

Covariant quantization of membrane dynamics

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A Lorentz covariant quantization of membrane dynamics is defined, which also leaves unbroken the full three dimensional diffeomorphism invariance of the membrane. This makes it possible to understand the reductions to string theory directly in terms of the Poisson brackets and constraints of the theories. Two approaches to the covariant quantization are studied, Dirac quantization and a quantization based on matrices, which play a role in recent work on \mathcal{M} theory. In both approaches the dynamics is generated by a Hamiltonian constraint, which means that all physical states are “zero energy.” A covariant matrix formulation may be defined, but it is not known if the full diffeomorphism invariance of the membrane may be consistently imposed. The problem is the non-area-preserving diffeomorphisms: they are realized nonlinearly in the classical theory, but in the quantum theory they do not seem to have a consistent implementation for finite N . Finally, an approach to a genuinely background independent formulation of matrix dynamics is briefly described. [S0556-2821(98)00306-3]

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I. INTRODUCTION

This paper studies the quantization of a theory of the embeddings of membranes in d dimensional spacetime, using methods that preserve the manifest Lorentz invariance of the theory. This problem is of interest first of all because the quantum theory of the super-membrane in $10+1$ dimensions [1] is intimately associated with current attempts to construct \mathcal{M} theory, which is a conjectured non-perturbative formulation of string theory [2]. The quantum theory of the super-membrane in the light cone gauge is known [3]. This gauge fixed version of the theory is elegantly described in terms of a theory of $N \times N$ matrices. One of the issues the present paper attempts to answer is whether there is a covariant quantization of the membrane that is also expressed in terms of matrices.

This is an important question because the same matrix quantum theory has been conjectured to give a description of \mathcal{M} theory in the infinite momentum frame [4]. This conjecture is motivated by another intriguing fact, which is that the same matrix quantum theory can be obtained as the dimensional reduction of supersymmetric quantum mechanics to zero spatial dimensions.

It is of great interest to know to what extent this triple correspondence, between the supermembrane, the reduction of supersymmetric quantum mechanics and \mathcal{M} theory is restricted only to the light cone gauge and the infinite momentum frame, or has a larger range of validity. To answer this question we need formulations of these theories which are Lorentz covariant.

In the present paper we study these problems by making a canonical quantization of membrane dynamics which is Lorentz covariant in the background spacetime. This turns out to be straightforward, so long as no gauge conditions are fixed on the membrane itself. This leads to a canonical formulation

of the membrane dynamics that has both manifest Lorentz covariance and complete invariance under the diffeomorphism group of the membrane.¹ For completeness we include also the coupling to the three form field $A_{\alpha\beta\gamma}$.

Beginning with this classical formalism, which is set out in the next section, we then study two different approaches to the quantization: Dirac quantization and quantization in terms of matrices as in the light cone gauge fixed theory.

One limitation of the present study is that most of the results reported below hold for any dimension d , and we have not so far completed the extension to the supermembrane. The extension to the $10+1$ dimensional supermembrane is expected to be straightforward, and will be carried out elsewhere. While we expect further insight from this extension, several of the results already found do apply directly to the $10+1$ dimensional supermembrane and are of immediate relevance to the question of the Lorentz covariant form of \mathcal{M} theory. Among them are:

The dynamics is given in terms of a Hamiltonian constraint, which is similar to the Hamiltonian of the light-cone gauge fixed formalism, except that all d matrices are present and the Lorentz metric ties up the indices, as a result of the manifest Lorentz covariance.

There are two important consequences of the fact that the dynamics is given by a constraint. The first is that all states are zero energy and the second is that as a consequence physical states are expected to be nonrenormalizable in the naive inner product. A new inner product on physical states must be chosen. This has implications for the construction and interpretation of zero energy states.²

A Lorentz covariant quantization in terms of matrices is given at a formal level (which means that the limit $N \rightarrow \infty$ is not understood) in Sec. VI. There are two more matrices in the Lorentz covariant formulation than in the light cone

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¹This formalism has been sketched, but less completely, in [5].

²These are discussed in the light-cone gauge fixed theory in [6].

gauge fixed formalism, but these are balanced by two new sets of constraints. One is the Hamiltonian constraint, the other is a set of the two dimensional diffeomorphism group of the constant time surfaces of the membrane that is broken in the light-cone gauge fixed formalism. These are the area-non-preserving diffeomorphisms. If the theory is going to be formulated in terms of matrices these must be realized non-linearly. I show that this can be done at the classical level, which tells us how to do it formally in the quantum theory. But whether it can actually be done depends on issues of regularization and operator ordering that have not yet been resolved. It is likely that these gauge symmetries can only be consistently imposed in the $N \rightarrow \infty$ limit. This is the main difficulty that must be solved if there is to be a Lorentz covariant formulation in terms of matrices.

It has been known for sometime that there is a limit in which the membrane theory reduces to the string theory [7]. I show in Sec. III that this can be understood completely at the level of the phase space and constraints of the canonical theory.

