Photon splitting in strong magnetic fields: S-matrix calculations

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The S-matrix approach to the treatment of photon splitting in a magnetized vacuum, with the electron propagators expressed in the Landau representation, is discussed critically. Although the analytic results of Mentzel, Berg and Wunner are confirmed, we propose that their available numerical results may be subject to two previously unidentified sources of error associated with the sum over principal quantum number n, leading to spurious contributions to the amplitude, and the extremely slow convergence of the sum for weak fields. It is shown how the sums may be rearranged to avoid the spurious contributions. If the Euler-Maclaurin summation formula is used to evaluate the infinite sums over n, the S-matrix approach then reproduces results derived by the effective Lagrangian and proper-time techniques in the weak-field, low-frequency limit. This method gives reliable results, for $B \gtrsim 0.01$ and $\omega \lesssim 0.1$, that reproduce those obtained by proper-time techniques. The S-matrix approach simplifies in the strong-field limit, $B \gg 1$, where the sum over n converges rapidly. Our results show that the branching ratio for the splittings $\perp \to \perp\perp$ and $\perp \to \parallel\parallel$ decreases from its known value ~ 3.4 for $B \ll 1$ towards zero for $B \gg 1$. For weak fields the S-matrix approach is unnecessarily cumbersome, and future numerical work should be based on the alternative approaches.
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I. INTRODUCTION

The third-order quantum electrodynamical process of photon splitting $\gamma \to \gamma' \gamma''$ in a strong magnetic field has become of renewed interest after two decades, following the publication of an S-matrix calculation of its rates by Mentzel, Berg and Wunner [1] (MBW). The numerical results of MBW suggest that the earlier work had seriously underestimated the strength of this process. MBW's analytic derivation is applicable to the non-dispersive regime below the pair creation threshold ($\omega = 2$ where natural units, $\hbar = c = 1$, are used and energies are in units of the electron rest energy of 511 keV), in which case the momentum vectors of the initial and final photons are collinear; the validity of MBW's analysis extends beyond the weak field ($B \ll 1$, where B is in units of the quantum critical field $B_c = 4.413 \times 10^{13}$ G) regime to arbitrary field strengths. As such, it is the first comprehensive presentation of the application of the S-matrix technique (using the Landau representation for the electron propagators) specifically to magnetic photon splitting, although Melrose and Parle [2,3] wrote down the S-matrix forms for splitting amplitudes. Prior to these works, splitting calculations used either effective Lagrangian [4–6] or variations of Schwinger's proper-time techniques [7–9], which yielded compact analytic forms for the rates R in the low energy ($R \propto \omega^5$) or low field ($R \propto B^6$) cases. MBW did not discuss these relevant limiting cases of their analytic results, and their conclusions are based on numerical evaluation of the complicated algebraic formulas. Wunner, Sang and Berg [10] (WSB) discussed these numerical results further. In particular, these authors found $\gamma \to \gamma' \gamma''$ rates just below the pair creation threshold comparable to and even exceeding the $\gamma \to e^+ e^-$ pair creation rates at low field strengths. The exceptionally large splitting rates implied by the numerical results of MBW and WSB are surprising given that $\gamma \to \gamma' \gamma''$ is a third-order process and pair production is first-order; hence photon splitting is expected to be of the order of α_f^2 weaker than $\gamma \to e^+ e^-$, where α_f is the fine structure constant. The MBW splitting rates also have a weak dependence on B, which is uncharacteristic for strong field QED processes. The claim of greatly enhanced splitting rates was questioned by Baier *et al.* [11] and Adler and Schubert [12]. This claim has since been retracted (Wilke and Wunner [13]), with a sign error in their numerical coding cited as the cause of the error in the rates of MBW and WSB. While the elimination of one coding error has gone a long way toward repairing their numerics, Wilke and Wunner's [13] S-matrix evaluations still do not coincide with the recent numerical computations based on Stoneham's proper-time rates [14] (specifically for the polarization mode $\perp \to \parallel\parallel$), which appear to be in excellent agreement with the recent alternative proper-time numerics of Baier *et al.* [11]. Differences by a factor of 2–3 emerge between the two data points given in [13] for $B \lesssim 1$ and [14], differences that are important in astrophysical applications of photon splitting.

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The astrophysical context in which photon splitting is of particular interest is pulsar magnetospheres. The standard model for pulsars has the magnetosphere of a strongly magnetized ($B \gtrsim 0.1$) neutron star populated by highly relativistic $e^{\pm}$ pairs generated through the process $\gamma \to e^+e^-$ in the polar cap regions [15–19]. According to the rates found in the 1970s [4–8], $\gamma \to \gamma' \gamma''$ can be neglected (to a first approximation) compared to $\gamma \to e^+e^-$. However, the enhanced rates for $\gamma \to \gamma' \gamma''$ found by MBW and WSB would imply that photons split before they could decay to produce pairs. This would undermine the standard model. At the next level of refinement, photon splitting at the accepted rates [4–8] is influential only for sufficiently strongly magnetized neutron star environments ($B \gtrsim 0.4$), and has recently been invoked [20,21] to explain the MeV cutoff in the spectrum of the gamma-ray pulsar PSR1509-58 and also in magnetar models of soft gamma repeaters [22–26]. Any substantial change in the accepted rate for $\gamma \to \gamma' \gamma''$ would require a major revision of these astrophysical theories.

The claims of MBW and WSB have raised a number of questions concerning the treatment of photon splitting: Can one show that the S-matrix formalism reproduces the low energy ($R \propto \omega^5$) and low field ($R \propto B^6$) limits derived by other techniques? How do the rates change for $B \gtrsim 1$? How is the remaining discrepancy between the numerical results of the S-matrix calculations of [13] and the proper-time calculations of [14] to be resolved? Which technique is the most convenient and reliable for numerical calculations? In this paper we report on a detailed investigation of all relevant aspects of the analytic treatment of photon splitting using the S-matrix approach with such questions in mind. We have rederived the general analytic expressions quoted in MBW and confirm them to be correct, specifically Eqs. (25) and (26) in MBW; we also note that they are consistent with the results of [2,3]. It is important to note that the starting points for the calculations using the effective Lagrangian approach of [4–6], the proper-time approach [7–9,11] and the S-matrix approach are strictly equivalent; the difference is in the use of different but equivalent [29] forms of the electron propagator in a magnetic field. Hence the results of [1,4–9,11] should all be equivalent (indeed the equivalence of [6,8,11] has been demonstrated in [12]). Therefore S-matrix computations should yield *identical* results to proper-time numerics. Note that the equivalence of the S-matrix and proper-time methods has been unequivocally demonstrated in the context of magnetic pair creation $\gamma \to e^+e^-$ [27,28]. The form of the propagator used in the proper-time (and effective Lagrangian) approach involves triple integrals over relatively simple (hyperbolic and exponential) functions, whereas the form of the propagator used in the S-matrix approach involves an integral over the parallel momentum p_z and sums over the spins and principal quantum numbers (n,n',n'') of the intermediate pair states, including a triple sum over triple products of generalized Laguerre functions. Inspection of the relevant expressions suggests that the former approach should be more convenient for numerical calculations and this inference is supported both by the difficulties already encountered in numerical calculations using the S-matrix approach and also by further pitfalls identified below.

In our analysis we follow earlier authors in neglecting the effects of dispersion, thereby restricting the splitting to collinear momenta for the incoming and outgoing photons. Without loss of generality one may then choose an inertial frame in which the axis (chosen to be the x axis) defined by the collinear momenta is orthogonal to the magnetic field (the z axis). The polarization modes of the magnetized vacuum are identified as $\parallel$ and $\perp$ depending on the direction of the electric vector relative to the magnetostatic field. We consider all the splittings permitted by the CP invariance symmetry, which are those that involve an even number of $\parallel$ modes: $\perp \to \parallel\parallel$, $\perp \to \perp\perp$, $\parallel \to \perp\parallel$. (When dispersive effects of the magnetized vacuum are taken into account, the latter two of these three allowed decay modes are forbidden [6].) A preliminary step in our rederivation is to simplify the general S-matrix expressions by summing over the spin states that are incorporated in the electron propagators: cf. Appendix A. In Sec. II we discuss the low frequency ($\omega \ll 1$) limits of these forms. The S-matrix amplitudes contain only odd powers of the frequency, and the accepted low-frequency dependence ($R \propto \omega^5$) requires that the terms linear in ω (referred to as linear-ω terms) in the S-matrix amplitudes vanish. In Sec. III the linear-ω terms are shown to sum identically zero. However, for the $\perp \to \parallel\parallel$ this sum involves a relabeling of the n values, and we show that if this relabeling is not performed, then a substantial nonzero component remains even when the sum over n is extended to very high values. We suggest that one feature of the numerical results of MBW is spurious due to failure to make this relabeling. In Sec. IV the terms cubic in frequency are analyzed and it is shown that all terms sum to zero except for the terms of the correct form ($\propto \omega\omega'\omega''$) to reproduce the results obtained by other QED techniques. It is shown that the S-matrix computations converge very rapidly for $B \gg 1$ where few terms in the summation are required. However for $B \lesssim 1$, an alternative numerical technique is needed. We use the Euler-Maclaurin summation formula to perform the sum over n. We show that this technique is particularly convenient for weak fields $B<1$, that it reproduces the numerical results of [13] (specifically for $\perp \to \parallel\parallel$ at $\omega = 0.1$ and $B>10$), and we discuss the breakdown of the technique for small $B<0.01$. Our conclusions are summarized in Sec. V.

II. ANALYTIC REDUCTION OF THE S-MATRIX FORMALISM

In deriving the rates for photon splitting within the S-matrix formulation, we adopt the convention of MBW to a large extent. Within this formalism, the electron propagators are expressed in terms of Landau spinors. The total S-matrix element $S_{fi}^{(3)}$ is the sum of the six component elements $S_{fi,1}^{(3)} - S_{fi,6}^{(3)}$. These six components involve only two independent elements, chosen to be $S_{fi,1}^{(3)}$ and $S_{fi,2}^{(3)}$ given by Eqs. (25) and (26) of MBW, with the others obtained from these two via the transformations presented in MBW's Eq. (15). These elements are simplified as follows:

(1) The propagation direction of the incoming (γ) and outgoing photons (γ', γ'') is chosen to be the x-direction, and then $k_z = k_z' = k_z'' = 0$ implies $p_z^2 = p_z'^2 = p_z''^2$.

