Universal massive spectral correlators and three-dimensional QCD

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Based on random matrix theory in the unitary ensemble, we derive the double-microscopic massive spectral correlators corresponding to the Dirac operator of QCD_3 with an even number of fermions N_f. We prove that these spectral correlators are universal, and demonstrate that they satisfy exact massive spectral sum rules of QCD_3 in a phase where flavor symmetries are spontaneously broken according to $U(N_f)\rightarrow U(N_f/2)\times U(N_f/2)$. [S0556-2821(98)01610-5]

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It has long been suspected that three-dimensional QCD (QCD_3), and even QED_3, with an *even* number of flavors $N_f\equiv 2\alpha$ may undergo spontaneous flavor symmetry breaking according to the pattern $U(N_f)\rightarrow U(N_f/2)\times U(N_f/2)$ [1]. This phenomenon can most easily be understood when one notices that for an even number of flavors the original two-spinors of (2+1)-dimensional fermions may be grouped pairwise into half as many four-spinors [2]. The resulting formalism has an uncanny resemblance to QCD_4 with $N_f/2$ flavors and *two* chiral symmetries, those associated with γ_4 and γ_5 rotations [1]. The global symmetry is, however, slightly unusual: $U(N_f)$, as follows directly from the original formulation in terms of two-spinors. The suggested flavor symmetry breaking can be directly understood in terms of the pseudo-chiral symmetry described above. Moreover, as has been remarked recently [3], the Coleman-Witten argument [4] applied to QCD_3 in the limit of many colors N_c leads to precisely this prediction.

An order parameter for the above symmetry breaking pattern is the absolute value of the chiral condensate, $\Sigma\equiv\Sigma_i\langle\bar{\psi}_i\psi_i\rangle/N_f$. By an analogue of the Banks-Casher relation [5], this condensate is related to the spectral density of the Dirac operator, evaluated at the origin, $\rho(0)=\Sigma/\pi$. A most remarkable and testable prediction of Verbaarschot and Zahed [3] is that the massless QCD_3 spectral density $\rho(\lambda)$ near the origin at $\lambda=0$, the microscopic spectral density

$$\rho_S(x)\equiv\lim_{V\rightarrow\infty}\frac{1}{V\Sigma}\rho\left(\frac{x}{V\Sigma}\right), \tag{1}$$

may be computed *exactly* in a unitarily invariant random matrix ensemble [3]. Here V denotes the three-volume, and the microscopic spectral density is therefore to be considered as a finite-volume scaling function. In this particular case the volume V translates directly into the size N of random matrices in the unitary ensemble.

The only required input for the above conjecture is the existence of a chiral condensate, and hence a non-vanishing $\rho(0)$. There is now substantial evidence that the analogous statements for (3+1)-dimensional theories [6] are correct, ranging from agreement with exact massless spectral sum rules of QCD and generalizations [7,8] to an explicit lattice gauge theory computation of the microscopic spectral density [9]. An essential ingredient in understanding how random matrix theory can provide exact statements about full-fledged quantum field theories is the proven universality, within random matrix theory, of the pertinent microscopic spectral densities [10].

Very recently the microscopic spectral densities of $SU(N_c\geqslant 3)$ gauge theories with N_f massive flavors in 3+1 dimensions have been computed from random matrix theory [11] (see also [12]). Such an extension is essential for future comparisons with lattice gauge theory beyond the quenched approximation. Remarkably, also these double-microscopic spectral densities (called so because both eigenvalues and masses need to be rescaled with volume V) are universal. Moreover, the double-microscopic massive spectral densities satisfy exact massive spectral sum rules of QCD [11,13].

In this Brief Report we shall extend the computation of double-microscopic massive spectral densities to the case of the random unitary invariant matrix ensemble, which, in view of the work of Verbaarschot and Zahed [3], can provide exact information about the Dirac operator spectrum in QCD_3. We shall prove that these double-microscopic spectral densities (and spectral correlators) are universal within the framework of random matrix theory, and show that they satisfy exact massive spectral sum rules of QCD_3. In doing this, we shall also provide an exact representation of the finite-volume partition function for QCD_3 in the so-called mesoscopic range of volumes [6].