In the special case of 2+1 dimensions we can find an exact physical state of the theory, which is an analogue of the Chern-Simons state [8] that plays an important role in quantum gravity [8–10]. I show in Sec. V how this may be interpreted as a semiclassical state associated to a certain class of solutions. This provides further evidence for the physical character of zero-energy states that are non-normalizable under the naive inner product.³

Before closing the Introduction we should remark that no Lorentz covariant theory can be more than a step on the road to the true, nonperturbative form of \mathcal{M} theory. Whatever it is, we know that \mathcal{M} theory cannot have its most fundamental formulation in terms of fields, strings, membranes or anything else moving in a classical spacetime manifold. This is so because \mathcal{M} theory must be a nonperturbative theory of quantum gravity and in any such theory the geometry of spacetime must emerge from a more fundamental quantum system that is not dependent on any background spacetime for its description. Such a theory may have gauge invariances such as diffeomorphism invariance or some extension of it; what it cannot have is any global symmetries that depend on the geometry of fixed background metrics.

Thus, the key question in \mathcal{M} theory is to find its background-independent, nonperturbative formulation. In Sec. VII a few steps towards such a theory is taken. I show that there are Lagrangian and Hamiltonian formulations of matrix dynamics in which the global symmetries are replaced by a matrix valued extension of diffeomorphism invariance. The relationship of this theory to \mathcal{M} theory is, however, unknown.

II. HAMILTONIAN REDUCTION WITHOUT GAUGE FIXING

We begin with the action for a 2+1 dimensional membrane \mathcal{M} embedded in d dimensional Minkowski spacetime in interaction with a three form field $A_{\alpha\beta\gamma}$:

$$S = \int_{\mathcal{M}} \sqrt{-g} + e \int_{\mathcal{M}} A. \quad (1)$$

Here $g = (\det g_{ij})$ where g_{ij} is the induced metric given in terms of the embedding coordinates $X^\alpha(t, \sigma^A)$ by

$$g_{ij} = \partial_i X^\alpha \partial_j X^\beta \eta_{\alpha\beta}. \quad (2)$$

Here $\eta_{\alpha\beta}$ is the Minkowski metric of the d dimensional background spacetime, so that $\alpha, \beta = 0, \dots, d-1$ and the three coordinates of the worldsheet, σ^i , $i = 0, 1, 2$ are broken down into $\sigma^0 = \tau$ and σ^A , $A = 1, 2$. $\eta^{\alpha\beta}$ will be used to raise and lower spacetime indices.

The action can then be written as

$$S = \int_{\mathcal{M}} \sqrt{\frac{1}{2} \dot{X}^\alpha \dot{X}^\beta G_{\alpha\beta}(X)} + e \int_{\mathcal{M}} A \quad (3)$$

where $G_{\alpha\beta}$ is a metric on the configuration space of the embeddings $X^\alpha(\sigma^A)$, which is given by

$$G_{\alpha\beta} = \eta_{\alpha\beta} h - 2h_{\alpha\beta} \quad (4)$$

where $h = \eta^{\alpha\beta} h_{\alpha\beta}$ and $h_{\alpha\beta}$ is the useful quantity

$$h_{\alpha\beta} = \langle X_\alpha, X_\gamma \rangle \langle X_\beta, X^\gamma \rangle \quad (5)$$

where $\langle X_\alpha, X_\gamma \rangle$ denotes the ‘‘manifold Poisson bracket,’’

$$\langle X_\alpha, X_\gamma \rangle = \epsilon^{AB} \partial_A X_\alpha \partial_B X_\gamma. \quad (6)$$

This of course has nothing to do with the phase space Poisson bracket we will shortly introduce.

We may note that the action is in the Barbour-Bertotti form [11] which shows that there is no intrinsic preferred time variable on the membrane. It also gives us an interpretation of the theory. We assume the topology of the membrane is fixed to be $\Sigma \times R$, with Σ a compact two manifold. Then the configuration space \mathcal{C} of the membrane consists of the embeddings of the two manifold Σ into d dimensional Minkowski spacetime. In coordinates this is given by $X^\alpha(\sigma)$. \mathcal{C} is an infinite dimensional manifold that has on it an indefinite metric given by $G(\delta_1 X^\alpha, \delta_2 X^\beta) = \int_\Sigma G_{\alpha\beta} \delta_1 X^\alpha \delta_2 X^\beta$. Then the action (3) tells us that when $A_{\alpha\beta\gamma} = 0$ the histories of the membrane trace out timelike geodesics of G .

This Barbour-Bertotti form also tells us how to construct the unconstrained Hamiltonian formulation, following the procedure used for that theory [12]. Without doing any gauge fixing we proceed directly to find the canonical momenta;⁴

$$p_\alpha(\sigma) = \frac{\partial S}{\partial \dot{X}_\alpha(\sigma)} = \frac{1}{\sqrt{-g}} G_{\alpha\beta} \dot{X}^\beta + e A_{\alpha\beta\gamma} \langle X^\beta, X^\gamma \rangle. \quad (7)$$

The elementary Poisson brackets are

$$\{X^\alpha(\tau, \sigma^A), p_\beta(\tau, \sigma^{A'})\} = \delta^2(\sigma, \sigma') \delta_\beta^\alpha. \quad (8)$$

³A similar state is studied in the 7 dimensional theory in [3].

