Rotating topological black holes

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A class of metrics solving Einstein's equations with a negative cosmological constant and representing rotating, topological black holes is presented. All such solutions are in the Petrov type-*D* class, and can be obtained from the most general metric known in this class by acting with suitably chosen discrete groups of isometries. First, by analytical continuation of the Kerr–de Sitter metric, a solution describing uncharged, rotating black holes whose event horizon is a Riemann surface of arbitrary genus $g>1$, is obtained. Then a solution representing a rotating, uncharged toroidal black hole is also presented. The higher genus black holes appear to be quite exotic objects; they lack global axial symmetry and have an intricate causal structure. The toroidal black holes appear to be simpler; they have rotational symmetry and the amount of rotation they can have is bounded by some power of the mass.

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I. INTRODUCTION

In the past months there has been an increasing interest in black holes whose event horizons have a nontrivial topology [1–3]. The solutions can be obtained with the least expensive modification of general relativity: the introduction of a negative cosmological constant. This is sufficient to avoid a few classic theorems forbidding nonspherical black holes [4–6], and comes as a happy surprise. Charged versions of these black holes were presented in [2]; they can form by gravitational collapse [7,8] of certain matter configurations, and all together form a sequence of thermodynamically well behaved objects, obeying the well known entropy-area law [3,9].

Up to now no rotating generalization of higher genus solutions has been known. Holst and Peldan recently showed that there does not exist any $(3+1)$-dimensional generalization of the rotating Banados-Teitelboim-Zanelli (BTZ) black hole [10]. Therefore, if we are looking for a rotating generalization of the topological black holes, we have to consider spacetimes with a nonconstant curvature. On the other hand, a charged rotating toroidal solution with a black hole interpretation has been presented by Lemos and Zanchin [11], following previous work on cylindrically symmetric solutions of Einstein's equations [12–18].

In this paper a rotating generalization of higher genus black holes together with another toroidal rotating solution will be presented. We do not present unique results, and apart from noticing that there is more than one nonisometric tori generating black holes, we satisfy ourselves with a discussion of some of the relevant properties they have.

We begin in Sec. II with the spacetime metric for the genus $g>1$ solution and give a proof that it solves Einstein's field equations with a negative Λ term.

In Sec. III we determine the black hole interpretation of the metric, and we consider in which sense mass and angular momentum are defined and conserved. We shall give a detailed description of the rather intricate causal structure and the related Penrose-Carter diagrams, but we do not discuss whether the black holes can result from gravitational collapse.

In Sec. IV we describe the rotating toroidal black hole's metric, together with an account of its main features, including the causal structure and the causal diagrams.

In this paper we shall use the curvature conventions of the Hawking-Ellis book [19] and employ Planck's dimensionless units.

II. SPACETIME METRIC FOR G>1 ROTATING BLACK HOLES

We begin by recalling the uncharged topological black holes discussed in [2,3]. The metric appropriate for genus $g>1$ reads

$$ds^2=-V(r)dt^2+V(r)^{-1}dr^2+r^2(d\theta^2+\sinh^2\theta d\phi^2) \tag{1}$$

with the lapse function $V(r)$ given by

$$V(r)=-1-\frac{\Lambda r^2}{3}-\frac{2\eta}{r}, \tag{2}$$

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where η is the mass parameter and $\Lambda=-3l^{-2}$ the cosmological constant. One notices that the (θ,ϕ) sector of the metric describes the two-dimensional noncompact space with constant, negative curvature. As is well known, this is the universal covering space for all Riemannian surfaces with genus $g>1$. Therefore, in order to get a compact event horizon, suitable identifications in the (θ,ϕ) sector have to be carried out, corresponding to the choice of some discrete group of isometries acting on hyperbolic 2-space properly discontinuously. After this has been done, the metric (1) will describe higher genus black holes. The $g=1$ case, with a toroidal event horizon, is given by the metric

$$ds^2=-V(r)dt^2+V(r)^{-1}dr^2+r^2d\sigma^2 \tag{3}$$

with the lapse function $V(r)$ given by

$$V(r)=-\frac{\Lambda r^2}{3}-\frac{2\eta}{r} \tag{4}$$

and $d\sigma^2$ is the line element of a flat torus. Its conformal structure is completely determined by a complex parameter in the upper complex half plane, τ, which is known as the Teichmüller parameter. A representative for the flat torus metric can then be written in the form

$$d\sigma^2=|\tau|^2dx^2+dy^2+2\,\mathrm{Re}\,\tau dx\,dy. \tag{5}$$

It is quite trivial to show that all such solutions have indeed a black hole interpretation, with various horizons located at roots of the algebraic equation $V(r)=0$, provided η is larger than some critical value depending on Λ. It can also be shown that for all genus, a ground state can be defined relative to which the Arnowitt-Deser-Misner (ADM) mass is a positive, concave function [3] of the black hole's temperature as defined by its surface gravity [20], and that the entropy obeys the area law [9,3].

We now determine at least one class of rotating generalizations of the above solutions starting with the higher genus case, namely when $g>1$. The toroidal rotating black hole will be described last. The metric (1) looks very similar to the Schwarzschild–de Sitter metric [21–23]

$$ds^2=-\left(1-\frac{\Lambda r^2}{3}-\frac{2\eta}{r}\right)dt^2+\left(1-\frac{\Lambda r^2}{3}-\frac{2\eta}{r}\right)^{-1}dr^2 +r^2(d\theta^2+\sin^2\theta d\phi^2). \tag{6}$$

(Here $\Lambda>0$).

For the latter, it is well known that a generalization to the rotating case exists, namely the Kerr–de Sitter spacetime [21–23], which describes rotating black holes in an asymptotically de Sitter space. Its metric reads, in Boyer-Lindquist-type coordinates,

$$\begin{aligned}ds^2=&\rho^2(\Delta_r^{-1}dr^2+\Delta_\theta^{-1}d\theta^2)+\rho^{-2}\Xi^{-2}\Delta_\theta\\ &\times[a\,dt-(r^2+a^2)d\phi]^2\sin^2\theta\\ &-\rho^{-2}\Xi^{-2}\Delta_r[dt-a\,\sin^2\theta d\phi]^2,\end{aligned} \tag{7}$$

where

$$\begin{aligned}\rho^2&=r^2+a^2\cos^2\theta,\\ \Delta_r&=(r^2+a^2)\left(1-\frac{\Lambda r^2}{3}\right)-2\eta r,\\ \Delta_\theta&=1+\frac{\Lambda a^2}{3}\cos^2\theta,\\ \Xi&=1+\frac{\Lambda a^2}{3},\end{aligned} \tag{8}$$

and a is the rotational parameter.

Now we note that Eq. (1) can be obtained from Eq. (6) by the analytical continuation

$$\begin{aligned}&t\to it,\quad r\to ir,\quad \theta\to i\theta,\quad \phi\to\phi,\\ &\eta\to -i\eta,\end{aligned} \tag{9}$$

thereby changing also the sign of Λ (this may be interpreted as an analytical continuation, too).

