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Dynamics of Bogomol'nyi-Prasad-Sommerfield dyons: Effective field theory approach

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Based on a detailed analysis of nonlinear field equations of the SU(2) Yang-Mills-Higgs system, we obtain the effective field theory describing the low-energy interaction of Bogomol'nyi-Prasad-Sommerfield (BPS) dyons and massless particles (i.e., photons and Higgs particles). Our effective theory manifests electromagnetic duality and spontaneously broken scale symmetry and reproduces the multimonopole moduli space dynamics of Manton in a suitable limit. Also given is a generalization of our approach to the case of BPS dyons in a gauge theory with an arbitrary gauge group that is maximally broken to $U(1)^k$. [S0556-2821(98)05206-0]

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I. INTRODUCTION

In certain spontaneously broken non-Abelian gauge theories we have magnetic monopoles as solitonic particles (in addition to the usual elementary field quanta) and, since their initial discovery by 't Hooft and Polyakov [1] in 1974, much effort has been made to clarify their physical role. Then, more recently, a number of exact results have been obtained in a class of supersymmetric gauge theories by exploiting the electromagnetic duality symmetry [2]. Magnetic monopoles relevant in this supersymmetric gauge theories are the so-called Bogomol'nyi-Prasad-Sommerfield (BPS) monopoles [3], i.e., magnetic-charge-carrying static solutions to the Yang-Mills-Higgs field equations in the BPS limit of vanishing Higgs potential. In the BPS limit, there is a Bogomol'nyi bound on the static energy functional and remarkably we have degenerate *static* multimonopole solutions that saturate the bound. Originally this was a semiclassical result at most; but, in the supersymmetric gauge theories, Witten and Olive [4] subsequently showed that this result may continue to be valid even after quantum corrections are included.

To study the duality and other issues, various authors discussed the interaction of slowly moving BPS monopoles, mainly following the work of Manton [5]. The central point is that the moduli space of (gauge inequivalent) static N-monopole solutions is finite dimensional and possesses a natural metric coming from the kinetic-energy terms of the Yang-Mills-Higgs Lagrangian. Manton suggested that the low-energy dynamics of a given set of monopoles and dyons may be approximated by geodesic motions on the moduli space. The metric for the two-monopole moduli space was determined by Atiyah and Hitchin [6] and has given information regarding the classical and quantum scattering processes of monopoles. More recently [7], the knowledge of the metric has been used in theories with extended supersymmetry to show the existence of some of the dyonic states required by the electromagnetic duality conjecture of Montonen and Olive [8].

While Manton's approach is believed to give a valid approximate description, it deviates from the viewpoint of modern effective field theory: it is *not* based on all relevant degrees of freedom at low energy. Dynamical freedoms in Manton's approach are restricted to collective coordinates of monopoles, but the freedoms associated with photons (γ) and massless Higgs particles (φ) are also relevant at low energy. We hope to remedy this in this article. Instead of looking into the dynamics of collective coordinates of *all* monopoles (this is Manton's moduli-space approximation), we will here obtain our effective field theory by studying how the collective coordinates of a *single* monopole or dyon get involved dynamically with soft electromagnetic and Higgs field excitations in the vicinity of the monopole or dyon. This effective theory can describe the low-energy interaction of monopoles with on-shell photons and Higgs particles, and in the appropriate limit produces the result of Manton as well. (Note that, in our approach, monopoles or dyons interact through the intermediary of electromagnetic and Higgs fields filling the space.) Moreover, it has the distinctive advantage that the underlying symmetries of the theory, the electromagnetic duality and spontaneously broken scale invariance, are clearly borne out, making our effective action unique.

The basic idea of our approach can be captured by considering the low-energy effective theory of massive vector particles in the BPS limit of the SU(2) Yang-Mills-Higgs model. In the unitary gauge with the Higgs fields aligned as $\phi^a(x)=\delta_{a3}(f+\varphi(x))$, the latter model is described by the Lagrange density[1]

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[1]We set $c=1$ and our metric convention is that with the signature $(-\ +\ +\ +)$.

$$\mathcal{L}=-\frac{1}{4}F^{\mu\nu}F_{\mu\nu}-\frac{1}{2}|(\mathcal{D}^{\mu}W^{\nu}-\mathcal{D}^{\nu}W^{\mu})|^{2}-\frac{1}{2}\partial_{\mu}\varphi\partial^{\mu}\varphi-e^{2}(f+\varphi)^{2}W^{\mu\dagger}W_{\mu}+ieF^{\mu\nu}W_{\mu}{}^{\dagger}W_{\nu}+\frac{e^{2}}{4}(W_{\mu}{}^{\dagger}W_{\nu}-W_{\nu}{}^{\dagger}W_{\mu})\times(W^{\mu\dagger}W^{\nu}-W^{\nu\dagger}W^{\mu}), \tag{1.1}$$