(2) The spatial integrals of Eq. (23) of MBW may be expressed in terms of the J functions of Melrose and Parle [29] [see Eq. (47) of their paper], with $k_\perp^2=\omega^2$, $k_\perp'^2=\omega'^2$ and $k_\perp''^2=\omega''^2$.
(3) The sum over the spin is performed.

The resulting expressions for $S_{fi,1}^{(3)}$–$S_{fi,6}^{(3)}$ for the $\perp\to\|\|$, $\perp\to\perp\perp$ and the $\|\to\perp\|$ splittings are given in Appendix A.

By taking the low-frequency limit, the S-matrix amplitudes can be further simplified such that an analytic solution can be obtained for the integral over p_z. In taking this limit, one expands each of the energy sums in the denominator of the S-matrix elements as a Taylor series. For example, for the $S_{fi,1}^{(3)}$ amplitude in Eq. (A1), the product of the two energy denominator terms is

$$\frac{1}{(E_0+\omega')(E_1+\omega)}=\frac{1}{E_0E_1}\,\frac{1}{(1+\omega'/E_0)(1+\omega/E_1)},\quad(1)$$

where $E_0=\varepsilon+\varepsilon''$ and $E_1=\varepsilon'+\varepsilon''$. For $\omega'/E_0\ll 1$ and $\omega/E_1\ll 1$ this becomes

$$\frac{1}{(E_0+\omega')(E_1+\omega)}\approx\frac{1}{E_0E_1}\left[1-\left(\frac{\omega'}{E_0}+\frac{\omega}{E_1}\right)+\left\{\left(\frac{\omega'}{E_0}\right)^2+\left(\frac{\omega}{E_1}\right)^2+\frac{\omega\omega'}{E_0E_1}\right\}-\cdots\right].\quad(2)$$

The frequency also appears in the arguments of the J functions, with each term in the S-matrix involving a product of such functions; cf. Appendix A. Within these J functions are the generalized Laguerre polynomials expressible as a power series in ω^2. Each term in this series expansion differs from its adjacent term by a factor of order $n\omega^2/2B$. Hence provided one has

$$\frac{n\omega^2}{2B}\ll 1,\quad(3)$$

the expansion of the generalized Laguerre polynomial converges rapidly and can be terminated at the desired order of photon energy. For example, in the linear-ω approximation to the S-matrix, the zero-order frequency terms in both the generalized Laguerre polynomials and the expansion in Eq. (2) of the energy denominators are coupled to linear frequency terms arising from factors that relate the J functions and the generalized Laguerre polynomials; cf. Eq. (A11).

The structure of the triple products of the J functions implies that only odd power combinations of photon energy are allowed in the $\mathcal{D}$ functions present in the S-matrix elements of Appendix A. This leads to restrictions on the values of the Landau quantum numbers n, n' and n''. These restrictions allow one to write n' and n'' in terms of n, so that only a single sum over n remains.

III. ANALYSIS OF THE LINEAR-ω COMPONENT

In the treatment of splitting using the effective Lagrangian and proper-time techniques it is known that the rates of all splittings are zero when only the terms of first order in ω, ω' or ω'' are retained (called linear-ω terms). This is not obviously the case in the S-matrix approach. In this section we prove that the linear-ω terms are indeed zero.

In the linear-ω terms the restrictions on the values of n, n' and n'' apply to the indices α, β, γ in the triple J product $J_\alpha^aJ_\beta^bJ_\gamma^c$; cf. Appendix A. The allowed values of α, β, γ are $n-n'$, $n''-n$, $n'-n''\pm1$ for $\perp\to\|\|$; $n-n'$, $n''-n\pm1$, $n'-n''$ for $\|\to\perp\|$; and $n-n'\pm1$, $n''-n\pm1$, $n'-n''\pm1$ for $\perp\to\perp\perp$. Only the combinations that satisfy

$$|\alpha|+|\beta|+|\gamma|=1\quad(4)$$

are allowed and we collect them into what we call arrangements which are treated separately. For $\perp\to\|\|$ and $\|\to\perp\|$ there are only three possible arrangements (with a linear-ω dependence) denoted A_1, A_2 and A_3: A_1 has $n'=n''=n$; A_2 has $n'=n+1$, $n''=n$ or $n'=n$, $n''=n+1$; and A_3 has $n'=n-1$, $n''=n$ or $n'=n$, $n''=n-1$. We show below how these arrangements combine to give zero for the $\perp\to\|\|$ splitting. The corresponding demonstration for $\|\to\perp\|$ is closely analogous, due to these two splittings being related by a crossing symmetry. We comment more briefly on the splitting $\perp\to\perp\perp$.

A. Analysis of the linear-ω component of the $\perp\to\|\|$ splitting

There are 16 terms to sum in each of the three arrangements for $\perp\to\|\|$ splitting. These give, for the S-matrix elements,

$$A_1=F_1\omega\sum_{n=0}^{\infty}n\int_{-\infty}^{\infty}dp_z\,\frac{3\varepsilon_{0,n}^2-2\varepsilon_n^2}{\varepsilon_n^5},\quad(5)$$

$$A_2=-F_1\omega\sum_{n=0}^{\infty}\int_{-\infty}^{\infty}dp_z\,\frac{n[2\varepsilon_{0,n}^2(2\varepsilon_n+\varepsilon_{n+1})-\varepsilon_n^2(3\varepsilon_n+\varepsilon_{n+1})]+\varepsilon_{0,n}^2(2\varepsilon_n+\varepsilon_{n+1})-\varepsilon_n^3}{\varepsilon_{n+1}\varepsilon_n^3(\varepsilon_n+\varepsilon_{n+1})^2},\quad(6)$$

$$A_3=-F_1\omega\sum_{n=1}^{\infty}\int_{-\infty}^{\infty}dp_z\,\frac{(n-1)[2\varepsilon_{0,n}^2(2\varepsilon_n+\varepsilon_{n-1})-\varepsilon_n^2(3\varepsilon_n+\varepsilon_{n-1})]-(2\varepsilon_n+\varepsilon_{n-1})(\varepsilon_n^2-\varepsilon_{0,n}^2)}{\varepsilon_{n-1}\varepsilon_n^3(\varepsilon_{n-1}+\varepsilon_n)^2},\quad(7)$$

where $\varepsilon_m=\sqrt{p_z^2+\varepsilon_{0,m}^2}$, $\varepsilon_{0,m}=\sqrt{1+2mB}$, and each term has the common factor

$$F_1=\frac{8\pi^2(4\pi\alpha_f)^{3/2}B}{\sqrt{\omega''\omega'\omega}}\,\delta(k_x''+k_x'-k_x)\,\delta(\omega''+\omega'-\omega)\,\frac{1}{(2V)^{3/2}},\tag{8}$$

where V is the volume associated with the interaction. The integral in Eq. (5) gives $A_1=0$ identically (cf. Sec. 3.241 [30]) and hence the only possible contributions to the linear-ω component are from A_2 and A_3.

The sum in Eq. (7) corresponds to $n\geqslant 1$. If the Landau levels are relabeled by making the transformation $n\rightarrow n+1$, then the sum in Eq. (7) is replaced by one over $n\geqslant 0$ and Eq. (7) becomes

$$A_3'=-F_1\omega\sum_{n=0}^{\infty}\int_{-\infty}^{\infty}dp_z\,\frac{n[2\varepsilon_{0,n+1}^2(2\varepsilon_{n+1}+\varepsilon_n)-\varepsilon_{n+1}^2(3\varepsilon_{n+1}+\varepsilon_n)]-(2\varepsilon_{n+1}+\varepsilon_n)(\varepsilon_{n+1}^2-\varepsilon_{0,n+1}^2)}{\varepsilon_n\varepsilon_{n+1}^3(\varepsilon_n+\varepsilon_{n+1})^2}.\tag{9}$$

Summing the A_2 and A_3' contributions, one obtains

$$A_2+A_3'=-F_1\omega\sum_{n=0}^{\infty}2(2n+1)\int_0^{\infty}dp_z\,\frac{1}{\varepsilon_{n+1}\varepsilon_n(\varepsilon_n+\varepsilon_{n+1})^2}\left[\frac{\varepsilon_{0,n}^2(2\varepsilon_n+\varepsilon_{n+1})}{\varepsilon_n^2}+\frac{\varepsilon_{0,n+1}^2(2\varepsilon_{n+1}+\varepsilon_n)}{\varepsilon_{n+1}^2}-2(\varepsilon_n+\varepsilon_{n+1})\right].\tag{10}$$

If one makes the substitution

$$s=\frac{\varepsilon_{n+1}}{\varepsilon_n}=\sqrt{\frac{p_z^2+\varepsilon_{0,n+1}^2}{p_z^2+\varepsilon_{0,n}^2}},\tag{11}$$

so that one has

$$\int_0^{\infty}dp_z\rightarrow\int_1^{(\varepsilon_{0,n+1}/\varepsilon_{0,n})}ds\,2sB\left(\frac{1}{s^2-1}\right)^{3/2}\frac{1}{\sqrt{\varepsilon_{0,n+1}^2-\varepsilon_{0,n}^2s^2}},\tag{12}$$

the integral in Eq. (10) is also zero (cf. [30], Secs. 2.26–2.28). This establishes that the linear-ω contribution to the amplitude for $\perp\rightarrow\parallel\parallel$ splitting is zero, as required. The proof that the linear-ω contribution to the amplitude for $\parallel\rightarrow\parallel\perp$ splitting vanishes is analogous, with the same integrals appearing with ω' in place of ω.

If the relabeling of the sum is not made, then the linear-ω contribution to the amplitude for these splittings does not vanish when the sum over n is cut off at any finite value. To see this, note that if one integrates A_2 and A_3' separately, using the substitution in Eq. (11), then provided that the relabeling indicated above is performed one obtains, at each n,

$$A_2=-A_3'.\tag{13}$$

However, if the relabeling is not performed, then when one integrates A_2 and A_3 (in its original form), one obtains, at each n,

$$A_2(n)=-A_3(n+1).\tag{14}$$

It follows that the sum to any maximum, $n_{\max}$, gives

$$A_2+A_3=A_2(n_{\max}).\tag{15}$$

We conclude that whenever a finite sum is performed without making the relabeling from Eq. (7) to (9), a spurious matrix element proportional to ω arises.

