Hypothesis of path integral duality. I. Quantum gravitational corrections to the propagator

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The action for a relativistic free particle of mass m receives a contribution $-m\mathcal{R}(x,y)$ from a path of length $\mathcal{R}(x,y)$ connecting the events x^i and y^i. Using this action in a path integral, one can obtain the Feynman propagator for a spinless particle of mass m in any background spacetime. If one of the effects of quantizing gravity is to introduce a minimum length scale L_P in the spacetime, then one would expect the segments of paths with lengths less than L_P to be suppressed in the path integral. Assuming that the path integral amplitude is invariant under the ''duality'' transformation $\mathcal{R}\rightarrow L_P^2/\mathcal{R}$, one can calculate the modified Feynman propagator in an arbitrary background spacetime. It turns out that the key feature of this modification is the following: The proper distance $(\Delta x)^2$ between two events, which are infinitesimally separated, is replaced by $\Delta x^2+L_P^2$; that is, the spacetime behaves as though it has a ''zero-point length'' of L_P. This equivalence suggests a deep relationship between introducing a ''zero-point length'' to the spacetime and postulating invariance of path integral amplitudes under duality transformations. In Schwinger's proper time description of the propagator, the weightage for a path with proper time s becomes $m(s+L_P^2/s)$ rather than as ms. As to be expected, the ultraviolet behavior of the theory is improved significantly and divergences will disappear if this modification is taken into account. Implications of this result are discussed. [S0556-2821(98)03810-7]

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I. INTRODUCTION AND SUMMARY

It has been conjectured for a long time that the spacetime structure at very small scales [close to $L_P\equiv(G\hbar/c^3)^{1/2}$] will be drastically affected by quantum gravitational effects. Since any quantum field has virtual excitations of arbitrary high energy—which probe arbitrary small scales—it follows that the conventional quantum field theory can only be an approximate description, valid at energies far smaller than Planck energies. The ''correct'' description of nature has to take into account the quantum nature of the spacetime geometry and should reduce to the conventional description at low energies. Can we say anything about the kind of modifications quantum gravitational effects will introduce into the description of other quantum fields? I investigate some aspects of this question in this paper.

Let us focus attention on a scalar field $\phi(x)$ of mass m in a D-dimensional Euclidean spacetime. Eventually we are interested (probably) in the case of $D=4$ Lorentzian spacetime, which can be achieved by suitable analytic continuation. Since all matter generates and couples to gravity, there is no such thing as a *free* scalar field; at the least, one should grant the fact that the scalar field is coupled to its own self-gravity. So, in general, the action $A[\phi,g_{ik}]$ describing the system will be a functional of both $\phi(x)$ *and* the metric $g_{ik}(x)$ of the spacetime. The full quantum field theory of such a system will be based on a formal path integral such as

$$\mathcal{G}=\sum_{g,\phi}\exp(-A[g,\phi]). \tag{1}$$

The Feynman propagator $G_F(x,y)$ for the scalar field (and higher-order n-point functions, all of which can be obtained from a path integral description) will contain information about the quantum mechanical properties of ϕ.

To the extent we can ignore the gravitational coupling, we can have a free scalar field in flat spacetime and the evaluation of $\mathcal{G}$ is trivial. At the next level, if we treat the background spacetime as curved but classical, one can ignore the sum over metrics in Eq. (1) and construct the propagator $G_F(x,y|g)$ in a given background metric g_{ik}. We do not have a closed form for this in an arbitrary background because the partial differential equation for $G_F(x,y|g)$ has no closed form solution in an arbitrary background. What is more important, *such a propagator cannot be trusted when* $(x-y)^2<L_P^2$ *since the quantum gravitational fluctuations of the background geometry cannot be ignored at these scales* and our approximation of working with a fixed background g_{ik} breaks down. We need to know how the quantum fluctuations of the metric affect the propagator $G_F(x,y|g)$ at these scales.

This is quite a different question from the one usually addressed on the subject of quantum fields in curved spacetime in which one worries how the quantum nature of the scalar field affects the background geometry (''back reaction''). Such an issue can be tackled, for example, by integrating out ϕ in Eq. (1) and obtaining an effective action for gravity, say. In contrast, we are interested in how the quantum fluctuations of g_{ik} affect the quantum propagator for the scalar field. Formally, if we write $g_{ik}=\overline{g}_{ik}+h_{ik}$, where $\overline{g}_{ik}$ is the, average, large scale spacetime metric and h_{ik} are the small scale quantum fluctuations, then we are interested in the effect of summing over the fluctuations h_{ik} in Eq. (1) to get low-energy quantum theory for the scalar field in the background metric g_{ik}. The resulting propagator $G_F(x,y|\overline{g})$, for example, can be thought of as the one found by averaging $G_F(x,y|g)$ over the quantum fluctuations in g_{ik} around $\overline{g}_{ik}$. In particular, $\overline{g}_{ik}$ could just be a flat spacetime metric η_{ik}.

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Even in this case, we expect the quantum fluctuations of gravity to modify the propagator for $(x-y)^2<L_p^2$ [or, in momentum space, for $(p^2+m^2)L_p^2>1$]. The concept of a free quantum field is an approximate, lower-energy, notion and we do have to change it for $(x-y)^2<L^2$. (In fact, even the description in terms of a field may be inadequate at short distances and we may need string theory or models based on Ashtekar variables.) Can we capture the key effects, quantum gravitational fluctuations, by invoking some general principle?

To address this question, it is convenient to write $G_F(x,y|g)$ in an alternative form. We know that the propagator in a given background metric can be expressed in two equivalent forms as

$$G_F(x,y|g)=\sum_{\text{paths}} e^{-m\mathcal{R}(x,y)}=\int_0^\infty d\tau e^{-m^2\tau}\int \mathcal{D}x \times\exp\left(-\frac{1}{4}\int_0^\tau g_{ik}\dot{x}^i\dot{x}^k d\eta\right). \tag{2}$$

In the first form, $\mathcal{R}(x,y|g)$is the proper length of a path connecting the events x and y, calculated with the background metric g_{ik}, and the sum is over all paths. The action $m\mathcal{R}$ has a square root in it but can be evaluated by standard lattice techniques (see the next section). It is also possible to show by these methods that the result is equivalent to the second expression. This expression, which is originally due to Schwinger, has a simple physical interpretation. By rescaling the time variable from η to $s\equiv m\eta$ and τ to $\tau'\equiv m\tau$ we can change the factor $\exp(-m^2\tau)$ to $\exp(-m\tau')$ and the path integral kernel to

$$K(x,y,\tau'|g)\equiv\int \mathcal{D}x\ \exp\left(-\frac{m}{4}\int_0^{\tau'} g_{ik}\dot{x}^i\dot{x}^k ds\right). \tag{3}$$

This kernel can be thought of as the probability amplitude for a particle to propagate from x to y in a proper time interval τ' in a given background spacetime. The amplitude for propagation with energy E (in the rest frame) is given by the Fourier transform of $K(x,y,\tau'|g)$ in the time variable τ', with respect to E in Lorentzian space; in the Euclidean space, it will be a Laplace transform. Setting the energy in the rest frame equal to m we obtain the expression in Eq. (2). (The physical interpretation of these expressions and their relationship to Jacobi action, etc., are explored in detail in Ref. [1]).

