Evolution of the Bianchi type I, Bianchi type III, and the Kantowski-Sachs universe: Isotropization and inflation

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We study the Einstein-Klein-Gordon equations for a convex positive potential in a Bianchi type I, a Bianchi type III, and a Kantowski-Sachs universe. After analyzing the inherent properties of the system of differential equations, the study of the asymptotic behaviors of the solutions and their stability is done for an exponential potential. The results are compared with those of Burd and Barrow. In contrast with their results, we show that for the Bianchi type I case isotropy can be reached without inflation and we find new critical points which lead to new exact solutions. On the other hand, we recover the result of Burd and Barrow, that if inflation occurs, then isotropy is always reached. The numerical integration is also done and all the asymptotical behaviors are confirmed. [S0556-2821(98)03510-3]

I. INTRODUCTION

Inflation, first introduced by Guth [1], was introduced in the standard cosmological model to solve the homogeneity, isotropy, and horizon problems. The latter is well explained due to the fact that inflation is characterized by an exponential or power-law expansion of the universe and at the same time a quasiconstant behavior of the Hubble horizon.

On the other hand, the homogeneity and isotropy problems are in fact not well explained because from the start the homogeneous and isotropic Friedman-Lemaître metric is used. To really solve the problem one should start with an arbitrary metric and show that inflation takes place and that the universe evolves towards a Friedman-Lemaître metric. The problem of the onset of inflation was considered numerically for spherical inhomogeneous cosmologies [2,3] and a seminumerical analysis was done for inhomogeneous, quasi-isotropic universes [4] using the long-wavelength iteration scheme [5,6]. They showed that a large initial inhomogeneity suppresses the inflation stage. Because of the analytical difficulties of the task, one can, as a first step, consider only a homogeneous but anisotropic metric and try to solve the isotropy problem.

This task was first undertaken by Collins and Hawking [7] who showed that, within the Bianchi-type universe filled by matter satisfying the dominant energy condition and positive pressure criterion, the isotropy problem can only be solved for types I, V, VII$_o$, and VII$_h$. They showed also that only a subclass of vanishing measure in the space of all homogeneous initial conditions can approach isotropy.

With the presence of an inflationary stage, when the dominant energy condition is violated, there was hope to obtain a cosmic no-hair theorem. The study done by Heusler [8] showed that there is no no-hair theorem for a real scalar field having a convex positive potential with a vanishing local minimum in a Bianchi-type universe. In fact, isotropy can only be approached if the underlying Lie group of the Bianchi-type metric is compatible with a Friedman-Lemaître model.

In addition to the Bianchi-type metrics, the Kantowski-Sachs model also describes a spatially homogeneous universe. This model with a perfect fluid description of matter and with or without a cosmological constant has been studied by a number of authors [9,10,11,12,13]. They found an anisotropic asymptotical behavior of the model. Burd and Barrow [14] analyzed this system with a real scalar field having an exponential potential as a source. Among others, they found an anisotropic asymptotical behavior or that inflation can occur depending on the value of the coupling constant entering in the definition of the potential. The exponential potential is motivated by the fact that it can be obtained, for example, by dimensional reduction of more fundamental theories [15] or in conformal equivalent theories of gravity whose Lagrangian is an arbitrary analytic function of the scalar curvature [16,17,18].

In this work, we will consider a real scalar field with a convex positive potential, but not necessarily with a local minimum, in a Bianchi type I, a Bianchi type III, and a Kantowski-Sachs model. These three model are in fact very closely related since it will be shown in Sec. II that the differential equations describing the evolution of the models are the same and that only the constraint equations differ. In Sec. III we study the inherent properties of the differential equations and we recover in the case of a local minimum of the potential a result of Heusler [8]. We also show why the solution of the dynamical system must be studied for a potential having a vanishing potential at infinity. Section IV is devoted to the special case of an exponential potential. The result obtained will be compared with the analysis done in Ref. [14]. In Sec. V the numerical integration of the system is done, and in Sec. VI the results will be summarized.

II. BASIC EQUATIONS

We consider a scalar field with a convex positive potential V in a homogeneous universe having the following metric:

$$ds^2 = g_{\mu\nu}\theta^\mu\theta^\nu = -\theta^0\theta^0 + \delta_{ij}\theta^i\theta^j, \tag{1}$$

where

$$\theta^0 = dt, \quad \theta^1 = a(t)dr, \quad \theta^2 = b(t)d\vartheta, \quad \theta^3 = b(t)f(\vartheta)d\phi,$$

with $f(\vartheta) = \vartheta$, $f(\vartheta) = \sinh(\vartheta)$ or $f(\vartheta) = \sin(\vartheta)$. We have, respectively, a Bianchi type I (BI), a Bianchi type III (BIII), and a Kantowski-Sachs (KS) metric.

The action is given by

$$S = \int \left(-\frac{R}{16\pi G} + \frac{1}{2} e_\mu \varphi e^\mu + V(\varphi) \right) \sqrt{-g} \; d^4x, \quad (2)$$

where g is the determinant of the metric and e_μ is the dual basis of θ^μ. By varying the action with respect to the metric, we get the Einstein equations

$$2H_a H_b + H_b^2 + \frac{k}{b^2} = 8\pi G \left(\frac{1}{2} \dot{\varphi}^2 + V(\varphi) \right), \quad (3)$$

$$2\dot{H}_b + 3H_b^2 + \frac{k}{b^2} = 8\pi G \left(-\frac{1}{2} \dot{\varphi}^2 + V(\varphi) \right), \quad (4)$$

$$\dot{H}_a + \dot{H}_b + H_a^2 + H_a H_b + H_b^2 = 8\pi G \left(-\frac{1}{2} \dot{\varphi}^2 + V(\varphi) \right), \quad (5)$$

with

$$H_a = \frac{\dot{a}}{a}, \quad H_b = \frac{\dot{b}}{b}, \quad k = -\frac{1}{f} \frac{d^2 f}{d\vartheta^2}. \quad (6)$$

For the BI, BIII and KS cases, we get, respectively, $k=0$, $-1, 1$.