where $F_{\mu\nu}=\partial_{\mu}A_{\nu}-\partial_{\nu}A_{\mu}$ is the electromagnetic field strength and $D_{\mu}W_{\nu}(\equiv\partial_{\mu}W_{\nu}+ieA_{\mu}W_{\nu})$ the covariant derivative of the charged vector field. The Higgs scalar φ, which is massless in the BPS limit, plays the role of dilaton. When the energy transfer ΔE is much smaller than the W-boson mass $m_v=ef$, the above theory may be substituted by an effective theory with the action $S_{\rm eff}$, whose dynamical variables consist of the positions $\mathbf{X}_n(t)$ of W bosons and two massless fields A_{μ} and φ. Ignoring contact interactions of ''heavy'' W fields and also relatively short-ranged magnetic moment interaction from Eq. (1.1), this low-energy action $S_{\rm eff}$ is easily identified, viz.,

$$S_{\rm eff}=\int d^4x\left\{\frac{1}{4}F^{\mu\nu}F_{\mu\nu}-\frac{1}{2}F^{\mu\nu}(\partial_{\mu}A_{\nu}-\partial_{\nu}A\mu)-\frac{1}{2}\partial_{\mu}\varphi\partial^{\mu}\varphi\right\}+\int dtL_{\rm eff}, \tag{1.2}$$

with $L_{\rm eff}$ given by

$$L_{\rm eff}=\sum_{n=1}^{N}\{-(m_v+g_s\varphi(\mathbf{X}_n,t))\sqrt{1-\dot{\mathbf{X}}_n^2}-q_n[A^0(\mathbf{X}_n,t)-\dot{\mathbf{X}}_n(t)\cdot\mathbf{A}(\mathbf{X}_n,t)]\}, \tag{1.3}$$

where $q_n=\pm e$ and $g_s=m_v/f=e>0$, denoting the electric and dilaton charges of the W particle, respectively. While we are eventually interested in the low-energy dynamics, it is also usuful to keep the full relativistic kinetic terms for particles and solitons. We remark that aside from the electromagnetic gauge invariance, this effective theory also inherits from the original theory the spontaneously broken scale invariance, which is described by

$$m_v+g_s\varphi'(x)=\frac{1}{\lambda}[m_v+g_s\varphi(x/\lambda)],$$
$$A'_{\mu}(x)=\frac{1}{\lambda}A_{\mu}(x/\lambda),\quad \mathbf{X}'_n(t)=\lambda\mathbf{X}_n(t/\lambda), \tag{1.4}$$

where λ is a real number.

From Eq. (1.3) we see that the low-energy dynamics of W particles are governed by the force law [here, $\mathbf{V}_n\equiv(d/dt)\,\mathbf{X}_n$][2]

$$\frac{d}{dt}\left[\{m_v+g_s\varphi(\mathbf{X}_n,t)\}\frac{\mathbf{V}_n}{\sqrt{1-\mathbf{V}_n^2}}\right]=q_n\mathbf{E}(\mathbf{X}_n,t)+q_n\mathbf{V}_n\mathbf{B}(\mathbf{X}_n,t)+g_s\mathbf{H}(\mathbf{X}_n,t)\sqrt{1-\mathbf{V}_n^2}, \tag{1.5}$$

where we have introduced the Higgs field strength $\mathbf{H}(x)\equiv-\boldsymbol{\nabla}\varphi(x)$ together with the electric and magnetic fields $(\mathbf{E},\mathbf{B})$. When nonrelativistic kinematics is appropriate, Eq. (1.5) reduces to

$$m_v\frac{d^2}{dt^2}\mathbf{X}_n=q_n[\mathbf{E}(\mathbf{X}_n,t)+\mathbf{V}_n\times\mathbf{B}(\mathbf{X}_n,t)]+g_s\mathbf{H}(\mathbf{X}_n,t) \tag{1.6}$$

and then, as was done in the classical electrodynamics [9], one may use this force law with field equations satisfied by A_{μ} and φ to discuss various low-energy processes. Associated with a uniformly accelerating W particle with acceleration $\mathbf{a}$, for instance, the usual near-zone fields will be accompanied by the radiation fields

$$\mathbf{E}(\mathbf{r},t)\sim\frac{q_n}{4\pi}\frac{\mathbf{R}\times(\mathbf{R}\times\mathbf{a})}{R^3},\quad \mathbf{B}(\mathbf{r},t)\sim-\frac{q_n}{4\pi}\frac{\mathbf{R}\times\mathbf{a}}{R^2},$$
$$\mathbf{H}(\mathbf{r},t)\sim\frac{g_s}{4\pi}\frac{(\mathbf{R}\cdot\mathbf{a})\mathbf{R}}{R^3},\quad H^0(\mathbf{r},t)\sim\frac{g_s}{4\pi}\frac{\mathbf{R}\cdot\mathbf{a}}{R^2}, \tag{1.7}$$

where $\mathbf{R}$ is the radial distance vector evaluated at the retarded time. Also the low-energy laboratory cross sections for the γW and φW scatterings are easily calculated to be

$$\left(\frac{d\sigma}{d\Omega}\right)_{\gamma W,\varphi W\to\gamma W}=\left(\frac{e^2}{4\pi m_v}\right)^2\sin^2\theta,$$
$$\left(\frac{d\sigma}{d\Omega}\right)_{\gamma W,\varphi W\to\varphi W}=\left(\frac{e^2}{4\pi m_v}\right)^2\cos^2\theta, \tag{1.8}$$

where θ is the angle between the direction of outgoing massless particles and that of the incident massless fields. Here we have neglected the spin of W particles. We have also taken care of the photon spin by averaging over the initial spin and summing over the final spin. Of course the same results may be obtained in the tree approximation of the full theory.

