mathrm{so}(2,2)$ [note that although $\mathcal{H}$ is not invariant under G_c, it is invariant under so(2,2)]. The standard computation on creators and annihilators shows that $[X, \varphi(f)] = \varphi(Xf)$. Then putting $\hat{X} = -iX$ one obtains

$$[\hat{X}, \varphi(f)] = -i\varphi(Xf),$$

which integrates into the desired formula.

C. Gauge covariance

Let us recall that the classical gauge change is in this context given by $\phi(x) \to \phi(x) + \lambda$. At the quantum level we define the gauge change by

$$U(\lambda) = e^{-i\lambda\varphi_S(i\sqrt{\pi}\phi_0)},$$

where we define $\varphi_S(\phi) = a(\phi) + a^\dagger(\phi)$. From the well-known formula (18)

$$e^{i\varphi_S(\psi)} \varphi_S(\psi') e^{-i\varphi_S(\psi)} = \varphi_S(\psi') + 2\,\mathrm{Im}\langle \psi', \psi\rangle,$$

we obtain

$$U(-\lambda)\varphi(f)U(\lambda) = \varphi(f) + 2\lambda\ \mathrm{Im}\langle p(f), i\sqrt{\pi}\phi_0\rangle Id$$
$$= \varphi(f) + \lambda\sqrt{4\pi}\ \mathrm{Im}(f, i\phi_0) Id$$
$$= \varphi(f) + \lambda\sqrt{4\pi}(f, \phi_0) Id.$$

That is to say the gauge transformation is given by

$$\varphi(f) \to \varphi(f) + \lambda\sqrt{4\pi}(f,\phi_0) Id,$$

or in the unsmeared form ($f = \delta_x$):

$$\varphi(x) \to \varphi(x) + \lambda Id.$$

VI. THE STRESS-TENSOR AND OTHER OBSERVABLES

Having shown that the field we constructed is causal and has all the covariance properties of the classical field, we can now turn to an investigation of the physical content of the theory.

We denote by $\underline{\mathcal{N}}$, $\underline{\mathcal{K}}$, and $\underline{\mathcal{H}}$ the Fock spaces built on $\mathcal{N}$, $\mathcal{K}$, $\mathcal{H}$ and by $\underline{\underline{\mathcal{N}}}$ the subspace of $\underline{\mathcal{K}}$ orthogonal to $\underline{\mathcal{K}}$:

$$\Psi \in \underline{\underline{\mathcal{N}}} \quad \text{iff} \quad \Psi \in \underline{\mathcal{K}} \quad \text{and} \quad \langle \Psi, \Phi\rangle = 0 \,\forall \Phi \in \underline{\mathcal{K}}. \tag{29}$$

To interpret the theory, we define its physical states and its observables.

First, the physical states are defined to be the elements of $\underline{\mathcal{K}}$. Note that the space $\underline{\underline{\mathcal{N}}}$ is strictly greater than $\underline{\mathcal{N}}$, for instance $\phi_0 \otimes \phi_1 + \phi_1 \otimes \phi_0$ belongs to $\underline{\underline{\mathcal{N}}}$ but not to $\underline{\mathcal{N}}$. We say that two physical states are physically equivalent if they differ by an element of $\underline{\underline{\mathcal{N}}}$. It is clear from the definition of $U(\lambda)$ that a gauge transformation maps a physical state into an equivalent state. As a consequence, the inner product of $\underline{\mathcal{K}}$

is gauge-independent. We have just defined the second-quantified Gupta-Bleuler triplet:

$$\underline{\mathcal{N}}\subset\underline{\mathcal{K}}\subset\underline{\mathcal{H}}.$$

Remark (quasiuniqueness of the vacuum): The space of de Sitter invariant states of $\underline{\mathcal{H}}$ is $\underline{\mathcal{N}}_0$ the subspace of finite length elements of $\underline{\mathcal{N}}$. This space is infinitely dimensional, hence the Fock vacuum is not the only de Sitter invariant state. Nevertheless one can see easily that all these states are physically equivalent to an element of the one-dimensional space generated by the vacuum state. In this sense we can say that the vacuum is unique.