It is likely that this spurious term arises in the numerical work of MBW, who did not make the foregoing relabeling and who chose $n_{\max}=30$. Consider Fig. 3 of MBW and the $\perp\rightarrow\parallel\parallel$ splitting mode for $\omega=0.01$. The ratio of the mean free paths at $B=0.1$ and $B=1$ is approximately 1.6. From our analysis, for $B=0.01$, 0.1 and 1.0, one obtains $A_2(30)=28.4F_1\omega$, $6.5F_1\omega$ and $0.74F_1\omega$, respectively. Hence, we estimate that the spurious term alone for $n_{\max}=30$ would give for this ratio [specifically, for $B^2A_2^2(B=1)$ to $B^2A_2^2(B=0.1)$] a value of 1.5, which is remarkably close to the 1.6 value obtained from Fig. 3 of MBW. Moreover, as n is increased, the absolute value of A_2 approaches a constant nonzero value. For example, increasing $n_{\max}$ to 60 for $B=0.1$ and $B=1$ changes $A_2(n_{\max})$ by only 6% and 1% respectively. Such a small change from a doubling of $n_{\max}$ could easily mislead one into believing that the expansion is converging to the correct result, and that the choice $n_{\max}=30$ gives good convergence. However, the result is entirely spurious. We suggest that such spurious contributions may be a hitherto unidentified source of error in the numerical work of MBW. Further, as this spurious linear contribution to the splitting rate vanishes as $\sim B^{-1/2}$, its effect is more pronounced at the lower fields.

B. Analysis of the linear-ω component of the $\perp\rightarrow\perp\perp$ splitting

For $\perp\rightarrow\perp\perp$ splitting, the linear-ω contribution to the amplitude contains 12 arrangements: $n''=n'=n+1$; $n''=n'=n-1$; $n''=n-1$, $n'=n$; $n''=n$, $n'=n-1$; $n''=n+1$, $n'=n$; $n''=n$, $n'=n+1$; $n''=n-1$, $n'=n+1$; $n''=n+1$, $n'=n-1$; $n''=n+2$, $n'=n+1$; $n''=n+1$, $n'=n+2$; $n''=n-1$, $n'=n-2$; and $n''=n-2$, $n'=n-1$. If the sums are relabeled in the same manner as for the $\perp\rightarrow\parallel\parallel$ splitting, these $\perp\rightarrow\perp\perp$ arrangements cancel to give a zero linear-ω dependence. In the low frequency regime of Fig. 3 in MBW, the $\perp\rightarrow\perp\perp$ splitting has no apparent linear-ω dependence but rather the correct cubic dependence. It is a little puzzling that there are no spurious linear-ω terms in the $\perp\rightarrow\perp\perp$ splitting results of MBW. This may, however, be due to the integrands, after appropriately relabeling the sums, canceling for the $\perp\rightarrow\perp\perp$ splitting case rather than the integrations

TABLE I. $T_{\rm total}$ at various fields B for the splittings $\perp\to\parallel\parallel$ and $\perp\to\perp\perp$. The attenuation coefficients R (cm^{-1}) can be obtained from these via Eq. (19).

B	$\Sigma_{n=0}^{200}T_{\perp\to\parallel\parallel}(n)$	$\Sigma_{n=0}^{\infty}T_{\perp\to\parallel\parallel}(n)$	$\Sigma_{n=0}^{200}T_{\perp\to\perp\perp}(n)$	$\Sigma_{n=0}^{\infty}T_{\perp\to\perp\perp}(n)$
1000	-0.33207	-0.33207	-6.587×10^{-4}	-6.570×10^{-4}
500	-0.33083	-0.33083	-1.302×10^{-3}	-1.299×10^{-3}
100	-0.32129	-0.32130	-6.062×10^{-3}	-6.046×10^{-3}
50	-0.31022	-0.31024	-0.01129	-0.01126
10	-0.24155	-0.24163	-0.03756	-0.03740
5	-0.18415	-0.18432	-0.05126	-0.05092
1	-0.03828	-0.03912	-0.03621	-0.03454
0.5	-9.208×10^{-3}	-0.01088	-0.01709	-0.01374
0.1	8.101×10^{-3}	-1.581×10^{-4}	-0.01698	-2.826×10^{-4}
0.05	0.01629	-2.041×10^{-5}	-0.03333	-3.735×10^{-5}
0.01	0.07349	-1.648×10^{-7}	-0.15742	-3.063×10^{-7}

over p_z which is the case for the $\perp\to\parallel\parallel$ splitting. As these integrations in MBW are evaluated numerically, the sums that require relabeling for the $\perp\to\parallel\parallel$ splitting may not be so obvious.

IV. CUBIC-ω TERMS

The S-matrix elements that are of third-order in photon energy are referred to as the cubic-ω terms. (The terms quadratic in photon energy are zero.) In this section we describe the form of these terms, and then discuss the sums over n, and specifically the use of the Euler-Maclaurin summation formula in evaluating these sums. For $\perp\to\parallel\parallel$ and $\perp\to\perp\perp$ splittings, the cubic-ω terms include terms $\propto\omega^3$ and $\propto\omega\omega'\omega''$. After appropriate relabeling of the sums over the Landau quantum numbers, the former sum to zero, consistent with the dependence on photon energy found by Stoneham [8].

The integrations over p_z in each of the possible $\omega\omega'\omega''$ dependent arrangements are carried out using Eq. (11) and similar substitutions. For each of the two splittings, the results of the integrations are summed together at each Landau quantum number n to give $T(n)$. The total S-matrix element is of the form

$$S_{fi}^{(3)}=F_3\sum_{n=0}^{\infty}T(n), \tag{16}$$

$$F_3=\frac{4\pi^2(4\pi\alpha_f)^{3/2}}{\sqrt{\omega\omega'\omega''}}\,\delta(k_x''+k_x'-k_x)\,\delta(\omega''+\omega'-\omega)\times\frac{1}{(2V)^{3/2}}\,\omega\omega'\omega''. \tag{17}$$

The different arrangements that make up the $T(n)$ term, and which are given explicitly in Appendix B, have no photon energy dependence. Photon energy dependence appears only in the factor F_3. On the other hand, there is no magnetic field dependence in the factor F_3; this appears only in the $T(n)$ component. The sum

$$T_{\rm total}=\sum_{n=0}^{n_{\rm max}}T(n) \tag{18}$$

appears in the expression for the attenuation coefficient, or inverse mean free path, as in MBW:

$$R=\frac{\alpha_f^3}{8\pi^2\lambda\!\!\!^{-}_c}\,\frac{\omega^5}{30}|T_{\rm total}|^2=4.245\left(\frac{\hbar\omega}{0.511\ \ \mathrm{MeV}}\right)^5|T_{\rm total}|^2\ \ \mathrm{cm}^{-1}. \tag{19}$$

We perform the sums to evaluate Eq. (19) explicitly for $\perp\to\parallel\parallel$ and $\perp\to\perp\perp$ and for various values of B, and our results are summarized in Table I.

The sum over n converges increasingly rapidly with increasing B. For example, for the high-B limit for $\perp\to\parallel\parallel$ splitting, the leading term, $T(0)\to-1/3$ for $B\gg1$, suffices. For $\omega=0.1$, this gives an attenuation coefficient of $R=4.72\times10^{-6}$ cm^{-1}, which reproduces the high field, low frequency limits obtained by Adler [6], Stoneham [8] and very recently by Baier *et al.* [11], and is in excellent agreement with Fig. 1 of Baring and Harding [14] and with the high field portion of Fig. 1 of Wilke and Wunner [13]. As B is decreased, $n_{\rm max}$ needs to be increased. In columns 2 and 4 of Table I, $T_{\rm total}$ is evaluated for $n_{\rm max}=200$ and a range of B values for $\perp\to\parallel\parallel$ and $\perp\to\perp\perp$ splittings respectively. However, this $n_{\rm max}$ is inadequate for the smaller values of B, and various computational difficulties arise (e.g., due to rapid variations in the logarithmic terms) in performing the sum directly.

A. Euler-Maclaurin summation formula

To circumvent these computational difficulties, we use the Euler-Maclaurin summation formula to extend $n_{\rm max}$ to ∞. This procedure involves terminating the sum after a finite (usually small) number of terms and estimating the residual in terms of an integral and odd derivatives of the function $T(n)$: namely,

$$T_{\rm total}=\sum_{n=0}^{\infty} T(n)=T(0)+\cdots+T(i-1)+\frac{1}{2}T(i)+\int_i^{\infty} dn T(n)-\frac{1}{12}T'(i)+\frac{1}{720}T'''(i)-\frac{1}{30240}T^{v}(i)+\frac{1}{1209600}T^{vii}(i)+\cdots , \quad (20)$$

where T', T''', T^{v} and T^{vii} denote the first, third, fifth and seventh derivatives of $T(i)$ and all the derivatives at the infinity limit are zero.

There is freedom to choose the values of i and of the number of derivatives of $T(i)$ to be retained, and these should be chosen such that the result is not sensitive to this choice. There is an added complication in the present application in that the sums in the arrangements may start at $n=0$, 1, 2 or 3, and one needs to decide whether to start counting at $n=0$ or at the first term in each of the sums. The latter option must be chosen to avoid spurious terms, of the same kind that appear in the linear-ω component discussed above. This is equivalent to relabeling the sums as for the linear-ω component. In this way, any spurious terms, such as the arrangements C_{12} and C_{13} for the $\perp\to\parallel\parallel$ splitting (see Appendix B), cancel. Once this relabeling is performed, the different arrangements are summed together to give $T(n)$ (see Appendix C).

The minimum value of i, at which $T(n)$ is well-behaved, is unity. However, the contributions from the higher order derivatives can be made smaller by choosing a higher value of i. For $B\geqslant 0.1$, derivatives of order greater than 5 are insignificant for $i=2$ or 3. For $B=0.01$, contributions from derivatives of order greater than 5 are negligible at the minimum $i=1$ value and hence $i=1$ was chosen for this B. Numerical instabilities occurred in the evaluation of $T_{\rm total}$ at the lowest field considered, that is 0.01, due to the natural logarithm terms in Eq. (20). When these logarithmic terms have arguments close to unity, as is the case for low fields, they produce terms that are only accurate to $\sim$10% of $T_{\rm total}$. The arguments of these logarithmic terms are ratios of rest mass energies, for example,

$$\frac{\varepsilon_{0,m+1}^2}{\varepsilon_{0,m}^2}=1+\frac{2B}{\varepsilon_{0,m}^2}, \quad (21)$$

the natural logarithm of which can be expanded as a power series using

$$\ln(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots , \quad -1<x\leqslant 1, \quad (22)$$

where $x=2B/\varepsilon_{0,m}^2$. If one uses expansions such as this in place of the natural logarithm terms in Eq. (20) for $B=0.01$, the numerical instabilities can be overcome and an accurate value of $T_{\rm total}$ is obtained for small i.