The above expressions assume that we have a classical background spacetime with a given, fixed, metric. As we said before, such a description is bound to break down when $(x-y)^2<L_P^2$. More generally, Eqs. (2), and (3) sum over paths which probe arbitrarily small scales at which the metric fluctuations are likely to be large. These fluctuations will affect the propagator $G_F(x,y|g)$ and will modify it. If we again write g_{ik} as $(\overline{g}_{ik}+h_{ik})$ and average over the fluctuations h_{ik}, then the effective propagator will be

$$G_F(x,y|\overline{g})\equiv\sum_h G_F(x,y|\overline{g}+h)\mathcal{P}(h), \tag{4}$$

where $\mathcal{P}(h)$ is the amplitude for a fluctuation h_{ik}, which will depend on the ''correct'' theory of gravity. We are interested in knowing the modified form of the propagator.

It is, of course, impossible to ''derive'' the correct propagator which takes into account quantum fluctuations of a metric. To do so, one needs a workable model for quantum gravity which will give us $\mathcal{P}(h)$. Since we do not have this, the best one can do is to take hints from various models for quantum gravity and come up with an ansatz [6]. This is what I propose to do along the following lines.

The strongest hint is the existence of the length $L_P\equiv(G\hbar/c^3)^{1/2}$, which is expected to play a vital role in the ''ultimate'' theory of quantum gravity. Simple thought experiments indicate that it is not possible to devise experimental procedures which will measure lengths with an accuracy greater than about $\mathcal{O}(L_P)$ [2]. This result suggests that one could think of the Planck length as some kind of ''zero-point length'' of spacetime. In some simple models of quantum gravity, L_P^2 does arise as a mean square fluctuation to spacetime intervals, due to quantum fluctuations of the metric [3]. In more sophisticated approaches, such as models based on string theory or Ashtekar variables, similar results arise in one guise or the other (see e.g., [4,5,7,9–12]). The existence of a fundamental length implies that processes involving energies higher than Planck energies will be suppressed and the ultraviolet behavior of the theory will be improved. All sensible models for quantum gravity provide some mechanism for good ultraviolet behavior, essentially through the existence of a fundamental length scale. One direct consequence of such an improved behavior will be that the Feynman propagator (in momentum space) will acquire a damping factor for energies larger than the Planck energy.

If the ultimate theory of quantum gravity has a fundamental length scale built into it, then it seems worthwhile to use this principle as the starting point to obtain a glimpse of the modifications introduced by quantum gravity effects at lower energies, provided we can introduce the quantum gravity effects through some powerful, general principle.

With this motivation in mind, let us ask how the propagation amplitude could be modified if there exists a fundamental zero-point length to the spacetime. In Eq. (2), the weightage given for a path of length $\mathcal{R}$ is $\exp(-m\mathcal{R})$ which is a monotonically decreasing function of $\mathcal{R}$. The existence of a fundamental length L_P would suggest that paths with length $\mathcal{R}\ll L_P$ should be suppressed in the path integral. This can, of course, be done in several different ways by arbitrarily modifying the expression in Eq. (2). In order to make a specific choice I shall invoke the following ''principle of duality.'' I will postulate that the weightage given for a path should be invariant under the transformation $\mathcal{R}\to L_P^2/\mathcal{R}$. Since the original path integral has the factor $\exp(-m\mathcal{R})$, we have to introduce the additional factor $\exp(-mL_P^2/\mathcal{R})$. We therefore modify Eq. (2) to

$$G_F(x,y|g)=\sum\ \exp\left[-m\left(\mathcal{R}+\frac{L_P^2}{\mathcal{R}}\right)\right]. \tag{5}$$

I will take this to be the basic postulate arising from the ''correct'' theory of quantum gravity. It may be noted that the ''principle of duality'' invoked here is similar to that which arises in string theories [7,9–12]. (It should, however, be stressed that the principle of duality used in string theories is not identical to this postulate, and nor is our postulate derivable from string theory. In the strict sense, the duality in the string theory operates in the internal space.) In fact we may think of Eq. (5) as a result of performing the averaging on the right-hand side of Eq. (4). Since I do not know $\mathcal{P}(h)$, this result is a postulate at present. It is also the simplest realization of duality for a free particle; we have demanded that the existence of a weightage factor $\exp(-ml)$ necessarily require the existence of another factor $\exp(-mL_P^2/l)$. We shall now study the consequences of the modifications we have introduced.

To do this we need to evaluate the path integral in Eq. (5). It turns out that this can indeed be done (see Sec. III) and the result is quite simple to state:

$$G_F(x,y|g)=\sum \exp\left[-m\left(\mathcal{R}+\frac{L_P^2}{\mathcal{R}}\right)\right]$$
$$=\int_0^\infty d\tau \exp\left(-m^2\tau-\frac{L_p^2}{\tau}\right)K(x,x',\tau|\overline{g}). \quad (6)$$

Our modification merely changes the weightage given to a path of proper time τ from $\exp(-m^2\tau)$ to $\exp(-m^2\tau-L_P^2/\tau)$ in Schwinger's prescription.

This result has an interesting interpretation. It is well known that the kernel $K(x,y;\tau|g)$ has a DeWitt-Schwinger expansion of the form

$$K(x,y;\tau|\overline{g})=\left(\frac{1}{4\pi\tau}\right)^{D/2}\exp\left(-\frac{(x-y)^2}{4\tau}\right)[1+\cdots], \quad (7)$$

where the ellipsis represents metric-dependent corrections. Using Eqs. (7) in Eq. (6) we can write our propagator as

$$G_F(x,y|\overline{g})=\int_0^\infty d\tau e^{-m^2\tau}\left(\frac{1}{4\pi\tau}\right)^{D/2}\exp\left(-\frac{(x-y)^2+4L_p^2}{4\tau}\right)\times[1+\cdots]. \quad (8)$$

Thus the net effect of our modification is to add a "zero-point length" $4L_p^2$ to $(x-y)^2$ in the exponential, thereby modifying the leading singular factor. *The postulate of duality used in the path integral is identical to the postulate of such a zero-point length*. This is one of the key results of this paper and—as far as I can see—this connection is far from obvious.

In the case of flat background spacetime, the terms indicated by the ellipsis vanish and the propagator is given by

$$G_F(x)=\left(\frac{1}{4\pi}\right)^{D/2}\int_0^\infty \frac{ds}{s^{D/2}}\exp\left(-m^2s-\frac{1}{4s}(x^2+l^2)\right), \quad (9)$$

where we have set $y=0$, $\tau=ms$ and defined $l\equiv 2L$. To see the effect of our new term, we may Fourier transform this expression with respect to x, giving

$$\widetilde{G}(\mathbf{p})=\int_0^\infty ds \exp\left(-(p^2+m^2)s-\frac{l^2}{4s}\right). \quad (10)$$

When $l=0$, this gives the conventional Feynman propagator in Fourier space $(p^2+m^2)^{-1}$. When $l\neq 0$ the integration can be performed to give

$$\widetilde{G}(\mathbf{p})=K_1(l\sqrt{p^2+m^2})\frac{l}{\sqrt{p^2+m^2}}, \quad (11)$$

where $K_1(z)$ is the modified Bessel function. The limiting forms of this expression are

$$\hat{G}(\mathbf{p})\rightarrow\begin{cases}(p^2+m^2)^{-1} & (\text{for } l\sqrt{p^2+m^2}\ll 1),\\ \dfrac{\exp(-l\sqrt{p^2+m^2})}{l^{1/2}(p^2+m^2)^{3/4}} & (\text{for } l\sqrt{p^2+m^2}\gg 1),\end{cases} \quad (12)$$

which clearly shows the suppression of energies higher than Planck energies.