As we have said before, in the Gupta-Bleuler triplets terminology, observables are defined by the property that they do not ''see'' the gauge states, as a consequence when Ψ and Ψ' are equivalent physical states (elements of $\underline{\mathcal{K}}$ such that $\Psi-\Psi'$ belongs to $\underline{\mathcal{N}}$), we must have

$$\langle\Psi|A|\Psi\rangle=\langle\Psi'|A|\Psi'\rangle,$$

for any observable A. Hence the field itself is not an observable as can be seen in the formula (17) because of the appearance of $a_s^\dagger$ and a_s. Nevertheless the operators $\partial_\mu\varphi$ no longer contain those terms [since $\phi_0(x)$ is constant], and the physically interesting observables are built with them.

The fact that φ is not an observable implies that the different two-point functions are gauge-dependent (except for W which is defined independently of the field and which, being a commutator, is gauge-invariant). As an example, the symmetric two-point function $G^{(1)}$ is not expected to have great meaning in our construction and a straightforward computation indeed shows that it vanishes:

$$G^{(1)}(x,x')=\langle 0|[\varphi(x),\varphi(x')]_+|0\rangle=2\,\mathrm{Re}\,W(x,x')=0.$$

This shows of course that it is de Sitter covariant, be it in a trivial way.

We will now deal with some observables and show that the ''negative frequency part'' of the field will realize a renormalization allowing a trivial computation of the mean values of the components of the stress tensor. At the classical level, the stress tensor is given by $(n-2=m=\xi=s=0)$,

$$T_{\mu\nu}=\partial_\mu\phi\partial_\nu\phi-\tfrac{1}{2}g_{\mu\nu}g^{\rho\sigma}\partial_\rho\phi\partial_\sigma\phi,$$

see for instance [19]. Hence

$$T_{00}=T_{11}=\tfrac{1}{2}[(\partial_\rho\phi)^2+(\partial_\alpha\phi)^2],$$

and

$$T_{01}=T_{10}=\partial_\rho\phi\partial_\alpha\phi.$$

Using $\nabla_\mu T^\mu{}_\nu=0$ one defines the Hamiltonian and momentum operators

$$P_0=\int_{\rho=0}T_{00}d\alpha,$$

$$P_1=-\int_{\rho=0}T_{01}d\alpha. \tag{30}$$

Using Eq. (17) one can readily quantize this expression into

$$P_0=\frac{1}{2}\sum_{k\neq 0}|k|(a_ka_k^\dagger+a_k^\dagger a_k)+\tfrac{1}{2}\sum_{k\neq 0}|k|(b_kb_k^\dagger+b_k^\dagger b_k)$$
$$-\sum_{k\neq 0}|k|a_kb_k-\sum_{k\neq 0}|k|a_k^\dagger b_k^\dagger+(a_0a_0^\dagger+a_0^\dagger a_0)$$
$$-a_0a_0-a_0^\dagger a_0^\dagger.$$

In this expression the usual renormalization (normal ordering) is now useless because $a_ka_k^\dagger+a_k^\dagger a_k+b_kb_k^\dagger+b_k^\dagger b_k=2(a_k^\dagger a_k+b_k^\dagger b_k)$, see Eq. (18). Hence one obtains

$$P_0=\sum_{k\neq 0}|k|a_k^\dagger a_k+\sum_{k\neq 0}|k|b_k^\dagger b_k-\sum_{k\neq 0}|k|a_kb_k$$
$$-\sum_{k\neq 0}|k|a_k^\dagger b_k^\dagger+\tfrac{1}{2}P^2,$$

where $P=P_p+P_n$. Then if Ψ is a physical state ($\Psi\in\underline{\mathcal{K}}_0$) one obtains

$$\langle\Psi|P_0|\Psi\rangle=\langle\Psi|\sum_{k\neq 0}|k|a_k^\dagger a_k|\Psi\rangle, \tag{31}$$

which is reassuringly positive, even though the operator P_0 contains negative definite terms.