The Klein-Gordon equation is obtained by varying the action with respect to the scalar field. We get

$$\ddot{\varphi} + (H_a + 2H_b)\dot{\varphi} + \frac{dV}{d\varphi} = 0. \quad (7)$$

In the following we will express every quantity in units of the Planck mass $m_p = 1/\sqrt{8\pi G}$. This can be achieved by setting $8\pi G = 1$ in the Einstein equations. The system is fully determined by the independent equations (3), (4), and (7). One can easily show that Eq. (5) follows from the others. This is a reflection of the Bianchi identities. After some algebraic manipulations, these equations can be written as a set of four first-order differential equations, which are k independent, and a constraint [Eq. (3)] which is conserved in the evolution.

This set of equations can also be written as a function of the expansion rate θ and the shear tensor $\sigma_{\mu\nu}$ of the hypersurface of constant time, Σ. We get, for the first-order equations,

$$\dot{\theta} = -\frac{1}{3} \theta^2 - 2\sigma^2 + V(\varphi) - \psi^2, \quad (8)$$

$$\dot{\sigma} = -\frac{1}{3\sqrt{3}} \theta^2 - \theta\sigma + \frac{1}{\sqrt{3}} \sigma^2$$
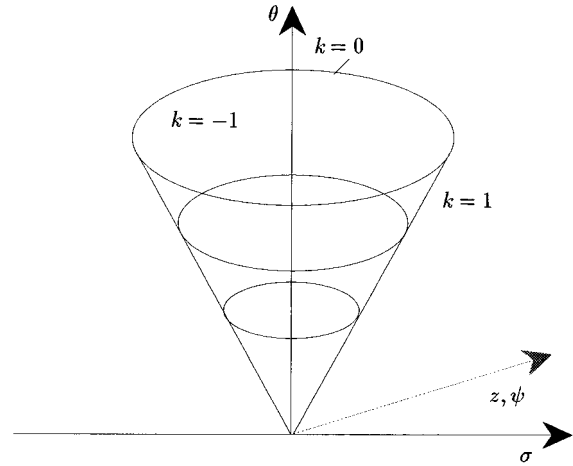

FIG. 1. Region of the space with coordinates $(\theta, \sigma, z, \psi)$ where the solutions lie as function of k. The solutions for the BI case are restricted in a submanifold of codimension 1, the "light cone."

$$+ \frac{1}{\sqrt{3}} \left(V(\varphi) + \frac{1}{2} \psi^2 \right), \quad (9)$$

$$\dot{\varphi} = \psi, \quad (10)$$

$$\dot{\psi} = -\theta\psi - \frac{dV}{d\varphi}, \quad (11)$$

where $\sigma = \frac{1}{2} \sigma_{\mu\nu} \sigma^{\mu\nu} = (\sqrt{1/3})(H_a - H_b)$ and $\theta = H_a + 2H_b$. ψ is defined by Eq. (10). As for the constraint equation, it reads

$$\frac{1}{3} \theta^2 + \frac{k}{b^2} = \sigma^2 + V(\varphi) + \frac{1}{2} \psi^2. \quad (12)$$

The solutions of Eqs. (8)–(11) do not depend on the type of the homogeneous model considered. Since Eq. (12) is conserved, we have to specify the homogeneous model only in the initial conditions. In the four-dimensional space with coordinates $(\theta, \sigma, z = \sqrt{V}, \psi)$ the BI solutions are on the "light cone," the BIII ones are in it, and the KS solutions remain outside the light cone (see Fig. 1).

The advantage of taking these geometrical meaningful quantities is obvious: the shear tensor expresses the direction-dependent deviation from the global expansion. Hence σ measures the anisotropy. As a criterium of isotropization, we will not use the vanishing of σ (as in [19]), but a stronger condition [7,14], which is

$$\frac{\sigma}{\theta} \to 0 \quad \text{as} \quad t \to \infty. \quad (13)$$

In this paper, we want to analyze the evolution of an expanding universe. Hence we will restrict ourselves to positive values of θ. We want to know under which conditions the isotropization of this model is generic and when inflation occurs. In some cases, general properties of the solutions can be found by just analyzing the differential equations.

III. INHERENT PROPERTIES OF THE DYNAMICAL SYSTEM

This section generalizes, in some respect, a result obtained by Heusler [8]. In the BI ($k=0$) and BIII ($k=1$) case, the asymptotic behavior of the solutions can be obtained directly by the study of Eqs. (8)–(12). Since we are interested in the evolution of an expanding universe, we suppose that θ is positive or equal to zero at some time t_0. We also restrict ourselves to positive and convex function V. From Eqs. (8) and (12), we get the following inequality relation ($k \leq 0$):

$$\dot{\theta} \leq V - \frac{1}{3}\theta^2 \leq 0. \tag{14}$$

Using the last inequality relation and again Eqs. (8),(12), it is easy to obtain

$$\dot{\theta} + \theta^2 = 3V - \frac{k}{b^2} \geq 0. \tag{15}$$

Setting $\mathcal{V} = ab^2$, which is positive, we can write

$$\theta = \frac{\dot{\mathcal{V}}}{\mathcal{V}}. \tag{16}$$

By hypothesis, $\dot{\mathcal{V}}$ is positive at t_0. Substituting Eq. (16) in Eq. (15), we have that $\dot{\mathcal{V}}/\mathcal{V} \geq 0$. Therefore, $\dot{\mathcal{V}}$ can only increase with time and θ will remain positive after t_0. We have shown that θ is a positive, monotonic decreasing function and, thus, we must have

$$\dot{\theta} \to 0 \quad \text{and} \quad \theta \to \theta_\infty \geq 0 \quad \text{as} \quad t \to \infty. \tag{17}$$

This function converges towards the critical value θ_∞. It is also easy to get

$$\dot{\theta} = \frac{k}{b^2} - 3\sigma^2 - \frac{3}{2}\psi^2 \leq 0. \tag{18}$$

Since the left-hand side (LHS) of the inequality is a sum of negative terms, we have that each of them must vanish as $t \to \infty$. Using this fact in the constraint equation, we obtain

$$\theta \to \sqrt{3V} \quad \text{as} \quad t \to \infty. \tag{19}$$

Equation (11) describes a damped harmonic oscillator, and so we know that the system must ''land'' at the minimum of the potential V_0 and thus

$$\theta_\infty = \sqrt{3V_0}. \tag{20}$$

For an exponential potential, the only minimum is at infinity and its value is zero. Let us summarize the results by the following theorem.