The rest of the paper is organized as follows. In Sec. II, I illustrate how the path integral can be rigorously defined using a D-dimensional lattice and limiting procedure. This "warm-up" exercise shows how the standard result (2) arises and sets the stage for the main analysis of the paper. In Sec. III, I evaluate the modified path integral using the same technique and obtain Eq. (6). Some of the implications are discussed in Sec. V.

II. WARM-UP: FEYNMAN PROPAGATOR FROM SUM OVER PATHS

A. Rigorous evaluation of the path integral

In defining the path integral in nonrelativistic quantum mechanics, we discretize the time axis, define the path integral with a nonzero spacing ϵ, and finally take the limit of ϵ going to zero. To define the path integral in D dimensions we can use a similar procedure. We will work in Euclidean space and introduce a cubic lattice with spacing ϵ. The path integral will be defined on the lattice and then we will take the limit of $\epsilon\rightarrow 0$. To obtain a finite value in the limit of $\epsilon\rightarrow 0$ we have to choose the measure and the mass parameter m, which varies in a specific fashion with ϵ. This can be done fairly easily and the final expression will agree with the standard Feynman propagator for a free scalar field. The calculation proceeds as follows.

We will work directly in Euclidean space of D dimensions. In this section we are primarily interested in the issues of principle, regarding the measure for the path integral, and will consider the path integral for a free particle. We have to, therefore, evaluate

$$\mathcal{G}_E(\mathbf{x_2},\mathbf{x_1};\mu_0)=\sum_{\text{all }\mathbf{x}(t)}\exp\{-m\; l[\mathbf{x}(t)]\} \quad (13)$$

in the Euclidean sector, where l is

$$l(\mathbf{x_2},\mathbf{x_1})=\int_0^1 ds\left|\left(\frac{d\mathbf{x}}{ds}\right)^2\right|^{1/2} \quad (14)$$

and is just the length of the curve $\mathbf{x}(s)$, connecting $\mathbf{x}(0)=\mathbf{x}_1$ and $\mathbf{x}(1)=\mathbf{x}_2$.

This quantity can be defined through the following limiting procedure: Consider a lattice of points in a

D-dimensional lattice with a uniform lattice spacing of ϵ. We will work out $\mathcal{G}_E$ in the lattice and will then take the limit of $\epsilon \to 0$ with a suitable measure $M(\epsilon)$. To obtain a finite result, it is also necessary to treat m (which is the only parameter in the problem) as a function $\mu(\epsilon)$ of the lattice spacing where the functional form is to be chosen in such a way to ensure finiteness of the continuum result. We will reserve the symbol m for the value of this function in the continuum limit. Thus we will define the path integral result as a limit:

$$\mathcal{G}(\mathbf{x_2},\mathbf{x_1};m) = \lim_{\epsilon \to 0} [M(\epsilon)\mathcal{G}(\mathbf{x_2},\mathbf{x_1};\mu(\epsilon))], \tag{15}$$

where the functions $M(\epsilon)$ and $\mu(\epsilon)$ are to be chosen so as to ensure finiteness. The rationale for this expression arises from the following point of view: We treat the continuum space as a limit of a lattice with the lattice spacing ϵ going to zero. We now construct a sequence of path integrals parametrized by the spacing ϵ by choosing certain functions $\mu(\epsilon)$ and $M(\epsilon)$ and define the continuum path integral as the limit of this sequence. We shall show later that this limit exists only if $\mu(\epsilon) \simeq (\ln 2D)/\epsilon$ and $M(\epsilon) \simeq (2D)^{-1}\epsilon^{-(D-2)}$ near $\epsilon \simeq 0$. The form of $\mu(\epsilon), M(\epsilon)$ for ϵ far away from zero, of course, makes no difference to the result we are after.

In a lattice with spacing of ϵ, Eq. (13) can be evaluated in a straightforward manner. Because of the translation invariance of the problem, $\mathcal{G}_E$ can only depend on $\mathbf{x}_2 - \mathbf{x}_1$; so we can set $\mathbf{x}_1 = 0$ and call $\mathbf{x}_2 = \epsilon \mathbf{R}$ where $\mathbf{R}$ is a D-dimensional vector with integral components: $\mathbf{R} = (n_1, n_2, n_3, \ldots, n_D)$. Let $C(N,\mathbf{R})$ be the number of paths of length $N\epsilon$ connecting the origin to the lattice point $\epsilon\mathbf{R}$. Since all the paths contribute a term $[\exp - \mu(\epsilon)(N\epsilon)]$ to Eq. (15), we get

$$\mathcal{G}_E(\mathbf{R};\epsilon) = \sum_{N=0}^{\infty} C(N;\mathbf{R}) \exp[-\mu(\epsilon) N\epsilon]. \tag{16}$$

The generating function determining $C(N;\mathbf{R}) = C(N;n_1,n_2,\ldots,n_D)$ can be calculated easily by the following arguments: Consider any particular path connecting the origin to the lattice point $\mathbf{R}$. Suppose that this path takes r_1 steps towards positive direction (''right'') in the first axis and l_1 steps towards negative direction (''left'') in the first axis. Then $n_1 = r_1 - l_1$; similarly $n_i = r_i - l_i$. The number of paths with a specified number of (r_i, l_i) for $i = 1, \ldots, D$ is just the number of ways of ordering the steps, specified by the integers $(r_1, \ldots, r_D, l_1, \ldots, l_D)$ with $\Sigma r_i + \Sigma l_i = N$. This is given by the coefficient of the polynomial expansion

$$(x_1 + x_2 + \cdots + x_D + y_1 + y_2 + \cdots + y_D)^N = \sum Q(N; r_i, l_i) x_1^{r_1} \cdots x_D^{r_D} y_1^{l_1} \cdots y_D^{l_D}. \tag{17}$$

In our problem, we allow (r_i, l_i) also to vary, keeping $r_i - l_i = n_i$ fixed for each i. The number of paths with this property is clearly given by using the above expression with $y_i = (1/x_i)$. Then we get

$$\left(x_1 + x_2 + \cdots + x_D + \frac{1}{x_1} + \frac{1}{x_2} + \cdots + \frac{1}{x_D} \right)^N = \sum C(N; n_1, n_2, \ldots, n_D) x_1^{n_1} \cdots x_D^{n_D}. \tag{18}$$

The expansion of the left-hand side gives the generating function for $C(N;\mathbf{R})$. For further manipulation, it is convenient to set $x_1 = e^{ik_1}, x_2 = e^{ik_2}, \ldots, x_D = e^{ik_D}$. Then we can write

$$F^N \equiv [e^{ik_1} + e^{ik_2} + \cdots + e^{ik_D} + e^{-ik_1} + \cdots + e^{-ik_D}]^N = \sum_{\mathbf{R}} C(N;\mathbf{R}) e^{i\mathbf{k}\cdot\mathbf{R}} \tag{19}$$

Therefore,

$$\sum_{\mathbf{R}} e^{i\mathbf{k}\cdot\mathbf{R}} \mathcal{G}_E(\mathbf{R};\epsilon) = \sum_{N=0}^{\infty} \sum_{\mathbf{R}} C(N;\mathbf{R}) e^{i\mathbf{k}\cdot\mathbf{R}} \exp[-\mu(\epsilon)N\epsilon] = \sum_{N=0}^{\infty} e^{-\mu(\epsilon)\epsilon N} F^N = \sum_{N=0}^{\infty} [F e^{-\mu(\epsilon)\epsilon}]^N = [1 - F e^{-\mu(\epsilon)\epsilon}]^{-1}. \tag{20}$$