The above effective theory may also be used to derive the effective Lagrangian for a system of slowly moving W particles. This effective particle Lagrangian results once we eliminate massless fields $A_{\mu}(x)$ and $\varphi(x)$ from the above effective Lagrangian by using their field equations in the near-zone approximation. For details on this procedure, see Appendix A. Assuming nonrelativistic kinematics for W particles, we then find the slow-motion Lagrangian of the form[3]

[2]As the force law for the n-th W particle, $\mathbf{E}$, $\mathbf{B}$, and $\mathbf{H}$ $(=-\boldsymbol{\nabla}\varphi)$ appearing here may be allowed to include only contributions that are really external to the very W particle.

[3]Our effective Lagrangian will lose its validity if two oppositely charged particles approach each other too closely.

$$L=\frac{1}{2}\sum_{n,m} g_{ij}^{(nm)}(\mathbf{X})\dot{\mathbf{X}}_n^i\dot{\mathbf{X}}_m^j+\sum_{n>m}\frac{g_s^2-q_nq_m}{4\pi|\mathbf{X}_n-\mathbf{X}_m|}, \qquad (1.9)$$

with the inertia metric

$$g_{ij}^{(nm)}(\mathbf{X})=m_v\delta_{nm}\delta_{ij}-\frac{g_s^2}{4\pi}\left[\delta_{nm}\left(\sum_{k\,(\neq n)}\frac{1}{|\mathbf{X}_k-\mathbf{X}_n|}\right)-\frac{1-\delta_{nm}}{|\mathbf{X}_n-\mathbf{X}_m|}\right]\delta_{ij}+\frac{q_nq_m-g_s^2}{8\pi|\mathbf{X}_n-\mathbf{X}_m|}\times\left[\delta_{ij}+\frac{(X_n^i-X_m^i)(X_n^j-X_m^j)}{|\mathbf{X}_n-\mathbf{X}_m|^2}\right](1-\delta_{nm}). \qquad (1.10)$$

In the special case of equally charged W particles only, the potential terms in Eq. (1.9) cancel, since $q_nq_m=g_s^2=e^2$, and the last term of the inertia metric (1.10) also cancels, with the metric

$$g_{ij}^{(nm)}(\mathbf{X})=m_v\delta_{nm}\delta_{ij}-\frac{g_s^2}{4\pi}\left\{\delta_{nm}\left(\Sigma_{k\,(\neq n)}\frac{1}{|\mathbf{X}_k-\mathbf{X}_n|}\right)-\frac{1-\delta_{nm}}{|\mathbf{X}_n-\mathbf{X}_m|}\right\}\delta_{ij}.$$

One may discuss, for instance, the low-energy scattering of two W particles on the basis of this effective Lagrangian.

In this paper we shall make a systematic study of the field equations of the Yang-Mills-Higgs system to establish the low-energy effective theory involving BPS monopoles or dyons. This will be much harder to analyze than the case of the W particles, for here we have to confront the problems associated with *nonlinear* nature of the given field equations. In Sec. II, static BPS dyon solutions are reviewed. Then, in Sec. III the force law analogous to Eq. (1.5) is derived for a BPS dyon and so are the appropriate generalizations of the results (1.7) and (1.8) when BPS dyons, rather than W particles, are involved. Two of us have considered parts of these problems earlier [10,11], but they did not encompass all the relevant processes (especially those involving massless Higgs particles). In Sec. IV, we formulate the effective field theory involving the dyon positions and two massless fields mentioned above in such a way that the results of Sec. III are fully accommodated. The resulting theory assumes the form corresponding to a duality-invariant generalization of the action (1.1). It is conceivable that our effective theory may have validity beyond tree level in the context of appropriately supersymmetrized models. Also, for a system of slowly moving BPS dyons (of the same sign), we obtain the effective Lagrangian analogous to Eq. (1.9) by the same procedure as above and show that it is closely related to Manton's moduli-space dynamics for well-separated monopoles. In Sec. V we discuss similar issues for BPS dyons in a gauge theory with an arbitrary gauge group that is maximally broken to $U(1)^k$. Here the appropriate monopole moduli space was recently obtained in Ref. [12]. Section VI is devoted to the summary and discussion of our work.