Theorem 3.1. Let be $k=0$ or $k=-1$ (Bianchi type I and Bianchi type III cases). If at a given time t_0 we have an expanding universe, that is, $\theta(t_0) \geq 0$, then (i) $\dot{\theta}(t) \leq 0$, $\theta(t) \geq 0$ for all $t \geq t_0$, (ii) $\sigma(t)$, $\psi(t)$, $k/b(t)^2 \to 0$ for $t \to \infty$, and (iii) $\theta \to \sqrt{3V_0}$, where V_0 is the minimum of the potential and in particular $\theta \to 0$ for an exponentially potential (i.e., $V = V_0 e^{-\lambda\varphi}$) for $t \to \infty$.

This theorem restates proposition 1 in [8] except than it can now also be applied for a convex potential V having its minimum at infinity. If $V_0 > 0$, we can use the known result that the BI and BIII universes converge exponentially to the isotropic de Sitter universe [20]. If $V_0 = 0$ at some finite value φ_0, then we only have isotropization in the BI case [8].

The above theorem gives no answer on isotropization for an exponential potential in a BI or a BIII model. Indeed, θ and σ vanish together. In this case, the asymptotic solutions of the dynamical system [Eqs. (8)–(11)] must be found to know under which conditions isotropy and/or inflation can occur. Since the equations of the dynamical system do not depend on k, the KS case will also be solved. This system has been actually studied in a paper of Burd and Barrow [14]. However, their analysis of the system was not complete. Their analysis covers a part of the phase space only. Indeed, only two of the critical points were found and some conclusions turn out to be incorrect.

In the next section, the analysis of the exponential potential case will be done in detail and we will compare our results with Ref. [14].

IV. STUDY OF THE DYNAMICAL SYSTEM WITH AN EXPONENTIAL POTENTIAL

A. Singular points

For a qualitative discussion of the system of differential equations (8)–(11), we have to determine the asymptotic behavior near the critical points. The usual methods of treating the problem rapidly turn out to be inadequate for this case: the singular points are highly nonhyperbolic so that the linearization or even the transformation to a normal form leads nowhere. Usually, the constraint equation can be used as a Ljapunov function. But in this case, Eq. (12) has no

TABLE I. Summary of results for isolated critical points.

λ	P_1				P_2				P_3			
	Type	Is	In	St	Type	Is	In	St	Type	Is	In	St
$\lambda = 0$	BIII	−	−	−	BI	+	e	↑	KS	−	e	−
$0 < \lambda < \sqrt{2}$	BIII	−	−	−	BI	+	p	↑	KS	−	p	−
$\lambda = \sqrt{2}$	BIII	−	−	−	BI	+	−	†	BI	+	−	†
$\sqrt{2} < \lambda < \sqrt{6}$	BIII	−	−	−	BI	+	−	†	BIII	−	−	↑
$\lambda = \sqrt{6}$	BIII	−	−	−	BI	+	−	?	BIII	−	−	↑
$\lambda > \sqrt{6}$	BIII	−	−	−	not defined				BIII	−	−	↑

TABLE II. Summary of results for critical points in Σ.

			P: $(S_0, 0, P_0) \in \Sigma$			
λ	P_0	S_0	Type	Is	In	St
$\lambda < \sqrt{6}$	$P_0 = \pm \sqrt{2/3}$	$S_0 = 0$	BI	+	−	$\downarrow_\Sigma$
	$-\sqrt{2/3} < P_0 < \sqrt{2/3}$	$S_0 \neq 0$	BI	−	−	$\downarrow_\Sigma$
$\lambda = \sqrt{6}$	$P_0 = -\sqrt{2/3}$	$S_0 = 0$	BI	+	−	$\downarrow_\Sigma$
	$-\sqrt{2/3} < P_0 < \sqrt{2/3}$	$S_0 \neq 0$	BI	−	−	$\downarrow_\Sigma$
	$P_0 = \sqrt{2/3}$	$S_0 = 0$	BI	+	−	?
$\lambda > \sqrt{6}$	$P_0 = -\sqrt{2/3}$	$S_0 = 0$	BI	+	−	$\downarrow_\Sigma$
	$-\sqrt{2/3} < P_0 < 2/\lambda$	$S_0 \neq 0$	BI	−	−	$\downarrow_\Sigma$
	$P_0 = 2/\lambda$	$S_0 \neq 0$	BI	−	−	?
	$2/\lambda < P_0 < \sqrt{2/3}$	$S_0 \neq 0$	BI	−	−	−
	$P_0 = \sqrt{2/3}$	$S_0 = 0$	BI	+	−	−

isolated root and therefore is of no help. There is no other obvious candidates for a Ljapunov function.

The problems originate from the fact that there are only quadratic terms on the right-hand side of Eqs. (8)–(11). There is a standard procedure for such systems: one has to ''divide'' all variables but one by the remaining variable and rewrite the system in terms of these new variables. Of course, this transformation is only nonsingular if the divisor does not vanish in the whole region of definition. In Ref. [14], the remaining variable was chosen to be $y = V(\Phi)$, which does not vanish in finite regions of the phase space. Here we prefer to distinguish the variable θ since it will allow us to find additional critical points (in finite regions of phase space). This transformation is well defined outside the origin.

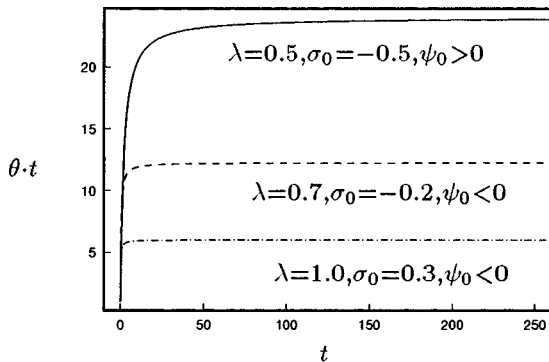Defining the variables S, U, and P by

$$\sigma = S\theta, \quad z = U\theta, \quad \psi = P\theta \tag{21}$$

and the time τ by

$$\frac{d}{d\tau} = \frac{1}{\theta}\frac{d}{dt}, \tag{22}$$

the differential equations (8)–(11) transform into

$$\theta' = \theta\left(-\frac{1}{3} - 2S^2 + U^2 - P^2\right), \tag{23}$$

$$S' = -\frac{1}{3\sqrt{3}} - \frac{2}{3}S + \frac{1}{\sqrt{3}}S^2 + \frac{1}{\sqrt{3}}U^2 + \frac{1}{2\sqrt{3}}P^2 + 2S^3 - SU^2$$
$$+ SP^2, \tag{24}$$

$$U' = U\left(\frac{1}{3} - \frac{\lambda}{2}P + 2S^2 - U^2 + P^2\right), \tag{25}$$

$$P' = -\frac{2}{3}P + \lambda U^2 + 2S^2P - U^2P + P^3, \tag{26}$$

and the constraint equation is

$$S^2 + U^2 + \frac{1}{2}P^2 + \frac{k}{b^2} = \frac{1}{3}. \tag{27}$$

A prime means the derivative with respect to τ. The dynamical system defined by Eqs. (24)–(26), which are independent of θ, will be referenced from now on by the tag (∗).

