PHYSICAL REVIEW D VOLUME 57, NUMBER 8 15 APRIL 1998

Comments on compositeness in the SU(2) linear σ model

M. D. Scadron*
Department of Physics, University of Tasmania, Hobart, Australia, 7005
(Received 24 March 1997; published 3 March 1998)

First we summarize the quark-level linear σ model compositeness conditions and verify that indeed $m_\sigma = 2m_q$ when $m_\pi = 0$ and $N_c = 3$, rather than in the $N_c \to \infty$ limit, as is sometimes suggested. Then we show that this compositeness picture also predicts a chiral symmetry restoration temperature $T_c = 2f_\pi$, where f_π is the pion decay constant. We contrast this self-consistent $Z=0$ compositeness analysis with prior studies of the compositeness problem. [S0556-2821(98)00408-1]

PACS number(s): 11.15.Pg, 11.30.Rd

Now that the scalar σ meson has been reinstated in the 1996 Particle Data Group (PDG) tables [1], it is appropriate to take seriously the various theoretical implications of a quark-level linear σ model (LσM) field theory. The original spontaneously broken LσM theory [2] was recently dynamically generated [3] at the quark level in the spirit of Nambu and Jona-Lasinio [4]. In this paper we summarize the color number N_c and compositeness properties of the above SU(2) quark-level LσM and comment on the recent LσM analysis of compositeness given by Lurie and Tupper [5].

First we display the interacting part of the standard LσM [2] (quark-level) Lagrangian density shifted around the true vacuum $\langle \vec{\pi} \rangle = \langle \sigma \rangle = 0$:

$$\mathcal{L}_{\text{int}} = g\bar{\psi}(\sigma + i\gamma_5 \vec{\tau}\cdot\vec{\pi})\psi + g'(\sigma^2 + \vec{\pi}^2)\sigma - (\lambda/4)(\sigma^2+\vec{\pi}^2)^2 - f_\pi g \bar{\psi}\psi, \tag{1a}$$

with (spontaneously broken) chiral couplings for $f_\pi \approx 93$ MeV

$$g = m_q/f_\pi, \quad g' = m_\sigma^2/2f_\pi = \lambda f_\pi. \tag{1b}$$

Once the LσM scalar field is shifted to $\langle \sigma \rangle = 0$, giving rise to the interacting but chiral-broken LσM Lagrangian (1), the Lee null-tadpole condition [6] depicted in Fig. 1 must be valid. Following Ref. [3], which exploits the dimensional regularization [7] characterization of these quadratic divergent tadpole graphs in Fig. 1 as $\int d^4p(p^2-m^2)^{-1} \sim m^2$, one expresses the Lee condition as

$$0 = -4m_q N_f N_c g m_q^2 + 0 + 3g' m_\sigma^2, \tag{2a}$$

where the zero on the right-hand side of Eq. (2a) corresponds to $m_\pi^2 = 0$ in the chiral limit. Upon using Eqs. (1b), this Lee null-tadpole condition (2a) becomes

$$\tfrac{1}{2} N_f N_c (2m_q)^4 = 3m_\sigma^4. \tag{2b}$$

Clearly, if the Nambu–Jona-Lasinio (NJL) relation [4]

$$m_\sigma = 2m_q \tag{3}$$

is valid, then Eq. (2b) requires

$$N_f N_c = 6 \tag{4}$$

or $N_c = 3$ when $N_f = 2$, the latter being an input in the SU(2) LσM .

It is well known that for $\pi^o \to 2\gamma$ decay, the $N_f = 2$ quark triangle empirically suggests $N_c = 3$ (also a LσM result). Moreover, Eq. (4) also follows from ''anomaly matching'' [8,9]. However, we shall not invoke here the stronger (but consistent) constraints due to dynamically generating the (quark-level) LσM as they follow from comparing quadratic and logarithmically divergent integrals using (compatible) regularization schemes [3].