We have included brief reviews of some relevant materials to make our paper reasonably self-contained. Presumably, various ideas developed in this work were previously anticipated by Manton [13] and others [12,14], who presented a simple derivation of the moduli-space metric for well-separated monopoles on the basis of closely related ideas. However, to our knowledge, the full story as presented here has not appeared before. In any case, our work might be viewed as a first-principle derivation of the effective field theory for the BPS monopoles and massless fields, in the sense that it has been extracted through a detailed study of time-dependent dynamics as implied by nonlinear field equations of the system.

II. STATIC BPS DYON SOLUTIONS IN SU(2) GAUGE THEORY

We shall recall here the basic construct of the BPS dyon solution in an SU(2) gauge theory spontaneously broken to U(1). For this discussion it is better not to work in the unitary gauge. The Lagrangian density is ($a=1,2,3$)

$$\mathcal{L}=-\frac{1}{4}G_a^{\mu\nu}G_{\mu\nu}^a-\frac{1}{2}(D_\mu\phi)_a(D^\mu\phi)_a, \qquad (2.1)$$

where

$$G_a^{\mu\nu}=\partial^\mu A_a^\nu-\partial^\nu A_a^\mu+e\epsilon_{abc}A_b^\mu A_c^\nu, \qquad (2.2)$$

$$(D_\mu\phi)_a=\partial_\mu\phi_a+e\epsilon_{abc}A_\mu^b\phi^c. \qquad (2.3)$$

The field equations read

$$(D_\mu G^{\mu\nu})_a=-e\epsilon_{abc}(D^\nu\phi)^b\phi^c, \qquad (2.4)$$

$$(D_\mu D^\mu\phi)_a=0. \qquad (2.5)$$

Without any nontrivial Higgs potential in the Lagrangian density, this is a classically scale-invariant system. For this system, spontaneous symmetry breaking is achieved by demanding the asymptotic boundary condition

$$|\phi|=\sqrt{\phi_a\phi_a}\to f>0 \quad \text{as} \quad r\to\infty. \qquad (2.6)$$

The unbroken U(1) will be identified with the electromagnetic gauge group below.

The above system admits static soliton solutions in the form of magnetic monopoles (or, more generally, dyons), the stability of which is derived from the topological argument. They will carry some nonzero charges with respect to long-range fields. To be explicit, we may define the electric and magnetic charges by

$$q=\oint_{r=\infty}dS_i\hat{\phi}^aE_i^a, \quad g=\oint_{r=\infty}dS_i\hat{\phi}^aB_i^a, \qquad (2.7)$$

with $E_i^a\equiv G_a^{0i}$, $B_i^a=\frac{1}{2}\epsilon_{ijk}G_a^{jk}$, and $\hat{\phi}^a=\phi^a/\sqrt{\phi_a\phi_a}$, and the dilaton charge[4] by

$$g_s=\oint_{r=\infty}dS_i\partial_i|\phi|=\oint_{r=\infty}dS_i\hat{\phi}^a(D_i\phi)^a. \qquad (2.8)$$

[4]This name is due to Harvey [14], who also emphasized the role of a Higgs scalar as a dilaton.

Then we have $g=4\pi n/e$ ($n\in\mathbf{Z}$) for a topological reason while q may take on classically any continuous value. Also, g_s is nothing but the mass of a static localized soliton up to a factor, viz.,

$$g_s=M/f, \tag{2.9}$$

with

$$M\equiv\int d^3r\ T^{00}=\int d^3r\frac{1}{2}\{E_i^aE_i^a+B_i^aB_i^a+(D_0\phi)^a(D_0\phi)^a +(D_i\phi)^a(D_i\phi)^a\}, \tag{2.10}$$

where T^{00} denotes the 00 component of the stress energy tensor

$$T^{\mu\nu}=G_a^{\mu\lambda}G_{a\lambda}^{\nu}+(D^\mu\phi)_a(D^\nu\phi)_a+\eta^{\mu\nu}\mathcal{L}. \tag{2.11}$$

The result (2.9), which seems to be not very well known, can be proved as follows. Consider the so-called improved stress energy tensor [15]

$$\tilde{T}^{\mu\nu}=T^{\mu\nu}+\frac{1}{6}(\eta^{\mu\nu}\partial^2-\partial^\mu\partial^\nu)|\phi|^2, \tag{2.12}$$

which is also conserved and satisfies at the same time the property of being traceless, after using the field equations. Then, for any static solution,

$$\int d^3r\ \tilde{T}^{00}=M-\frac{1}{6}\int d^3r\ \nabla^2|\phi|^2 =M-\frac{1}{3}fg_s, \tag{2.13}$$

using Eq. (2.10) and the asymptotic behavior $|\phi|\sim f - g_s/4\pi r$. On the other hand, since the traceless tensor $\tilde{T}^{\mu\nu}$ is also divergence-free, we have

$$\int d^3r\ \tilde{T}^{00}=\int d^3r\ \tilde{T}^{ii}=\int d^3r\ \partial_i(\tilde{T}^{ij}x^j) = \oint_{r=\infty}dS_i\frac{x^j}{6}(\delta_{ij}\nabla^2-\partial^i\partial^j)|\phi|^2=\frac{2}{3}fg_s. \tag{2.14}$$