⁴We use signature + - - - -

We find immediately three primary constraints, which follow only from the definition of the momenta (7). The diffeomorphism constraints are

$$D_A(\sigma) = (\partial_A x^\alpha) p_\alpha - e A_{\alpha\gamma\delta} \langle X^\gamma, X^\delta \rangle = 0 \quad (9)$$

and it is easy to see that acting on the embedding coordinates $X^\alpha(\sigma)$ they generate $\text{Diff}(\Sigma)$. Then there is the Hamiltonian constraint

$$\begin{aligned} \mathcal{H}(\sigma) &= \frac{1}{2} G^{-1\alpha\beta} (p_\alpha - e A_{\alpha\gamma\delta} \langle X^\gamma, X^\delta \rangle) (p_\beta - e A_{\beta\rho\sigma} \langle X^\rho, X^\sigma \rangle) \\ &- 1 = 0 \end{aligned} \quad (10)$$

where $G^{-1\alpha\beta}$ is the inverse of $G_{\alpha\beta}$, i.e., $G^{-1\alpha\beta} G_{\beta\gamma} = \delta_\gamma^\alpha$. It is straightforward to verify that the vanishing of these constraints follows from the definition of p_α .

$G^{-1\alpha\beta}$ may be constructed as a power series as

$$G^{-1\alpha\beta} = \frac{1}{h} \left(\eta^{\alpha\beta} + \frac{2h^{\alpha\beta}}{h} + \dots \right). \quad (11)$$

However, the nonlinear terms actually do not affect the evolution on the constraint surface because

$$h^{\alpha\beta} p_\beta = 2 \det(q) q^{AC} (\partial_A x^\alpha) D_C. \quad (12)$$

Here q_{AB} is the two dimensional induced metric on Σ defined by $q_{AB} = \partial_A X_\alpha \partial_B X^\alpha$. [Note that $h = \frac{1}{2} \det(q)$ is negative as the induced metric has Minkowskian signature.] This means that

$$\mathcal{H} = \mathcal{H}_0 + D_A R^{AB} D_B \quad (13)$$

where we have thus a new linear combination of constraints

$$\mathcal{H}_0 = \frac{1}{2} p_\alpha \frac{\eta^{\alpha\beta}}{h} p_\beta - 1 \quad (14)$$

and

$$R^{AB} = \frac{2}{h} q q^{AB} + \dots \quad (15)$$

Thus, on the diffeomorphism constraint surface $D_A = 0$ we have

$$\mathcal{H} \approx \mathcal{H}_0. \quad (16)$$

It is easy to verify that these constraints close to give an algebra very like the (2+1)-dimensional Arnowitt-Deser-Misner (ADM) algebra,

$$\{D(v), D(w)\} = D([v, w]) \quad (17)$$

with $D(v) = \int_\Sigma v^A D_A$. We also have

$$\{\mathcal{H}_0(N), \mathcal{H}_0(M)\} = \int_\Sigma (M \partial_A N - N \partial_A M) \frac{\mathcal{H}_0}{h} q q^{AB} D_B \quad (18)$$

and

$$\{D(v), \mathcal{H}_0(N)\} = \mathcal{H}_0(\mathcal{L}_v(N)). \quad (19)$$

Finally, it is convenient to densitize the constraint (14) to make the constraint polynomial, which gives us⁵

$$\tilde{\mathcal{H}}_0 = h \mathcal{H}_0 = \frac{1}{2} p_\alpha \eta^{\alpha\beta} p_\beta - h. \quad (20)$$

III. RELATION OF THE MEMBRANE TO STRING THEORY AT THE CLASSICAL LEVEL

It is easy to demonstrate within the canonical framework that string theory may be recovered from a particular limit of membrane theory. Let us consider a membrane whose spatial sections have topology $S^1 \times S^1$ with coordinates σ on the first S^1 and a periodic coordinate $\rho \in [-1, 1]$ on the second S^1 , which satisfies $\oint d\rho = 1$. We then take an ansatz for an evolution of a membrane in D dimensional spacetime of the form

$$Z^\alpha(\tau, \sigma, \rho) = Z^\alpha(\tau, \sigma) + \epsilon \rho W^\alpha(\tau, \sigma). \quad (21)$$