Such numerical instabilities, generated for very small fields B or very large n, limit the use of the Euler-Maclaurin summation approach to $B\gtrsim 0.01$ and i not too large. The results of the Euler-Maclaurin summation approach are given in columns 3 and 5 of Table I for the $\perp\to\parallel\parallel$ and $\perp\to\perp\perp$ splittings respectively and a range of B values.

TABLE II. R/ω^5 at various fields B for the splittings $\perp\to\parallel\parallel$ and $\perp\to\perp\perp$ calculated from the Euler-Maclaurin summation results of Table I.

B	$R/\omega^5(\perp\to\parallel\parallel)$	$R/\omega^5(\perp\to\perp\perp)$
1000	4.681×10^{-1}	1.832×10^{-6}
500	4.646×10^{-1}	7.163×10^{-6}
100	4.382×10^{-1}	1.552×10^{-4}
50	4.086×10^{-1}	5.382×10^{-4}
10	2.478×10^{-1}	5.938×10^{-3}
5	1.442×10^{-1}	1.101×10^{-2}
1	6.496×10^{-3}	5.064×10^{-3}
0.5	5.025×10^{-4}	8.014×10^{-4}
0.1	1.061×10^{-7}	3.390×10^{-7}
0.05	1.768×10^{-9}	5.922×10^{-9}
0.01	1.153×10^{-13}	3.983×10^{-13}

B. Value of $n_{\rm max}$

In the present analysis for low photon energies, $n_{\rm max}$ must be finite. As already noted, for $B\gg 1$, a high accuracy can be obtained by retaining only the first few n; for example, for $B=10$ we estimate that $n_{\rm max}$=20 suffices. By Eq. (3), for $B\geqslant 10$, such an $n_{\rm max}$ validates the results for $\omega\leqslant 0.1$. However, with decreasing B one needs to choose larger $n_{\rm max}$. Moreover, the requirement of Eq. (3) then imposes a limit on the frequency for which the sum is valid. For $B=1$, an $n_{\rm max}\sim 1000$ is suitable (corresponding to a frequency limit of $\omega\leqslant 0.01$), but for $B<1$, $n_{\rm max}$ should increase as $\sim 1/B$. Specifically, for $B=1$, $B=0.1$ and $B=0.01$, this means $n_{\rm max}$ should be $\sim 10^3$, $\sim 10^4$ and $\sim 10^5$ which, conservatively from Eq. (3), requires $\omega\leqslant 0.01$, $\omega\leqslant 0.001$ and $\omega\leqslant 0.0001$ respectively. For $B\lesssim 1$, this results in a difference of about 1% between $T_{\rm total}$ as evaluated via the sum from 0 to $n_{\rm max}$ and $T_{\rm total}$ as evaluated from the Euler-Maclaurin summation approach. Hence the Euler-Maclaurin results are accurate to at least 1% for the range of B values presented and the corresponding frequency limits. These frequency limits for $B\lesssim 1$ can be increased [but not beyond $\omega\leqslant 0.1$ as required by Eq. (2)], if one accepts a corresponding increased error in $T_{\rm total}$.

MBW chose $n_{\rm max}$=30 for the two fields $B=1$ and $B=0.1$, and clearly these are not large enough. In the retraction paper [13], Wilke and Wunner stated that $n_{\rm max}$ had been increased in their new evaluations. According to the foregoing estimates one needs $n_{\rm max}\sim 10^3-10^4$ for the smallest value, $B=0.5$, for which results are presented by [13], and it is unlikely that they chose $n_{\rm max}$ large enough for their results to be accurate.

C. Decay rate

In Table II, R/ω^5 is evaluated via Eq. (19) from the Euler-Maclaurin sums presented in Table I. The results are plotted in Fig. 1 for magnetic fields strengths $B\geqslant 0.1$. Consider first the $\perp\to\parallel\parallel$ splitting. The graph is similar in shape to that found by Wilke and Wunner [13], and it asymptotes to the high-B limit of Baier *et al.* [11]. For $B\geqslant 10$ and $\omega=0.1$, our

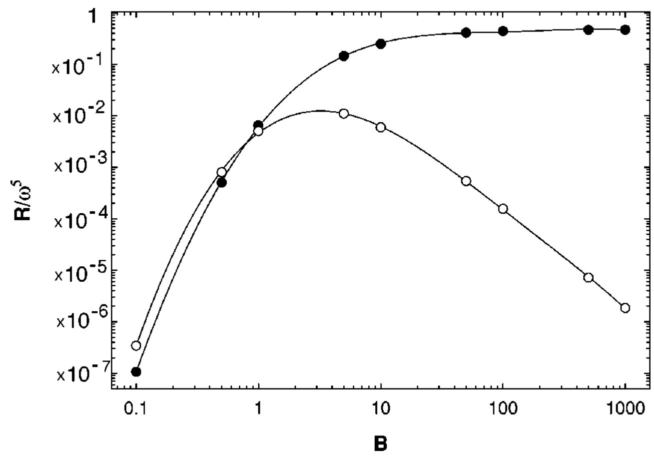

FIG. 1. R/ω^5 versus field B for the splittings $\perp \to \parallel\parallel$ (denoted by solid circles) and $\perp \to \perp\perp$ (denoted by open circles).

results are in close agreement with those in Fig. 1 of [13]. We are unable to make comparisons for $B \lesssim 1$ where we expect the results of [13] to be overestimates due to their n_{max} not being sufficiently large. Our results are however in excellent agreement with those of Baring and Harding [14] at the two weakest fields, $B=1$ and $B=0.1$, that they consider. These authors also found significant discrepancies (factors of 2–3) between their proper-time computations and the results of Wilke and Wunner when $B<1$. Furthermore, our S-matrix results and the proper-time results of [14] converge on the $B \ll 1$ limit of Adler [6].

As is apparent from Table II and Fig. 1, the channel $\perp \to \perp\perp$ is favored compared to the channel $\perp \to \parallel\parallel$ for $B \ll 1$, but there is a changeover such that $\perp \to \perp\perp$ becomes negligible for $B \gg 1$. The weak-B and high-B results are embodied in the low frequency asymptotic limits obtained by Adler [6] and Stoneham [8], the branching ratios for which are given explicitly in [21]. The high-B results are also consistent with those of Baier *et al.* [11] who found that only $\perp \to \parallel\parallel$ splitting is allowed for $B \gg 1$. The exact behavior of the branching ratios for the two channels between these two B-limits is fully described in this analysis. In the next section we comment briefly on the possible astrophysical significance of this result.

A comparison of our results with those of Stoneham [8] provides a useful check on the range of validity of the Euler-Maclaurin summation approach for $B \ll 1$. Stoneham's results imply R/ω^5 values of 1.16×10^{-7} and 1.16×10^{-13} for $B=0.1$ and $B=0.01$ respectively for $\perp \to \parallel\parallel$ splitting and 3.94×10^{-7} and 3.94×10^{-13} for $\perp \to \perp\perp$ splitting. These are comparable with the corresponding R/ω^5 values presented in Table II obtained using the Euler-Maclaurin summation approach. For $\perp \to \perp\perp$ compared with $\perp \to \parallel\parallel$ splitting, Stoneham [8] calculated a branching ratio of $1.85^2=3.42$ [Eq. (43) of [8]] for $B \ll 1$, $\omega \ll 1$, and our results give a branching ratio increasing with decreasing B and equal to $1.86^2=3.46$ for the lowest value $B=0.01$ for which we found that the Euler-Maclaurin summation approach did not encounter noticeable numerical difficulties. We conclude that the Euler-Maclaurin summation approach gives reliable results at least for fields as weak as $B=0.01$.

V. CONCLUSIONS

Our investigation of the S-matrix treatment of photon splitting confirms the analytic results of MBW [1] in detail, and establishes the equivalence of the S-matrix approach to the alternative treatments based on the effective Lagrangian and proper time techniques. Several new results emerge from our investigation in the low frequency regime:

The expansion of the S-matrix elements leads to elements that involve sums over spin states and principal quantum number states; we perform the sums over the spin states explicitly. For $\omega \ll 1$, we show how the sums of principal quantum numbers reduce to a set of what we call arrangements, each of which involves a sum over a single principal quantum number n.

This expansion of the S-matrix elements includes terms that are linear in the frequencies (linear-ω terms), but such terms are known not to contribute to the transition rates evaluated using the effective Lagrangian or proper time techniques. We show that these contributions either integrate to zero for each n or cancel due to the terms in one arrangement at n canceling with the terms in another arrangement at $n+1$.

We point out that if the relevant sums are not relabeled appropriately, truncating the sum at any given $n=n_{max}$ leaves a spurious residual term $\propto\omega$. We estimate the magnitude of this term for the value (n_{max}=30) chosen by MBW, and note that it appears that some of their numerical results exhibit (an incorrect) frequency dependence that is expected from such a spurious contribution.

We evaluate the S-matrix elements that are cubic in the frequencies (cubic-ω terms) for the two splittings $\perp \to \parallel\parallel$ and $\perp \to \perp\perp$, and note that only terms $\propto \omega\omega'\omega''$ contribute to the transition rates evaluated using the effective Lagrangian or proper time techniques. We show that the terms in the S-matrix treatment which are not of this form integrate or sum to zero. The integrations over p_z for the $\propto \omega\omega'\omega''$ contributions are evaluated analytically.

We estimate the value of n_{max} needed to give reliable results; for $B \gg 1$ an n_{max} of a few suffices, but for $B \lesssim 1$ we suggest as a conservative rule of thumb that $n_{max}=10^3/B$ is needed. This casts doubt on all existing numerical estimates of the splitting rates for $B \lesssim 1$ based on the S-matrix approach. The S-matrix approach, however, is useful for calculating photon splitting rates for strong fields where the sum over n converges rapidly.