The relation (2.9) follows immediately from Eqs. (2.13) and (2.14).[5]

Based on Eq. (2.10), it is not difficult to show that the mass of configurations with given g and q satisfies the inequality (called the Bogomol'nyi bound) [3,16]

$$M\geqslant f\sqrt{g^2+q^2}. \tag{2.15}$$

Moreover, to obtain static solutions to field equations (2.4) and (2.5) with the lowest possible energy $M=f\sqrt{g^2+q^2}$ for given $g=\mp4\pi n/e$ (n is a positive integer) and $q=g\tan\beta$, it suffices to consider solutions to the first-order Bogomol'nyi equations [16]

$$B_i^a=\mp\cos\beta(D_i\phi)^a,\quad E_i^a=\mp\sin\beta(D_i\phi)^a,\quad (D_0\phi)^a=0. \tag{2.16}$$

These are equations relevant to BPS dyons and for $\beta=0$ reduce to the Bogomol'nyi equations for uncharged monopoles:

$$B_i^a=\mp(D_i\phi)^a,\quad A_0^a=0. \tag{2.17}$$

Actually all dyon solutions to Eq. (2.16), denoted as $(\bar{\phi}^a(\mathbf{r};\beta),\bar{A}_i^a(\mathbf{r};\beta),\bar{A}_0^a(\mathbf{r};\beta))$, can be obtained from the static monopole solutions $(\bar{\phi}^a(\mathbf{r};\beta=0),\bar{A}_i^a(\mathbf{r};\beta=0))$ satisfying Eq. (2.17). This is achieved by the simple substitution [18]

$$\begin{aligned}\bar{\phi}_a(\mathbf{r};\beta)&=\bar{\phi}_a(\mathbf{r}\cos\beta;0),\\ \bar{A}_i^a(\mathbf{r};\beta)&=\cos\beta\bar{A}_i^a(\mathbf{r}\cos\beta;0),\\ \bar{A}_0^a(\mathbf{r};\beta)&=\mp\sin\beta\bar{\phi}^a(\mathbf{r}\cos\beta;0).\end{aligned} \tag{2.18}$$

The $n=\pm1$ solutions to Eq. (2.17) are well known [3]:

$$\begin{aligned}\bar{A}_a^i(\mathbf{r};0)&=\epsilon_{aij}\frac{\hat{r}_j}{er}\left(1-\frac{m_vr}{\sinh m_vr}\right),\\ \bar{\phi}_a(\mathbf{r};0)&=\pm\hat{r}_af\left(\coth m_vr-\frac{1}{m_vr}\right).\end{aligned} \tag{2.19}$$

These describe the BPS one-(anti-)monopole solution, centered at the spatial origin, with $g=\mp4\pi/e$ and mass $M=g_sf=4\pi f/e$. If the substitution (2.18) is made with these solutions, the results are the (classical) BPS dyon solutions with $g=\mp4\pi/e$, $q=\mp4\pi\tan\beta/e$, and mass $M=g_sf=4\pi f/e\cos\beta$. Being a Bogomol'nyi system, there are also static multi-monopole solutions satisfying Eq. (2.17). However, physically, they may be viewed as representing configurations involving several of the fundamental $n=\pm1$ monopoles described above. The latter interpretation is supported by the observation that the dimension of the moduli space of solutions with $g=\mp4\pi n/e$ is $4n$ [17]; this is precisely the number one would expect for configurations of n monopoles, each of which is specified by three position coordinates and a U(1) phase angle associated with dyonic excitations.

III. TIME-DEPENDENT SOLUTIONS BASED ON FIELD EQUATIONS

A. Summary of our previous analyses

We now turn to the study of low-energy dynamics involving BPS dyons, as dictated by the time-dependent field equations of the Yang-Mills-Higgs system. Particularly important processes are those in which a single BPS dyon interacts with electromagnetic and Higgs fields: they give the most

[5] If the new tensor $\tilde{T}^{\mu\nu}$ were used to define the soliton mass, one would end up with the mass value $2M/3$, but we adhere to our definition (2.10) for the soliton mass since this mass also enters the equation of motion for a soliton (see Secs. III and IV); the *physical* mass is equal to M.

direct information on the nature of effective interaction vertices involving these freedoms. Some of these processes were previously analyzed by two of us [10,11] and in this subsection we shall recall the results obtained there.