Therefore we are led to apply the analytical continuation (9) also to Kerr–de Sitter spacetime (7), additionally replacing a by ia. This leads to the metric

$$\begin{aligned}ds^2=&\rho^2(\Delta_r^{-1}dr^2+\Delta_\theta^{-1}d\theta^2)+\rho^{-2}\Xi^{-2}\Delta_\theta\\ &\times[a\,dt-(r^2+a^2)d\phi]^2\sinh^2\theta\\ &-\rho^{-2}\Xi^{-2}\Delta_r[dt+a\,\sinh^2\theta d\phi]^2,\end{aligned} \tag{10}$$

where now

$$\begin{aligned}\rho^2&=r^2+a^2\cosh^2\theta,\\ \Delta_r&=(r^2+a^2)\left(-1-\frac{\Lambda r^2}{3}\right)-2\eta r,\\ \Delta_\theta&=1-\frac{\Lambda a^2}{3}\cosh^2\theta,\\ \Xi&=1-\frac{\Lambda a^2}{3},\end{aligned} \tag{11}$$

and $\Lambda<0$.

One observes that Eq. (10) describes a spacetime which reduces, in the limit $a=0$, to the static topological black holes (1). For our further purpose it is convenient to write Eq. (10) in the form

$$\begin{aligned}ds^2=&-\frac{\rho^2\Delta_\theta\Delta_r}{\Xi^2\Sigma^2}dt^2+\frac{\rho^2}{\Delta_r}dr^2+\frac{\rho^2}{\Delta_\theta}d\theta^2\\ &+\frac{\Sigma^2\,\sinh^2\theta}{\Xi^2\rho^2}[d\phi-\omega dt]^2,\end{aligned} \tag{12}$$

where we introduced

$$\Sigma^2=(r^2+a^2)^2\Delta_\theta-a^2\sinh^2\theta\Delta_r \tag{13}$$

and the angular velocity

$$\omega = \frac{a[(r^2+a^2)\Delta_\theta + \Delta_r]}{\Sigma^2}. \tag{14}$$

Next we show how to compactify the (θ,ϕ) sector into a Riemann surface while preserving the differentiability of the metric. The timelike 3-surfaces at fixed coordinate radius r are foliated by surfaces at fixed coordinate time t into a family of spacelike 2-surfaces, and we would like these to be Riemann surfaces with genus $g>1$. The metric induced on such surfaces is

$$d\sigma^2 = \frac{\rho^2}{\Delta_\theta} d\theta^2 + \frac{\Sigma^2 \sinh^2\theta}{\Xi^2 \rho^2}\, d\phi^2. \tag{15}$$

Note that the Gaussian curvature of this metric is no longer constant, as it was in the case $a=0$.

In order to get a Euclidean metric we have to require that $\Sigma^2>0$. This is the case for every $r\in\mathbb{R}$, $\theta\geqslant 0$, if $-al^{-2}(a^2+l^2)<\eta<al^{-2}(a^2+l^2)$ [or for every $r\geqslant 0$, $\theta\geqslant 0$, if $\eta>-al^{-2}(a^2+l^2)$]. Outside the prescribed η interval, the metric may become singular or may change the signature.

The compactification is now performed in the same way as for a Riemann surface of constant curvature (i.e., for $a=0$; in this case see e.g. [24]). That is, we have to identify opposite sides of a properly chosen regular geodesic $4g$-gon centered at the origin $\theta=0$. The geodesics have to be computed from the metric (15), and therefore they are different from those in the case of constant curvature. The size of the $4g$-gon is determined by the requirement that the sum of the polygon angles be equal to 2π [24], in order to avoid conical singularities. Indeed, the local version of the Gauss-Bonnet theorem yields

$$\int_B K\, dA = 2\pi - \sum_{i=1}^{4g} (\pi - \beta_i), \tag{16}$$

where B is the interior of the geodesic polygon, K the Gaussian curvature of the (θ,ϕ) surface, dA the area element of the metric (15), and β_i the ith polygon angle. (Of course, the β_i are all equal, as the polygon is regular.) From Eq. (16) we see that the requirement $\Sigma_i\beta_i=2\pi$ fixes the size of B. Equation (16) then gives

$$\int_B K\, dA = 2\pi(2-2g), \tag{17}$$

which is the Gauss-Bonnet theorem for a Riemann surface of genus g. *A priori*, it is not obvious that a polygon which satisfies Eq. (17) with $g>1$ really exists. Therefore let us sketch a short existence proof. If the polygon is very small, the sum of the interior angles is larger than 2π, since the metric (15) approaches a flat metric for $\theta\to 0$. On the other hand, enlarging the polygon, the sum of the angles decreases until it is zero at a certain limit. [This is the limit when the polygon vertices lie on the border of the Poincaré disk on which Eq. (15) can also be defined by a proper coordinate transformation. The geodesics meet this border orthogonally and therefore the angle sum is zero.] As $\Sigma_i\beta_i$ is a continuous function of the distance of the vertices from the origin $\theta=0$, we deduce that the desired polygon indeed exists.

The next question which arises is that of the differentiability of the metric after the compactification. Identifying geodesics assures that the metric is in C^1. (Use ''Fermi coordinates'' [25] in neighborhoods of the geodesics which have to be identified. In these coordinates one has on the geodesics $\sigma_{ij}=\delta_{ij}$ and $\sigma_{ij,k}=0$.) As the second derivatives of the metric are bounded, the metric is even in $C^{1,\alpha}$ (which means that the first derivatives are Hoelder continuous with exponent α). Now we note that one obtains the Gaussian curvature K by applying a quasilinear elliptic operator L of second order to the metric σ,

$$K = L[\sigma]. \tag{18}$$

L can be written as

$$L = \sum_{\beta\leqslant 2} a_\beta(x, \partial^l\sigma)\partial^\beta, \tag{19}$$

where x stands for the coordinates on the surface, β is a multi-index, the coefficients a_β are matrices, and $l\leqslant 1$. We now express the zeroth and first derivatives of the metric in $a_\beta(x,\partial^l\sigma)$ as functions of the coordinates x. This makes the operator L linear with coefficients in $C^{0,\alpha}$. As the Gaussian curvature is also in $C^{0,\alpha}$ on the compactified surface, we conclude from the regularity theorem for solutions of linear elliptic equations [26,27] that the metric σ is (at least) in $C^{2,\alpha}$.

As we have now compactified the (θ,ϕ) sector to a Riemann surface S_g, the topology of the manifold is that of $\mathbb{R}^2\times S_g$.