Our starting point is the random matrix ensemble of the partition function

$$\mathcal{Z}=\int dM\prod_{f=1}^{N_f}\det(M+im_f)\mathrm{e}^{-N\,\mathrm{tr}\,V(M^2)}. \tag{2}$$

The integration is over Hermitian $N\times N$ matrices M with the associated Haar measure, and we parametrize the potential in a general way by $V(M^2)=\Sigma_{k\geqslant 1}[g_k/(2k)]M^{2k}$. Masses are grouped pairwise with opposite signs [3]: $\{m_f\}=\{m_1,-m_1,m_2,-m_2,\ldots\}$. Introducing the eigenvalues λ_i of the Hermitian matrix M we have, discarding an irrelevant overall factor,

$$\mathcal{Z}=\int_{-\infty}^{\infty}\prod_{i=1}^{N}\left(d\lambda_i\prod_{f=1}^{N_f/2}(\lambda_i^2+m_f^2)\mathrm{e}^{-NV(\lambda_i^2)}\right)|\det_{ij}\lambda_j^{i-1}|^2. \quad (3)$$

We thus seek polynomials $P_n^{[\alpha]}(\lambda;m_1,\ldots,m_\alpha)$ orthogonal with respect to the weight functions

$$w(\lambda)=\prod_{f=1}^{\alpha}(\lambda^2+m_f^2)\mathrm{e}^{-NV(\lambda^2)}. \quad (4)$$

Since the weight functions are even in λ due to the pairwise assignment of masses, the polynomials split into even and odd sectors. We treat these two sectors separately, until we combine them to construct the kernel. In Ref. [10] it was proved that when all $m_f=0$ the orthogonal polynomials have, for fixed $x=N\lambda$ and $t=2n/N$, a universal limiting behavior. Normalized according to $P_{2n}^{[\alpha]}(0)=P_{2n+1}^{[\alpha]}{}'(0)=1$ the limit is

$$\mathcal{P}_+^{[\alpha]}(t;x)\equiv\lim_{N\to\infty}P_{2n}^{[\alpha]}\left(\frac{x}{N}\right)\Bigg|_{n=Nt/2}=\Gamma\left(\alpha+\frac{1}{2}\right)\frac{J_{\alpha-1/2}(u(t)x)}{(u(t)x/2)^{\alpha-1/2}}, \quad (5a)$$

$$\mathcal{P}_-^{[\alpha]}(t;x)\equiv\lim_{N\to\infty}N\,P_{2n+1}^{[\alpha]}\left(\frac{x}{N}\right)\Bigg|_{n=Nt/2}=x\,\Gamma\left(\alpha+\frac{3}{2}\right)\frac{J_{\alpha+1/2}(u(t)x)}{(u(t)x/2)^{\alpha+1/2}}. \quad (5b)$$

Here $u(t)$ is determined by

$$u(t)=\int_0^t\frac{dt'}{2\sqrt{r(t')}},\qquad t=\sum_k\frac{g_k}{2}\binom{2k}{k}r(t)^k, \quad (6)$$

and $u(1)=\pi\rho(0)$ [where $\rho(0)$ is the large-N spectral density at the origin].

We are now in a position to generalize this result to the case of massive fermions, following the method developed in Ref. [11]. We use the following lemma to construct the required polynomials:

Lemma 1 (Christoffel). If $\{P_n(\lambda)\}_{n=0,1,\ldots}$ is a set of polynomials orthogonal with respect to an *even* weight function $w(\lambda)$,

$$\widetilde{P}_n(\lambda)=\frac{P_n(\lambda)P_{n+2}(\lambda')-P_{n+2}(\lambda)P_n(\lambda')}{\lambda^2-\lambda'^2} \quad (7)$$

are polynomials orthogonal with respect to the weight $(\lambda^2-\lambda'^2)\,w(\lambda)$ [14].