The first case concerns an accelerating BPS dyon in the presence of a weak, uniform, electromagnetic field asymptotically [11], viz., under the condition that

$$\frac{\phi^a}{|\phi|}B_i^a\to(\mathbf{B}_0)_i,\quad \frac{\phi^a}{|\phi|}E_i^a\to(\mathbf{E}_0)_i\quad \text{for } r\to\infty. \tag{3.1}$$

This generalizes the problem originally considered by Manton [18] some time ago. Due to the uniform asymptotic fields present, the center of dyon is expected to undergo a constant acceleration, namely, $\mathbf{X}(t)=\frac{1}{2}\mathbf{a}t^2$ (the acceleration $\mathbf{a}$ to be fixed posteriorly) in the reference frame with respect to which the dyon has zero velocity at $t=0$. To find the appropriate solution to the field equations (2.4) and (2.5), the following ansatz has been chosen in Ref. [11]:

$$\begin{aligned}\phi^a(\mathbf{r},t)&=\tilde{\phi}^a(\mathbf{r}';\beta),\\ A_i^a(\mathbf{r},t)&=-ta_i\overline{A_0^a}(\mathbf{r}';\beta)+\tilde{A}_i^a(\mathbf{r}';\beta),\\ A_0^a(\mathbf{r},t)&=-ta_i\overline{A_i^a}(\mathbf{r}';\beta)+\tilde{A}_0^a(\mathbf{r}';\beta),\end{aligned} \tag{3.2}$$

with

$$\begin{aligned}\tilde{\phi}^a(\mathbf{r}';\beta)&=\overline{\phi^a}(\mathbf{r}';\beta)+\Pi^a(\mathbf{r}';\beta),\\ \tilde{A}_i^a(\mathbf{r}';\beta)&=\overline{A_i^a}(\mathbf{r}';\beta)+\alpha_i^a(\mathbf{r}';\beta),\\ \tilde{A}_0^a(\mathbf{r}';\beta)&=\mp\sin\beta\overline{\phi^a}(\mathbf{r}';\beta)+\alpha_0^a(\mathbf{r}';\beta),\end{aligned} \tag{3.3}$$

where $\mathbf{r}'\equiv\mathbf{r}-\mathbf{X}(t)$, the functions $(\overline{\phi^a}(\mathbf{r};\beta),\overline{A_\mu^a}(\mathbf{r};\beta))$ represent the static dyon solution given by (2.18) (with $g=\mp4\pi/e$ and $q=g\tan\beta$), and the yet-to-be-determined functions (Π^a,α_μ^a) are assumed to be $O(a)$ [or $O(B_0)$ or $O(E_0)$]. Terms beyond $O(a)$ are ignored. Note that the functions (Π^a,α_μ^a) will account for the long-range electromagnetic and Higgs fields as well as the field deformations near the dyon core.

It then follows that the field equations (2.4) are fulfilled if the functions (Π^a,α_μ^a) satisfy the equations

$$\tilde{B}_i^a=\mp(\tilde{D}_i+a_i)^{ab}(\cos\beta\tilde{\phi}^b\pm\tan\beta\alpha_0^b), \tag{3.4}$$

$$(\overline{D}_i\overline{D}_i\alpha_0)^a=-e^2\cos^2\beta\epsilon_{abc}\epsilon_{bdf}\overline{\phi^c}\,\overline{\phi^f}\alpha_0^d, \tag{3.5}$$

where $\tilde{D}_i^{ab}\equiv(D_i^{ab})_{A^a\to\tilde{A}^a}$, $\tilde{G}_c^{ji}\equiv(G_c^{ji})_{A^a\to\tilde{A}^a}$, and the suppressed dependent variable is $\mathbf{r}'$. At the same time, the field strength E_i^a to $O(a)$ is given by

$$E_i^a(\mathbf{r},t)=-ta_j\overline{G_a^{ij}}+(\tilde{D}_i+a_i)^{ab}\tilde{A}_0^b. \tag{3.6}$$

From these equations and the condition (3.1), one finds that the acceleration $\mathbf{a}$ should have the value given by

$$M\mathbf{a}=g\mathbf{B}_0+q\mathbf{E}_0\quad\left(M=\frac{4\pi f}{e\cos\beta}\right), \tag{3.7}$$

while the function α_0^a behaves asymptotically such as

$$\alpha_0^a(\mathbf{r}';\beta)\to\mp\cos\beta(\sin\beta\mathbf{B}_0-\cos\beta\mathbf{E}_0)\cdot\mathbf{r}'\hat{r}^a\quad\text{for } r\to\infty. \tag{3.8}$$

Note that Eq. (3.7) is the equation of motion in the dyon instantaneous rest frame and the corresponding covariant generalization

$$\frac{d}{dt}\left(\frac{M\mathbf{V}}{\sqrt{1-\mathbf{V}^2}}\right)=g(\mathbf{B}_0-\mathbf{V}\times\mathbf{E}_0)+q(\mathbf{E}_0+\mathbf{V}\times\mathbf{B}_0)$$

$$\left(\mathbf{V}\equiv\frac{d}{dt}\mathbf{X}\right) \tag{3.9}$$

can also be secured by further considering the implication as the Lorentz boost of our ansatz (3.2) is performed.