For the Bianchi type I case ($k = 0$), Eq. (27) describes the surface of an ellipsoid, say, E, which separates the Kantowski-Sachs (inside) from the Bianchi type III (outside) solutions. The Bianchi type I model ($k = 0$) has to be treated carefully because in this case the system can only evolve on a two-dimensional submanifold W in the space of the variables (S, U, P), defined by
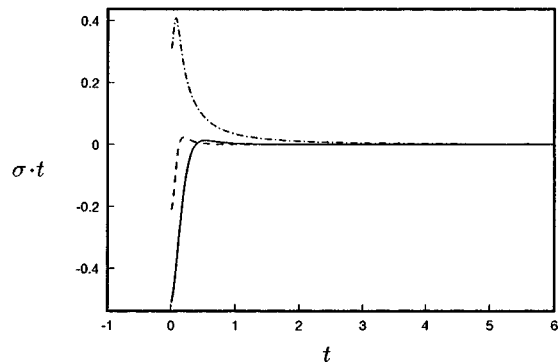

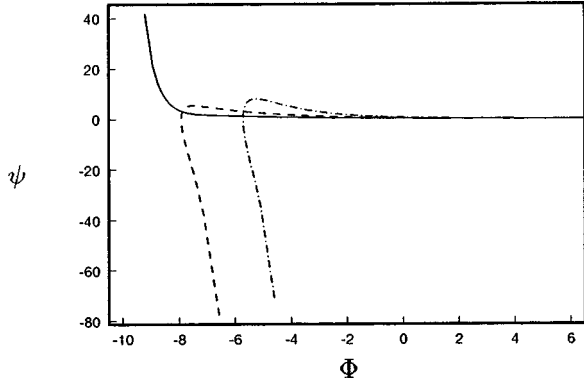FIG. 2. Evolution of Kantowski-Sachs solutions with $\lambda < \sqrt{2}$.

FIG. 3. The corresponding phase portraits for the KS case with $\lambda < \sqrt{2}$.

$$W = f^{-1}(\{\tfrac{1}{3}\}) \quad \text{with} \quad f(S,U,P) = S^2 + U^2 + \tfrac{1}{2}P^2. \quad (28)$$

It may happen that W coincides with the stable manifold of a critical point which is unstable with respect to the whole system (∗). In this case the point is nevertheless stable for the Bianchi type I metric.

We will concentrate on determining the asymptotic behavior of this reduced system. θ is then given by a simple integration of Eq. (23). After a short calculation we find for the system (∗) three discrete critical points and a one-parameter family of critical points:
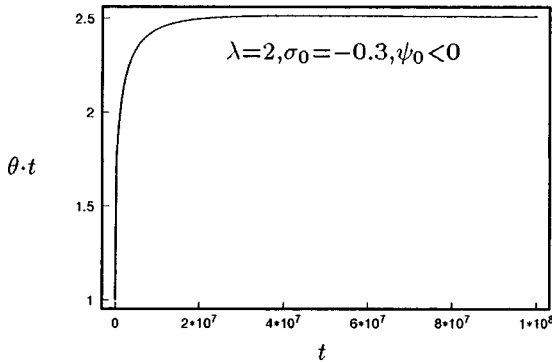
$$P_1: \quad S = -\frac{1}{2\sqrt{3}}, \quad U = 0, \quad P = 0, \quad (29)$$

$$P_2: \quad S = 0, \quad U = \sqrt{\frac{6 - \lambda^2}{18}},$$

$$P = \frac{\lambda}{3} \quad \text{for} \quad \lambda \leqslant \sqrt{6}, \quad (30)$$

$$P_3: \quad S = \frac{1}{2\sqrt{3}} \frac{2 - \lambda^2}{\lambda^2 + 1}, \quad U = \frac{\sqrt{\lambda^2 + 2}}{\sqrt{2}(\lambda^2 + 1)},$$

$$P = \frac{\lambda}{\lambda^2 + 1}, \quad (31)$$

$$\Sigma: \quad U = 0, \quad 3S^2 + \frac{3}{2}P^2 = 1$$

with $S \in [-\sqrt{1/3}, \sqrt{1/3}]$. $\quad (32)$

Comparing these critical points with the results of Burd and Barrow [14], one easily verifies that P_2 and P_3 correspond to their critical points (IV) and (III), whereas P_1 and the points of Σ have no counterpart in their paper. Using the variables in Ref. [14], P_1 and Σ can be found as critical points lying at infinity in their phase space.

To be sure that there are no additional critical points at infinity in our coordinate system either, we first introduce spherical coordinates

$$S = r \sin \vartheta, \quad U = r \cos \varphi \cos \vartheta, \quad P = r \sin \varphi \cos \vartheta \quad (33)$$

and then map the points at infinity ($r = \infty$) to the surface of a unit sphere by the transformations

$$r \rightarrow \rho = \frac{r}{r+1}, \quad d\tau \rightarrow d\eta = \frac{d\tau}{1 - \rho}. \quad (34)$$

At infinity ($\rho = 1$) we thus find

$$\frac{d\rho}{d\eta} = 2 - \cos^2 \vartheta (1 + 2\cos^2 \varphi), \quad (35)$$

$$\frac{d\varphi}{d\eta} = \frac{\lambda}{2} \cos \varphi \cos \vartheta (2\cos^2 \varphi + \sin^2 \varphi), \quad (36)$$

$$\frac{d\vartheta}{d\eta} = \frac{\cos \vartheta}{2\sqrt{3}} (2 - \cos^2 \vartheta + \cos^2 \varphi \cos^2 \vartheta$$
$$- \sqrt{3}\lambda \sin \varphi \cos^2 \varphi \sin \vartheta \cos \vartheta). \quad (37)$$

For a critical point at infinity, expressions (35)–(37) have to vanish simultaneously, which is easily seen to be impossible. Therefore, the points P_1, P_2, P_3 and the points of Σ are really the only critical points of the system (∗).