Thus the condition (4) depends on the NJL relation (3) being true also in the LσM. The latter assertion follows when one dynamically generates [3] the entire LσM Lagrangian (1) starting from a simpler chiral quark model (CQM) Lagrangian, as well as dynamically generating the two additional equations

$$m_\sigma = 2m_q, \quad g = 2\pi/\sqrt{N_c}. \tag{5}$$

For $N_c = 3$, the latter pion-quark coupling in Eq. (5) is $g = 2\pi/\sqrt{3} \approx 3.63$, near the anticipated value found from the πNN coupling $g_{\pi NN} \sim 13.4$ so that $g \approx g_{\pi NN}/3g_A \sim 3.5$. Then the nonstrange constituent quark mass is $m_q = f_\pi 2\pi/\sqrt{3} \approx 326$ MeV, near $M_N/3$ as expected. However, rather than repeating Ref. [3] in detail, we offer an easier derivation of $m_\sigma = 2m_q$ following only from the quark loops induced by the CQM Lagrangian. This naturally leads to the notion of ''compositeness.''

To this end, we invoke the log-divergent gap equation from Fig. 2:

$$1 = -i4\tfrac{1}{2}N_f N_c g^2 \int \bar{d}^4p(p^2 - m_q^2)^{-2}, \tag{6}$$

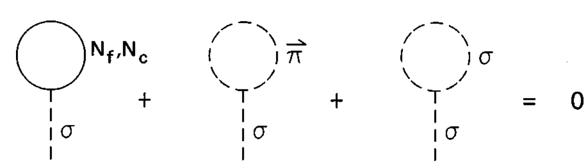

FIG. 1. Quark and meson tadpole loops summing to zero.

*Permanent address: Physics Department, University of Arizona, Tucson, AZ 85721.

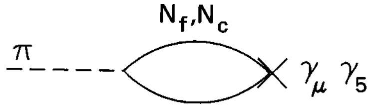

FIG. 2. Quark loops for the axial vector current matrix element $\langle 0|A_\mu|\pi\rangle$.

where $\bar{d}^4p=(2\pi)^{-4}d^4p$. Equation (6) is the chiral-limiting one-loop nonperturbative expression of the pion decay constant $f_\pi=m_q/g$ with the quark mass m_q canceling out. This LσM log-divergent gap equation (6) also holds in the context of the four-quark NJL model [10]. Then the one-loop-order $g_{\sigma\pi\pi}$ coupling depicted in Fig. 3 is

$$g_{\sigma\pi\pi}=2gm_q\left[-i4\tfrac{1}{2}N_fN_cg^2\int \bar{d}^4p(p^2-m_q^2)^{-2}\right]=2gm_q. \tag{7}$$

The one-loop $g_{\sigma\pi\pi}$ in Eq. (7) ''shrinks'' to the tree-order meson-meson coupling in Eq. (1b), $g'=m_\sigma^2/2f_\pi$, only if $m_\sigma=2m_q$ is valid along with the quark-level Goldberger-Treiman relation (GTR) $f_\pi g=m_q$. This is a $Z=0$ compositeness condition [11], stating that the loosely bound σ meson could be treated either as a $\bar{q}q$ bound state (as in the NJL picture) or as an elementary particle as in the LσM framework of Fig. 3. However, in either case $m_\sigma=2m_q$ must hold and therefore the additional LσM Lee condition (2) requires $N_c=3$ when $N_f=2$ in Eq. (4).

It is also possible to appreciate the one-loop order $Z=0$ compositeness condition in the context of the LσM [3] in a different manner. Our version of the $Z=0$ compositeness condition is that the log-divergent gap equation (6) can be expressed in terms of a four-dimensional UV cutoff as