What we are doing is reducing the embeddings of the membrane \mathcal{M} to the embedding of a worldsheet \mathcal{S} defined by the condition $\epsilon = 0$. Z^α is the embedding coordinate of the worldsheet and it and W^α are then fields on the worldsheet. In the limit $\epsilon \rightarrow 0$ the membrane goes over into the worldsheet; we want to see if the dynamics of the membrane goes over into bosonic string theory in the same limit. To accomplish this we need a condition on the W^α so that they are restricted by the Z^α . To see what it should be we look at the action for the membrane in the presence of the ansatz (21) and in limit of small ϵ . We have, for $I, J, K = \tau, \sigma$ coordinates on the worldsheet,

$$\begin{aligned} \det(g_{ij}) &= \epsilon^2 (\det(q_{IJ}) W_\alpha W^\alpha - 2 \det(q_{IJ}) q^{KL} (W_\alpha \partial_K Z^\alpha) \\ &\quad \times (W_\beta \partial_L Z^\beta)) + O(\epsilon^3) \end{aligned} \quad (22)$$

where q_{IJ} is the induced metric on \mathcal{S} . (This should be distinguished from q_{AB} , the induced metric on the constant time surfaces of the membrane, which we called Σ .) Hence we see that the conditions we require are

$$W_\alpha W^\alpha = 1, W_\alpha \partial_\sigma Z^\alpha = W_\alpha \partial_\tau Z^\alpha = 0 \quad (23)$$

so that

$$\det(g_{ij}) = \epsilon^2 \det(q_{IJ}) + O(\epsilon^3). \quad (24)$$

Given the $Z^\alpha(\tau, \sigma)$ this gives three equations at each point (τ, σ) to determine the three $W^\alpha(\tau, \sigma)$. Once we have done this we have that

$$S^{\text{membrane}} = \epsilon \int d\tau d\sigma \sqrt{-\det(q_{IJ})} = \epsilon S^{\text{Nambu}}(Z^\alpha). \quad (25)$$

⁵We employ, inconsistently, the convention that density weights are marked with a tilde. Note that the restriction of $\tilde{\mathcal{H}}_0$ to the transverse coordinates is minus the usual light cone gauge Hamiltonian.

This correspondence goes through in the equations of motion as well. To show this we look at the definition of the momenta for the membrane (7), which gives us

$$p_\alpha = \epsilon \frac{(\partial_\sigma Z^\alpha)^2}{\sqrt{\det(q)}} \eta_{\alpha\beta} \dot{Z}^\beta + O(\epsilon) \quad (26)$$

On the other hand, the definition of the momentum of the string (in the presence of the momentum constraint, $\partial_\tau Z^\alpha \partial_\sigma Z_\alpha = 0$) is

$$p_\alpha^{str} = \frac{(\partial_\sigma Z^\alpha)^2}{\sqrt{\det(q)}} \eta_{\alpha\beta} \dot{Z}^\beta. \quad (27)$$

Thus, we have

$$p_\alpha^{str}(\tau, \sigma) = \lim_{\epsilon \rightarrow 0} \frac{1}{\epsilon} \oint_{S^1} d\rho p_\alpha(\tau, \sigma, \rho). \quad (28)$$

From the densitized Hamiltonian constraint (20), we have, for the potential energy of the membrane,

$$\mathcal{V}^{mem} = \langle X_\alpha, X_\beta \rangle \langle X^\alpha, X^\beta \rangle = \frac{2\epsilon^2}{l_{pl}^2} (\partial_\sigma Z^\alpha)^2 + O(\epsilon^3). \quad (29)$$

Thus, the Hamiltonian constraint of the membrane theory is of the form

$$\tilde{\mathcal{H}}_0 = \epsilon^2 \mathcal{H}^{string} + O(\epsilon^3) \quad (30)$$

where

$$\mathcal{H}^{string} = \frac{1}{2} p_\alpha^{str} p_\beta^{str} \eta^{\alpha\beta} + (\partial_\sigma Z^\alpha)(\partial_\sigma Z^\alpha) \eta_{\alpha\beta}. \quad (31)$$

Similarly, we have, for the diffeomorphism constraints,

$$\begin{aligned} D(v) &= \int d\sigma \oint_{S^1} d\rho v^A p_\alpha \partial_A X^\alpha(\sigma, \rho) \\ &= \epsilon \int d\sigma v^\sigma p_\alpha^{str} \partial_\sigma Z^\alpha(\sigma) = \epsilon D(v^\sigma)^{string}. \end{aligned} \quad (32)$$

Thus, we have shown that there is a limit in which a sector of the membrane theory goes over into the bosonic theory.

IV. REALIZATION OF THE TWO DIMENSIONAL DIFFEOMORPHISMS

One goal of the present work is to construct a quantization of the Lorentz covariant Hamiltonian dynamics described here in terms of a matrix representation similar to that used in the light-cone gauge fixed formalism. The main obstacle to doing this is that it is only the subgroup of the two dimensional diffeomorphism group that preserve the area element of the induced metric that are represented in the matrix formalism by $SU(N)$ transformations, in the limit of large N . This is fine for the light-cone gauge fixed formalism, because there the full diffeomorphism group of Σ has been broken down to the area preserving ones [3]. But if we want to quantize the covariant formalism we have to represent all of $\text{Diff}(\Sigma)$. In order to understand how to do this we must first

study how the non-area preserving diffeomorphisms act on the embedding coordinates and momenta.