It is worthwhile to note that the critical points correspond to exact solutions of our original system [Eqs. (8)–(11)]. Indeed, if we consider the pullback of a critical point (S_k, U_k, P_k) of (∗) for a nonconstant potential ($\lambda > 0$), the volume expansion θ is given by

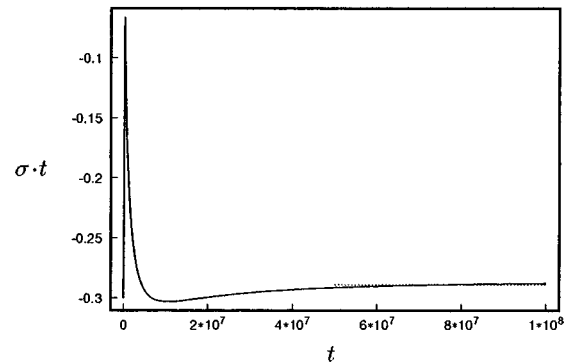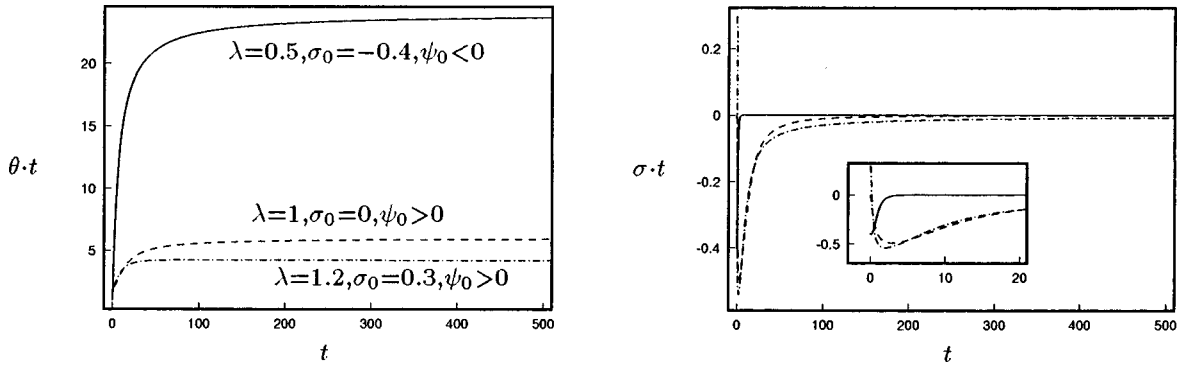$$\theta = \theta_0 e^{A_k \tau}, \quad (38)$$

FIG. 4. Evolution of a Kantowski-Sachs solution with $\lambda > \sqrt{2}$.

FIG. 5. Evolution of Bianchi type III solutions with $\lambda < \sqrt{2}$.

with $A_k = -1/3 - 2S^2 + U^2 - P^2|_{(S,U,P)=(S_k,U_k,P_k)}$, and as a function of the original time t, we obtain the following exact solutions:

$$\theta = -\frac{1}{A_k t}, \quad \sigma = -\frac{S_k}{A_k t}, \quad \psi = -\frac{P_k}{A_k t}, \quad V(\Phi) = \frac{U_k^2}{A_k^2 t^2}. \quad (39)$$

The stability analysis of the critical points will give the stability of the above exact solutions.

B. Stability analysis of the singular points

1. Critical point P_1

The linearization of $(*)$ around the critical point $P_1:(S,U,P)=[-(2\sqrt{3})^{-1},0,0]$ is diagonal and has eigenvalues $\varepsilon_{S,P} = -1/2$ and $\varepsilon_U = 1/2$. So P_1 is unstable both to the future and to the past. Starting around P_1, the critical point can never be reached by any solutions of the dynamical system.

Using Eq. (39), the critical point P_1 will correspond to the unstable solution

$$\theta = \frac{2}{t}, \quad \sigma = -\frac{1}{\sqrt{3}t}, \quad \psi = 0, \quad V(\Phi) = 0. \quad (40)$$

2. Critical point P_2

In P_2 the linearization of $(*)$ has the eigenvalues $\varepsilon_1 = -1 + \lambda^2/6$ (twice) and $\varepsilon_2 = -2/3 + \lambda^2/3$, and so the critical point is asymptotically stable [with respect to all the system $(*)$ and for $\tau \to \infty$] as long as $\lambda < \sqrt{2}$ and unstable for $\sqrt{2} < \lambda < \sqrt{6}$.

Since P_2 lies on the ellipsoid which separates the KS from the BIII domain, its neighborhood intersects all three types of universe. More precisely, starting from a point P, near P_2, defined by

$$S = \delta S, \quad U = \sqrt{\frac{6-\lambda^2}{18}} + \delta U, \quad P = \frac{\lambda}{3} + \delta P, \quad (41)$$

the constraints equation (27) expanded to first order tells us to which universe P belongs since we have

$$\sqrt{\frac{2(6-\lambda^2)}{9}}\,\delta U + \frac{\lambda}{3}\,\delta P = -\frac{k}{b^2}. \quad (42)$$

Depending from which point we start the integration, the solution can be in either type of universe.

As mentioned earlier, the Bianchi type I ($k=0$) metric has to be considered very carefully because the system is constrained by Eq. (27) to the two-dimensional submanifold W. The tangential space to W in the point P_2 is given by

$$T_{P_2}W = \{\mathbf{X} \in R^3 : \mathbf{X} \perp \nabla f(P_2)\}. \quad (43)$$

But $\nabla f(P_2) = (0,\sqrt{2(6-\lambda^2)}/3,\lambda/3)$ is an eigenvector for the eigenvalue ε_2, and so $T_{P_2}W$ coincides with the eigenspace to the eigenvalue ε_1. That is, W is the stable submanifold through P_2 for $\lambda < \sqrt{6}$ and consequently P_2 is in this entire range an asymptotically stable critical point for Bianchi type I solutions, contrary to the Bianchi type III and Kantowski-Sachs solutions. This particular behavior for the BI case was overseen in Ref. [14].