$$1=\ln(1+\Lambda^2/m_q^2)-(1+m_q^2/\Lambda^2)^{-1}, \tag{8}$$

where we have substituted only $g=2\pi/\sqrt{N_c}$ and $N_f=2$ into Eq. (6) in order to deduce Eq. (8). The numerical solution of Eq. (8) is the dimensionless ratio $\Lambda/m_q\approx 2.3$, which is slightly *larger* than the NJL ratio in Eq. (3) or in Eq. (5), $m_\sigma/m_q=2$. Introducing the above dynamically generated quark mass of 326 MeV, the UV cutoff inferred from Eq. (8) [i.e., from Eq. (6)] is $\Lambda\approx 2.3m_q\approx 750$ MeV. This 750-MeV cutoff in turn suggests (in the LσM) that lighter masses signal elementary particles, such as $m_\pi=0$, $m_q\approx 325$ MeV, and $m_\sigma=2m_q\approx 650$ MeV. Heavier meson masses than 750-MeV signal $\bar{q}q$ bound states, such as $\rho(770)$, $\omega(783)$, and $A_1(1260)$. This is the essence of the $Z=0$ compositeness conditions of Ref. [11].

Given the above equations (3)–(8), we are now prepared to comment in detail on the LσM compositeness analysis of Ref. [5]. Again using the log-divergent cutoff condition (8), the LσM renormalization constant Z_3 computed in Eq. (3) of Ref. [5] can be expressed as

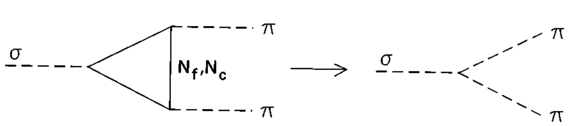

FIG. 3. Chiral quark model loops for $\sigma\rightarrow\pi\pi$.

$$Z_3=1-\frac{N_cg^2}{4\pi^2}. \tag{9}$$

Then the dynamically generated LσM meson-quark coupling in Eq. (5) indeed corresponds to $Z_3=0$ from Eq. (9), as anticipated.

However, the renormalization constant Z_4 in Ref. [5] then becomes, using Eq. (8),

$$Z_4=1+\left[3\lambda-\frac{2N_cg^4}{\lambda}\right]\frac{1}{4\pi^2}. \tag{10}$$

Ignoring for the moment the second term in Eq. (10) proportional to 3λ, we note that the log-divergent gap equation (6) requires the $\pi\pi\rightarrow\pi\pi$ quark box (dynamically generated by the CQM Lagrangian) ''shrinking'' [as in Eq. (7) and in Fig. 3] to a point contact term λ provided that [3]

$$\lambda=2g^2. \tag{11}$$

Equation (11) also follows from both LσM couplings [2] in Eq. (1b) combined with $g_{\sigma\pi\pi}=2gm_q$ from Eq. (7). Substituting Eq. (11) into the third (quark loop) term in Eq. (10), one finds

$$Z_4=1+0-\frac{N_cg^2}{4\pi^2} \tag{12}$$

(where the middle zero term corresponds to the neglected meson loop in contrast to Ref. [5]). Equation (12) parallels the Z_3 renormalization constant in Eq. (9). In these two cases

$$Z_3=Z_4=1-\frac{N_cg^2}{4\pi^2} \tag{13}$$

and then the resulting compositeness conditions $Z_3=Z_4=0$ *both* reconfirm that $g=2\pi/\sqrt{N_c}$, as earlier dynamical generated in Eqs. (5).

The reason why one must neglect the second meson loop term proportional to 3λ in Eq. (10) is because, e.g., $\pi_\alpha\pi_\beta\rightarrow\pi_\gamma\pi_\delta$ scattering has tree-level (or one-loop) graphs that must *vanish* in the strict zero momentum chiral limit. This fact was emphasized on pp. 324–327 of the text by de Alfaro, Fubini, Furlan, and Rossetti (DFFR) in Ref. [2]. Specifically, the quartic LσM contact term $-\lambda$ is canceled by the cubic σ pole term $2g'^2/m_\sigma^2\rightarrow\lambda$ by virtue of the Gell-Mann–Lévy LσM meson chiral couplings in Eq. (1b). After the (tree-level) lead term cancellation between contact term λ and s,t,u,σ meson poles in the LσM, DFFR obtain the amplitude

$$T_{\pi\pi}\propto\frac{1}{f_\pi^2}(s\,\delta_{\alpha\beta}\delta_{\gamma\delta}+t\,\delta_{\alpha\gamma}\delta_{\beta\delta}+u\,\delta_{\alpha\delta}\delta_{\beta\gamma}). \tag{14}$$