To do this we split the vector fields v^A into the area preserving and area-non-preserving part, each of which is given by a scalar field. We call them a and n for area preserving and non-area preserving. The decomposition is

$$v^A = \frac{1}{\sqrt{q}} \epsilon^{AB} \partial_B a + q^{AB} \partial_B n. \quad (33)$$

We have

$$\mathcal{L}_v \sqrt{q} = \partial_A \sqrt{q} v^A = \sqrt{q} \nabla^2 n \quad (34)$$

where

$$\nabla^2 = \frac{1}{\sqrt{q}} \partial_A \sqrt{q} q^{AB} \partial_B \quad (35)$$

showing that a parametrizes the area preserving subgroup of $\text{Diff}(\Sigma)$, which we call $\text{Diff}_{\sqrt{q}}(\Sigma)$ while n parametrizes the coset $\text{Diff}(\Sigma)/\text{Diff}_{\sqrt{q}}(\Sigma)$.

The action of the area preserving part defines a vector density $\tilde{a}^A = \epsilon^{AB} \partial_B a$ whose action on functions is embedded in the Poisson algebra of functions

$$\tilde{a}^A \partial_A \phi = \langle \phi, a \rangle. \quad (36)$$

Thus, the map $\phi: \tilde{a}^A \rightarrow a$ of divergence free vector fields to scalars defines an embedding of the Lie algebra of area preserving diffeomorphisms into the Poisson algebra on Σ given by \langle, \rangle . It is this Poisson algebra which is mapped to $SU(N)$ in the limit $N \rightarrow \infty$ in the quantization of the membrane in which the embedding coordinates $X^\alpha(\tau, \sigma, \rho)$ are mapped to matrices X_j^{al} [3].

What about the non-area preserving part? This is given also by functions, but the action does not map linearly into the Poisson algebra on Σ . However, we can find a nonlinear realization of the generators of $\text{Diff}(\Sigma)/\text{Diff}_{\sqrt{q}}(\Sigma)$ on the embedding coordinates X^α and their conjugate momenta \tilde{p}_α . If we consider the undensitized non-area preserving vector field,

$$N^A = q^{AB} \partial_B n, \quad (37)$$

then using the definition of the induced metric we have, for any function f and density $\tilde{\omega}$ on Σ ,

$$\mathcal{L}_N f = \frac{\langle f, X^\alpha \rangle \langle n, X_\alpha \rangle}{\langle X_\mu, X_\nu \rangle \langle X^\mu, X^\nu \rangle} \quad (38)$$

$$\mathcal{L}_N \tilde{\omega} = \partial_A (\tilde{\omega} N^A) = \left\langle \tilde{\omega} \frac{\langle n, X_\alpha \rangle}{\langle X_\mu, X_\nu \rangle \langle X^\mu, X^\nu \rangle}, X^\alpha \right\rangle. \quad (39)$$

These equations apply, in particular to the X^α and \tilde{p}_α (which is, of course, a density on Σ .) The first gives a nonlinear realization of $\text{Diff}_q(\Sigma)/\text{Diff}(\Sigma)$:

$$\mathcal{L}_N X^\beta = \frac{\langle X^\beta, X^\alpha \rangle \langle n, X_\alpha \rangle}{\langle X_\mu, X_\nu \rangle \langle X^\mu, X^\nu \rangle}. \quad (40)$$

The second gives the transformation of the momenta

$$\mathcal{L}_N \tilde{p}_\alpha = \partial_A (\tilde{p}_\alpha N^A) = \left\langle \tilde{p}_\alpha \frac{\langle n, X_\beta \rangle}{\langle X_\mu, X_\nu \rangle \langle X^\mu, X^\nu \rangle}, X^\beta \right\rangle. \quad (41)$$

As these are diffeomorphisms, by (19) they must leave the constraint surface $\mathcal{H}_0 = 0$ invariant. Thus, the theory has *two* gauge invariances, each given by a mapping of $\text{Diff}(\Sigma)$ into the algebra of functions on Σ . The first is the linear action (36) of the area preserving transformations. The second is the nonlinear representation of $\text{Diff}(\Sigma)/\text{Diff}_{\bar{q}}(\Sigma)$ which is given by (40) and (41). Both must be represented in a quantization of the covariant theory.

V. DIRAC QUANTIZATION

We can now discuss the quantization of the membrane theory. I will discuss briefly two methods of quantization. We start with Dirac quantization. This is straightforward, but makes so far no connection with the matrix models. We do find one interesting result which is that in the particular case of 2+1 dimensions we can find an exact physical state that describes the reduction of the membrane to the string. After describing this we will turn to the question of the existence of a matrix representation of the covariant membrane.