Finally, we remark that Eq. (10) is a limit case of the metric of Plebanski and Demianski [28], which is the most general known Petrov type-D solution of the source-free Einstein-Maxwell equations with cosmological constant. In the case of zero electric and magnetic charge it reads

$$ds^2 = \frac{1}{(1-pq)^2}\left\{\frac{p^2+q^2}{\mathcal{P}}dp^2 + \frac{\mathcal{P}}{p^2+q^2}(d\tau + q^2 d\sigma)^2 + \frac{p^2+q^2}{\mathcal{L}}dq^2 - \frac{\mathcal{L}}{p^2+q^2}(d\tau - p^2 d\sigma)^2\right\}, \tag{20}$$

where the structure functions are given by

$$\mathcal{P} = \left(-\frac{\Lambda}{6}+\gamma\right) + 2np - \epsilon p^2 + 2\eta p^3 + \left(-\frac{\Lambda}{6}-\gamma\right)p^4,$$

$$\mathcal{L} = \left(-\frac{\Lambda}{6}+\gamma\right) - 2\eta q + \epsilon q^2 - 2nq^3 + \left(-\frac{\Lambda}{6}-\gamma\right)q^4. \tag{21}$$

Λ is the cosmological constant, η and n are the mass and nut parameters, respectively, and ϵ and γ are further real parameters. (For details cf. [28].) Rescaling the coordinates and the constants according to

$$p\to L^{-1}p,\ \ q\to L^{-1}q,\ \ \tau\to L\tau,\ \ \sigma\to L^{3}\sigma,$$
$$\eta\to L^{-3}\eta,\ \ \epsilon\to L^{-2}\epsilon,\ \ n\to L^{-3}n,$$
$$\gamma\to L^{-4}\gamma+\frac{\Lambda}{6},\ \ \Lambda\to\Lambda \tag{22}$$

and taking the limit as $L\to\infty$, one obtains

$$ds^2=\frac{p^2+q^2}{\mathcal{P}}dp^2+\frac{\mathcal{P}}{p^2+q^2}(d\tau+q^2d\sigma)^2+\frac{p^2+q^2}{\mathcal{L}}dq^2$$
$$-\frac{\mathcal{L}}{p^2+q^2}(d\tau-p^2d\sigma)^2, \tag{23}$$

where now

$$\mathcal{P}=\gamma+2np-\epsilon p^2-\frac{\Lambda}{3}p^4,$$
$$\mathcal{L}=\gamma-2\eta q+\epsilon q^2-\frac{\Lambda}{3}q^4. \tag{24}$$

Setting now

$$q=r,\ \ p=a\cosh\theta,\ \ \tau=\frac{t-a\phi}{\Xi},\ \ \sigma=-\frac{\phi}{a\Xi},$$
$$\epsilon=-1-\frac{\Lambda a^2}{3},\ \ \gamma=-a^2,\ \ n=0, \tag{25}$$

one gets our solution (10). As we said, the metric (10) turns out to be a limit case of the more general solution (20) of Einstein's equation. This formally shows that Eq. (10) should solve Einstein's field equations with cosmological constant, i.e., the analytical continuation (9) of the Kerr–de Sitter metric should yield again a solution. Anyhow, one could doubt the procedure as it involves an infinite limit of some parameters in the initial solution of Einstein's equations. Therefore, let us sketch a short independent proof of the fact that Eq. (10) still satisfies Einstein's equations.

Generally speaking, all functions which appear in the left hand side of Einstein's equations containing the cosmological constant are polynomial in metric tensor components, components of the inverse metric tensor and derivatives of metric tensor components. Considering all these functions as independent variables, the left-hand side (LHS) of Einstein's equations defines analytic functions in these variables. Let us consider Kerr–de Sitter spacetime defined above. Then the metric, its inverse and its derivatives define locally analytic functions of the (generally complex) variables $t,r,\theta,\phi,\eta,a,\Lambda$. We conclude that the LHS of Einstein's equations defines analytic functions of $(t,r,\theta,\phi,\eta,a,\Lambda)$ in open connected domains away from singularities corresponding to zeros of Δ_r and the determinant of the analytically continued metric $[g=-\Xi^{-4}(r^2+a^2\cos^2\theta)^2]$. Moreover, we know that, for real values of $(t,r,\theta,\phi,\eta,a,\Lambda)$, $\Lambda>0$, these functions vanish because the Kerr–de Sitter metric is a solution of Einstein's equations. Hence, due to the theorem of uniqueness of the analytical continuation of a function of several complex variables, they must vanish concerning all (generally complex) remaining values of $(t,r,\theta,\phi,\eta,a,\Lambda)$, provided they belong to the same domain of analyticity of the previously considered real values. In particular, we can pick out the set of values determining the metric (10) as final values. Notice that these values belong to the same analyticity domain of the values determining the Kerr–de Sitter metric because one can easily find piecewise smooth trajectories in the space $\mathbb{C}^7$, connecting Kerr–de Sitter parameters to parameters appearing in the metric (10), and skipping all singularities.

III. SOME PROPERTIES OF $G>1$ ROTATING BLACK HOLES

We shall briefly discuss now the black hole interpretation of the proposed solutions and some of their physical properties, starting with the case $g>1$.

A. Curvature

Let us begin by looking at the curvature of the spacetime metric (10). The only nonvanishing complex tetrad component of the Weyl tensor is given by

$$\Psi_2=-\frac{2\eta}{(r+ia\cosh\theta)^2}. \tag{26}$$

(The Ψ_i, $i=0,\ldots,4$, are the standard complex tetrad components describing the conformal curvature. For details, cf. [28,29].) For $\eta=0$ the Weyl tensor vanishes and, since $R_{ij}=\Lambda g_{ij}$, our manifold is a space of constant curvature, $k=-l^{-2}$, i.e., a quotient space of the universal covering of anti–de Sitter space. This situation is comparable to that of the Kerr metric, which, for vanishing mass parameter, is simply the Minkowsky metric written in oblate spheroidal coordinates.

One further observes that Ψ_2 is always nonsingular, in particular the curvature singularity in Kerr–de Sitter space at $\rho^2=0$, i.e., $r=0,\theta=\pi/2$ vanishes after the analytical continuation, as $r^2+a^2\cosh^2\theta$ is always positive. Hence the manifold may be extended to values $r<0$, and closed timelike curves will always be present. This is similar to the BTZ black hole [30], where no curvature singularity occurs (see also [31] for an exhaustive determination of $(2+1)$-black holes and their topology). On the other hand, all nonrotating solutions with $\eta\neq0$ found so far have curvature singularities at the origin, but do not violate the strong causality condition.

B. Singularity structure and horizons

The metric (10) becomes singular at $\Delta_r=0$. With $\Lambda=-3l^{-2}$ this equation reads

$$(r^2+a^2)\left(\frac{r^2}{l^2}-1\right)-2\eta r=0. \tag{27}$$

There are several cases, in all of which Δ_r is positive for r smaller than the left most zero or larger than the right most zero. Here are the various cases.