We show that the use of the Euler-Maclaurin summation formula allows one to estimate the sum for $B \lesssim 1$; our results based on this technique reproduce those evaluated using the effective Lagrangian or proper time techniques for $B \gtrsim 0.01$. However, the technique develops numerical fluctuations that limit its use to $B \gtrsim 0.01$.

One result that we find comfirms the weak-B and high-B branching ratios for the splittings $\perp \to \perp\perp$ and $\perp \to \parallel\parallel$: this ratio decreases from its value ~ 3.4 for $B \ll 1$ with increasing B, and approaches zero for $B \gg 1$; cf. Fig. 1. The behavior of the branching ratio between these two extremes is fully described herein.

We comment briefly on the possible astrophysical significance of this final result. As discussed in Sec. I, for $B \gtrsim 0.4$, photon splitting is important in the context of the population of pulsar magnetospheres by pairs. In strongly magnetized neutron stars ($B \gtrsim 0.1$), photons form bound pairs (positronium) rather than free pairs [31,32]. One possible implication of the change in branching ratio is on the state in positronium that is formed. Only ∥-polarized photons decay to form the ground state of positronium whereas ⊥-polarized photons can decay into positronium with either the electron or the positron in its first Landau level [33,34]. (The latter decay to the ground state with emission of a cyclotron photon which may give an observable signature.) Thus the variation of the branching ratio with field B affects the population of the Landau levels of the resulting positronium. For $0.4 \lesssim B \lesssim 1$, the branching ratio for the splittings $\perp \to \perp\perp$ and $\perp \to \parallel\parallel$ is of the order of unity, and for $B \gtrsim 1$, this ratio decreases rapidly with increasing B ($\propto B^{-2}$ for $B \gtrsim 10$). Hence as the field B increases, the photons produced in the photon splitting process are increasingly more ∥-polarized, favoring the formation of positronium in the ground state.

A general conclusion is that the S-matrix approach is much more cumbersome than the approaches based on the effective Lagrangian or proper time techniques, especially for numerical purposes. We have simplified the S-matrix approach by summing over the spin states, and by introducing the Euler-Maclaurin summation technique to sum over n. However, these difficulties simply do not arise in the other approaches. It should be emphasized that the different approaches are formally equivalent, as may be shown by starting from the propagator used in the S-matrix approach, and performing the sums and the integral over p_z to reconstruct the propagator used in the other approaches; cf. [3]. Future numerical calculations for photon splitting are better based on the simpler alternative approach, rather than the S-matrix approach for fields $B \lesssim 1$.

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APPENDIX A

In this appendix, the simplified forms for the S-matrix amplitudes $S^{(3)}_{fi,j}$ for $j=1-6$ are presented for the three splittings permitted by CP invariance. These amplitudes [with (1)–(3) of Sec. II implemented] are expressed, using the scaling factor S_0, as

$$S^{(3)}_{fi,1}=S_0\sum_{nn'n''}\int_{-\infty}^{\infty}dp_z\,\frac{(-)^{n''-n'}}{\varepsilon\varepsilon'\varepsilon''}\,\frac{1}{\varepsilon+\varepsilon''+\omega'-i\epsilon}\,\frac{1}{\varepsilon'+\varepsilon''+\omega-i\epsilon'}\,\mathcal{D}_1,$$

$$S^{(3)}_{fi,3}=S_0\sum_{nn'n''}\int_{-\infty}^{\infty}dp_z\,\frac{(-)^{n''-n'}}{\varepsilon\varepsilon'\varepsilon''}\,\frac{1}{\varepsilon+\varepsilon''-\omega'-i\epsilon}\,\frac{1}{\varepsilon'+\varepsilon''-\omega-i\epsilon'}\,\mathcal{D}_1,$$

$$S^{(3)}_{fi,2}=S_0\sum_{nn'n''}\int_{-\infty}^{\infty}dp_z\,\frac{(-)^{n''-n'}}{\varepsilon\varepsilon'\varepsilon''}\,\frac{1}{\varepsilon''+\varepsilon'+\omega-i\epsilon'}\,\frac{1}{\varepsilon+\varepsilon'+\omega''-i\epsilon}\,\mathcal{D}_2,$$

$$S^{(3)}_{fi,4}=S_0\sum_{nn'n''}\int_{-\infty}^{\infty}dp_z\,\frac{(-)^{n''-n'}}{\varepsilon\varepsilon'\varepsilon''}\,\frac{1}{\varepsilon''+\varepsilon'-\omega-i\epsilon'}\,\frac{1}{\varepsilon+\varepsilon'-\omega''-i\epsilon}\,\mathcal{D}_2,$$

$$S^{(3)}_{fi,5}=S_0\sum_{nn'n''}\int_{-\infty}^{\infty}dp_z\,\frac{(-)^{n''-n'}}{\varepsilon\varepsilon'\varepsilon''}\,\frac{1}{\varepsilon'+\varepsilon-\omega''-i\epsilon}\,\frac{1}{\varepsilon''+\varepsilon+\omega'-i\epsilon'}\,\mathcal{D}_3,$$

$$S^{(3)}_{fi,6}=S_0\sum_{nn'n''}\int_{-\infty}^{\infty}dp_z\,\frac{(-)^{n''-n'}}{\varepsilon\varepsilon'\varepsilon''}\,\frac{1}{\varepsilon'+\varepsilon+\omega''-i\epsilon}\,\frac{1}{\varepsilon''+\varepsilon-\omega'-i\epsilon'}\,\mathcal{D}_3, \tag{A1}$$

where

$$S_0=2\pi^2\frac{(4\pi\alpha_f)^{3/2}B}{\sqrt{\omega\omega'\omega''}}\,\frac{1}{(2V)^{3/2}}\,\delta(k''_x+k'_x-k_x)\times\delta(\omega''+\omega'-\omega), \tag{A2}$$

$$\varepsilon=(p_z^2+\varepsilon_0^2)^{1/2}\quad\text{and}\quad\varepsilon_0^2=(1+2nB)^{1/2},$$
$$\varepsilon'=(p_z^2+\varepsilon_0'^2)^{1/2}\quad\text{and}\quad\varepsilon_0'^2=(1+2n'B)^{1/2},$$
$$\varepsilon''=(p_z^2+\varepsilon_0''^2)^{1/2}\quad\text{and}\quad\varepsilon_0''^2=(1+2n''B)^{1/2}. \tag{A3}$$

The $\mathcal{D}_1$, $\mathcal{D}_2$ and $\mathcal{D}_3$ components differ for each of the splittings and are defined as follows:

(1) $\perp\to\parallel\parallel$

$$\mathcal{D}_1=\sqrt{8nn'n''B^3}\Delta_1+\sqrt{2n''B}[\varepsilon\varepsilon'+p_z^2-1]\Delta_2+\sqrt{2n'B}[\varepsilon\varepsilon''-p_z^2+1]\Delta_3+\sqrt{2nB}[\varepsilon'\varepsilon''+p_z^2+1]\Delta_4,$$

$$\mathcal{D}_2=\sqrt{8nn'n''B^3}\Delta_1-\sqrt{2n''B}[\varepsilon\varepsilon'-p_z^2+1]\Delta_2-\sqrt{2n'B}[\varepsilon\varepsilon''+p_z^2-1]\Delta_3+\sqrt{2nB}[\varepsilon'\varepsilon''+p_z^2+1]\Delta_4,$$

$$\mathcal{D}_3=\sqrt{8nn'n''B^3}\Delta_1-\sqrt{2n''B}[\varepsilon\varepsilon'-p_z^2+1]\Delta_2+\sqrt{2n'B}[\varepsilon\varepsilon''-p_z^2+1]\Delta_3-\sqrt{2nB}[\varepsilon'\varepsilon''-p_z^2-1]\Delta_4, \tag{A4}$$

where, using the J-notation of Melrose and Parle [29],

$$\Delta_1=J_{n-n'}^{n'-1}(\chi'')J_{n''-n}^{n}(\chi')J_{n'-n''+1}^{n''-1}(\chi)-J_{n-n'}^{n'}(\chi'')J_{n''-n}^{n-1}(\chi')J_{n'-n''-1}^{n''}(\chi),$$

$$\Delta_2=J_{n-n'}^{n'-1}(\chi'')J_{n''-n}^{n-1}(\chi')J_{n'-n''-1}^{n''}(\chi)-J_{n-n'}^{n'}(\chi'')J_{n''-n}^{n}(\chi')J_{n'-n''+1}^{n''-1}(\chi),$$

$$\Delta_3=J_{n-n'}^{n'-1}(\chi'')J_{n''-n}^{n-1}(\chi')J_{n'-n''+1}^{n''-1}(\chi)-J_{n-n'}^{n'}(\chi'')J_{n''-n}^{n}(\chi')J_{n'-n''-1}^{n''}(\chi),$$

$$\Delta_4=J_{n-n'}^{n'-1}(\chi'')J_{n''-n}^{n}(\chi')J_{n'-n''-1}^{n''}(\chi)-J_{n-n'}^{n'}(\chi'')J_{n''-n}^{n-1}(\chi')J_{n'-n''+1}^{n''-1}(\chi). \tag{A5}$$