Inverting the Fourier transform, we have

$$\mathcal{G}_E(\mathbf{R};\epsilon) = \int \frac{d^D\mathbf{k}}{(2\pi)^D} \frac{e^{-i\mathbf{k}\cdot\mathbf{R}}}{(1 - e^{-\mu(\epsilon)\epsilon} F)} = \int \frac{d^D\mathbf{k}}{(2\pi)^D} \frac{e^{-i\mathbf{k}\cdot\mathbf{R}}}{\left(1 - 2e^{-\mu(\epsilon)\epsilon} \sum_{j=1}^{D} \cos k_j\right)}. \tag{21}$$

Converting to the physical length scales $\mathbf{x} = \epsilon\mathbf{R}$ and $\mathbf{p} = \epsilon^{-1}\mathbf{k}$ we get

$$\mathcal{G}_E(\mathbf{x};\epsilon) = \int \frac{\epsilon^D d^D\mathbf{p}}{(2\pi)^D} \frac{e^{-i\mathbf{p}\cdot\mathbf{x}}}{\left(1 - 2e^{-\mu(\epsilon)\epsilon} \sum_{j=1}^{D} \cos p_j \epsilon\right)}. \tag{22}$$

We are now ready to take the limit of the zero lattice spacing. As $\epsilon \to 0$, the denominator of the integrand becomes

$$1 - 2e^{-\epsilon\mu(\epsilon)} \left(D - \frac{1}{2}\epsilon^2 |\mathbf{p}|^2 \right) = 1 - 2De^{-\epsilon\mu(\epsilon)} + \epsilon^2 e^{-\epsilon\mu(\epsilon)} |\mathbf{p}|^2 = \epsilon^2 e^{-\epsilon\mu(\epsilon)} \left[|\mathbf{p}|^2 + \frac{1 - 2De^{-\epsilon\mu(\epsilon)}}{\epsilon^2 e^{-\epsilon\mu(\epsilon)}} \right], \tag{23}$$

so that we will get, for small ϵ,

$$\mathcal{G}_E(\mathbf{x};\epsilon) \simeq \int \frac{d^D\mathbf{p}}{(2\pi)^D} \frac{A(\epsilon) e^{-i\mathbf{p}\cdot\mathbf{x}}}{|\mathbf{p}|^2 + B(\epsilon)}, \tag{24}$$

where

$$A(\epsilon)=\epsilon^{D-2}e^{\epsilon\mu(\epsilon)},\quad B(\epsilon)=\frac{1}{\epsilon^2}[e^{\epsilon\mu(\epsilon)}-2D]. \tag{25}$$

[The convergence of the integral over all p should be interpreted in a distributional sense as we have used a Taylor expansion of the denominator of Eq. (22).] The continuum theory has to be defined in the limit of $\epsilon\to 0$ with some measure $M(\epsilon)$; that is, we want to obtain

$$\mathcal{G}_E(\mathbf{x};m)|_{\text{continuum}}=\lim_{\epsilon\to 0}\{M(\epsilon)\mathcal{G}_E(\mathbf{x};\epsilon)\}. \tag{26}$$

The choice of the measure is dictated by the requirement that the right-hand side should be finite in this limit. We now demand

$$\lim_{\epsilon\to 0}\left[\frac{1}{\epsilon^2}(e^{\epsilon\mu(\epsilon)}-2D)\right]=m^2 \tag{27}$$

and

$$\lim_{\epsilon\to 0}[M(\epsilon)\epsilon^{D-2}e^{\epsilon\mu(\epsilon)}]=1. \tag{28}$$

The first condition implies that, near $\epsilon\approx 0$,

$$\mu(\epsilon)\approx\frac{\ln 2D}{\epsilon}+\frac{m^2}{2D}\epsilon\approx\frac{\ln 2D}{\epsilon}. \tag{29}$$

Using this in the second condition (28), we can determine the measure as

$$M(\epsilon)=\frac{1}{2D}\,\frac{1}{\epsilon^{D-2}}. \tag{30}$$

With this choice, we get

$$G_E(\mathbf{x};m)\equiv\lim_{\epsilon\to 0}\mathcal{G}_E(\mathbf{x};\epsilon)M(\epsilon)=\int\frac{d^D\mathbf{p}}{(2\pi)^D}\,\frac{e^{-i\mathbf{p}\cdot\mathbf{x}}}{|\mathbf{p}|^2+m^2}, \tag{31}$$

which is the standard Feynman propagator. This analysis gives a rigorous meaning to the nonquadratic path integral with a square root and also illustrates the role played by the choice of the measure. In the continuum limit, we have only one length scale m^{-1}; this fact suggests that the right-hand side of Eq. (27) should scale as m^2. Setting the proportionality constant to unity should be thought of a (partial) choice of measure. Similarly, $M(\epsilon)$ can be multiplied by any finite quantity. The choice in Eq. (28) should also be considered as part of the definition of measure.

To connect this expression with Schwinger's proper time representation is easy. By writing $(|\mathbf{p}|^2+m^2)^{-1}$ as

$$\frac{1}{|\mathbf{p}|^2+m^2}=\int_0^\infty d\tau\quad e^{-\tau(m^2+|\mathbf{p}|^2)} \tag{32}$$

and doing the integration over $\mathbf{p}$, we get

$$G_E(\mathbf{x};m)=\sum\ \exp(-m\mathcal{R})=\int_0^\infty\frac{d\tau}{(4\pi\tau)^{D/2}}e^{-m^2\tau}\times\exp\left(-\frac{|x^2|}{4\tau}\right). \tag{33}$$

Part of the integrand can be expressed as an ordinary quadratic path integral:

$$K(\mathbf{x},\mathbf{y};\tau)\equiv\int\mathcal{D}x\ \exp\left(-\frac{1}{4}\int_0^\tau\dot{x}^i\dot{x}_i ds\right)=\left(\frac{1}{4\pi\tau}\right)^{D/2}\exp\left(-\frac{(\mathbf{x}-\mathbf{y})^2}{4\tau}\right), \tag{34}$$

where we have shifted the origin to $\mathbf{y}$. Using this in Eq. (33), we get the final result, quoted in Eq. (2):

$$\sum\ \exp[-m\mathcal{R}(x,y)]=\int_0^\infty d\tau e^{-m^2\tau}\int\mathcal{D}x\ \exp\left(-\frac{1}{4}\int_0^\tau\dot{x}_i\dot{x}^i ds\right). \tag{35}$$

B. Physical interpretation

The above analysis relates a nonquadratic path integral (containing a square root) to a standard quadratic path integral. This result has a simple physical interpretation, which is worth emphasizing. Consider the standard path integral kernel $K(\mathbf{x},\mathbf{y},\tau)$ in quantum mechanics, defined through the Hamiltonian form of the action:

$$K(\mathbf{x},\mathbf{y},t)=\sum_{\mathbf{x}(t)}\sum_{\mathbf{p}(t)}\exp\left(\frac{i}{\hbar}\int_0^t dt'[\mathbf{p}\cdot\dot{\mathbf{x}}-H(\mathbf{p},\mathbf{x})]\right], \tag{36}$$

with $H\geqslant 0$. From the principles of quantum mechanics, we would expect the Fourier transform

$$B(\mathbf{x},\mathbf{y};E)\equiv\int_0^\infty K(\mathbf{x},\mathbf{y},;E)e^{iEt}dt \tag{37}$$

to give the amplitude for the particle to propagate from $\mathbf{y}$ to $\mathbf{x}$ with energy E. [Only $t\geqslant 0$ is relevant in the Fourier transform (37), since K is taken to vanish for $t<0$.] But the trajectory of a classical particle with fixed energy can be described using the Jacobi action