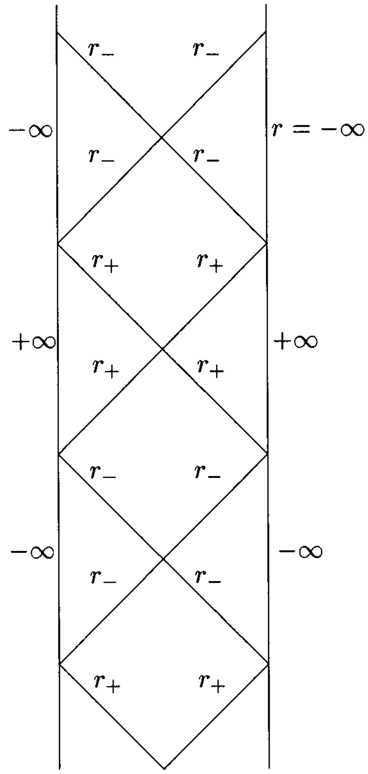

FIG. 1. $\theta=0$ Penrose-Carter diagram for the $g>1$ black hole in the case where Δ_r has one positive root r_+ and one negative root r_-, and for the toroidal black hole in the case where Δ_r has two distinct positive roots r_- and r_+. For the latter, a timelike double ring singularity occurs at $r=0$, $P=0$. The lines at $r=\pm\infty$ represent the spatial infinity which is asymptotically AdS and thus timelike. These lines are infinitely far from internal points of the manifold when the distance is measured along geodesics. The intersection between horizons and lines $r=\pm\infty$ represent the timelike and light-like future (or past) for the stationary regions confining with $r=\pm\infty$. These intersections represent also the spatial infinity for the internal regions bounded by horizons. In all cases, these intersections are infinitely far from internal points when the distance is measured along corresponding geodesics. The lines at $r=r_\pm$ are future and past event horizons respectively for regions confining with $r=\pm\infty$ and Cauchy horizons for Cauchy surfaces belonging to the internal diamond-shaped globally hyperbolic regions. The patch repeats itself infinitely in the vertical direction.

(i) If $l^2<a^2(7+4\sqrt{3})$ and $\eta\in\mathbb{R}$ there is only one positive solution r_+ of Eq. (27) and only one negative solution r_-. For $r>r_+$ and $r<r_-$, Δ_r is positive and ∂_r is spacelike. For $r_-<r<r_+$, ∂_r becomes timelike. r_- and r_+ are first order zeros. The causal structure on the axis is given in Fig. 1.

(ii) If $l^2=a^2(7+4\sqrt{3})$ and (a) $\eta\neq\pm\eta_0$, $\eta_0=(4l/3)(26\sqrt{3}-45)^{1/2}>0$, then the solutions behave as in case (i), (b) $\eta=-\eta_0$, then there is a first order root r_- for $r<0$ and a third order root $r_+=[(l^2-a^2)/6]^{1/2}$ for $r>0$ and Δ_r changes sign by crossing the roots, and (c) $\eta=\eta_0$, then there is a first order root r_+ for $r>0$ and a third order root $r_-=-[(l^2-a^2)/6]^{1/2}$ for $r<0$. Again Δ_r changes sign crossing the roots. The Penrose-Carter diagrams in the cases (b) and (c) are also given by Fig. 1.

(iii) If $l^2>a^2(7+4\sqrt{3})$ and $\eta\leqslant 0$ we have again several subcases. Let

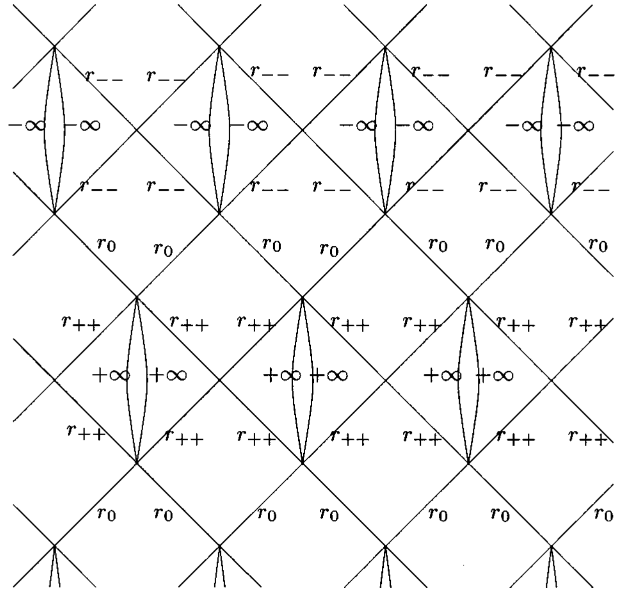

FIG. 2. $\theta=0$ Penrose-Carter diagram for the $g>1$ black hole in one extreme case where r_- and r_+ coincide [case (iii,b), $r_-=r_+=:r_0$]. The infinities are not joined together. The patch repeats itself infinitely in the horizontal and the vertical direction.

$$R_\pm=\sqrt{\frac{1}{6}(l^2-a^2)\mp\frac{1}{6}\sqrt{(l^2-a^2)^2-12l^2a^2}},$$

$$\eta_\pm=-\frac{2R_\pm}{3l^2}\left[(l^2-a^2)\pm\frac{1}{2}\sqrt{(l^2-a^2)^2-12l^2a^2}\right]. \tag{28}$$

(Note that $\eta_-<\eta_+<0$). (a) for $0\geqslant\eta>\eta_+$ Δ_r behaves as in (i); (b) for $\eta=\eta_+$, Δ_r has two positive zeros $r_+=R_+$ and $r_{++}>r_+$ and a negative zero r_{--}. At $r=r_+$ the graph of Δ_r versus r is tangent to the r axis and Δ_r does not change sign (r_+ is a second order zero), whereas at $r=r_{++}$ and $r=r_{--}$, Δ_r changes sign from negative to positive values and from positive to negative values respectively, as r increases. These are first order zeros. The causal structure is shown in Fig. 2; (c) for $\eta_-<\eta<\eta_+$, Δ_r has three positive zeros r_-, r_+, r_{++} and one negative zero r_{--} where Δ_r changes sign. These zeros are first order; the causal structure is shown in Fig. 3; (d) in the case $\eta=\eta_-$ one obtains again two positive roots r_- and $r_{++}=R_->r_-$, and a negative root r_{--}. At r_- and r_{--}, which are first order roots, Δ_r changes sign from $-$ to $+$ and from $+$ to $-$ respectively, whereas at r_{++}, which is a second order root, Δ_r does not change sign. For the corresponding Penrose-Carter diagram see Fig. 4; and (e) for $\eta<\eta_-$ we get again the same behavior as in (i).