For the special values $\lambda = \sqrt{2}$ and $\lambda = \sqrt{6}$, at least one eigenvalue vanishes and the linearization does not contain enough information to determine the stability of P_2. By projecting the system $(*)$ on a center manifold through P_2 [21,22], we find that P_2 is unstable for $\lambda = \sqrt{2}$. In the case $\lambda = \sqrt{6}$ the center manifold is two dimensional and therefore the stability analysis is much more difficult. We did not find the asymptotical properties of the solutions around this singular point for this value of λ.

The critical point P_2 transforms back to the exact solution

$$\theta = \frac{6}{\lambda^2 t}, \quad \sigma = 0, \quad \psi = \frac{2}{\lambda t}, \quad V(\Phi) = \frac{2(6-\lambda^2)}{\lambda^4 t^2}. \quad (44)$$
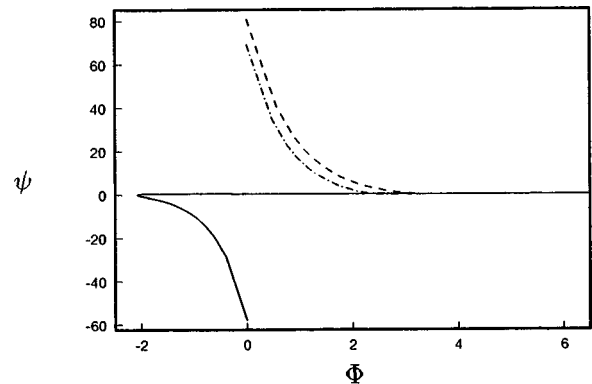
FIG. 6. The corresponding phase portraits for the BIII case with $\lambda < \sqrt{2}$.
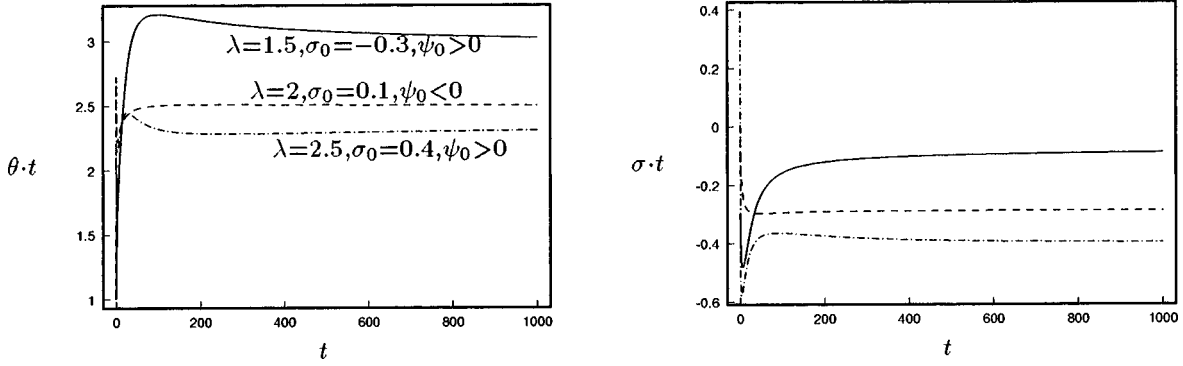
FIG. 7. Evolution of Bianchi type III solutions with $\lambda > \sqrt{2}$.

For $\lambda < \sqrt{2}$ this solution violates the dominant energy condition ($\psi^2 - V < 0$) and thus inflation occurs. This can be directly seen since from the above equation we obtain

$$a = a_0 t^{2/\lambda^2}. \qquad (45)$$

For the asymptotically stable range ($\lambda < \sqrt{2}$ for KS and BIII and $\lambda < \sqrt{6}$ for BI) of P_2, the solution is also asymptotically stable in the sense that solutions which start near it converge to it.

3. Critical point P_3

The eigenvalues of the linearization of $(*)$ in P_3 are

$$\varepsilon_1 = -\frac{\lambda^2 + 2}{2(\lambda^2 + 1)}, \quad \varepsilon_{2,3}$$

$$= \frac{-(\lambda^2 + 2) \pm \sqrt{(18 - 7\lambda^2)(\lambda^2 + 2)}}{4(\lambda^2 + 1)}. \qquad (46)$$

For $\lambda < \sqrt{2}$, all eigenvalues are real, and ε_2 and ε_3 have different signs; so P_3 is unstable. For $\lambda = \sqrt{2}$, P_3 coincides with P_2, and so we conclude as before that the critical point is unstable. In the range $\sqrt{2} < \lambda \leq \sqrt{18/7}$, all eigenvalues are real and negative, while, for $\lambda > \sqrt{18/7}$, ε_2 and ε_3 become complex with the same negative real part. So P_3 is asymptotically stable for $\lambda > \sqrt{2}$.

Again, we find the following exact solution after the pullback of P_3:

$$\theta = \frac{2(\lambda^2 + 1)}{\lambda^2 t}, \quad \sigma = \frac{2 - \lambda^2}{\sqrt{3}\lambda^2 t}, \quad \psi = \frac{2}{\lambda t},$$

$$V(\Phi) = \frac{2(\lambda^2 + 2)}{\lambda^4 t^2}, \qquad (47)$$

which is asymptotically stable for $\lambda > \sqrt{2}$.

4. Critical points of Σ

Since the critical points in Σ are on the ellipsoid E, we find as before (Sec. IV B 2) that there are points belonging to all three types of universe near Σ.

For the linearization of $(*)$ around a point $P_\Sigma = S_0$, 0, P_0 of Σ, we find the eigenvalues

$$\varepsilon_1 = 0, \quad \varepsilon_2 = 1 - \frac{\lambda}{2} P_0, \quad \varepsilon_3 = \frac{2}{3}(2 + \sqrt{3}S_0). \qquad (48)$$

If $P_0 > 2/\lambda$, ε_2 and ε_3 have different signs and the critical point is unstable both to the future and to the past.