Note in particular that, if Ψ is a physical state, then $\langle\Psi|P^2|\Psi\rangle=0$, even though $P|\Psi\rangle\neq 0$ and $\langle 0|0\rangle=1$, something that is of course not possible in a Hilbert space. This shows that the zero modes do not contribute to the energy of the physical states at all and should be compared with the approach of [3] recalled in Sec. III, where the vacuum is not normalizable as the zero eigenvector of P_p.

Moreover if $\Psi-\Psi'$ belongs to $\underline{\mathcal{N}}$, we have

$$\langle\Psi|P_0|\Psi\rangle=\langle\Psi'|P_0|\Psi'\rangle,$$

and P_0 is therefore an observable. The same is true for P_1 which has the explicit form

$$P_1=\sum_{k\neq 0}ka_k^\dagger a_k+\sum_{k\neq 0}kb_k^\dagger b_k-\sum_{k\neq 0}ka_kb_k-\sum_{k\neq 0}ka_k^\dagger b_k^\dagger.$$

The stress tensor itself is an observable. In order to compute mean values we first compute $\langle 0|\partial_\mu\varphi(x)\partial_\nu\varphi(x)|0\rangle$. The terms containing $a_0,a_s,a_0^\dagger,a_s^\dagger$ do not contribute and we obtain

$$\langle 0|\partial_\mu\varphi(x)\partial_\nu\varphi(x)|0\rangle=\sum_{k\neq 0}\partial_\mu\phi_k(x)\partial_\nu\overline{\phi_k}(x)$$
$$-\sum_{k\neq 0}\partial_\mu\overline{\phi_k}(x)\partial_\nu\phi_k(x)$$
$$=2i\,\mathrm{Im}\sum_{k\neq 0}\partial_\mu\phi_k(x)\partial_\nu\overline{\phi_k}(x)$$
$$=0.$$

The cancellation is due to the unusual second term of the right hand side of the first line which comes from the terms of the field containing b_k and $b_k^\dagger$. As a consequence we obtain

$$\langle 0|T_{\mu\nu}(x)|0\rangle = 0,$$

and there is an automatic renormalization of the stress tensor.

Similarly we can compute the mean values of the stress tensor on physical states of the following type:

$$|k_1^{n_1}\ldots k_j^{n_j}\rangle = \frac{1}{\sqrt{n_1!\ldots n_j!}}(a_{k_1}^\dagger)^{n_1}\ldots(a_{k_j}^\dagger)^{n_j}|0\rangle.$$

The same cancellation holds and one obtains

$$\langle k_1^{n_1}\ldots k_j^{n_j}|\partial_\mu\varphi(x)\partial_\nu\varphi(x)|k_1^{n_1}\ldots k_j^{n_j}\rangle = \frac{1}{2\pi}\sum_{i=1}^{j} n_i \ \mathrm{Re}(\partial_\mu\phi_{k_i}(x)\partial_\nu\overline{\phi_{k_i}}(x)).$$

As a consequence we obtain

$$\langle k_1^{n_1}\ldots k_j^{n_j}|T_{00}(x)|k_1^{n_1}\ldots k_j^{n_j}\rangle = \frac{1}{2\pi}\sum_{i=1}^{j} n_i|k_i|, \quad (32)$$

in agreement with Eqs. (30) and (31), and once again no infinite terms appears in this computation. Note that the formula (32) shows that the use of negative norm solutions in the definition of the field does not yield negative energy.

As said before, the ''negative norm part'' of the field allows a renormalization of the stress tensor. Indeed, no infinite term appears in the previous computation, and the conformal anomaly also disappears. In order to better understand this fact let us consider the ''positive norm part'' of the field:

$$\varphi_p(x) = \sum_{k\neq 0}\phi_k(x)a_k + \psi_p a_p + \sum_{k\neq 0}\overline{\phi_k}(x)a_k^\dagger + \overline{\psi_p}a_p^\dagger,$$

where $a_p = a(\psi_p)$ and $a_p^\dagger = a^\dagger(\psi_p)$. As explained in Sec. III, this field is the two-dimensional analogue of the field defined in (3). The mean value of the stress tensor of this field in the Gupta-Bleuler vacuum state is given by

$$\langle 0|T_{\mu\nu}|0\rangle = \sum_{k\neq 0} T_{\mu\nu}(\phi_k,\overline{\phi_k}) + T_{\mu\nu}(\psi_p,\overline{\psi_p}).$$