(iv) If $l^2>a^2(7+4\sqrt{3})$ and $\eta>0$ the discussion of the roots is symmetric to that for the case (iii), considering the symmetry of Eq. (27) under the combined inversion $r\rightarrow -r$, $\eta\rightarrow-\eta$. In this case one has in general one positive first order zero and up to three negative zeros. All zeros of Δ_r in the examined cases are merely coordinate singularities, similar to the Schwarzschild case. They represent horizons, as the normals to the constant t and constant r surfaces become null when r is a root of $\Delta_r=0$. The pair of outermost

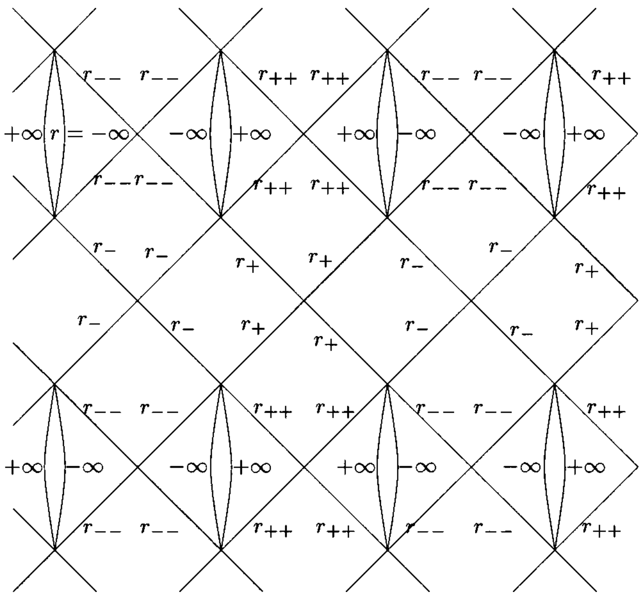

FIG. 3. $\theta=0$ Penrose-Carter diagram for the $g>1$ black hole in the case where Δ_r has four distinct real roots r_{--}, r_-, r_+, r_{++}. The infinities $r=+\infty$ and $r=-\infty$ are not joined together. The patch repeats itself infinitely in the horizontal and the vertical direction.

horizons r_H [e.g., $r_H=r_{++}$ and $r_H=r_{--}$ in case (iii,d)] are also event horizons as the Killing trajectories in the exterior stationary domains never intersect the surfaces $r=r_H$. The future parts of these event horizons are the boundary of the causal past of all timelike inextendible geodesics contained in the respective stationary regions which reach the future timelike infinity (see the Penrose-Carter diagrams).

However, the resulting causal structure is rather intricate. We notice the complete absence of metric singularities at $r=0$. This allows one to consider the coordinate r in the complete range $(-\infty,+\infty)$ as we did above.

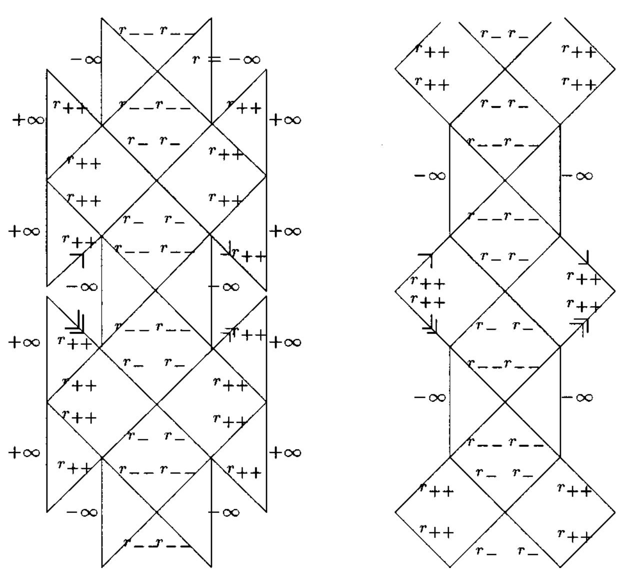

FIG. 4. $\theta=0$ Penrose-Carter diagram for the $g>1$ black hole in the other extreme case where r_{++} and r_+ coincide [case (iii,d)]. Now it is no longer possible to cover the plane with the elementary patches in a usual manner. Therefore one has to make some identifications as indicated by the arrows. The patch as well as the identifications repeat themselves infinitely in the vertical direction.

We remark that there is an extreme case [(iii,d)] which for $a\to 0$ gives the naked singularity discussed in [2,3], but for $a>0$ still represents a black hole. Hence the nonrotating naked singularity is unstable, as it turns into a black hole by an infinitesimal addition of angular momentum. This seems to lend some support to the cosmic censorship conjecture.

In all cases discussed above, the outermost zeros represent event horizons. Their Gaussian curvature is given by

$$K=-\frac{1}{\rho_H^6}\left[(\rho_H^2-4a^2\cosh^2\theta)\left((r_H^2+a^2)\Delta_\theta+\frac{\rho_H^2a^2}{l^2}\sinh^2\theta\right)+\frac{4a^2\rho_H^4}{l^2}\cosh^2\theta\right], \tag{29}$$

where the index H indicates that the corresponding quantities are to be evaluated on the event horizon r_H. K is no more constant as in the nonrotating case, because the horizon has been warped by the rotation.

C. Angular velocity and surface gravity

At least for $\eta>-al^{-2}(a^2+l^2)$, the positive event horizon (as well as any $r=\text{const}>0$ surface) rotates relative to the stationary frame at infinity, where ∂_t is timelike, with angular velocity $\Omega_H=\omega(r_H,\theta)$, where ω is given by Eq. (14), which yields

$$\Omega_H=\frac{a}{r_H^2+a^2}. \tag{30}$$

Notice that $\omega(r,\theta)$ is just given by $d\phi/dt$ along timelike trajectories with fixed values for r and θ, t being proportional to the proper time τ according to $t=(\Xi\Sigma/\rho\sqrt{\Delta_\theta\Delta_r})\tau$. These are trajectories of corotating observers.

There also exists a dragging effect at infinity, as ω is nonvanishing there, its value being $\Omega_\infty=a/(a^2+l^2)$.

The surface gravity κ is another important property of the event horizon. It is normally defined in terms of the null, future pointing generators of the horizon, using

$$l^c\nabla_c l^a=\kappa l^a, \quad l^a=\partial_t+\Omega_H\partial_\phi. \tag{31}$$

However, although in the present case ∂_t still is a global Killing field, the vector ∂_ϕ is only a local Killing field, because of the procedure used to build up S_g. This agrees with the known result that Riemann surfaces with $g>1$ admit no global Killing fields, nor even global conformal Killing fields. Nevertheless, the surface gravity can still be defined as the acceleration per unit coordinate time which is necessary to hold in place a corotating particle (i.e., one at some fixed r and θ) near the event horizon. Such a particle will move on the trajectories considered above, where $\omega=d\phi/dt$. These trajectories are integral curves of the vector field $u=N^{-1}(\partial_t+\omega\partial_\phi)$, which is timelike everywhere in the $r>0$ exterior domain bounded by the outermost event horizon. Notice that Nu is a timelike Killing field and thus the exterior domain is stationary. The function N normalizing the four-velocity is the lapse function of the foliation determined by the Killing coordinate time t, and is

$$N^2 = \frac{\rho^2 \Delta_\theta \Delta_r}{\Xi^2 \Sigma^2}. \quad (32)$$

By computing the four-acceleration, one obtains in this way

$$\kappa = \frac{1}{2(a^2+l^2)(r_H^2+a^2)} \left[3r_H^3 + (a^2-l^2)r_H + \frac{a^2 l^2}{r_H} \right]. \quad (33)$$

Remarkably, this is constant over the event horizon even in the absence of a true rotational symmetry. In view of this last fact, the meaning of the surface gravity as the quantum temperature of the black hole remains a little bit obscure. The fact is that, although one can define a conserved mass by using the time translation symmetry of the metric, one cannot define a strictly conserved angular momentum, but only a conserved angular momentum with respect to a special choice of the observers at infinity. Hence the status of the first law for such black holes certainly needs further clarifications. As we will see, the situation will be rather different for toroidal black holes, which behave quite similarly to the Kerr solution. This also suggests that higher genus rotating black holes may be a kind of stable soliton solution in anti–de Sitter gravity.