By replacing $\lambda'\to im$, we can use this procedure to incorporate the factor λ^2+m^2 due to a pair of fermions of masses $\pm m$ into the weight function. By iterating this procedure, we can construct polynomials orthogonal with respect to the weight (4). In the large-n, N limit, the difference in n in the numerator of Eq. (7) is replaced by the differential in t. Then the next lemma allows us to express the polynomials in a closed form:

Lemma 2. Let $P^{[\alpha]}(t;\lambda_0,\lambda_1,\ldots,\lambda_\alpha)$, $\alpha=0,1,2,\ldots$, be a set of functions generated by the iteration

$$P^{[\alpha+1]}(t;\lambda_0,\lambda_1,\ldots,\lambda_{\alpha+1})=\frac{P^{[\alpha]}(t;\lambda_0,\lambda_1,\ldots,\lambda_\alpha)P_t^{[\alpha]}(t;\lambda_{\alpha+1},\lambda_1\ldots,\lambda_\alpha)-P_t^{[\alpha]}(t;\lambda_0,\lambda_1,\ldots,\lambda_\alpha)P^{[\alpha]}(t;\lambda_{\alpha+1},\lambda_1\ldots,\lambda_\alpha)}{\lambda_0^2-\lambda_{\alpha+1}^2}. \quad (8)$$

Then they are given by

$$P^{[\alpha]}(t;\lambda_0,\lambda_1,\ldots,\lambda_{\alpha-1},\lambda_\alpha)=c(t;\lambda_1,\ldots,\lambda_\alpha)\frac{\det_{i,j}P^{(i)}(t;\lambda_j)}{\prod_{i=1}^{\alpha}(\lambda_0^2-\lambda_i^2)} \quad (9)$$

where $P^{(i)}(t;\lambda)=(\partial^i/\partial t^i)\,P^{[0]}(t;\lambda)$, and $c(t;\lambda_1,\ldots,\lambda_\alpha)$ is a function in t and in $\{\lambda_1,\ldots,\lambda_\alpha\}$.

We refer the reader to Ref. [11] for the proof of these lemmas. Now we replace $\lambda_i\to x_i/N$ and $P^{[0]}(t;\lambda)$ by its microscopic limit (5), $P_\pm^{[0]}(t;x/N)\to u(t)^{\pm1/2}x^{1/2}J_{\mp1/2}[u(t)x]$. Here the upper and lower signs stand for polynomials in the even and odd sectors, respectively. Then we can prove by induction that its t-derivatives are expressed as

$$P_\pm^{(i)}\left(t;\frac{x}{N}\right)\to\sum_{k=0}^{i}d_{i,k}(t)x^{k+1/2}\,J_{k\mp1/2}[u(t)x]. \quad (10)$$

Once it is substituted inside the determinant det $P^{(i)}(\lambda_j)$, only the top term proportional to $x^{i+1/2}\,J_{i\mp1/2}[u(t)x]$ contributes. Thus the determinant in Eq. (9) is replaced by

$$d(t)\det_{0\leq i,j\leq\alpha}x_j^{i+1/2}\,J_{i\mp1/2}[u(t)x_j]. \quad (11)$$

Performing the analytical continuation of $(\zeta_1,\ldots,\zeta_\alpha)$ to imaginary $(i\mu_1,\ldots,i\mu_\alpha)$, we thus obtain the microscopic limit of the orthogonal polynomials:

$$\mathcal{P}_{\pm}^{[\alpha]}(t;x,\{\mu_f\})\equiv \lim_{N\to\infty} P_{\pm}^{[\alpha]}\left(t;\frac{x}{N},\left\{\frac{\mu_f}{N}\right\}\right)$$