Then DFFR in [2] note that Eq. (14) above is just the Weinberg $\pi\pi$ amplitude [12] when $m_\pi^2=0$, found instead via the model-independent current algebra and partial conservation of axial vector current rather than from the LσM). Also note that Eq. (14) indeed vanishes in the strict zero-momentum chiral limit. A similar chiral cancellation of the 3λ term in Eq. (10) also holds in one-loop order.

When computing the one-loop order renormalization constant Z_4 as done by Lurie and Tupper in Ref. [5] leading to Eq. (10) above, one must be careful to (a) account for DFFR's cancellation due to the soft chiral symmetry relation $2g'^2/m_\sigma^2 \to \lambda$ and (b) reorganize the perturbation theory using the log-divergent gap equation (6) to shrink quark loops to a contact meson term λ with $\lambda = 2g^2$ as found in Eq. (11). Then, even in one-loop order one must recover the Weinberg form for $\pi\pi$ scattering Eq. (14) in a model-independent fashion.

This means that the meson loop graph with quartic couplings proportional to $3\lambda^2$ contributing to λZ_4 as $3\lambda^2/4\pi^2$ in Eq. (10) will be canceled by fermion box graphs that are of higher loop order. Although our nonperturbative approach mixes perturbation theory loops of different order, both DFFR's and our use of the Gell-Mann–Lévy chiral symmetry meson relation $2g'^2/m_\sigma^2 \to \lambda$ have the bonus of our nonperturbative approach retaining the consistent chiral symmetry compositeness condition $Z_3 = Z_4 = 0$ from Eq. (13).

Keeping instead the middle term in Eq. (10) proportional to 3λ, Lurie and Tupper [5] conclude that the resulting $Z_4 = 0$ (then different) compositeness condition requires that the NJL limit $m_\sigma \to 2m_q$ is recovered only when $N_c \to \infty$. Akama and Zinn Justin [13] reach the same conclusion, although they are not working with SU(2) chiral mesons $(\sigma, \vec{\pi})$. In our opinion, however, the chiral SU(2) LσM (1) already has $N_c = 3$ and not $N_c \to \infty$ built in via the Lee condition in Eqs. (2) but only when $m_\sigma = 2m_q$ in the chiral limit. We obtain these satisfying results only by canceling the middle 3λ meson term in Eq. (10) against higher quark loop graphs. Reference [5] does not account for the above cancellation of DFFR.

Finally, we extend the above zero-temperature $(T=0)$ chiral symmetry absence of quartic meson loops in Eqs. (10), (12), and (14) to finite temperature. Again following Ref. [5] we write the tadpole equation in the mean-field approximation at high temperatures for the quark-level SU(2) LσM as

$$v[(3 + N_f^2 - 1)\lambda T^2/12 + N_f N_c g^2 T^2/12 + \lambda(v^2 - f_\pi^2)] = 0 \tag{15}$$

for flavor $N_f = 2$ and $v = v(T)$ with $v(0) = f_\pi \sim 90$ MeV in the chiral limit. The first two terms in Eq. (15) represent quartic σ and $\vec{\pi}$ loops, while the third term involving N_c is the u and d quark bubble loop. The temperature factors of $T^2/12$ in Eq. (15) were originally obtained from finite-temperature field theory Feynman rules [14].

Now in fact there should be *no* quartic meson loop contributions surviving in Eq. (15) due to the above DFFR-type argument or the resulting Weinberg $\pi\pi$ amplitude in Eq. (14), even at finite temperatures. So the nontrivial solution of Eq. (15) at the chiral symmetry restoration temperature T_c [where $v(T_c) = 0$] is for $N_f = 2$, $N_c = 3$, and $\lambda = 2g^2$, with the first two meson loop terms in Eq. (15) proportional to $(3 + N_f^2 - 1)\lambda$ consequently omitted:

$$T_c = 2f_\pi \sim 180 \text{ MeV}. \tag{16}$$

While this predicted temperature scale in Eq. (16) had been obtained earlier [15,16], Lurie and Tupper [5] also noted Eq. (16) above but rejected it because of the meson loop contributions in Eq. (15).