From the metric (10) we may read off

$$g_{tt} = \frac{a^2 \Delta_\theta \sinh^2\theta - \Delta_r}{\rho^2 \Xi^2}. \quad (34)$$

From this expression, one recognizes that g_{tt} may change sign within all regions where $\Delta_r > 0$. For that reason one cannot define ''static'' comoving observers with the coordinates t, r, θ, ϕ near the outermost horizons but only ''nonstatic'' corotating observers as we did above. Anyhow, $g_{tt} < 0$ for $|r|$ sufficiently large. The surface where $g_{tt} = 0$ inside any region where $\Delta_r > 0$ is one of the boundaries of an ergoregion in which both ∂_t and ∂_r are spacelike. This is therefore a stationary limit surface, locally determined by

$$a^2 \Delta_\theta \sinh^2\theta = \Delta_r. \quad (35)$$

The remaining boundaries of this ergoregion are event horizons located at roots of Δ_r. These are general features of rotating black hole metrics. Furthermore, similarly to the Kerr solution, the event horizon and the surrounding stationary limit surface meet at $\theta = 0$, where they are smoothly tangent to each other provided Δ_r vanishes in a first-order zero.

D. Mass and angular momentum

The two conserved charges which are associated with a rotating self-gravitating system are the mass and the angular momentum.

One approach to a general and sensible definition of conserved charges associated to a given spacetime, is the canonical Arnowitt-Deser-Misner (ADM) analysis appropriately extended to include non-asymptotically flat solutions. This led to the introduction of the more general concept of quasilocal energy [32] for a spatially bounded self-gravitating system, and more generally, to various other quasilocal conserved charges. These may be obtained as follows. One considers a spacetime enclosed into a timelike three-boundary B, which is assumed to be orthogonal to a family of spatial slices, Σ_t, foliating spacetime. The slices foliate the boundary into a family of 2-surfaces $\mathcal{B}_t = \Sigma_t \cap B$ (which need not be connected), and these will have outward pointing spacelike normals in Σ_t, denoted ξ^a, and future pointing normals in B, denoted u^a. The validity of vacuum Einstein's equations in the inner region, with or without cosmological constant, then implies along B the usual diffeomorphism constraint of general relativity:

$$D_a(\Theta^{ab} - b^{ab}\Theta) = 0, \quad \Theta = \Theta^a{}_a, \quad (36)$$

where b_{ab} is the boundary three-metric, D_a the associated covariant derivative along B, and Θ_{ab} its extrinsic curvature. If now the boundary three-metric admits a Killing vector K^a, then contracting Eq. (36) with K_a and integrating over B from $\mathcal{B}_{t_1}$ to $\mathcal{B}_{t_2}$ one obtains the conservation law $Q_K(t_1) = Q_K(t_2)$, where the conserved charge is

$$Q_K(t) = -\frac{1}{8\pi} \int_{\mathcal{B}_t} [\Theta_{ab} - b_{ab}\Theta] K^a u^b \sqrt{\sigma}. \quad (37)$$

The quasilocal mass is then defined to be the charge associated with the time evolution vector field of the foliation Σ_t, when this is a symmetry of the boundary geometry. This field will be $K^a = N u^a + V^a$, with lapse function N and the shift vector V^a constrained to be tangent to B. In this way the time evolution of the three-geometry on Σ_t induces a well defined time evolution of the two-geometry of $\mathcal{B}_t$ along B. The quasilocal energy is defined for observers which travel orthogonally to $\mathcal{B}_t$ in B, i.e., for $K^a = u^a$ and is, from Eq. (37),

$$E(\mathcal{B}_t) = -\frac{1}{8\pi} \int_{\mathcal{B}_t} [\Theta_{ab} u^a u^b + \Theta] \sqrt{\sigma}. \quad (38)$$

It can be shown that $E(\mathcal{B}_t)$ is minus the rate of change of the on-shell gravitational action per unit of *proper time* along the timelike boundary B [32], a fact which motivates the definition. However, as u^a is not in general a symmetry of the boundary, the quasilocal energy in not conserved, e.g., gravitational waves may escape from the region of interest, and it can also be negative (binding energy, cf. [33]). In our non-asymptotically flat context, where the lapse function diverges at infinity, one can define the quasilocal energy by measuring the rate of change of the action per unit of *coordinate time*. Then one uses $K^a = N u^a$ and the energy is as in Eq. (38) but with a further factor N under the integral, so we denote it by $E_N(\mathcal{B}_t)$. Similarly, the angular momentum will be the charge associated to a rotational symmetry, generated by a spacelike Killing field $\tilde{K}^a$.

It is very important that York and Brown's quasilocal charges be functions of the canonical data alone. If background subtractions were necessary, these ought to be chosen appropriately to achieve this requirement. The quasilocal mass can also be arrived at by a careful handling of the boundary terms in the Hamiltonian for general relativity. Then one arrives at the equivalent expression for the mass [32,34,35], as measured from infinity:

$$M=-\frac{1}{8\pi}\int_{S_g(R)}[N(\widetilde{\Theta}-\widetilde{\Theta}_0)-16\pi(P_{ab}V^a\xi^b-(P_{ab}V^a\xi^b)_{|0})]\sqrt{\sigma}\,d^2x, \tag{39}$$

where quantities with a subscript 0 denote background subtractions, chosen so that M is a function of the canonical data alone [32], and the limit $R\to\infty$ is understood.

In our case, $S_g(R)$ is an asymptotic Riemann surface at $r=R$ embedded in a $t=$const slice, with outward pointing normal ξ^a and extrinsic curvature $\widetilde{\Theta}$, P_{ab} is the momentum canonically conjugate to the metric induced on the slice, and (N,V^a) are the lapse function and the shift vector of the $t=$const foliation.