If $P_0 < 2/\lambda$, ε_2 and ε_3 are both positive, and so P_Σ is unstable to the future. The stability for $\tau \to -\infty$ is determined by the behavior of the system on a center manifold [21,22]. It is easy to see that Σ itself is a center manifold and that the restriction of $(*)$ to Σ is just the trivial system

$$S' = 0, \quad U' = 0, \quad P' = 0. \qquad (49)$$

That means that, for $\tau \to -\infty$, P_Σ is an attractor. It is also easy to see that, for $\lambda < \sqrt{6}$, Σ as a whole is a past attractor for $(*)$. Indeed, ε_2 is then positive for all points P_Σ.
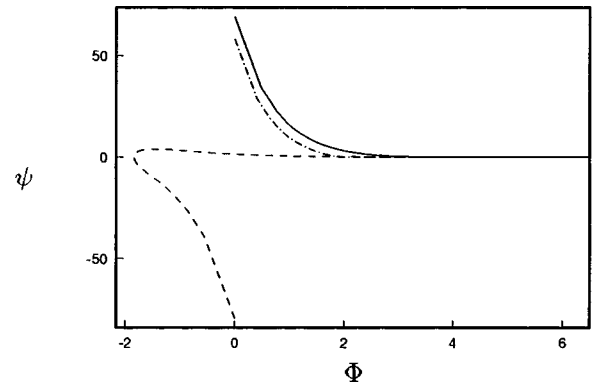
The pullback of points out of Σ gives an exact solution of the form

$$\theta = \frac{1}{t}, \quad \sigma = \frac{\sigma_0}{t}, \quad \psi = \frac{\psi_0}{t}, \quad V(\Phi) = 0, \qquad (50)$$

with the condition $\frac{2}{3} - 2\sigma_0^2 - \psi_0^2 = 0$.

C. Summary

The cases with $\lambda = 0$ (constant potential) can be treated analogously. For the critical points P_2 and P_3 this has been done by Burd and Barrow [14], while for P_1 and Σ the results found are still valid since the asymptotic behavior does not depend on λ.

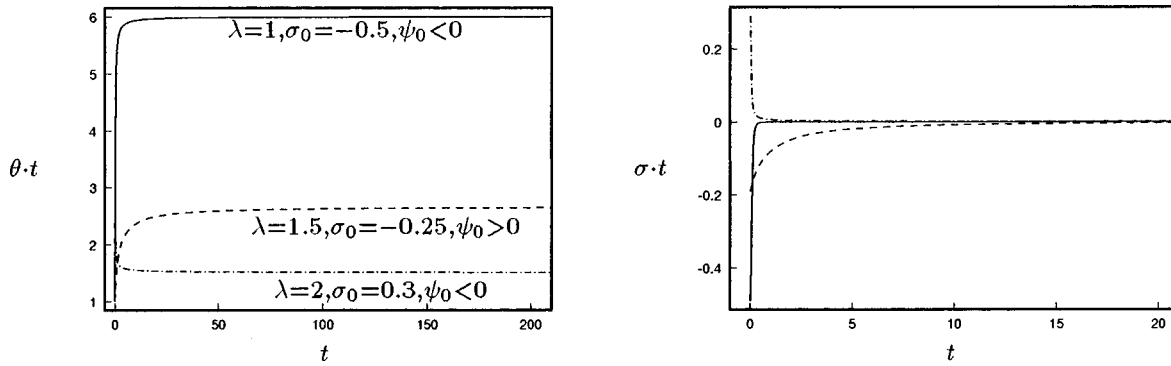FIG. 8. Corresponding phase portraits for the BIII case with $\lambda > \sqrt{2}$.

FIG. 9. Evolution of Bianchi type I solutions with $\lambda < \sqrt{6}$.

The results of this section are summarized in Table I (isolated critical points) and Table II (family Σ). They show for every critical point the type of the corresponding exact solution (BI, BIII, or KS), its isotropization (Is) and inflation (In) (e, exponential inflation; p, power-law inflation; or $-$, no inflation), and the stability (St) of the critical points ($\uparrow$, asymptotic future stable: $\dagger$, asymptotic future stable for Bianchi type I solutions and unstable otherwise; $\downarrow_\Sigma$, past convergence to Σ; and $-$, unstable).

From Table I, it follows that for $\lambda < \sqrt{2}$ the unique asymptotical stable solution is represented by the point P_2. This solution describes an inflationary and isotropic universe.

For $\sqrt{2} < \lambda < \sqrt{6}$, isotropy can be reached without inflation if we restrict ourselves to the BI universe. This fact is compatible with a Collins-Hawking result [7], which states that for ordinary matter, within the BI universe, isotropy can be reached without inflation. Indeed, for $\sqrt{2} < \lambda < \sqrt{6}$ the equation of state of the matter field corresponding to the P_2 solutions is given by

$$p = \omega \rho, \qquad (51)$$

with $\omega \in \,]-1/3,1[$, and contains the ordinary matter case ($0 \leq \omega < 1$).

For $\lambda > \sqrt{6}$, the unique asymptotical stable solution is represented by the point P_3. This BIII solution describes an anisotropic universe.
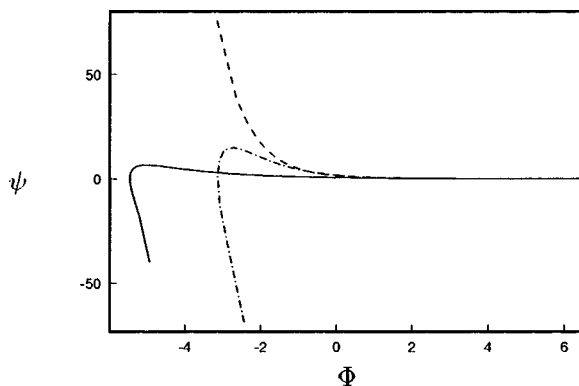
There is also a unique attractor in the past: the Σ manifold. As a consequence, we must start the numerical integration near Σ to have all the history of the evolution of the model.

The next section will be devoted to the numerical integration.

V. NUMERICAL RESULTS

In this section we will follow numerically the evolution of the system (8)–(11). We have seen in Sec. IV that the points of the surface Σ are the only past attractor of the system, and so it seems obvious to study the development of solutions starting near the corresponding exact solution

$$\theta = \frac{1}{t}, \quad \sigma = \sigma_0 \frac{1}{t}, \quad \psi = \psi_0 \frac{1}{t}, \quad V(\Phi) = 0, \qquad (52)$$

with $t \rightarrow 0$ and with $3\sigma_0^2 + 3/2\psi_0^2 = 1$. Since these exact solutions are in the BI universe, the numerical solutions starting near Σ can be in any three types of universe (e.g., BI, BIII, or KS).

To distinguish the different future attractors, it is convenient not to plot the variables θ and σ themselves (they always vanish as $1/t$), but the quantities $\theta \cdot t$ and $\sigma \cdot t$, which converge to constant values at the critical points.