Direct computation proves that $T_0^0(\psi_p,\overline{\psi_p}) + T_1^1(\psi_p,\overline{\psi_p}) = 0$ and also that $T_{01}(\psi_p,\overline{\psi_p}) = T_{10}(\psi_p,\overline{\psi_p}) = 0$. The other term $\Sigma_{k\neq 0}T_{\mu\nu}(\phi_k,\overline{\phi_k})$ is exactly the quantity computed in a general setting in (13), and this is where the conformal anomaly appears. Hence the absence of a conformal anomaly in the Gupta-Bleuler vacuum is due to cancellations between the positive norm and negative norm part of the field, and this renormalization seems to be very different from the other ones which all present this anomaly. Of course, it is not very surprising that our field which is conformally covariant in a rather strong sense does not present any conformal anomaly which, after all, can appear only by breaking this conformal invariance.

ACKNOWLEDGMENTS

The authors would like to thank R. Balbinot for helpful discussion and J-P. Gazeau and M. Takook for their interest in this work.

APPENDIX: THE FOCK SPACE

Let us recall that for any Hilbert space $\mathcal{A}$ one defines the corresponding Fock space $\underline{\mathcal{A}}$ by

$$\underline{\mathcal{A}} = \bigoplus_{n\geq 0} S_n(\mathcal{A}),$$

where $S_n(\mathcal{A})$ is the nth symmetrical tensor product of $\mathcal{A}$. When $\mathcal{A}$ is realized as a space $L^2(\mathbb{R}^d, d\mu)$, one can realize $S_n(\mathcal{A})$ as the space of square integrable symmetric functions of n variables on $\mathbb{R}^d$. The one-dimensional space $S_0(\mathcal{A})$ is written $|0\rangle$ and called the vacuum state. As is well-known the creators $a_k^\dagger$ and annihilators a_k create and annihilate, respectively, the mode ϕ_k. They can be realized on the Fock space in the following way. Let Ψ be in $S_n(\mathcal{A})$, we have

$$(a_k\Psi)(x_1,\ldots,x_{n-1}) = \sqrt{n}\int\overline{\phi_k(x)}\Psi(x,x_1,\ldots,x_{n-1})d\mu(x),$$

and

$$(a_k^\dagger\Psi)(x_1,\ldots,x_{n+1}) = \frac{1}{\sqrt{n+1}}\sum_{i=1}^{n}\phi_k(x_i)\times\Psi(x_1,\ldots,\check{x}_i,\ldots,x_{n+1}),$$

where $\check{x}_i$ means that this term is omitted. It is clear from this definition that one can define the annihilator and creator of any element ϕ of $\mathcal{A}$ by

$$(a(\phi)\Psi)(x_1,\ldots,x_{n-1}) = \sqrt{n}\int\overline{\phi(x)}\Psi(x,x_1,\ldots,x_{n-1})\times d\mu(x),$$

and

$$(a^\dagger(\phi)\Psi)(x_1,\ldots,x_{n+1}) = \frac{1}{\sqrt{n+1}}\sum_{i=1}^{n}\phi(x_i)\times\Psi(x_1,\ldots,\check{x}_i,\ldots,x_{n+1}).$$

On can see easily that a is antilinear as a function of ϕ and that $a^\dagger$ is linear. Moreover one has

$$[a(\phi),a^\dagger(\phi')] = \int\overline{\phi(x)}\phi'(x)d\mu(x).$$

This gives of course the usual commutation relations when applied to the modes ϕ_k.