The charge associated to a rotational Killing symmetry generated by $\widetilde{K}^a$ can also be written as a function of the canonical data, and is

$$J=-2\int_{S_g}[P_{ab}\widetilde{K}^a\xi^b-(P_{ab}\widetilde{K}^a\xi^b)_{|0}]\sqrt{\sigma}\,d^2x. \tag{40}$$

Unlike the case of nonrotating topological black holes, where a natural choice for the background can be made, no distinctive background metric has been found in the present case. The best we are able to do is to define the mass relative to some other solution with the same topology and rotation parameter. In spite of the dragging effect at infinity and the intricate form of the metric, what we get is the very simple result

$$M=\frac{\eta-\eta_0}{4\pi\Xi(r_H^2+a^2)}\mathcal{A}_H, \tag{41}$$

where $\mathcal{A}_H$ is the horizon area. η can be expressed as a function of the outermost horizon location r_H, by using $\Delta_r(r_H,\eta)=0$. Thus η really is related to the Hamiltonian mass, albeit in a relative sense. The quasilocal energy is not equal to the quasilocal mass and is not even equal to the mass in the limit $R\to\infty$, a consequence of the dragging effect at infinity. Indeed, we obtain the trivial result that $E(R)-E_0(R)=0$, if the background has the same rotation parameter but different η. Thus all solutions with equal a have the same quasilocal energy.

Concerning the angular momentum we are in a different position, since there is no global rotational Killing symmetry. However, the vector $\widetilde{K}=\partial_\phi$, although it is not a Killing vector, obeys locally the condition $\nabla_{(a}\widetilde{K}_{b)}=0$ and is therefore a kind of approximate symmetry, we could say a locally exact symmetry. We may try to compute J using Eq. (40) with $\widetilde{K}=\partial_\phi$. Then one finds that J is already finite without any subtraction and we get $J=\Xi^{-2}\eta a\,\mathcal{I}$, where the integral

$$\mathcal{I}=\frac{3}{8\pi}\int_{S_g}\sinh^3\theta\,d\theta\,d\phi \tag{42}$$

has to be performed over a fundamental domain of the Riemann surface S_g (we were unable to do this, however). This is weakly conserved in the sense that it depends on the choice of a spatial slice in the three-boundary at infinity. Due to these facts, the ''first law'' and the full subject of black hole thermodynamics needs further clarifications here. In this connection, one should use a kind of quasilocal formalism for black hole thermodynamics, along the lines of Brown *et al.* [36] for asymptotically anti–de Sitter black holes.

One may note, among other things, that $J=0$ for the locally anti–de Sitter solution corresponding to $\eta=0$, in agreement with Holst's and Peldan's theorem [10]. Physically, in this case $\Omega_H=\Omega_\infty$ and the horizon does not rotate relative to the stationary observers at infinity.

IV. THE ROTATING TOROIDAL BLACK HOLE

We discuss now another black hole solution in anti–de Sitter gravity which represents a rotating torus hidden by an event horizon. The first solution of this kind has been discovered by Lemos and Zanchin [11] by compactifying a charged open black string. This is a solution that can be obtained from the nonrotating toroidal metric by mixing time-angle variables into new ones. This is not a permissible coordinate transformation in the large, as angles, unlike time, are periodic variables. This is why the solutions one obtains are globally different, as clearly shown by Stachel while investigating the gravitational analogue of the Aharanov-Bohm effect [37].

The metric we shall present cannot be obtained by forbidden coordinate mixing, but it can be obtained from the general Petrov type-*D* solution already presented by a simple choice of parameters. By requiring the existence of the nonrotating solution (which we know to exist) and the time inversion symmetry, $t\to-t$, $\phi\to-\phi$, we get the following metric tensor:

$$ds^2=-N^2dt^2+\frac{\rho^2}{\Delta_r}dr^2+\frac{\rho^2}{\Delta_P}dP^2+\frac{\Sigma^2}{\rho^2}(d\phi-\omega\,dt)^2, \tag{43}$$

where P is a periodic variable with some period T, ϕ is another angular variable with period 2π and

$$\rho^2=r^2+a^2P^2,\quad \Delta_P=1+\frac{a^2}{l^2}P^4, \tag{44}$$

$$\Delta_r=a^2-2mr+l^{-2}r^4,\quad \Sigma^2=r^4\Delta_P-a^2P^4\Delta_r. \tag{45}$$

Finally, the angular velocity and the lapse are given by, respectively

$$\omega=\frac{\Delta_rP^2+r^2\Delta_P}{\Sigma^2}a,\quad N^2=\frac{\rho^2\Delta_P\Delta_r}{\Sigma^2}. \tag{46}$$

The solution is obtained as a limit case of the Plebanski-Demianski metric by setting $\varepsilon=0$, $\gamma=a^2$ and rescaling $p=aP$ (this last to have the limit $a\to0$).

The metric induced on the spacelike two-surfaces at some constant r and t is then

$$d\sigma^2=\frac{a^2P^2+r^2}{\Delta_P}dP^2+\frac{\Sigma^2}{a^2P^2+r^2}d\phi^2. \tag{47}$$

As long as $\Sigma^2>0$, this is a well defined metric on a cylinder, but as it stands it cannot be defined on the torus which one gets identifying some value of P, say $P=T/2$, with $P=-T/2$. This is because the components of the metric are even, rational functions of P but have unequal derivatives at $\pm T/2$. Thus we need to cover $S^1\times S^1$ with four coordinate patches, and set $P=\lambda\,\sin\theta$ in a neighborhood of $\theta=0$ and $\theta=\pi$ and $P=\lambda\,\cos\theta$ in a neighborhood of $\theta=\pi/2$ and $\theta=3\pi/2$, where λ is a constant needed to match the length of the circle to the chosen value T. On the overlap $\cos\theta$ is a C^∞ function of $\sin\theta$ and vice versa, so now the metric is well defined and smooth on a torus.

Even on the cylinder, the metric (43) represents a rotating cylindrical black hole not isometric to the one discussed by Lemos [11,18] or Santos [17], which are stationary generalizations of the general static cylindrical solution found by Linet [16]. Thus in this case we have not a unique solution, but rather a many-parameter family of stationary, locally static metrics. This was to be expected as whenever the first Betti number of a static manifold is nonvanishing, there exists in general a many-parameter family of locally static, stationary solutions of Einstein's equations, a fact which can be regarded as a gravitational analogue of the Aharanov-Bohm effect [37].

We shall study now the metric (43) for $m>0$. Notice the symmetry under the combined inversion $r\to -r$, $m\to -m$. The metric coefficients are functions of (r,P) and P is identified independently of ϕ. Therefore the metric has a global rotational symmetry (unlike the higher genus solutions) and is stationary. We shall consider mostly the region $r\geqslant 0$ which has a black hole interpretation and is the physically relevant region for black holes forming by collapse. Anyhow, the metric (43) admits a sensible continuation to $r<0$.