$$A_{\text{Jacobi}}=\int_0^t dt'\sqrt{2m_0(E-V)|\dot{\mathbf{x}}|^2}. \tag{38}$$

We will therefore expect the relation

$$\sum_{\text{paths}}\exp\left(i\int_0^t dt'\sqrt{2m_0(E-V)|\dot{\mathbf{x}}|^2}\right)=B(\mathbf{x},\mathbf{y};E)=\int_0^\infty dt\quad e^{iEt}\int\mathcal{D}\mathbf{x}\ \exp\left(i\int_0^t\left(\frac{1}{2}m_0\dot{x}^2-V\right)dt'\right) \tag{39}$$

to hold, thereby expressing a square root path integral in terms of a quadratic path integral. Taking $m_0=(m/2)$, $V=0$, $E=m$, and switching to the Euclidean sector gives the result

$$\sum_{\text{paths}} \exp[-m\mathcal{R}(\mathbf{x},\mathbf{y})]=\int_0^\infty dt\, e^{imt}\int \mathcal{D}x \times\exp\left[-\frac{m}{4}\int_0^t \dot{x}^2 dt'\right], \quad (40)$$

which is the same as Eq. (2) after the rescaling $t=m\tau$ and continuing to the Euclidean sector. The choice of $E=mc^2$ shows that the energy of the particle in the rest frame is on the mass shell.

To prove the result (39), we need the following path integral identities:

$$\delta(f(t))=\sum_{\lambda(t)} \exp\left(i\int dt\lambda(t)f(t)\right), \quad (41)$$

$$\sum_{\mathbf{p}} \exp\left(i\int dt[\mathbf{p}\cdot\dot{\mathbf{x}}+a(t)p^2]\right)=\exp i\left(\int dt\frac{\dot{x}^2}{4a(t)}\right), \quad (42)$$

$$\sum_{\lambda(t)} \exp\left[i\int dt\left(\lambda(t)a(t)+\frac{b(t)}{\lambda(t)}\right)\right]=\exp\left(i\int dt[-4ab]^{1/2}\right). \quad (43)$$

The first result is merely the definition of the delta functional; the second and third can be obtained in the Euclidean sector by standard time slicing techniques and can be analytically continued. They are direct generalizations of the corresponding results of ordinary integrals. (Equation (43) is the generalization of the ordinary integral

$$\int_0^\infty dx\exp\left(-ax^2-\frac{b}{x^2}\right)=\frac{1}{2}\sqrt{\frac{\pi}{a}}\exp([-4ab]^{1/2}) \quad (44)$$

with appropriate definition of the measure.)

Introducing into the integrand of Eq. (36) the ''expansion of unity'' in the form:

$$1=\int_0^\infty dE\,\delta(E-H(\mathbf{p},\mathbf{x})), \quad (45)$$

we get

$$K(\mathbf{x},\mathbf{y};t)=\int_0^\infty dE\sum_{\mathbf{x}}\sum_{\mathbf{p}}\delta(E-H(\mathbf{p},\mathbf{x}))\times\exp\left(\frac{i}{\hbar}\int_0^t dt'(\mathbf{p}\cdot\dot{\mathbf{x}}-H(\mathbf{p},\mathbf{x}))\right)$$
$$=\int_0^\infty dE\sum_{\mathbf{x}}\sum_{\mathbf{p}}\delta(E-H)e^{-iEt}\times\exp\left(i\int_0^t dt'(\mathbf{p}\cdot\dot{\mathbf{x}})\right). \quad (46)$$

So

$$\int_0^\infty K(\mathbf{x},\mathbf{y};t)e^{iEt}dt\equiv B(\mathbf{x},\mathbf{y};E) =\sum_{\mathbf{x}}\sum_{\mathbf{p}}\delta\left(\frac{\mathbf{p}^2}{2m}+V(\mathbf{x})-E\right)\times\exp\left(i\int dt\mathbf{p}\cdot\dot{\mathbf{x}}\right). \quad (47)$$

We now express the delta functional using Eq. (41):

$$\delta\left(\frac{\mathbf{p}^2}{2m}+V(\mathbf{x})-E\right)=\sum_{\lambda(t)}\exp\left(i\int \lambda(t)\left[\frac{p^2}{2m}+V(\mathbf{x})-E\right]dt\right). \quad (48)$$

Then

$$B(\mathbf{x},\mathbf{y};E)=\sum_{\mathbf{x}}\sum_{\lambda(t)}\exp i\left(\int dt\lambda(t)[V(\mathbf{x})-E]\right)\times\sum_{\mathbf{p}}\exp i\left(\int dt\left[\mathbf{p}\cdot\dot{\mathbf{x}}+\frac{\lambda(t)}{2m}p^2\right]\right)$$
$$=\sum_{\mathbf{x}}\sum_{\lambda(t)}\exp\left(i\int dt\lambda(t)[V(\mathbf{x})-E]\right)\times\exp\left(i\int dt\frac{1}{2}\frac{m}{\lambda(t)}\dot{\mathbf{x}}^2\right)$$
$$=\sum_{\mathbf{x}}\sum_{\lambda(t)}\exp\left(i\int dt\left[\lambda(t)[V(\mathbf{x})-E]+\frac{m}{2\lambda(t)}\dot{\mathbf{x}}^2\right]\right)$$
$$=\sum_{\mathbf{x}}\exp\left(i\int dt\sqrt{2m(E-V)|\dot{\mathbf{x}}|^2}\right). \quad (49)$$

In arriving at the second equality, we have used Eq. (42) and in arriving at the last equality we have used Eq. (43). This proves the result quoted above.

To summarize, we have demonstrated how path integrals involving square roots can be given a rigorous definition—using a lattice regularization scheme—in Sec. II A. This definition of the path integral is given a more intuitive interpretation in Sec. II B. We shall now work out the modified path integral along the same lines.

III. FEYNMAN PROPAGATOR WITH DUALITY INVARIANT PATH INTEGRAL

We shall now turn to the main task of the paper, viz., evaluation of the modified path integral in Eq. (5). It is easy to see that the lattice version now becomes

$$\mathcal{G}(\mathbf{R},\epsilon)=\sum_{N=0}^{\infty}C(N,\mathbf{R})\exp\left[-\mu(\epsilon)\epsilon N-\frac{\lambda(\epsilon)}{\epsilon N}\right], \quad (50)$$

where $\lambda(\epsilon)$ is a lattice parameter which will play the role of (mL_P^2) in the continuum limit. This replaces Eq. (17) of previous analysis. After evaluating $G(\mathbf{R},\epsilon)$, we multiply it by a measure $\mathcal{M}(\epsilon)$ and take the limit $\epsilon\to 0$. The functions $\mathcal{M}(\epsilon)$, $\mu(\epsilon)$, $\lambda(\epsilon)$ are to be chosen so that, in the continuum limit, μ corresponds to the mass m and λ to (mL_p^2). Since we expect the result to have the correct limit as $L_p\to 0$, we anticipate that the form of $\mu(\epsilon)$ will be as given by Eq. (30).