The explicit, closed-form solution to Eqs. (3.4) and (3.5) has been given in Ref. [11]. Because of its rather complicated structure, we shall here describe its characteristic features only. It is everywhere regular, with the fields near the dyon core (i.e., at a distance $d\sim1/m_v$) deformed suitably to match smoothly the long-range fields having simple physical interpretation. The physical contents of the long-range electromagnetic field is given in terms of $B_i^{\rm em}\equiv(\phi^a/|\phi|)B_i^a$ and $E_i^{\rm em}\equiv(\phi^a/|\phi|)E_i^a$ and that of the long-range Higgs field by $H^\mu\equiv-(\phi^a/|\phi|)(D^\mu\phi)^a$. These quantities are conveniently expressed using the *retarded distance* $\mathbf{R}=\mathbf{r}-\frac{1}{2}\mathbf{a}t_{\rm ret}^2$, with $t_{\rm ret}$ determined (for a given $\mathbf{r}$ and t) through the implicit equation $t-t_{\rm ret}=|\mathbf{r}-\frac{1}{2}\mathbf{a}t_{\rm ret}^2|\equiv R$. Explicitly, in the region $m_vr'\gg1$,

$$\begin{aligned}\mathbf{B}^{\rm em}(\mathbf{r},t)\sim\mathbf{B}_0&+\frac{g}{4\pi}\frac{\hat{\mathbf{R}}-\mathbf{v}_{\rm ret}}{(1-\hat{\mathbf{R}}\cdot\mathbf{v}_{\rm ret})^3R^2}-\frac{q}{4\pi}\frac{\hat{\mathbf{R}}\times\mathbf{v}_{\rm ret}}{R^2}\\&+\left\{\frac{g}{4\pi}\frac{\hat{\mathbf{R}}\times(\hat{\mathbf{R}}\times\mathbf{a})}{R}-\frac{q}{4\pi}\frac{\hat{\mathbf{R}}\times\mathbf{a}}{R}\right\},\end{aligned} \tag{3.10}$$

$$\begin{aligned}\mathbf{E}^{\rm em}(\mathbf{r},t)\sim\mathbf{E}_0&+\frac{q}{4\pi}\frac{\hat{\mathbf{R}}-\mathbf{v}_{\rm ret}}{(1-\hat{\mathbf{R}}\cdot\mathbf{v}_{\rm ret})^3R^2}+\frac{g}{4\pi}\frac{\hat{\mathbf{R}}\times\mathbf{v}_{\rm ret}}{R^2}\\&+\left\{\frac{q}{4\pi}\frac{\hat{\mathbf{R}}\times(\hat{\mathbf{R}}\times\mathbf{a})}{R}+\frac{g}{4\pi}\frac{\hat{\mathbf{R}}\times\mathbf{a}}{R}\right\},\end{aligned} \tag{3.11}$$

$$\mathbf{H}(\mathbf{r},t)\sim\frac{g_s}{4\pi}\frac{\hat{\mathbf{R}}-\mathbf{v}_{\rm ret}}{(1-\hat{\mathbf{R}}\cdot\mathbf{v}_{\rm ret})^3R^2}+\left\{\frac{g_s}{4\pi}\frac{(\hat{\mathbf{R}}\cdot\mathbf{a})\hat{\mathbf{R}}}{R}\right\}, \tag{3.12}$$

$$H^0(\mathbf{r},t)\sim\frac{g_s}{4\pi}\frac{\hat{\mathbf{R}}\cdot\mathbf{v}_{\rm ret}}{(1-\hat{\mathbf{R}}\cdot\mathbf{v}_{\rm ret})^3R^2}+\left\{\frac{g_s}{4\pi}\frac{(\hat{\mathbf{R}}\cdot\mathbf{a})}{R}\right\}, \tag{3.13}$$

where $\mathbf{a}$ is given by Eq. (3.7), $\mathbf{v}_{\rm ret}\equiv\mathbf{a}t_{\rm ret}$, and $g_s=4\pi/e\cos\beta$ (i.e., equal to the dilaton charge of the dyon).

Note that expressions (3.10) and (3.11) are fully consistent with the electromagnetic fields of a pointlike dyon in motion and exhibit the manifest electromagnetic duality. [See Eq. (1.7) for a comparison.] This statement applies to both near-zone fields of $O(R^{-2})$ and radiation fields [the $O(R^{-1})$ terms marked by the curly brackets in Eqs. (3.10)–(3.13)]. Now the radiation energy flux, measured by the $0i$ component of the stress energy tensor, is given as

$$T^{0i}_{\rm rad}=G^{0k}_a G^{ik}_a+(D^0\phi)^a(D^i\phi)^a=\epsilon^{ijk}E^{\rm em}_j B^{\rm em}_k+H^0H^i$$
$$=\frac{g_s^2}{16\pi^2R^2}(|\mathbf{a}\times\hat{\mathbf{R}}|+|\mathbf{a}\cdot\hat{\mathbf{R}}|),\tag{3.14}$$

where we used the relation $g_s^2=g^2+q^2$.