A. Singularity and horizons

The event horizons arise from the zeros of Δ_r. In the case $m>0$ that we are considering, all zeros may appear in the region $r\geqslant 0$ only (see Fig. 1 for the causal structure in the nonextreme case). Considering the metric (43), one finds that there is a critical value, a_c, for the rotation parameter a, such that for $a>a_c$ the solution is a naked singularity. For $0\leqslant a<a_c$ there are two positive first order roots, r_+ and r_- with $r_+\geqslant r_-$, which coalesce at the second order root $r_+=r_-=(ml^2/2)^{1/3}$ when $a=a_c$. This critical value is

$$a_c=\sqrt{3}(m/2)^{2/3}l^{1/3}. \tag{48}$$

The event horizon is located at the larger value r_+, and has a surface gravity

$$\kappa=\frac{2r_+^3-ml^2}{l^2r_+^2}. \tag{49}$$

The surface gravity vanishes when $a=a_c$ and the metric describes an extreme black hole (see Fig. 5). Finally, there is a curvature singularity at $\rho^2=0$, namely at $r=P=0$. As a point set at fixed time, this is $\{p,q\}\times S^1$, where $\{p,q\}$ are the two points on the torus at $r=0$ which correspond to $P=0$, and looks like a pair of disjoint ring singularities. Another point of interest is that $g_{tt}>0$ and $\Sigma^2<0$ in a neighborhood

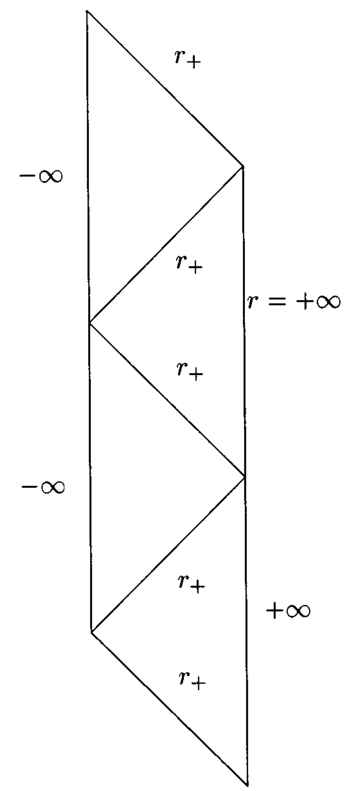

FIG. 5. Penrose-Carter diagram for the toroidal black hole in the extreme case where r_+ and r_- coincide. Again, a timelike double ring singularity occurs at $r=0$, $P=0$. The patch repeats itself infinitely in the vertical direction.

of $r=0$. Therefore the torus turns into a Lorentzian submanifold with ϕ becoming a timelike coordinate. Evidently there are closed timelike curves around the origin. As we can see, the situation is quite similar to the Kerr metric, except that the Euler characteristic of the horizon now vanishes. To check this, notice that the metric on the horizon is, locally,

$$d\sigma^2=\frac{a^2P^2+r_+^2}{\Delta_P}\,dP^2+\frac{r_+^4\Delta_P}{a^2P^2+r_+^2}\,d\phi^2. \tag{50}$$

This metric can be written in conformally flat form by factoring out the $\phi\phi$ component, which is smooth and positive. The conformal metric has $\sigma_{\phi\phi}=1$ and it turns out to be flat. The actual metric is thus conformally flat and defined on a compact domain. The scalar curvature of a conformally flat manifold is a total divergence and vanishes when integrated over a closed manifold. Therefore the Euler characteristic vanishes and the horizon, which we assumed to be compact and orientable, must be a torus. Furthermore, by rescaling the metric with a constant parameter μ, we can see that the periods scale as $2\pi\to 2\pi\mu$, $T\to\mu T$. Therefore it is the ratio of the periods that is conformally invariant. This ratio determines the conformal class of the torus and is the analogue of the more familiar Teichmüller parameter. Since all surfaces at constant r take on the topology of a torus, $\mathcal{T}^2$, the topology of the external region (the domain of outer communication in Carter's language) is that of $\mathbb{R}^2\times\mathcal{T}^2$. Finally, a few comments on the presence of ergoregions are in order. Consideration of the metric (43) lead us to

$$g_{tt}=\frac{a^2\Delta_P-\Delta_r}{\rho^2}. \tag{51}$$

From this expression, we see that $g_{tt}>0$ within the regions where $\Delta_r<0$. Conversely, within regions where $\Delta_r>0$, outside the external horizon in particular, g_{tt} may change its sign becoming positive for $|r|$ sufficiently small and negative for large $|r|$. In fact, as in the previously considered case, ergotori appear within the regions $\Delta_r>0$. In particular, this happens outside the outermost event horizon. Ergoregions, where both ∂_t and ∂_r are spacelike, are bounded by event horizons and the surfaces at $g_{tt}=0$, given by the implicit equation

$$P^4=\frac{r^4-2ml^2r}{a^4}. \tag{52}$$

Differently from the previously examined class of topological rotating black holes, surfaces at $g_{tt}=0$ and horizons do not meet in the case of a toroidal rotating black hole. Indeed, surfaces at $g_{tt}=0$, in the region outside the external horizon fill the interval $[r_m,r_e]$ where $r_+<r_m=(2ml^2)^{1/3}$ and r_e is the positive root of $r^4-2ml^2r-\lambda^4a^4=0$. (There is another surface at $g_{tt}=0$ for $r<0$, filling the interval $[r'_e,0]$ where r'_e is the remaining negative solution of the above equation.)

B. Mass and angular momentum

With the given choice of the periods (T for P and 2π for ϕ), we determine the area of the event horizon to be

$$\mathcal{A}=2\pi Tr_+^2 \tag{53}$$

and the angular velocity, $\Omega_H=ar_+^{-2}$. The Hamiltonian mass of the given spacetime, relative to the background solution with toroidal topology but $m=a=0$, can be computed by carefully handling the divergent terms appearing when the boundary of spacetime is pushed to spatial infinity. Also, the Killing observers at infinity relative to which the mass is measured have a residual, P-dependent angular velocity

$$\Omega_\infty=al^{-2}P^2 \tag{54}$$

and this also must be taken into account. All calculations done, we get a conserved mass M and a conserved angular momentum J according to

$$M=\frac{mT}{2\pi}, \quad J=Ma, \tag{55}$$

giving to a the expected meaning. As a is bounded from above by a_c, in order for the solution to have a black hole interpretation, we see that the angular momentum is bounded by a power $M^{5/3}$ of the mass.

V. CONCLUSION

We have presented a class of exact solutions of Einstein's equations with negative cosmological constant, having many of the features which are characteristic of black holes. All these solutions are of Petrov-type D and the horizons, when they exist, have the topology of Riemann surfaces and therefore they lack rotational symmetry for genus $g>1$. Among the solutions there is also a toroidal black hole, still different from the Lemos-Zanchin solution. The toroidal metric has an exact rotational symmetry and a well-defined mass and angular momentum. From this perspective, it is more promising as a thermodynamical object and one may hope to find suitable generalizations of the ''four laws of black hole mechanics'' [38]. Apart from this, the solutions seem to be interesting in their own right; they have intriguing properties and may provide further ground to test string theory ideas in black hole physics and the character of singularities in general relativity.

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