We in turn claim that the first two σ and $\vec{\pi}$ loop terms in Eq. (15) [and the middle term in Eq. (10) proportional to 3λ] are all zero due to chiral cancellations as by DFFR [2]. Then Eq. (15) reduces to the nontrivial solution $N_c g^2 T_c^2/6 = \lambda f_\pi^2$ (leading to $T_c = 2f_\pi$) or to a quark box loop shrinking to a meson-meson quartic point [3] due to the log-divergent gap equation (6), itself a version of the $Z=0$ compositeness condition.

Although we concur with Lurie and Tupper's [5] choice of the finite-temperature quark bubble sign in Eq. (15) (as opposed to the studies in Ref. [15]), there is an easier way to deduce $T_c = 2f_\pi$ by studying the single fermion loop propagator dynamically generating the quark mass [3]. Then, with no sign ambiguity arising at finite temperature one finds [17]

$$m_q(T) = m_q + \frac{8N_c g^2 m_q}{-m_\sigma^2}\frac{T^2}{24}, \tag{17}$$

where the $-m_\sigma^2$ factor in Eq. (17) indicates the σ meson tadpole propagator generating the quark mass. When $T = T_c$ the quark mass ''melts,'' $m_q(T_c) = 0$, and Eq. (17) reduces to

$$m_\sigma^2 = g^2 T_c^2 \quad \text{or} \quad T_c = 2f_\pi \tag{18}$$

provided $N_c = 3$ and $m_\sigma = 2m_q = 2f_\pi g$.

We believe it significant that recent numerical simulations of lattice gauge theories find [18] $T_c = 150 \pm 30$ MeV, which is consistent with Eqs. (16) and (18). In fact, the zero-temperature quark-level LσM theory in Ref. [3] is likewise compatible with the reinstated scalar σ in the PDG tables [1] or in Ref. [19], the latter deducing a broad nonstrange σ scalar as f_0 (400–900) with mean mass $m_\sigma \approx 650$ MeV. This latter scale is in fact predicted in Ref. [3] as $m_\sigma = 2f_\pi(2\pi/\sqrt{3}) \approx 650$ MeV.

Rather than starting at $T=0$, an alternative approach to generating a realistic low-energy chiral field theory begins at the chiral restoration temperature [with $m_q(T_c) = 0$] involving bosons $\vec{\pi}$ and σ alone [20] and later adds in the fundamental meson-quark interaction in Eq. (1). Only then does one deduce the quark-level LσM field theory [21]. While issues of $N_c = 3$ and compositeness are then postponed, the resulting LσM theory in Ref. [21] starting at $T = T_c \sim 200$ MeV with $\lambda \sim 20$ appears quite similar to the $T=0$ LσM field theory in Refs. [2,3] with $\lambda \approx 26$ from Eq. (11) and $T_c \approx 180$ MeV from Eq. (16).

This research was partially supported by the Australian Research Council. M. D. S. appreciates hospitality of the University of Western Ontario and the University of Tasmania. He also is grateful to V. Elias, D. McKeon, R. Mendel, V. Miransky, and especially R. Delbourgo for insightful comments.

[1] Particle Data Group, R. M. Barnett *et al.*, Phys. Rev. D **54**, 1 (1996).

[2] M. Gell-Mann and M. Lévy, Nuovo Cimento **16**, 705 (1960); see also V. de Alfaro, S. Fubini, G. Furlan, and C. Rossetti, *Currents in Hadron Physics* (North-Holland, Amsterdam, 1973), Chap. 5.