(2) $\perp\to\perp\perp$

$$\mathcal{D}_1=\sqrt{8nn'n''B^3}\Delta_1+\sqrt{2n''B}[\varepsilon\varepsilon'-p_z^2-1]\Delta_2+\sqrt{2n'B}[\varepsilon\varepsilon''+p_z^2+1]\Delta_3+\sqrt{2nB}[\varepsilon'\varepsilon''+p_z^2+1]\Delta_4,$$

$$\mathcal{D}_2=\sqrt{8nn'n''B^3}\Delta_1-\sqrt{2n''B}[\varepsilon\varepsilon'+p_z^2+1]\Delta_2-\sqrt{2n'B}[\varepsilon\varepsilon''-p_z^2-1]\Delta_3+\sqrt{2nB}[\varepsilon'\varepsilon''+p_z^2+1]\Delta_4,$$

$$\mathcal{D}_3=\sqrt{8nn'n''B^3}\Delta_1-\sqrt{2n''B}[\varepsilon\varepsilon'+p_z^2+1]\Delta_2+\sqrt{2n'B}[\varepsilon\varepsilon''+p_z^2+1]\Delta_3-\sqrt{2nB}[\varepsilon'\varepsilon''-p_z^2-1]\Delta_4, \tag{A6}$$

where

$$\Delta_1=J_{n-n'+1}^{n'-1}(\chi'')J_{n''-n+1}^{n-1}(\chi')J_{n'-n''+1}^{n''-1}(\chi)-J_{n-n'-1}^{n'}(\chi'')J_{n''-n-1}^{n}(\chi')J_{n'-n''-1}^{n''}(\chi),$$

$$\Delta_2=J_{n-n'+1}^{n'-1}(\chi'')J_{n''-n-1}^{n}(\chi')J_{n'-n''-1}^{n''}(\chi)-J_{n-n'-1}^{n'}(\chi'')J_{n''-n+1}^{n-1}(\chi')J_{n'-n''+1}^{n''-1}(\chi),$$

$$\Delta_3=J_{n-n'+1}^{n'-1}(\chi'')J_{n''-n-1}^{n}(\chi')J_{n'-n''+1}^{n''-1}(\chi)-J_{n-n'-1}^{n'}(\chi'')J_{n''-n+1}^{n-1}(\chi')J_{n'-n''-1}^{n''}(\chi),$$

$$\Delta_4=J_{n-n'+1}^{n'-1}(\chi'')J_{n''-n+1}^{n-1}(\chi')J_{n'-n''-1}^{n''}(\chi)-J_{n-n'-1}^{n'}(\chi'')J_{n''-n-1}^{n}(\chi')J_{n'-n''+1}^{n''-1}(\chi). \tag{A7}$$

(3) $\parallel\to\perp\parallel$

$$\mathcal{D}_1=\sqrt{8nn'n''B^3}\Delta_1+\sqrt{2n''B}[\varepsilon\varepsilon'+p_z^2-1]\Delta_2+\sqrt{2n'B}[\varepsilon\varepsilon''+p_z^2+1]\Delta_3+\sqrt{2nB}[\varepsilon'\varepsilon''-p_z^2+1]\Delta_4,$$

$$\mathcal{D}_2=\sqrt{8nn'n''B^3}\Delta_1-\sqrt{2n''B}[\varepsilon\varepsilon'-p_z^2+1]\Delta_2-\sqrt{2n'B}[\varepsilon\varepsilon''-p_z^2-1]\Delta_3+\sqrt{2nB}[\varepsilon'\varepsilon''-p_z^2+1]\Delta_4,$$

$$\mathcal{D}_3=\sqrt{8nn'n''B^3}\Delta_1-\sqrt{2n''B}[\varepsilon\varepsilon'-p_z^2+1]\Delta_2+\sqrt{2n'B}[\varepsilon\varepsilon''+p_z^2+1]\Delta_3-\sqrt{2nB}[\varepsilon'\varepsilon''+p_z^2-1]\Delta_4, \tag{A8}$$

where

$$\Delta_1=J_{n-n'}^{n'-1}(\chi'')J_{n''-n-1}^{n}(\chi')J_{n'-n''}^{n''}(\chi)-J_{n-n'}^{n'}(\chi'')J_{n''-n+1}^{n-1}(\chi')J_{n'-n''}^{n''-1}(\chi),$$

$$\Delta_2=J_{n-n'}^{n'-1}(\chi'')J_{n''-n+1}^{n-1}(\chi')J_{n'-n''}^{n''-1}(\chi)-J_{n-n'}^{n'}(\chi'')J_{n''-n-1}^{n}(\chi')J_{n'-n''}^{n''}(\chi),$$

$$\Delta_3=J_{n-n'}^{n'-1}(\chi'')J_{n''-n+1}^{n-1}(\chi')J_{n'-n''}^{n''}(\chi)-J_{n-n'}^{n'}(\chi'')J_{n''-n-1}^{n}(\chi')J_{n'-n''}^{n''-1}(\chi),$$

$$\Delta_4=J_{n-n'}^{n'-1}(\chi'')J_{n''-n-1}^{n}(\chi')J_{n'-n''}^{n''-1}(\chi)-J_{n-n'}^{n'}(\chi'')J_{n''-n+1}^{n-1}(\chi')J_{n'-n''}^{n''}(\chi). \tag{A9}$$

The arguments of the J functions are defined via

$$\chi=\frac{\omega^2}{2B},\quad \chi'=\frac{\omega'^2}{2B},\quad \chi''=\frac{\omega''^2}{2B}, \tag{A10}$$

where the J functions are

$$J_\nu^n(x)=\left\{\frac{n!}{(n+\nu)!}\right\}^{1/2}e^{-x/2}x^{\nu/2}L_n^\nu(x)=(-1)^\nu J_{-\nu}^{n+\nu} \tag{A11}$$

and $L_n^\nu(x)$ are the generalized Laguerre functions. These functions are related to the $I_{n,n'}(x)$ functions defined by Sokolov and Ternov [35] by

$$I_{n,n'}(x)=J_{n-n'}^{n'}(x). \tag{A12}$$

APPENDIX B

The third order photon energy terms can be obtained in only three possible ways:

(1) by coupling the linear photon energy dependent terms to the quadratic terms in the low energy expansion of the product of the energy denominators [denoted by (102)]

(2) by coupling the linear photon energy dependent terms to the quadratic component of the Laguerre polynomial for each J in turn out of the three in the triple J product and the zero-order energy component of the low energy expansion of the product of the energy denominators [denoted by (120)]

(3) by selecting those triple J products, $J^a_\alpha J^b_\beta J^c_\gamma$, which give

$$|\alpha|+|\beta|+|\gamma|=3 \tag{B1}$$

(for example, combinations such as $\alpha=\pm 2$, $\beta=\pm 1$, $\gamma=0$), where the coupling is to the zero-order components of both the Laguerre polynomial and the low energy expansion of the product of the energy denominators [denoted (300)].

In the low frequency regime, the total S-matrix is given by Eq. (16), where

$$\sum_{n=0}^{\infty} T(n)=\sum_{k=1}^{k_{\max}} C_k \tag{B2}$$

and $k_{\max}$ is 13 and 30 respectively for the $\perp\Rightarrow\parallel\,\parallel$ and $\perp\Rightarrow\perp\,\perp$ splittings. For these two splitting modes, the C_k components at each of the allowable arrangements are presented below in their entirety.

1. $\omega''\omega'\omega$ dependent terms of the $\perp\Rightarrow\parallel\,\parallel$ splitting mode

There are three (102) type terms corresponding to the three arrangements C_1 with $n''=n'=n$; C_2 with $n'=n+1$, $n''=n$ or $n'=n$, $n''=n+1$; and C_3 with $n'=n-1$, $n''=n$ or $n'=n$, $n''=n-1$, where

$$C_1=-\frac{4B}{3}\sum_{n=0}^{\infty}\frac{n}{\varepsilon_{0,n}^4}, \tag{B3}$$

$$C_2=\frac{1}{2B}\sum_{n=0}^{\infty}\left\{\frac{3}{2}(6n+1)+\frac{2B}{3\varepsilon_{0,n}^2}(4n-1)+\frac{5\varepsilon_{0,n}^2}{2B}(2n+1)-\ln\left(\frac{\varepsilon_{0,n+1}^2}{\varepsilon_{0,n}^2}\right)\left[5(2n+1)\left(\frac{\varepsilon_{0,n}^2}{2B}\right)^2+2(7n+2)\frac{\varepsilon_{0,n}^2}{2B}+5n\right]\right\}, \tag{B4}$$

$$C_3=\frac{1}{2B}\sum_{n=1}^{\infty}\left\{\frac{3}{2}(6n-1)-\frac{2B}{3\varepsilon_{0,n}^2}(4n+1)-\frac{5\varepsilon_{0,n}^2}{2B}(2n-1)+\ln\left(\frac{\varepsilon_{0,n}^2}{\varepsilon_{0,n-1}^2}\right)\left[5(2n-1)\left(\frac{\varepsilon_{0,n}^2}{2B}\right)^2-2(7n-2)\frac{\varepsilon_{0,n}^2}{2B}+5n\right]\right\}. \tag{B5}$$

Similarly, there are three (120) type terms corresponding to the three arrangements C_4 with $n''=n'=n$; C_5 with $n'=n+1$, $n''=n$ or $n'=n$, $n''=n+1$; and C_6 with $n'=n-1$, $n''=n$ or $n'=n$, $n''=n-1$, where

$$C_4=0, \tag{B6}$$

$$C_5=-\frac{1}{2B}\sum_{n=0}^{\infty}n\left[n+\frac{2B}{\varepsilon_{0,n}^2}-\frac{1}{2}\left\{\frac{n\varepsilon_{0,n}^2}{B}+(n-1)\right\}\times\ln\left(\frac{\varepsilon_{0,n+1}^2}{\varepsilon_{0,n}^2}\right)\right], \tag{B7}$$

and

$$C_6=\frac{1}{2B}\sum_{n=1}^{\infty}\left[(n^2-4n+2)+\frac{2Bn}{\varepsilon_{0,n}^2}-\frac{1}{2}\left\{\frac{(n^2-4n+2)\varepsilon_{0,n}^2}{B}-n(n-3)\right\}\ln\left(\frac{\varepsilon_{0,n}^2}{\varepsilon_{0,n-1}^2}\right)\right]. \tag{B8}$$