$$=c(t,\{\mu_f\})\,\frac{\begin{vmatrix} x^{1/2}J_{\mp 1/2}(u(t)x) & x^{3/2}J_{1\mp 1/2}(u(t)x) & \cdots & x^{\alpha+1/2}J_{\alpha\mp 1/2}(u(t)x)\\ \mu_1^{1/2}I_{\mp 1/2}(u(t)\mu_1) & -\mu_1^{3/2}I_{1\mp 1/2}(u(t)\mu_1) & \cdots & (-)^{\alpha}\mu_1^{\alpha+1/2}I_{\alpha\mp 1/2}(u(t)\mu_1)\\ \vdots & \vdots & \cdots & \vdots\\ \mu_\alpha^{1/2}I_{\mp 1/2}(u(t)\mu_\alpha) & -\mu_\alpha^{3/2}I_{1\mp 1/2}(u(t)\mu_\alpha) & \cdots & (-)^{\alpha}\mu_\alpha^{\alpha+1/2}I_{\alpha\mp 1/2}(u(t)\mu_\alpha)\end{vmatrix}}{\prod_{f=1}^{\alpha}(x^2+\mu_f^2)}. \tag{12}$$

The microscopic kernel and spectral density are constructed out of $\mathcal{P}_{\pm}^{[\alpha]}(t=1)$ as

$$K^{(\alpha)}(x,x';\{\mu_f\})=C(\{\mu_f\})\sqrt{\prod_{f=1}^{\alpha}(x^2+\mu_f^2)(x'^2+\mu_f^2)}\;\frac{\mathcal{P}_{+}^{[\alpha]}(1;x,\{\mu_f\})\mathcal{P}_{-}^{[\alpha]}(1;x',\{\mu_f\})-\mathcal{P}_{-}^{[\alpha]}(1;x,\{\mu_f\})\mathcal{P}_{+}^{[\alpha]}(1;x',\{\mu_f\})}{x-x'}, \tag{13}$$

$$\rho_S^{(\alpha)}(x;\{\mu_f\})$$

$$=\frac{1}{\pi\rho(0)}K^{(\alpha)}\left(\frac{x}{\pi\rho(0)},\frac{x}{\pi\rho(0)};\left\{\frac{\mu_f}{\pi\rho(0)}\right\}\right)$$

$$=\frac{\mathcal{C}(\{\mu_f\})}{\prod_{\alpha=1}^{f}(x^2+\mu_f^2)}\left(\begin{vmatrix} x^{1/2}J_{-1/2}(x) & \cdots & x^{\alpha+1/2}J_{\alpha-1/2}(x)\\ \mu_1^{1/2}I_{-1/2}(\mu_1) & \cdots & (-)^{\alpha}\mu_1^{\alpha+1/2}I_{\alpha-1/2}(\mu_1)\\ \vdots & \cdots & \vdots\\ \mu_\alpha^{1/2}I_{-1/2}(\mu_\alpha) & \cdots & (-)^{\alpha}\mu_\alpha^{\alpha+1/2}I_{\alpha-1/2}(\mu_\alpha)\end{vmatrix}\begin{vmatrix} x^{1/2}J_{-1/2}(x) & \cdots & x^{\alpha+1/2}J_{\alpha-1/2}(x)\\ \mu_1^{1/2}I_{1/2}(\mu_1) & \cdots & (-)^{\alpha}\mu_1^{\alpha+1/2}I_{\alpha+1/2}(\mu_1)\\ \vdots & \cdots & \vdots\\ \mu_\alpha^{1/2}I_{1/2}(\mu_\alpha) & \cdots & (-)^{\alpha}\mu_\alpha^{\alpha+1/2}I_{\alpha+1/2}(\mu_\alpha)\end{vmatrix}\right.$$