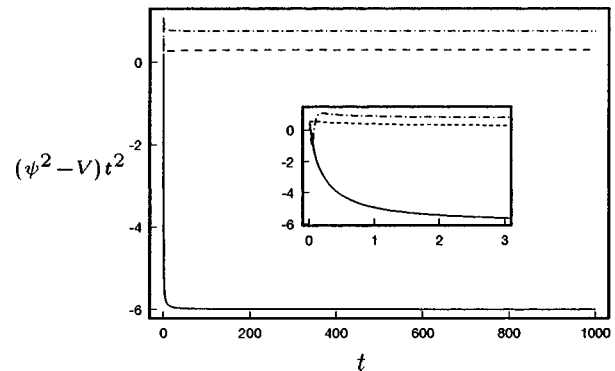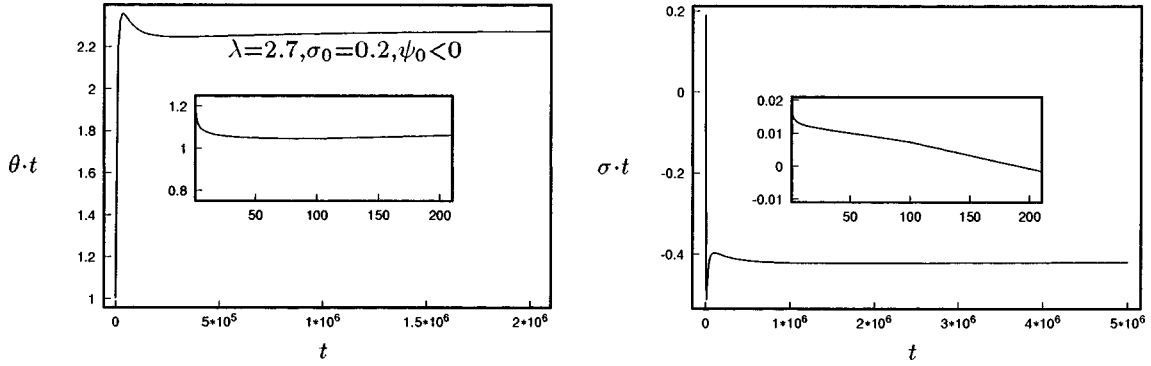FIG. 10. Corresponding phase portraits for the BI case with $\lambda < \sqrt{6}$.

FIG. 11. Inflation can only occur when the dominant energy condition is violated, that is, when $(\psi^2 - V)t^2$ is negative. $(\psi^2 - V)t^2$ is plotted for the corresponding numerical solutions described by Figs. 9 and 10.

FIG. 12. Evolution of a Bianchi type I solution with $\lambda > \sqrt{6}$.

A. Kantowski-Sachs solutions

For the initial asymptotic behavior

$$\theta(t_0) = \frac{1}{t_0}, \quad \sigma(t_0) = \frac{\sigma_0 + \delta\sigma}{t_0}, \quad u(t_0) = \frac{\delta u}{t_0},$$

$$\psi(t_0) = \frac{\psi_0 + \delta\psi}{t_0}, \tag{53}$$

we will have a Kantowski-Sachs solutions if the following inequality is satisfied:

$$2\sigma_0\delta\sigma + \psi_0\delta\psi > 0. \tag{54}$$

Figure 2 shows the results for some values of λ less then the critical value $\sqrt{2}$. The solution clearly converges towards the exact solution corresponding to the critical point P_2. The corresponding phase portraits are given in Fig. 3.

The development of a solution with $\lambda > \sqrt{2}$ is shown in Fig. 4. The solution approaches again the (now unstable) attractor P_2, is repelled, and finally converges towards the exact solution corresponding to the attractor P_3.

B. Bianchi type III solutions

For a Bianchi type III solution of the form (53), the integration constants have to satisfy the condition

$$2\sigma_0\delta\sigma + \psi_0\delta\psi < 0. \tag{55}$$

The solutions with $\lambda < \sqrt{2}$ (Fig. 5) initially tend to the Bianchi type III solution corresponding to P_1. But since this is an unstable attractor, the system finally evolves towards the solution P_2. Figure 6 shows the corresponding phase portraits for these cases.

For $\lambda > \sqrt{2}$ the system again approaches the unstable P_1 solution in the beginning, is repelled, and converges towards the P_3 solution, which is the only attractor in this range of values (Fig. 7, phase portraits in Fig. 8).

C. Bianchi type I solutions

For a Bianchi type I solution, the perturbations of the initial state (53) have to satisfy the equation

$$2\sigma_0\delta\sigma + \psi_0\delta\psi = 0. \tag{56}$$

For Bianchi type I solutions, the exact solution corresponding to P_2 is attractive not only for $\lambda < \sqrt{2}$, but for the whole range for which P_2 is defined, that is, for $\lambda < \sqrt{6}$ (Fig. 9, phase portraits in Fig. 10). The numerical calculation confirms our analysis for $\sqrt{2} < \lambda < \sqrt{6}$: the solutions still isotropize, but are no longer inflationary as one can see from Fig. 11 where we have plotted $\psi^2 - V(\Phi)$, which violates the dominant energy condition when negative.

For $\lambda > \sqrt{6}$ we have again the solution to P_3 as the only attractor. An example for this case is shown in Fig. 12. The system quickly evolves towards the now unstable solution P_2, is repelled, and converges towards the solution to P_3.

VI. CONCLUSIONS

We have studied the Einstein-Klein-Gordon (EKG) equations for a convex positive potential in a Bianchi type I, a type Bianchi III, and a Kantowsky-Sachs universe.

After analyzing the inherent properties of the equations, it was shown in Sec. III why a detailed analysis of the solutions of the EKG equations was needed for a vanishing potential at infinity. By taking an exponential potential ($V = V_0 e^{-\lambda\varphi}$), it was shown for which values of λ inflation and/or isotropy were reached asymptotically. We recovered the results of Ref. [14], but also found new asymptotical behaviors and new exact solutions represented by the singular points P_1 and the submanifold Σ.

We also found that for some values of λ isotropy can be reached without inflation in a Bianchi type I universe. But when inflation occurs ($\lambda < \sqrt{2}$), then isotropy is always reached. We also integrated the equations numerically to obtain the entire history of the evolution of the model. All the founded asymptotical behaviors were confirmed numerically.

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