Under the procedure of Dirac quantization one begins with some kinematical hilbert space \mathcal{H}^{kin} and establishes the canonical commutation relations associated to the Poisson brackets (8). The natural representation to use is the configuration space representation $\Psi[X^\alpha]$, where the kinematical configuration space \mathcal{C}^{kin} consists of maps $X^\alpha(\sigma, \rho): \Sigma \rightarrow M^N$ from the two surface Σ to N dimensional Minkowski space-time. The operator assignments are the natural ones in which

$$p_\alpha \Psi = i\hbar \frac{\delta \Psi}{\delta X^\alpha}. \quad (42)$$

On this we impose first the diffeomorphism constraints (9) in the form

$$\hat{D}(v)\Psi[X^\alpha] = \int_\Sigma (\mathcal{L}_v X^\beta) \frac{\delta \Psi}{X^\beta} [X^\alpha]. \quad (43)$$

This is solved in general by the requirement that

$$\Psi[X^\alpha] = \Psi[\phi \circ X^\alpha] \quad (44)$$

where $\phi \in \text{Diff}(\Sigma)$ so that the states become functionals on $\mathcal{C}^{diff} = \mathcal{C}^{kin}/\text{Diff}(\Sigma)$. The problem is then to invent a regularization so that the solutions to

$$\mathcal{H}(N)\Psi = 0 \quad (45)$$

can be extracted. Once this is done a physical inner product is to be picked on the space of solutions to both sets of constraints.

In particular cases some exact solutions can be found. For example, for the case of $N=3$ we can split the Hamiltonian into self-dual and anti-self-dual parts

$$H_0 = \frac{1}{2} P_\alpha^- P^{+\alpha} \quad (46)$$

where

$$P_\alpha^\pm = p_\alpha \pm \epsilon_{\alpha\beta\gamma} \langle X^\beta, X^\gamma \rangle. \quad (47)$$

An analogue of the Chern-Simons state for quantum gravity [8] can be construct using

$$Y[X^\alpha] = \frac{1}{3} \int_\Sigma \epsilon_{\alpha\beta\gamma} X^\alpha \langle X^\beta, X^\gamma \rangle \quad (48)$$

so that

$$\frac{\delta Y}{\delta X^\alpha} = \epsilon_{\alpha\beta\gamma} \langle X^\beta, X^\gamma \rangle. \quad (49)$$

If we define the ‘‘Chern-Simons state’’ by

$$\Psi_{CS}[X^\alpha] = e^{iY[X^\alpha]} \quad (50)$$

it follows directly that

$$P_\alpha^+ \Psi_{CS}[X^\alpha] = 0. \quad (51)$$

Since this state is manifestly invariant under $\text{Diff}(\Sigma)$ this is a well defined physical state.

It may be objected that the state is not-normalizable. However, this is only the case in a naive Fock inner product, which might be established on the kinematical state space \mathcal{H}^{kin} . This objection rules out the consideration of an analogous state in the case of Yang-Mills theory. However, this objection does not hold in the case of theories whose dynamics is governed by constraints, because *all* physical states, being zero energy states of the Hamiltonian *constraint* are expected to be non-normalizable in this kinematical inner product. The inner product on physical states must be constructed on the space of solutions to the constraints. Since we do not have a full space of physical states we are not yet in a position to do this, on the other hand, at the present stage there can be no objection to taking the state $\Psi_{CS}[X^\alpha]$ to be physical as a working hypothesis and seeing where it leads. We may note that in the case of quantum gravity there are good arguments that the analogous state is in fact the full nonperturbative vacuum state for the theory in the presence of a cosmological constant. In this case both the exact Planck scale description and semiclassical limit are understood. For small cosmological constant the state has a semiclassical interpretation which describes fluctuations around de Sitter spacetime [8,10], while the exact description of the state is as the Kauffman invariant of quantum spin networks at level $k = 6\pi/G^2\Lambda$ [13].

In fact the Chern-Simons state in the present context must also have a semiclassical interpretation, since it is of the form of a WKB state. To find that interpretation we note that treating $Y[X^\alpha]$ as a Hamilton Jacobi function we have

$$p_\alpha = \frac{\partial Y}{\partial X^\alpha} = \epsilon_{\alpha\beta\gamma} \langle X^\beta, X^\gamma \rangle. \quad (52)$$

We may note that this satisfies the classical Hamiltonian and momentum constraints. To find the velocities we may use the time defined by the densitized Hamiltonian constraint (20), so that

$$\dot{X}_\alpha = \{X_\alpha, \widetilde{\mathcal{H}}_0\} = p_\alpha = \epsilon_{\alpha\beta\gamma} \langle X^\beta, X^\gamma \rangle. \quad (53)$$

The state (50) then is a semiclassical state that describes fluctuations around the solutions to this equation.

We may note that a similar state can be constructed in seven dimensions using the octonions [3], by replacing $\epsilon_{\alpha\beta\gamma}$ in (50) by the structure constants for the seven imaginary octonions. In fact, the octonions can be used to give a compact expression to M(atr)ix theory, which will be described in [14].

VI. IS THERE A MATRIX FORMULATION OF THE COVARIANT THEORY?

It would be very convenient if the regularization of the light cone gauge fixed theory in terms of $N \times N$ Hermitian matrices could be carried out as well for the covariant version of the theory. To investigate this we may consider states of the form $\Psi[\hat{X}^\alpha]$ where the \hat{X}^α are $d \times N \times N$ Hermitian matrices in d dimensional spacetime. The momenta \widetilde{p}_α are then represented as $\partial/\partial\hat{X}^\alpha$. The algebra of functions on Σ under \langle, \rangle is then taken over to the matrix algebra, so that $\langle X^\alpha, X^\beta \rangle \rightarrow [\hat{X}^\alpha, \hat{X}^\beta]$. The area element preserving subgroup of the diffeomorphism group $\text{Diff}_{\sqrt{q}}(\Sigma)$ then map to the group $\text{SU}(N)$, which becomes the gauge group.