$$\left.-\begin{vmatrix} x^{-1/2}J_{-1/2}(x)+x^{1/2}J_{-3/2}(x) & \cdots & x^{\alpha-1/2}J_{\alpha-1/2}(x)+x^{\alpha+1/2}J_{\alpha-3/2}(x)\\ \mu_1^{1/2}I_{-1/2}(\mu_1) & \cdots & (-)^{\alpha}\mu_1^{\alpha+1/2}I_{\alpha-1/2}(\mu_1)\\ \vdots & \cdots & \vdots\\ \mu_\alpha^{1/2}I_{-1/2}(\mu_\alpha) & \cdots & (-)^{\alpha}\mu_\alpha^{\alpha+1/2}I_{\alpha-1/2}(\mu_\alpha)\end{vmatrix}\begin{vmatrix} x^{1/2}J_{1/2}(x) & \cdots & x^{\alpha+1/2}J_{\alpha+1/2}(x)\\ \mu_1^{1/2}I_{1/2}(\mu_1) & \cdots & (-)^{\alpha}\mu_1^{\alpha+1/2}I_{\alpha+1/2}(\mu_1)\\ \vdots & \cdots & \vdots\\ \mu_\alpha^{1/2}I_{1/2}(\mu_\alpha) & \cdots & (-)^{\alpha}\mu_\alpha^{\alpha+1/2}I_{\alpha+1/2}(\mu_\alpha)\end{vmatrix}\right). \tag{14}$$

The constant $\mathcal{C}(\{\mu_f\})=C(\{\mu_f\})/[\pi\rho(0)]^{\alpha^2+1}$ is determined to be

$$\mathcal{C}(\{\mu_f\})^{-1}=2\det_{1\leqslant i,j\leqslant\alpha}[(-)^i\mu_j^{i-1/2}I_{i-3/2}(\mu_j)]\times\det_{1\leqslant i,j\leqslant\alpha}[(-)^i\mu_j^{i-1/2}I_{i-1/2}(\mu_j)] \tag{15}$$

by requiring the matching between the $x\to\infty$ limit of the microscopic density [normalized as in Eq. (14)] and the macroscopic density at $\lambda=0$:

$$\rho_S^{(\alpha)}(x\to\infty;\{\mu_f\})\to 1/\pi. \tag{16}$$

For convenience we exhibit the first two examples of $\rho_S^{(\alpha)}$ (with degenerate masses μ):

$$\pi\rho_S^{(1)}(x;\mu)=1+\frac{\mu}{x^2+\mu^2}\,\frac{\cos 2x-\cosh 2\mu}{\sinh 2\mu}, \tag{17a}$$

$$\pi\rho_S^{(2)}(x;\mu,\mu)=1-\frac{\mu\{4\mu(x^2-\mu^2)(1-\cos 2x\,\cosh 2\mu)+2[(x^2+\mu^2)(\cos 2x-\cosh 2\mu)+4x\mu^2\,\sin 2x]\sinh 2\mu\}}{(x^2+\mu^2)^2(4\mu^2-\sinh^2 2\mu)}. \tag{17b}$$

It follows directly from the above construction and the universality proof of Ref. [10] that we have simultaneously proved that the orthogonal polynomials (12), the kernel (13) (as well as all higher spectral correlators), and the microscopic spectral density itself (14) are *universal*, i.e. insensitive to the potential $V(M^2)$ in this limit.

By using the minor expansion of the determinants and Hankel's asymptotic formula for the Bessel functions, we can easily check that the microscopic kernels (13) and densities (14) for arbitrary α satisfy a sequence of decoupling relations for heavy fermions [11]:

$$\rho_S^{(\alpha)}(x;\mu_1,\ldots,\mu_{\alpha-1},\mu_\alpha) \xrightarrow{\mu_\alpha\to\infty} \rho_S^{(\alpha-1)}(x;\mu_1,\cdots,\mu_{\alpha-1}) \xrightarrow{\mu_{\alpha-1}\to\infty} \rho_S^{(\alpha-2)}(x;\mu_1,\cdots,\mu_{\alpha-2})\to\cdots. \tag{18}$$

We similarly verify that when all masses vanish, $\rho_S^{(\alpha)}(x;0,\ldots,0)$ agrees with the result obtained directly from the massless case [3],

$$\rho_S^{(\alpha)}(x;0,\ldots,0)=\frac{x}{4}[J_{\alpha+1/2}(x)^2+J_{\alpha-1/2}(x)^2 - J_{\alpha+1/2}(x)J_{\alpha-3/2}(x) - J_{\alpha-1/2}(x)J_{\alpha+3/2}(x)]. \tag{19}$$