[3] R. Delbourgo and M. D. Scadron, Mod. Phys. Lett. A **10**, 251 (1995), regularize the LσM using dimensional regularization. This was recently extended to analytic and Pauli-Villars regularization by R. Delbourgo, A. Rawlinson and M. Scadron (unpublished).

[4] Y. Nambu and G. Jona-Lasinio, Phys. Rev. **122**, 345 (1961).

[5] D. Lurie and G. B. Tupper, Phys. Rev. D **47**, 3580 (1993).

[6] B. W. Lee, *Chiral Dynamics* (Gordon and Breach, New York, 1972), p. 12.

[7] See, e.g., G. 't Hooft and M. Veltman, Nucl. Phys. **B44**, 189 (1972); see also the review by R. Delbourgo, Rep. Prog. Phys. **39**, 345 (1976).

[8] S. Adler, Phys. Rev. **177**, 2426 (1969); J. Bell and R. Jackiw, Nuovo Cimento **60**, 47 (1969); see also J. Schwinger, Phys. Rev. **82**, 664 (1951).

[9] G. 't Hooft, in *Recent Developments in Gauge Theories*, edited by G. 't Hooft *et al.* (Plenum, New York, 1980); Y. Frishman, A. Schwimmer, T. Banks, and S. Yankielowicz, Nucl. Phys. **B177**, 157 (1981); S. Coleman and B. Grossman, *ibid.* **B203**, 205 (1982); see also K. Huang, in *Quarks, Leptons and Gauge Fields* (World Scientific, Singapore, 1992).

[10] See, e.g., V. Dmitrasinovic, H. Schulze, R. Tegen, and R. Lemmer, Phys. Rev. D **52**, 2855 (1995). Combine their Eqs. (8) and (15) to obtain our gap equation (6).

[11] A. Salam, Nuovo Cimento **25**, 224 (1962); S. Weinberg, Phys. Rev. **130**, 776 (1963).

[12] S. Weinberg, Phys. Rev. Lett. **17**, 616 (1966).

[13] K. Akama, Phys. Rev. Lett. **76**, 184 (1966); J. Zinn Justin, Nucl. Phys. **B367**, 105 (1991), consider a four-quark $U(1)_L \times U(1)_R$ simple NJL model finding $\lambda = g^2$ and $m_\sigma \to 2m_q$ when $N_c \to \infty$. However, this does not correspond to the SU(2) LσM with $m_\sigma = 2m_q$ when $N_c = 3$.

[14] See, e.g., L. Dolan and R. Jackiw, Phys. Rev. D **9**, 3320 (1974).

[15] D. Bailin, J. Cleymans, and M. D. Scadron, Phys. Rev. D **31**, 164 (1985); J. Cleymans, A. Kocić, and M. D. Scadron, *ibid.* **39**, 323 (1989).

[16] See, e.g., the review by T. Hasuda, Nucl. Phys. **A544**, 27 (1992); see also M. Asaka and K. Yazaki, *ibid.* **A504**, 668 (1989); A. Barducci and R. Casalfuoni, Phys. Rev. D **41**, 1610 (1990).

[17] N. Bilić, J. Cleymans, and M. D. Scadron, Int. J. Mod. Phys. A **10**, 1169 (1995).

[18] See, e.g., reviews by B. Petersson, in *Lattice '92*, Proceedings of the International Symposium, Amsterdam, The Netherlands, edited by J. Smit and P. van Baal [Nucl. Phys. (Proc. Suppl.) **B30**, 66 (1993)]; K. M. Bitar, *ibid.*, p. 315.

[19] N. Tornqvist and M. Roos, Phys. Rev. Lett. **76**, 1575 (1996).

[20] K. Rajagopal and F. Wilczek, Nucl. Phys. **B404**, 577 (1993); M. Asakawa, Z. Huang, and X.-N. Wang, Phys. Rev. Lett. **74**, 3126 (1995); J. Randrup, *ibid.* **77**, 1226 (1996).

[21] L. P. Csernai and I. N. Mishustin, Phys. Rev. Lett. **74**, 5005 (1995).