This is sufficient for the light cone gauge theory, where the area element preserving diffeos are the only gauge symmetry, but will it work for the covariant formulation, where the gauge symmetry is expanded to the full 3 dimensional diffeomorphism group of the membrane? To do this we must implement on the $\text{SU}(N)$ invariant functionals of the membrane two additional constraints, which are, formally, the Hamiltonian constraint

$$\hat{\mathcal{H}}_0 \Psi[\hat{X}^\mu] = \left[-\frac{\partial^2}{\partial \hat{X}^\alpha \partial \hat{X}^\alpha} + [\hat{X}^\alpha, \hat{X}^\beta][\hat{X}_\alpha, \hat{X}_\beta] \right] \Psi[\hat{X}^\mu] = 0 \quad (54)$$

and the area nonpreserving part of the diffeomorphisms of Σ . We may note that the counting is right; this formalism has two more matrix degrees of freedom than the light cone gauged fixed theory, but these are balanced by two additional matrix valued constraints. Presumably the Hamiltonian constraint can be implemented, as it differs only by some signs from the Hamiltonian operator that has been studied in the light cone gauge fixed theory. The difficulty is with the remaining non-area preserving diffeomorphisms; at present the author is unaware of any method for implementing them.

To have a chance of succeeding we can multiply the vector field by \hbar to get polynomial transformation laws. [This step is implicit in writing the area preserving diffeomorphisms in terms of $\text{SU}(N)$ transformations, so we use it here as well.] Using symmetric ordering to preserve the Hermiticity of the matrices we find transformation laws of the form

$$\delta \hat{X}^\mu = [\hat{n}, \hat{X}^\alpha][\hat{X}_\alpha, \hat{X}_\mu] + [\hat{X}_\alpha, \hat{X}_\mu][\hat{n}, \hat{X}^\alpha] \quad (55)$$

$$\delta \hat{p}_\alpha = [\hat{p}_\alpha[\hat{n}, \hat{X}_\mu], \hat{X}^\nu] + [[\hat{n}, \hat{X}_\mu], \hat{p}_\alpha, \hat{X}^\nu]. \quad (56)$$

Equivalently, up to an $\text{SU}(N)$ transformation these can be replaced by a corresponding set of double commutator transformations:

$$\delta \hat{X}^\mu = [[\hat{n}, \hat{X}_\alpha], \hat{X}^\mu] \hat{X}^\alpha + \hat{X}^\alpha [[\hat{n}, \hat{X}_\alpha], \hat{X}^\mu]. \quad (57)$$

Acting on quantum states these should generate the constraint,

$$\hat{\mathcal{D}}[\hat{n}] \Psi[\hat{X}^\rho] = ([[\hat{n}, \hat{X}_\alpha], \hat{X}^\mu] \hat{X}^\alpha + \hat{X}^\alpha [[\hat{n}, \hat{X}_\alpha], \hat{X}^\mu]) \frac{\delta \Psi[\hat{X}^\rho]}{\delta \hat{X}^\mu}. \quad (58)$$

Unfortunately, at least for finite N , these do not appear to generate a symmetry of the Hamiltonian constraint (54). It seems likely that if these symmetries can be implemented exactly, it will be only in the $N \rightarrow \infty$ limit.⁶ It is also possible to speculate that this additional symmetry has something to do with the ‘‘hidden’’ symmetries in supergravity and string theory, however there is little more that can be said unless a way is found to implement them in the quantum theory.

VII. TOWARDS A GENUINELY NONPERTURBATIVE FORM OF \mathcal{M} THEORY

Before closing this paper, we turn briefly to the key problem of finding a fundamental, background independent formulation of \mathcal{M} theory. Such a formulation may have no dependence on a particular classical spacetime. Nor can it have any global symmetries, as those arise in general relativity and other gravitational theories only as symmetries of particular solutions. A theory that has diffeomorphism invariance, or some extension of it as the fundamental gauge symmetry cannot have any global symmetries associated with particular spacetime manifolds.

This follows from general arguments about the role of diffeomorphism invariance in theories in which the spacetime geometry is a dynamical field. Other arguments, coming directly from string theory lead to the same conclusion. For example, T duality and the other dualities tell us that string theories defined as expansions around different spacetime backgrounds are sometimes completely equivalent to each other [15]. There are further arguments that these dualities are to be considered to be gauge symmetries of \mathcal{M} theory. In that case the gauge invariant description cannot be given in terms of fixed classical backgrounds.