It finally remains to compare these universal matrix model results with exact massive spectral sum rules of QCD_3 in the phase of broken flavor symmetry. In Ref. [3] it was argued that the relevant finite-volume partition function for QCD_3 can be written

$$\mathcal{Z}(\mathcal{M})=\int dU\ \exp[N\Sigma\ \mathrm{tr}(\mathcal{M}U\Gamma_5U^\dagger)], \tag{20}$$

where the integration has been extended from the coset $U(N_f)/U(N_f/2)\times U(N_f/2)$ to $SU(N_f)$. The mass matrix $\mathcal{M}$ takes the form $\mathrm{diag}(m_1,\ldots,m_{N_f/2},-m_1,\ldots,-m_{N_f/2})$. The other matrix is $\Gamma_5=\mathrm{diag}(\mathbb{1},-\mathbb{1})$, where $\mathbb{1}$ is an $(N_f/2)\times(N_f/2)$ unit matrix. As could have been guessed by comparison with the case of QCD_4 [15], the partition function (20) is an example of the Harish-Chandra–Itzykson-Zuber integral [16], now for Hermitian matrices. The only slight complication arises from the fact that Γ_5 has two sets of $N_f/2$-fold degenerate eigenvalues, which makes the standard expression for the integral indeterminate. One can take care of this by regularizing the Γ_5 matrix in any way that removes the degeneracy, performing the integral, and subsequently taking the degenerate limit. We define $\mu_i\equiv N\Sigma m_i$. Using the prescription above, the integral (20) can be performed explicitly, and one gets, up to an irrelevant normalization factor,

$$\mathcal{Z}(\mathcal{M})=\frac{\det\begin{pmatrix}\mathbf{A}(\{\mu_i\}) & \mathbf{A}(\{-\mu_i\})\\ \mathbf{A}(\{-\mu_i\}) & \mathbf{A}(\{\mu_i\})\end{pmatrix}}{\Delta(\mathcal{M})} \tag{21}$$

where $\Delta(\mathcal{M})$ is the Vandermonde determinant of the mass matrix $\mathcal{M}$. The $(N_f/2)\times(N_f/2)$ matrix $\mathbf{A}(\{\mu_i\})$ is defined by $\mathbf{A}_{ij}\equiv\mu_i^{j-1}e^{\mu_i}$. An analogous procedure applies to the mass matrix if one insists on getting the result with some or all of the $N_f/2$ mass eigenvalues being equal. Massive spectral sum rules can now be derived by taking derivatives with respect to one or more of the mass eigenvalues [13]. For example, for 2 and 4 fermion species of degenerate (up to a sign, see the discussion above) masses m, this gives, for the simplest sum rules (summing over *positive* eigenvalues only),

$$N_f=2:\ \left\langle \sum_n{}' \frac{1}{\lambda_n^2+m^2}\right\rangle=\frac{\Sigma^2N^2}{2\mu}\left(\coth 2\mu-\frac{1}{2\mu}\right), \tag{22a}$$

$$N_f=4:$$

$$\left\langle \sum_n{}' \frac{1}{\lambda_n^2+m^2}\right\rangle =\frac{\Sigma^2N^2}{2\mu^2}\,\frac{\sinh^2 2\mu-\mu\sinh 2\mu\cosh 2\mu-2\mu^2}{4\mu^2-\sinh^2 2\mu}. \tag{22b}$$

We note that in the limit $\mu\to0$ these sum rules reduce correctly to those of the massless case [3], where the right hand sides above are replaced by $\Sigma^2N^2N_f/[2(N_f^2-1)]$.

We can now check these massive spectral sum rules by means of the identity

$$\frac{1}{N^2\Sigma^2}\left\langle \sum_n{}' \frac{1}{\lambda_n^2+m^2}\right\rangle=\int_0^\infty dx\ \frac{\rho_S^{(N_f/2)}(x;\mu,\ldots,\mu)}{x^2+\mu^2} \tag{23}$$

and the general expression (14). The integrals are elementary, and we find that the massive spectral sum rules are exactly satisfied.

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