Suppose now that we have a Krein space $\mathcal{H}$ equipped with a nonpositive inner product $\langle\,,\,\rangle$. There is on $\mathcal{H}$ a (nonunique) Hilbert space structure. We can now define the Fock space $\underline{\mathcal{H}}$

using this positive product and also define annihilators and creators using the nonpositive inner product. For instance if $\langle\, ,\rangle$ is as in Eq. (2), one has

$$(a(\phi)\Psi)(x_1,\ldots,x_{n-1})=\sqrt{n}i\int_{\rho=0}\overline{\phi(\rho,\alpha)}\overleftrightarrow{\partial}_\rho \times\Psi((\rho,\alpha),x_1,\ldots,x_{n-1})d\alpha$$

for any square integrable n-symmetric function Ψ. The creator is also defined by

$$(a^\dagger(\phi)\Psi)(x_1,\ldots,x_{n+1})=\frac{1}{\sqrt{n+1}}\sum_{i=1}^{n+1}\phi(x_i)\times\Psi(x_1,\ldots,\check{x}_i,\ldots,x_{n+1}).$$

These operators have very similar properties as the usual ones (see [16] for details), in particular they are conjugate to one of the other with respect to $\langle\, ,\rangle$ and one has

$$[a(\phi),a^\dagger(\phi)]=\langle\phi,\phi'\rangle.$$

Note that any unitary (for the nonpositive inner product) operator V on $\mathcal{H}$ yields a unitary operator $\underline{V}$ on $\underline{\mathcal{H}}$, but this operator is not always bounded (even though it is unitary) and is defined on a dense subspace of $\underline{\mathcal{H}}$ containing $\underline{\mathcal{H}}_0$, the space of finite length elements. One can easily verify that

$$\underline{V}_a(\phi)\underline{V}^{-1}=a(V\phi),$$

and a similar equality holds for $a^\dagger$ and for the so-called Segal field $\varphi_S(\phi)=a(\phi)+a^\dagger(\phi)$.

[1] B. Allen, Phys. Rev. D **32**, 3136 (1985).
[2] B. Allen and A. Folacci, Phys. Rev. D **35**, 3771 (1987).
[3] K. Kirsten and J. Garriga, Phys. Rev. D **48**, 567 (1993).
[4] J-P. Gazeau, J. Renaud, and M. Takook (in preparation).
[5] L. H. Ford and C. Pathinayake, Phys. Rev. D **39**, 3642 (1989).
[6] R. Floreanini, C. T. Hill, and R. Jackiw, Ann. Phys. (N.Y.) **175**, 345 (1987).
[7] T. S. Bunch and P. C. W. Davies, Proc. R. Soc. London **A360**, 117 (1978).
[8] F. Mandl and G. Shaw, *Quantum Field Theory* (Wiley, New York, 1993).
[9] B. Binegar, C. Fronsdal, and W. Heidenreich, J. Math. Phys. **24**, 2828 (1983).
[10] J-P. Gazeau, J. Math. Phys. **26**, 1847 (1985).
[11] S. De Bièvre and J. Renaud, J. Math. Phys. **35**, 3775 (1994).
[12] J. Bognar, *Indefinite Inner Product Spaces* (Springer-Verlag, Berlin, 1974).
[13] P. C. W. Davies and S. A. Fulling, Proc. R. Soc. London **A354**, 59 (1977).
[14] J. Bros, J-P. Gazeau, and U. Moschella, Phys. Rev. Lett. **73**, 1746 (1994).
[15] C. J. Isham, in *Differential Geometrical Methods in Mathematical Physics II*, edited by K. Bleuler *et al.*, Lecture Notes in Mathematics Vol. 676 (Springer, Berlin, 1978).
[16] M. Mintchev, J. Phys. A **13**, 1841 (1990).
[17] M. Sugiura, *Unitary Representations and Harmonic Analysis* (North-Holland, Amsterdam, 1990).
[18] O. Bratteli and D. W. Robinson, *Operator Algebras and Quantum Statistical Mechanics II* (Springer-Verlag, New York, 1981).
[19] N. D. Birrell and P. C. W. Davies, *Quantum Fields in Curved Space* (Cambridge University Press, Cambridge, England, 1982).