To evaluate this path integral on the lattice we again begin with the generating function for $C(N,\mathbf{R})$, given by Eq. (20):

$$F^N \equiv \sum_{\mathcal{R}} C(N;\mathcal{R}) e^{i\mathbf{k}\cdot\mathcal{R}} = (e^{ik_1} + e^{ik_2} + \cdots + e^{ik_D} + e^{-ik_1} + e^{-ik_2} + \cdots + e^{-ik_D})^N. \tag{51}$$

This now leads to the expression

$$\sum_{\mathcal{R}} e^{i\mathbf{k}\cdot\mathcal{R}} \mathcal{G}(\mathcal{R},\epsilon) = \sum_{N=0}^{\infty} e^{-\mu\epsilon N - (\lambda/\epsilon N)} \sum_{\mathcal{R}} C(N,\mathcal{R}) e^{i\mathbf{k}\cdot\mathcal{R}} = \sum_{N=0}^{\infty} e^{-N(\mu\epsilon - \ln F) - (\lambda/\epsilon N)}. \tag{52}$$

Thus, our problem reduces to evaluating a sum of the form

$$S(a,b) \equiv \sum_{n=0}^{\infty} \exp\left(-a^2 n - \frac{b^2}{n}\right) = \sum_{n=1}^{\infty} \exp\left(-a^2 n - \frac{b^2}{n}\right), \tag{53}$$

which is more complicated than the geometric progression of Eq. (21). Fortunately this expression can be evaluated by some algebraic tricks (see the Appendix) and the answer is

$$S(a,b) = \int_0^\infty \frac{q\,dq}{2b^2} \frac{J_0(q) e^{-(a^2+q^2/4b^2)}}{[1-e^{-(a^2+q^2/4b^2)}]^2} = \frac{1}{(1-e^{-a^2})} - \int_0^\infty dq \frac{J_1(q)}{[1-e^{-(a^2+q^2/4b^2)}]}, \tag{54}$$

where $J_\nu(x)$ is a Bessel function of order ν. The first form of the integral shows that the expression is well defined while the second form has the advantage of separating out the b-independent part as the first term. [Note that the two summations in Eq. (53) will differ by unity if $b=0$; the results in Eq. (54) will go over to the second summation in Eq. (53) if the limit $b\to 0$ is taken.] In our case, $b^2 = (\lambda/\epsilon)$ and $a^2 = \mu\epsilon - \ln F$: So we get

$$S(a,b) = \int_0^\infty q\,dq\,J_0(q) \left\{ \frac{\epsilon}{2\lambda} \frac{F\exp\left(-\mu\epsilon - \frac{q^2\epsilon}{4\lambda}\right)}{\left[1 - F\exp\left(-\mu\epsilon - \frac{q^2\epsilon}{4\lambda}\right)\right]^2} \right\}. \tag{55}$$

This gives

$$G(\mathbf{R}) = \int \frac{d^D k}{(2\pi)D} \int_0^\infty q\,dq\,J_0(q) e^{-i\mathbf{k}\cdot\mathbf{R}} \times \left\{ \frac{\epsilon}{2\lambda} \frac{F\exp\left(-\mu\epsilon - \frac{q^2\epsilon}{4\lambda}\right)}{\left[1 - F\exp\left(-\mu\epsilon - \frac{q^2\epsilon}{4\lambda}\right)\right]^2} \right\}. \tag{56}$$

Rescaling back to $\mathbf{x} = \epsilon R$, $\mathbf{p} = \epsilon^{-1}\mathbf{k}$, we find

$$G(\mathbf{x}) = \int \frac{\epsilon^D d^D p}{(2\pi)^D} \int_0^\infty q\,dq\,J_0(q) \left\{ \frac{\epsilon}{2\lambda} \frac{H e^{-i\mathbf{px}}}{(1-H)^2} \right\}, \tag{57}$$

with

$$H \equiv F\exp(-\alpha\epsilon) \equiv F\exp\left\{-\mu\epsilon - \frac{q^2}{4}\frac{\epsilon}{\lambda}\right\},$$

$$\alpha\epsilon = \mu(\epsilon)\epsilon + \frac{q^2}{4}\frac{\epsilon}{\lambda}. \tag{58}$$

This expression is dimensionless; we now take the $\epsilon\to 0$ limit, to get

$$1 - H = 1 - 2e^{-\epsilon\alpha} \sum_{i=1}^{D} \cos \epsilon p_i \approx 1 - 2e^{-\epsilon\alpha}\left[D - \frac{1}{2}\epsilon^2 p^2\right] = \epsilon^2 e^{-\alpha\epsilon}\left[p^2 + \frac{e^{\alpha\epsilon}}{\epsilon^2}(1 - 2De^{-\alpha\epsilon})\right]. \tag{59}$$

So we can write, retaining leading terms

$$G(\mathbf{x}) \cong \int \frac{\epsilon^D d^D p}{(2\pi)^D} \int_0^\infty q\,dq\,J_0(q) e^{-i\mathbf{p}\cdot\mathbf{x}} \left[\frac{e^{\alpha\epsilon}}{2\lambda\epsilon^3} \frac{2D}{(p^2+B)^2} \right], \tag{60}$$

where $B(\epsilon)$ is defined as

$$B = \frac{1}{\epsilon^2}\{e^{\mu\epsilon + (q^2/4)(\epsilon/\lambda)} - 2D\}. \tag{61}$$

Consider now the $\epsilon\to 0$ limit of B; using Eq. (29), we have $e^{\mu\epsilon} \approx 2D + m^2\epsilon^2$ for small ϵ. So

$$B \approx \frac{1}{\epsilon^2}\{2D(e^{(q^2/4)(\epsilon/\lambda)} - 1) + m^2\epsilon^2 e^{(q^2/4)(\epsilon/\lambda)}\}. \tag{62}$$

For the first term to be finite at yjr $\epsilon\to 0$ limit, we need the small-ϵ dependence to be of the form

$$\exp\left(\frac{q^2\epsilon}{4\lambda}\right) - 1 \approx A_1(q)\epsilon^2. \tag{63}$$

This implies that

$$\exp\left(\frac{q^2\epsilon}{4\lambda}\right) \approx 1 + A_1\epsilon^2 \approx 1 + \frac{q^2}{4}\frac{\epsilon}{\lambda}, \tag{64}$$

giving

$$B\approx\frac{1}{\epsilon^2}\{2DA_1\epsilon^2+m^2\epsilon^2[1+\mathcal{O}(\epsilon^2)\cdots]\}$$

$$=(2DA_1+m^2). \tag{65}$$

Further, since $A_1(q)\epsilon^2=(q^2/4)(\epsilon/\lambda)$, we need λ to scale as $\lambda\approx(q^2/4A_1)(1/\epsilon)$ as $\epsilon\to 0$. Since $\lambda(\epsilon)$ is to be independent of q, we must have $A_1(q)=(L^{-2}q^2/2D)$ with $L=$const. Then $\lambda(\epsilon)\approx(L^2/4)(1/\epsilon)(2D)$ as $\epsilon\to 0$. We thus find that, near $\epsilon\cong 0$, we need

$$B\cong m^2+\frac{q^2}{L^2},\quad e^{\alpha\epsilon}=e^{\mu\epsilon}e^{q^2\epsilon/4\lambda}\cong 2D,\quad \lambda\epsilon^3\approx\frac{L^2}{4}\epsilon^2. \tag{66}$$

Putting everything into Eq. (57), we get

$$G(\mathbf{x})=\int\frac{\epsilon^D d^D p}{(2\pi)^D}e^{-i\mathbf{p}\cdot\mathbf{x}}\int_0^\infty\frac{q\,dq\,J_0(q)}{L^2\epsilon^2}\frac{2(2D)^2}{(p^2+m^2+q^2/L^2)^2}$$

$$=4\epsilon^{D-2}D^2\int\frac{d^Dp}{(2\pi)^D}e^{-i\mathbf{p}\cdot\mathbf{x}}\int_0^\infty\frac{2qJ_0(q)L^2dq}{[L^2(p^2+m^2)+q^2]^2}. \tag{67}$$