Finally, there are seven (300) type terms, three of which, C_7, C_8 and C_9, belong to the set $n''=n+1$, $n'=n-1$ or $n''=n-1$, $n'=n+1$; $n''=n-1$, $n'=n-2$ or $n''=n-2$, $n'=n-1$; and $n''=n+2$, $n'=n+1$ or $n''=n+1$, $n'=n+2$ respectively. The other four are C_{10} with $n''=n'=n-1$; C_{11} with $n''=n'=n+1$; C_{12} with $n''=n$, $n'=n-2$ or $n''=n-2$, $n'=n$; and C_{13} with $n''=n+2$, $n'=n$ or $n''=n$, $n'=n+2$. These are as follows:

$$C_7=-\frac{1}{2B}\sum_{n=1}^{\infty}n\left[2\left(\frac{n\varepsilon_{0,n}^2}{B}-1\right)\ln\left(\frac{\varepsilon_{0,n}^2}{\varepsilon_{0,n-1}^2}\right)-\left(\frac{n\varepsilon_{0,n}^2}{B}+(n-1)\right)\ln\left(\frac{\varepsilon_{0,n+1}^2}{\varepsilon_{0,n-1}^2}\right)\right], \tag{B9}$$

$$C_8=\frac{1}{4B}\sum_{n=2}^{\infty}(n-1)\left[\left(\frac{(n-1)\varepsilon_{0,n}^2}{B}-n\right)\ln\left(\frac{\varepsilon_{0,n-1}^2}{\varepsilon_{0,n-2}^2}\right)-\frac{(n-1)\varepsilon_{0,n}^2}{2B}\ln\left(\frac{\varepsilon_{0,n}^2}{\varepsilon_{0,n-2}^2}\right)\right], \tag{B10}$$

$$C_9=\frac{1}{4B}\sum_{n=0}^{\infty}(n+1)\left[\left(\frac{(n+1)\varepsilon_{0,n}^2}{B}+n\right)\ln\left(\frac{\varepsilon_{0,n+1}^2}{\varepsilon_{0,n}^2}\right)-\left(\frac{(n+1)\varepsilon_{0,n}^2}{2B}+n\right)\ln\left(\frac{\varepsilon_{0,n+2}^2}{\varepsilon_{0,n}^2}\right)\right], \tag{B11}$$

$$C_{10}=\frac{1}{B}\sum_{n=1}^{\infty}(n-1)\left[\frac{1}{\varepsilon_{0,n-1}^2}-2n+\frac{\varepsilon_{0,n}^2(2n-1)}{2B}\ln\left(\frac{\varepsilon_{0,n}^2}{\varepsilon_{0,n-1}^2}\right)\right], \tag{B12}$$

$$C_{11}=\frac{1}{B}\sum_{n=0}^{\infty}(n+1)\left[\frac{1}{\varepsilon_{0,n+1}^2}+2n-\frac{\varepsilon_{0,n}^2(2n+1)}{2B}\ln\left(\frac{\varepsilon_{0,n+1}^2}{\varepsilon_{0,n}^2}\right)\right], \tag{B13}$$

$$C_{12}=-\frac{1}{2B}\sum_{n=2}^{\infty}(n-1)\left[(n-1)+\frac{1}{2}\left(n-\frac{(n-1)\varepsilon_{0,n}^2}{2B}\right)\times\ln\left(\frac{\varepsilon_{0,n}^2}{\varepsilon_{0,n-2}^2}\right)\right], \tag{B14}$$

$$C_{13}=\frac{1}{2B}\sum_{n=0}^{\infty}(n+1)\left[(n+1)-\frac{1}{2}\left(n+\frac{(n+1)\varepsilon_{0,n}^2}{2B}\right)\times\ln\left(\frac{\varepsilon_{0,n+2}^2}{\varepsilon_{0,n}^2}\right)\right]. \tag{B15}$$

2. $\omega''\omega'\omega$ dependent terms of the $\perp\Rightarrow\perp\perp$ splitting mode

There are seven (102) type terms comprising C_1 with $n''=n'=n+1$; C_2 with $n''=n-1$, $n'=n$ or $n''=n$, $n'=n-1$; C_3 with $n''=n+1$, $n'=n$ or $n''=n$, $n'=n+1$; C_4 with $n''=n'=n-1$; C_5 with $n''=n-1$, $n'=n+1$ or $n''=n+1$, $n'=n-1$; C_6 with $n''=n+2$, $n'=n+1$ or $n''=n+1$, $n'=n+2$; and C_7 with $n''=n-1$, $n'=n-2$ or $n''=n-2$, $n'=n-1$, which are as follows:

$$C_1=\frac{2}{B}\sum_{n=0}^{\infty}(n+1)\left[\left\{n+\frac{\varepsilon_{0,n}^2(3n+1)}{2B}\right\}\ln\left(\frac{\varepsilon_{0,n+1}^2}{\varepsilon_{0,n}^2}\right)+\frac{B(n+1)}{\varepsilon_{0,n+1}^2}-(3n+1)\right], \tag{B16}$$

$$C_2=\frac{1}{2B}\sum_{n=1}^{\infty}n\left[\left\{2(n-1)+\frac{(12n-7)\varepsilon_{0,n-1}^2}{2B}\right\}\ln\left(\frac{\varepsilon_{0,n}^2}{\varepsilon_{0,n-1}^2}\right)-(12n-7)+\frac{4B^2n}{\varepsilon_{0,n}^4}+\frac{B(8n-3)}{\varepsilon_{0,n}^2}\right], \tag{B17}$$

$$C_3=\frac{1}{2B}\sum_{n=0}^{\infty}n\left[-\left\{5(2n+1)+\frac{(12n+7)\varepsilon_{0,n}^2}{2B}\right\}\ln\left(\frac{\varepsilon_{0,n+1}^2}{\varepsilon_{0,n}^2}\right)+(12n+7)-\frac{4B^2n}{\varepsilon_{0,n}^4}+\frac{B(8n+3)}{\varepsilon_{0,n}^2}\right], \tag{B18}$$

$$C_4=\frac{2}{B}\sum_{n=1}^{\infty}(n-1)\left[\frac{B(n-1)}{\varepsilon_{0,n-1}^2}+(3n-1)-\left\{(2n-1)+\frac{(3n-1)\varepsilon_{0,n-1}^2}{2B}\right\}\ln\left(\frac{\varepsilon_{0,n}^2}{\varepsilon_{0,n-1}^2}\right)\right], \tag{B19}$$

$$C_5=-\frac{1}{B}\sum_{n=1}^{\infty}n\left[-2+\frac{(n+1)}{2}\left(1+\frac{\varepsilon_{0,n}^2}{B}\right)\ln\left(\frac{\varepsilon_{0,n+1}^2}{\varepsilon_{0,n}^2}\right)-\frac{(n-1)}{2}\left(1+\frac{\varepsilon_{0,n-1}^2}{B}\right)\ln\left(\frac{\varepsilon_{0,n}^2}{\varepsilon_{0,n-1}^2}\right)\right], \tag{B20}$$

$$C_6=-\frac{1}{2B}\sum_{n=0}^{\infty}(n+1)\left[-\frac{(4n+1)}{4}\left(1+\frac{\varepsilon_{0,n+1}^2}{B}\right)\ln\left(\frac{\varepsilon_{0,n+2}^2}{\varepsilon_{0,n+1}^2}\right)+\frac{(n-5)}{4}\left(1+\frac{\varepsilon_{0,n}^2}{2B}\right)\ln\left(\frac{\varepsilon_{0,n+2}^2}{\varepsilon_{0,n}^2}\right)+\frac{5}{4}\left(1+\frac{\varepsilon_{0,n}^2}{B}\right)\times\ln\left(\frac{\varepsilon_{0,n+1}^2}{\varepsilon_{0,n}^2}\right)+\frac{3n}{2}+\frac{1}{2}\right], \tag{B21}$$

$$C_7=-\frac{1}{2B}\sum_{n=2}^{\infty}(n-1)\left[\frac{(4n-1)}{4}\left(1+\frac{\varepsilon_{0,n-2}^2}{B}\right)\ln\left(\frac{\varepsilon_{0,n-1}^2}{\varepsilon_{0,n-2}^2}\right)-\frac{(n+5)}{4}\left(1+\frac{\varepsilon_{0,n-2}^2}{2B}\right)\ln\left(\frac{\varepsilon_{0,n}^2}{\varepsilon_{0,n-2}^2}\right)+\frac{5}{4}\left(1+\frac{\varepsilon_{0,n-1}^2}{B}\right)\times\ln\left(\frac{\varepsilon_{0,n}^2}{\varepsilon_{0,n-1}^2}\right)-\frac{3n}{2}+\frac{1}{2}\right]. \tag{B22}$$

Similarly, there are seven (120) type terms comprising C_8 with $n''=n'=n+1$; C_9 with $n''=n-1$, $n'=n$ or $n''=n$, $n'=n-1$; C_{10} with $n''=n+1$, $n'=n$ or $n''=n$, $n'=n+1$; C_{11} with $n''=n'=n-1$; C_{12} with $n''=n+2$, $n'=n+1$ or $n''=n+1$, $n'=n+2$; C_{13} with $n''=n-1$, $n'=n+1$ or $n''=n+1$, $n'=n-1$; and C_{14} with $n''=n-1$, $n'=n-2$ or $n''=n-2$, $n'=n-1$, which are as follows:

$$C_8=\sum_{n=0}^{\infty}4n(n+1)\left[\frac{(n+1)}{\varepsilon_{0,n+1}^2}-\frac{n}{2B}\ln\left(\frac{\varepsilon_{0,n+1}^2}{\varepsilon_{0,n}^2}\right)\right], \tag{B23}$$

$$C_9=-\sum_{n=1}^{\infty}n(n-1)\left[\frac{n}{\varepsilon_{0,n}^2}-\frac{(n-1)}{2B}\ln\left(\frac{\varepsilon_{0,n}^2}{\varepsilon_{0,n-1}^2}\right)\right], \tag{B24}$$

$$C_{10}=-\sum_{n=0}^{\infty}n(n-3)\left[\frac{(n+1)}{2B}\ln\left(\frac{\varepsilon_{0,n+1}^2}{\varepsilon_{0,n}^2}\right)-\frac{n}{\varepsilon_{0,n}^2}\right], \tag{B25}$$

$$C_{11}=\sum_{n=1}^{\infty}4(n-1)^2\left[\frac{n}{2B}\ln\left(\frac{\varepsilon_{0,n}^2}{\varepsilon_{0,n-1}^2}\right)-\frac{(n-1)}{\varepsilon_{0,n-1}^2}\right], \tag{B26}$$

$$C_{12}=-\frac{1}{2B}\sum_{n=0}^{\infty}n(n+1)\left[\frac{(n+2)}{4}\ln\left(\frac{\varepsilon_{0,n+2}^2}{\varepsilon_{0,n}^2}\right)-\frac{(n+1)}{2}\ln\left(\frac{\varepsilon_{0,n+1}^2}{\varepsilon_{0,n}^2}\right)\right], \tag{B27}$$

$$C_{13}=\frac{2}{B}\sum_{n=1}^{\infty}n(2n-1)\left[\frac{(n+1)}{4}\ln\left(\frac{\varepsilon_{0,n+1}^2}{\varepsilon_{0,n-1}^2}\right)-\frac{n}{2}\ln\left(\frac{\varepsilon_{0,n}^2}{\varepsilon_{0,n-1}^2}\right)\right], \tag{B28}$$

$$C_{14}=-\frac{1}{2B}\sum_{n=2}^{\infty}(n-1)(n-4)\left[\frac{n}{4}\ln\left(\frac{\varepsilon_{0,n}^2}{\varepsilon_{0,n-2}^2}\right)-\frac{(n-1)}{2}\ln\left(\frac{\varepsilon_{0,n-1}^2}{\varepsilon_{0,n-2}^2}\right)\right]. \tag{B29}$$