Whatever else it has accomplished, the studies of nonperturbative quantum gravity [16–18,22] and topological quantum field theory, and their inter-relations [22] have shown us that it is possible to construct background independent, diffeomorphism invariant quantum field theories, even to the level of mathematical rigor reached by ordinary constructive quantum field theory [19]. This should give us the confidence to attempt the same for \mathcal{M} theory.

One strategy to construct such a theory would be to construct a dynamics of $N \times N$ matrices which has no global

⁶Djordje Minic has kindly informed me that Hidetoshi Awata and he have considered similar issues in the context of a covariant Lagrangian matrix theory.

symmetries, but instead a group of gauge symmetries larger than $SU(N)$. The simplest way to do this is to find an action which is a functional of a set of matrices that does not depend on a background metric. This is easy to do, as the following example illustrates.

A theory depending on d , $N \times N$ matrices, X_a , $a = 1, \dots, d$ that does not depend on a background metric is described by the action,

$$S^d = \epsilon^{a_1 \dots a_d} \text{Tr}[X_{a_1} \dots X_{a_d}]. \quad (59)$$

This vanishes trivially for even d , as a result of the Jacobi identity. This simple fact is analogous to the fact that in the continuum

$$S^d_{cont} = \int \text{Tr}[F \wedge F \dots \wedge F] \quad (60)$$

is a topological invariant, as the Bianchi identity reduces to the Jacobi identity of the matrices. But for odd d the action S^d does not vanish. Instead, one has a kind of matrix analogue of Chern-Simons theory. Interestingly, higher dimensional Chern-Simons theories have local degrees of freedom [24,25], and the structure of their constraints and equations of motion can be intricate.

For odd $d = 2n + 1$ the equations of motion are

$$\frac{\delta S^{2n+1}}{\delta X_a} = \epsilon^{ab_1 \dots b_{2n}} X_{b_1} \dots X_{b_{2n}} = 0. \quad (61)$$

The solution spaces of these theories include the solution spaces of the background dependent theories in which $[X_a, X_b] = 0$ for all a, b . At the same time, the global symmetry of the background dependent matrix models, $X_a \rightarrow X'_a = X_a + V_a I$, where I is the identity matrix and V_a 's are constants, is replaced by a *gauge invariance*

$$X_a \rightarrow X'_a = X_a + V_a(X) I \quad (62)$$

where the $V_a(X)$ are now *functions* on the space of matrices. To see this note that

$$\delta S^{2n+1} = \epsilon^{ab_1 \dots b_{2n}} V_a \text{Tr}[X_{b_1} \dots X_{b_{2n}}] = 0. \quad (63)$$

We can see these features as well from the canonical formalism. We may introduce a continuous time by representing explicitly the time dimension. The $2n + 1$ component we represent as time, so we write $X_{2n+1} = A_0$. We then have

$$S'^{2n+1} = \int ds \epsilon^{b_1 \dots b_{2n}} V_a \text{Tr}[(\mathcal{D}_0 X_{b_1}) X_{b_2} \dots X_{b_{2n}}] = 0. \quad (64)$$

The canonical momenta are

$$\Pi^a = \epsilon^{ab_2 \dots b_{2n}} [X_{b_2} \dots X_{b_{2n}}]. \quad (65)$$

There is a gauge constraint,

$$G = [X_a, \Pi^a]. \quad (66)$$

In addition, there are $2n$ constraints,

$$D^a = \text{Tr} \Pi^a = 0 \quad (67)$$

that follow from the vanishing of S^{2n} . These generate the $2n$, ‘‘spatial’’ components of the gauge symmetry (62).

More structure may be introduced by following the strategy of CDJ [20] and introducing Lagrange multipliers into the action. This will be discussed elsewhere.

Of course, this is not the only possible approach to a background independent dynamics of matrices. The new path integral formulations of spin network evolution may be interpreted as a dynamics for matrices, if the spin networks are taken to be not embedded in any background manifold, as is advocated in [21]. A general formulation of background independent theories of quantum geometry, which may have some relevance to membrane dynamics is also proposed in [22,23]. Of course, the relevance of any of these models to \mathcal{M} theory remains to be shown.

VIII. CONCLUSIONS

Put briefly, we have made some progress towards a covariant formulation of membrane dynamics. The crucial issues left so far unsolved are

The choice of the physical inner product for the physical states, which is unlikely to be the same as in the light-cone gauge fixed theory. This opens up the issue of the physical interpretation of the quantum states of the membrane as well as the consistency of a non-perturbative quantization of the membrane in any dimensions.

The possibility of a matrix representation of the covariant theory rests on the implementation of a nonlinear realization of the non-area preserving diffeomorphisms of the membrane. This gauge symmetry, together with the Hamiltonian constraint, is necessary to balance the increase in the number of matrices from $d - 2$ to d which moving from the light cone gauge to a covariant formalism requires.

Further work in this subject will also include the extension to the supermembrane, which will involve also the study of special dimensions such as $d = 10 + 1$. But the results found so far in this general study tell us what those more specific studies will have to accomplish if there is to be a Lorentz covariant formulation of \mathcal{M} theory arising from the dynamics of membranes.

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