We now choose the measure $M(\epsilon)$ such that

$$\lim_{\epsilon\to 0}4\epsilon^{D-2}D^2M(\epsilon)=1. \tag{68}$$

Then we get the final result

$$G(\mathbf{x})=\int\frac{d^Dp}{(2\pi)^D}e^{-i\mathbf{p}\cdot\mathbf{x}}\int_0^\infty\frac{2qJ_0(q)L^2dq}{[L^2(p^2+m^2)+q^2]^2}. \tag{69}$$

We have thus successfully defined the path integral in Eq. (5) using a lattice regularization procedure. Note that we now needed three functions $M(\epsilon)$, $\mu(\epsilon)$. and $\lambda(\epsilon)$. Of these, $M(\epsilon)$ and $\mu(\epsilon)$ were required even in the standard free particle case, discussed in Sec. II. In fact, we are using the same functional form $(M(\epsilon)\propto\epsilon^{2-D},\mu(\epsilon)\propto\epsilon^{-1})$ for these functions near $\epsilon\approx 0$. The new entity needed now is $\lambda(\epsilon)$ which should correspond to (mL_p^2) in the continuum limit. This function scales as $\lambda(\epsilon)\propto(L^2/\epsilon)$ near $\epsilon\approx 0$. At this stage we can only say that $L\propto L_p$; the proportionality constant, as usual, cannot be determined by considerations of measure; we shall say more about this later.

Our result can be recast in more useful forms. To begin with, the momentum space propagator is given by

$$G(\mathbf{p})\equiv\int d^D\mathbf{x}G(\mathbf{x})e^{i\mathbf{p}\cdot\mathbf{x}}=\int_0^\infty\frac{2qJ_0(q)L^2dq}{[q^2+L^2(p^2+m^2)]^2}. \tag{70}$$

Using the identity

$$\int_0^\infty dz\frac{zJ_0(z)}{(z^2+Q^2)^2}=\frac{K_1(Q)}{2Q}, \tag{71}$$

where $K_1(Q)$ is the modified Bessel function, we get

$$G(\mathbf{p})=\frac{L}{\sqrt{p^2+m^2}}K_1(L\sqrt{p^2+m^2})$$

$$=\begin{cases}(p^2+m^2)^{-1} & (\text{as } L\to 0),\\ \dfrac{e^{-L\sqrt{p^2+m^2}}}{L^{1/2}(p^2+m^2)^{3/4}} & (\text{as } L\to\infty).\end{cases} \tag{72}$$

Clearly, the propagator reduces to the standard form $(p^2+m^2)^{-1}$ obtained earlier, when $L^2(p^2+m^2)\to 0$. By setting $q=L\lambda$ we get

$$G(\mathbf{p})=2\int_0^\infty\frac{\lambda J_0(L\lambda)d\lambda}{[\lambda^2+p^2+m^2]^2}=-\frac{\partial}{\partial m^2}\int_0^\infty\frac{2\lambda J_0(L\lambda)d\lambda}{[\lambda^2+p^2+m^2]}. \tag{73}$$

Expressing the denominator using the identity (32), and differentiating with respect to m^2, it is easy to show that

$$G(p)=\int_0^\infty 2\tau d\tau e^{-\tau(p^2+m^2)}W(L,\tau), \tag{74}$$

where

$$W(L,\tau)=\int_0^\infty\lambda d\lambda J_0(L\lambda)e^{-\tau\lambda^2}=\frac{1}{2\tau}\exp\left(-\frac{L^2}{2\tau}\right), \tag{75}$$

with the last equality following from a standard identity related to Bessel functions. Using this, we can write

$$G(p)=\int_0^\infty d\tau\,\exp\left[-\tau(p^2+m^2)-\frac{L^2}{4\tau}\right]. \tag{76}$$

which has the same form as Eq. (10). Fourier transforming with respect to $\mathbf{p}$, we get the key result

$$G(\mathbf{x})=\int_0^\infty d\tau\left(\frac{1}{4\pi\tau}\right)^{D/2}\exp\left[-\tau m^2-\frac{x^2+L^2}{4\tau}\right]$$

$$=\int_0^\infty d\tau\exp\left(-\tau m^2-\frac{L^2}{4\tau}\right)K(x;\tau). \tag{77}$$

This is the result quoted in Eqs. (9) and (6), if we identify $L^2=4L_p^2$. Our definition of the limiting procedure only shows that $L\propto L_p$. The actual proportionality constant depends on the definition of measure and we shall see in the next section why $L^2=4L_p^2$ is natural.

IV. GENERALIZATION TO CURVED SPACETIME

A rigorous way of evaluating Eq. (5), viz., to define the path integral on a lattice and use a limiting procedure, this was done in the above for flat background spacetime. It is possible that this procedure can be generalized to curved spacetime. Unfortunately, this procedure hides the extreme simplicity of the result in Eq. (77) and does not make transparent the origin of several intermediate results. Here, I shall follow a different and simpler route and rederive the result. This rederivation suggests a generalization to curved spacetime.

The key idea is that the new factor in the path integral, $\exp(-a^2/\mathcal{R})$, can be expressed in terms of a factor like $\exp(-b^2\mathcal{R})$ by performing a Gaussian integral. The latter factor, of course, can be evaluated in the path integral. The Gaussian integration will also produce a $\mathcal{R}^{1/2}$ factor in front which needs to be taken care of by doing a two-dimensional Gaussian integration and a differentiation. With such elementary algebraic tricks, one can prove Eq. (77).

We start with a slight generalization of Eq. (2):

$$\sum e^{-(m+\alpha)\mathcal{R}} = \int_0^\infty d\tau K(x',x;\tau|\overline{g}) \exp[-m(m+\alpha)\tau]. \tag{78}$$

This can be easily proved by the lattice techniques used in Sec. II. [See Eq. (27); redefining the right-hand side to be $m(m+\alpha)$ will lead to Eq. (78).] More precisely, this equation defines the measure used on the left-hand side of Eq. (78). [This definition is nonstandard in the sense that we have replaced m by $(m+\alpha)$ in the functional on the left-hand side but changed m^2 to $m(m+\alpha)$ on the right-hand side. But it is a perfectly valid definition for the measure. In fact we can define the right-hand side of Eq. (27) to be in general of the form $m^2F(\alpha/m)$ where F is an arbitrary, dimensionless function. This is possible because we now have two-dimensional constants m and α.] We now introduce a two real variables (k_1,k_2) with $k^2 \equiv k_1^2 + k_2^2$ and set $\alpha = k^2/m$ to get

$$\sum \exp\left[-\left(m+\frac{k^2}{m}\right)\mathcal{R}\right] = \int_0^\infty d\tau K(x',x;\tau|\overline{g}) \times \exp[-(m^2+k^2)\tau]. \tag{79}$$

Differentiating this equation with respect to k^2 gives

$$\sum \left(\frac{\mathcal{R}}{m} e^{-(k^2/m)\mathcal{R}}\right) e^{-m\mathcal{R}} = \int_0^\infty d\tau (\tau e^{-k^2\tau}) e^{-m^2\tau} K(x',x;\tau). \tag{80}$$

Fourier transforming on the variables (k_1,k_2) with respect to two new variables (l_1,l_2), we find

$$\sum \int \frac{d^2k}{\pi} \frac{\mathcal{R}}{m} e^{-m\mathcal{R}} \exp\left(i\mathbf{k}\cdot\mathbf{l} - \frac{k^2}{m}\mathcal{R}\right) = \int_0^\infty d\tau \left(\int \frac{d^2k}{\pi} e^{i\mathbf{k}\cdot\mathbf{l} - k^2\tau}\right) \tau e^{-m^2\tau} K \tag{81}$$

or, equivalently,

$$\sum \exp\left(-m\mathcal{R} - \frac{ml^2}{4\mathcal{R}}\right) = \int_0^\infty d\tau K \exp\left(-m^2\tau - \frac{l^2}{4\tau}\right). \tag{82}$$

Defining $l^2 = 4L_p^2$, we get the final result

$$G_F(x,y|\overline{g}) = \sum \exp - m\left(\mathcal{R} + \frac{L_p^2}{\mathcal{R}}\right) = \int_0^\infty d\tau K(x,x',\tau|\overline{g}) \exp\left[-\left(m^2\tau + \frac{L_p^2}{\tau}\right)\right]. \tag{83}$$

The above approach gives a surprisingly quick derivation of our result (77) provided we accept the definition of measure in Eq. (78) and set $L = 2L_p$. The above analysis suggests a possible way of interpreting the path integral duality in an arbitrary curved background spacetime.