Finally, there are 16 (300) type terms comprising C_{15} with $n''=n'=n$; C_{16} with $n''=n+1$, $n'=n-1$ or $n''=n-1$, $n'=n+1$; C_{17} with $n''=n-1$, $n'=n-2$ or $n''=n-2$, $n'=n-1$; C_{18} with $n''=n+2$, $n'=n+1$ or $n''=n+1$, $n'=n+2$; C_{19} with $n''=n$, $n'=n-1$ or $n''=n-1$, $n'=n$; C_{20} with $n''=n'=n+1$; C_{21} with $n''=n$, $n'=n+1$ or $n''=n+1$, $n'=n$; C_{22} with $n''=n'=n-1$; C_{23} with $n'=n$, $n''=n+2$ or $n'=n+2$, $n''=n$; C_{24} with $n''=n'=n-2$; C_{25} with $n''=n'=n+2$; C_{26} with $n''=n-2$, $n'=n$ or $n'=n-2$, $n''=n$; C_{27} with $n'=n+1$, $n''=n+3$ or $n'=n+3$, $n''=n+1$; C_{28} with $n''=n-3$, $n'=n-2$ or $n''=n-2$, $n'=n-3$; C_{29} with $n''=n+3$, $n'=n+2$ or $n'=n+3$, $n''=n+2$; and C_{30} with $n'=n-3$, $n''=n-1$ or $n''=n-3$, $n'=n-1$, which are as follows:

$$C_{15}=-\sum_{n=0}^{\infty}\frac{2n^2}{\varepsilon_{0,n}^2}\left[\frac{2}{\varepsilon_{0,n}^2}+1\right], \tag{B30}$$

$$C_{16}=\frac{1}{2B}\sum_{n=1}^{\infty}n^2\left[(n+1)\ln\left(\frac{\varepsilon_{0,n+1}^2}{\varepsilon_{0,n-1}^2}\right)-2n\ln\left(\frac{\varepsilon_{0,n}^2}{\varepsilon_{0,n-1}^2}\right)\right], \tag{B31}$$

$$C_{17}=\frac{1}{4B}\sum_{n=2}^{\infty}n(n-1)\left[\ln\left(\frac{\varepsilon_{0,n}^2}{\varepsilon_{0,n-1}^2}\right)\right], \tag{B32}$$

$$C_{18}=\frac{1}{4B}\sum_{n=0}^{\infty}n(n+1)\left[\ln\left(\frac{\varepsilon_{0,n+1}^2}{\varepsilon_{0,n}^2}\right)\right], \tag{B33}$$

$$C_{19}=\sum_{n=1}^{\infty}n(n-1)\left[\frac{n}{\varepsilon_{0,n}^2}-\frac{(2n-1)}{4B}\ln\left(\frac{\varepsilon_{0,n}^2}{\varepsilon_{0,n-1}^2}\right)\right], \tag{B34}$$

$$C_{20}=\sum_{n=0}^{\infty}n(n+1)\left[\frac{2(n+1)}{\varepsilon_{0,n+1}^2}-\frac{(2n+1)}{2B}\ln\left(\frac{\varepsilon_{0,n+1}^2}{\varepsilon_{0,n}^2}\right)\right], \tag{B35}$$

$$C_{21}=\sum_{n=0}^{\infty}n(n+1)\left[\frac{(2n+1)}{4B}\ln\left(\frac{\varepsilon_{0,n+1}^2}{\varepsilon_{0,n}^2}\right)-\frac{n}{\varepsilon_{0,n}^2}\right], \tag{B36}$$

$$C_{22}=\sum_{n=1}^{\infty}n(n-1)\left[\frac{(2n-1)}{2B}\ln\left(\frac{\varepsilon_{0,n}^2}{\varepsilon_{0,n-1}^2}\right)-\frac{2(n-1)}{\varepsilon_{0,n-1}^2}\right], \tag{B37}$$

$$C_{23}=-\sum_{n=0}^{\infty}n(n+1)\left[\frac{(n+2)}{2B}\ln\left(\frac{\varepsilon_{0,n+2}^2}{\varepsilon_{0,n}^2}\right)-\frac{2n}{\varepsilon_{0,n}^2}\right], \tag{B38}$$

$$C_{24}=-\sum_{n=2}^{\infty}(n-1)(n-2)\left[\frac{n}{4B}\ln\left(\frac{\varepsilon_{0,n}^2}{\varepsilon_{0,n-2}^2}\right)-\frac{(n-2)}{\varepsilon_{0,n-2}^2}\right], \tag{B39}$$

$$C_{25}=-\sum_{n=0}^{\infty}(n+1)(n+2)\left[\frac{(n+2)}{\varepsilon_{0,n+2}^2}-\frac{n}{4B}\ln\left(\frac{\varepsilon_{0,n+2}^2}{\varepsilon_{0,n}^2}\right)\right], \tag{B40}$$

$$C_{26}=\sum_{n=2}^{\infty}n(n-1)\left[\frac{(n-2)}{2B}\ln\left(\frac{\varepsilon_{0,n}^2}{\varepsilon_{0,n-2}^2}\right)-\frac{2n}{\varepsilon_{0,n}^2}\right], \tag{B41}$$

$$C_{27}=\frac{1}{4B}\sum_{n=0}^{\infty}(n+1)(n+2)\left[\frac{(n+3)}{3}\ln\left(\frac{\varepsilon_{0,n+3}^2}{\varepsilon_{0,n}^2}\right)-(n+1)\ln\left(\frac{\varepsilon_{0,n+1}^2}{\varepsilon_{0,n}^2}\right)\right], \tag{B42}$$

$$C_{28}=\frac{1}{8B}\sum_{n=3}^{\infty}(n-1)(n-2)\left[\frac{n}{3}\ln\left(\frac{\varepsilon_{0,n}^2}{\varepsilon_{0,n-3}^2}\right)-(n-2)\ln\left(\frac{\varepsilon_{0,n-2}^2}{\varepsilon_{0,n-3}^2}\right)\right], \tag{B43}$$

$$C_{29}=\frac{1}{4B}\sum_{n=0}^{\infty}(n+1)(n+2)\left[\frac{(n+3)}{3}\ln\left(\frac{\varepsilon_{0,n+3}^2}{\varepsilon_{0,n}^2}\right)-\frac{(n+2)}{2}\ln\left(\frac{\varepsilon_{0,n+2}^2}{\varepsilon_{0,n}^2}\right)\right], \tag{B44}$$

$$C_{30}=\frac{1}{2B}\sum_{n=3}^{\infty}(n-1)(n-2)\left[\frac{n}{3}\ln\left(\frac{\varepsilon_{0,n}^2}{\varepsilon_{0,n-3}^2}\right)-\frac{(n-1)}{2}\ln\left(\frac{\varepsilon_{0,n-1}^2}{\varepsilon_{0,n-3}^2}\right)\right]. \tag{B45}$$

APPENDIX C

In this appendix, the $T(n)$ values used in the Euler-Maclaurin summation formula for the two splittings, $\perp\to\parallel\parallel$ and $\perp\to\perp\perp$, are presented.

1. The $\perp\to\parallel\parallel$ splitting

The $T(n)$ term in Eq. (20) for the $\perp\to\parallel\parallel$ mode is

$$T(n)=\frac{1}{2B}\left\{-\frac{8B^2n}{3\varepsilon_{0,n}^4}+\frac{2B(n-1)}{3\varepsilon_{0,n}^2}+\frac{2n}{\varepsilon_{0,n}^2}-\frac{2B(n+2)}{3\varepsilon_{0,n+1}^2}+\frac{2(n+1)}{\varepsilon_{0,n+1}^2}+3(2n+1)-\left[2n^3+n^2\left(12+\frac{1}{B}\right)\right.\right.$$
$$\left.\left.+n\left(11+\frac{7}{B}\right)+1+\frac{7}{2B}\right]\ln\left(\frac{\varepsilon_{0,n+1}^2}{\varepsilon_{0,n}^2}\right)+\left[n^3+\frac{n^2}{2}\left(7+\frac{1}{B}\right)+\frac{n}{2}\left(7+\frac{2}{B}\right)+1+\frac{1}{2B}\right]\ln\left(\frac{\varepsilon_{0,n+2}^2}{\varepsilon_{0,n}^2}\right)\right\},\quad n\geqslant 0. \tag{C1}$$

2. The $\perp\to\perp\perp$ splitting

For the $\perp\to\perp\perp$ splitting, the $T(n)$ term in Eq. (20) is

$$T(n)=\frac{1}{2B}\left\{-\frac{6B(n+1)(n+2)^2}{\varepsilon_{0,n+2}^2}+\frac{4B^2(n+1)^2}{\varepsilon_{0,n+1}^4}+\frac{B(n+1)(12n^2+24n+9)}{\varepsilon_{0,n+1}^2}-\frac{4Bn^2(B+2)}{\varepsilon_{0,n}^4}+\frac{Bn(-6n^2+2n+3)}{\varepsilon_{0,n}^2}-12n\right.$$
$$-3-\frac{1}{4}\left[23n^3+58n^2+n\left(73-\frac{24}{B}\right)+38-\frac{6}{B}\right]\ln\left(\frac{\varepsilon_{0,n+1}^2}{\varepsilon_{0,n}^2}\right)-\frac{1}{2}\left[11n^2+n\left(27+\frac{6}{B}\right)+16+\frac{6}{B}\right]\ln\left(\frac{\varepsilon_{0,n+2}^2}{\varepsilon_{0,n+1}^2}\right)$$
$$\left.+\frac{1}{2}\left[\frac{7n^3}{2}+\frac{37n^2}{2}+n\left(25+\frac{3}{B}\right)+10+\frac{3}{B}\right]\ln\left(\frac{\varepsilon_{0,n+2}^2}{\varepsilon_{0,n}^2}\right)+\frac{1}{2}\left[\frac{3n^3}{2}+9n^2+\frac{33n}{2}+9\right]\ln\left(\frac{\varepsilon_{0,n+3}^2}{\varepsilon_{0,n}^2}\right)\right\},\quad n\geqslant 0. \tag{C2}$$

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