Given the kernel $K(x,y;\tau|\overline{g})$ for a particle to propagate from x to y in proper time τ (in some background metric $\overline{g}_{ik}$), one would have originally evaluated the Feynman propagator by giving a weightage $\exp(-m^2\tau)$ and integrating over τ. The effect of our modification is to change this weightage to $\exp(-m^2\tau - L_p^2/\tau)$. In deriving this result, we have not bothered to specify explicitly the measure in Eq. (5). To this extent, the derivation is formal and not rigorous.

V. CONCLUSIONS

One immediate consequence of this result is the interpretation in terms of the ''zero-point length'' mentioned in the Introduction. We know that the kernel $K(x,y;\tau|g)$ has an expansion of the form

$$K(x,y;\tau|\overline{g}) = \left(\frac{1}{4\pi\tau}\right)^{D/2} \exp\left(-\frac{(x-y)^2}{4\tau}\right)[1+\cdots], \tag{84}$$

where the ellipsis represents metric-dependent corrections. Using Eq. (84) in Eq. (83) we can write our propagator as

$$G_F(x,y|\overline{g}) = \int_0^\infty d\tau e^{-m^2\tau} \left(\frac{1}{4\pi\tau}\right)^{D/2} \exp\left(-\frac{(x-y)^2 + 4L_p^2}{4\tau}\right) \times [1+\cdots]. \tag{85}$$

Thus the net effect of our modification is to add a ''zero-point length'' $4L_p^2$ to $(x-y)^2$ in the exponent, modifying the leading singular behavior of the original propagator. In other words, *the modification of the path integral based on the principle of duality leads to results which are identical to adding a ''zero-point length'' in the spacetime interval*.

I wish to argue that the connection shown above is nontrivial; I know of no simple way of guessing this result. The standard Feynman propagator of quantum field theory can be obtained either through a lattice regularization of a path integral or from Schwinger's proper time representation. By adding a zero-point length in the Schwinger's representation we obtain a modified propagator. Alternatively, using the principle of duality, we could modify the expression for the path integral amplitude on the lattice and obtain—in the continuum limit—a modified propagator. Both these constructions are designed to suppress energies larger than Planck energies. *However, there is absolutely no reason for these two expressions to be identical*. The fact that they are identical suggests that the principle of duality is connected in some deep manner with the spacetime intervals having a zero-point length. Alternatively, one may conjecture that any approach which introduces a minimum length scale in spacetime (like in string models) will lead to some kind of principle of duality. This conjecture seems to be true in conventional string theories though it must be noted that the term duality is used in a somewhat different manner in string theories. (The concept of duality in string theory is reviewed in several articles; see. e.g., Refs. [7–12] and the references cited therein. The closest to our approach seems to be the T duality.)

The second obvious point, of course, is the improved ultraviolet behavior in the theory which is studied in a forthcoming paper [14]. For example, this ultraviolet finiteness allows a renormalization procedure to be carried out without the need for regularization in $\lambda\phi^4$ theory and QED. Renormalized coupling constants now have no divergent pieces and depend on the Planck length. In this sense, the Planck length acts as a natural cutoff, as to be expected.

The third issue is related to anomalies (like the trace anomaly) in curved spacetime. The conventional calculations do depend on the need to regularize the expressions in one way or the other [13]. With ultraviolet finiteness it is not clear whether the anomalies will survive or not. A detailed calculation [14] shows that the trace anomaly, for example, is finite and depends on the Planck length.

There is another implication of this result which requires study. To begin with a Planck length cutoff is equivalent to changing the density of states at high energies. The number of quantum states accessible to field theoretic systems becomes effectively finite. In the case of a black hole—for example—the number of microstates will be finite and will lead to a finite value for its entropy. This issue is under investigation.

APPENDIX: EVALUATION OF THE SUM

We need to evaluate the sum

$$S(a,b)\equiv\sum_{n=1}^{\infty} e^{-a^2n-b^2/n}=\sum_{n=0}^{\infty} e^{-a^2n-b^2/n} \quad (b\neq 0). \tag{A1}$$

To do this, we introduce two real variables (x,y) and write $b^2\equiv(x^2+y^2)/4$. Then we have the identity

$$\exp\left(-\frac{b^2}{n}\right)=\frac{\sqrt{n}}{\sqrt{\pi}}\int_{-\infty}^{\infty}dk_x e^{-nk_x^2+ik_x x}\frac{\sqrt{n}}{\sqrt{\pi}}\times\int_{-\infty}^{\infty}dk_y e^{-nk_y^2+ik_y y}=\int\frac{d^2\mathbf{k}}{\pi}ne^{-nk^2+i\mathbf{k}\cdot\mathbf{x}}. \tag{A2}$$

So the sum we need is

$$S(a,\mathbf{x})=\int\frac{d^2\mathbf{k}}{\pi}e^{i\mathbf{k}\cdot\mathbf{x}}\sum_{n=1}^{\infty} ne^{-n(a^2+\mathbf{k}^2)}, \quad |\mathbf{x}|=2b, \tag{A3}$$

with $\mathbf{x}=(x,y)$ being a two-dimensional vector. Now

$$\sum_{n=0}^{\infty} ne^{-\mu n}=-\frac{\partial}{\partial\mu}\left(\frac{1}{1-e^{-\mu}}\right)=\frac{e^{-\mu}}{(1-e^{-\mu})^2}, \tag{A4}$$

giving

$$S(a,\mathbf{x})=\int\frac{d^2\mathbf{k}}{\pi}e^{i\mathbf{k}\cdot\mathbf{x}}\frac{e^{-(a^2+k^2)}}{(1-e^{-(a^2+k^2)})^2} \quad (|\mathbf{x}|=2b)$$
$$=\int_0^{2\pi}\frac{d\theta}{\pi}\int_0^{\infty}kdke^{2ikb\cos\theta}\frac{e^{-(a^2+k^2)}}{[1-e^{-(a^2+k^2)}]^2}. \tag{A5}$$

To do the θ integration, we need the result

$$I=\int_0^{2\pi}d\theta e^{i\mu\cos\theta}=2\pi J_0(\mu). \tag{A6}$$

Using this we get

$$S(a,b)=2\int_0^{\infty}kdkJ_0(2kb)\frac{e^{-(a^2+k^2)}}{[1-e^{-(1^2+k^2)}]^2}$$
$$=\int_0^{\infty}\frac{qdq}{2b^2}\frac{J_0(q)e^{-(a^2+q^2/4b^2)}}{[1-e^{-(a^2+q^2/4b^2)}]^2}. \tag{A7}$$

This is the result quoted in the text.

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