In Ref. [10], an analogous perturbative scheme was used to study light scattering off a neutral BPS monopole in the long-wavelength limit. Here the incident electromagnetic wave is assumed to have magnetic field given as

$$\mathbf{B}^{\rm em}_{\rm in}=\mathrm{Re}\left[\frac{iM\omega^2}{g}\mathbf{a}e^{i\mathbf{k}\cdot\mathbf{x}-i\omega t}\right]\quad(\omega=|\mathbf{k}|,\mathbf{k}\cdot\mathbf{a}=0),\tag{3.15}$$

where the frequency ω and the magnitude a are taken to be sufficiently small so that $\omega/m_v\ll 1$ and $\omega a\ll 1$. The center of the monopole is then expected to undergo a nonrelativistic motion

$$\mathbf{X}(t)=\mathrm{Re}[-i\mathbf{a}e^{-i\omega t}]\tag{3.16}$$

[with the initial condition $\mathbf{X}(0)=0$]. So, in this case, the solution to the field equations (2.4) may be sought on the basis of the ansatz

$$\phi^a(\mathbf{r},t)=\overline{\phi^a}(\mathbf{r}-\mathbf{X})+\mathrm{Re}[\Pi^a(\mathbf{r},t)],$$
$$[\Pi^a(\mathbf{r},t)=\widetilde{\Pi}^a(\mathbf{r}-\mathbf{X})e^{-i\omega t}],$$
$$A^a_i(\mathbf{r},t)=\overline{A^a_i}(\mathbf{r}-\mathbf{X})+\mathrm{Re}[\alpha^a_i(\mathbf{r},t)],$$
$$[\alpha^a_i(\mathbf{r},t)=\widetilde{\alpha}^a_i(\mathbf{r}-\mathbf{X})e^{-i\omega t}],\tag{3.17}$$
$$A^a_0(\mathbf{r},t)=\mathrm{Re}[\widetilde{\alpha}^a_0(\mathbf{r}-\mathbf{X})e^{-i\omega t}],$$

where $\overline{A^a_i}(\mathbf{r})$ and $\overline{\phi^a}(\mathbf{r})$ represent the static BPS monopole solution in Eq. (2.19). The function $(\widetilde{\Pi}^a,\ \widetilde{\alpha}^a_\mu)$ are assumed to be $O(a\omega)$ and in the asymptotic region should account for the incident wave and outgoing radiations.

Using the ansatz (3.17) with field equations (2.4) and (2.5) give rise to complicated differential equations for the functions $(\widetilde{\Pi}^a,\ \widetilde{\alpha}^a_\mu)$. However, as noted in Ref. [10], a great simplification is achieved with the introduction of the functions $\beta^a_i(\mathbf{r},t)$ by the equation

$$G^{ij}_a(\mathbf{r},t)=\mp\epsilon_{ijk}[(D_k\phi)^a(\mathbf{r},t)+\beta^a_k(\mathbf{r},t)].\tag{3.18}$$

The field equations are fulfilled if the β^a_i satisfy the equation

$$[(\overline{D_k}\,\overline{D_k}+\omega^2)\widetilde{\beta}_i]^a+e^2\epsilon_{abc}\epsilon_{bde}\widetilde{\beta}^d_i\overline{\phi^e}\,\overline{\phi^c}=0\tag{3.19}$$

[here, $\overline{D^{ac}_i}\equiv\partial_i\delta_{ac}+e\epsilon_{abc}\overline{A^b_i}(\mathbf{r}-\mathbf{X})$], and then the functions $\widetilde{\phi}^a$ and $\widetilde{A}^a_i$ can be found using

$$\widetilde{\Pi}^a=\frac{1}{\omega^2}[(\overline{D_i}\widetilde{\beta}_i)_a-ie\omega\epsilon_{abc}\widetilde{\alpha}^b_0\overline{\phi^c}-i\omega^2a_j\partial_j\overline{\phi^a}],\tag{3.20}$$

$$\widetilde{\alpha}^a_i=\frac{1}{\omega^2}[\mp\epsilon_{ijk}(D_j\widetilde{\beta}_k)^a+e\epsilon_{abc}\widetilde{\beta}^b_i\overline{\phi^c}-i\omega(\overline{D_i}\widetilde{\alpha}_0)^a$$
$$-i\omega^2a_j\partial_j\overline{A^a_i}].\tag{3.21}$$

So what remains nontrivial is to solve Eq. (3.19). There is no equation to fix the functions α^a_0, but this just reflects arbitrariness in the choice of gauge.

The solution to Eq. (3.19), found in Ref. [10], reads

$$\widetilde{\beta}^a_i(\mathbf{r}')=\pm i\omega^2a_if\ \coth m_vr'e^{i\mathbf{k}\cdot\mathbf{r}'}\hat{r}'^a\mp i\omega^2a_i\frac{e^{i\omega r'}}{er'}\hat{r}'^a$$
$$+O(a\omega^3),\tag{3.22}$$

where $\mathbf{r}'\equiv\mathbf{r}-\mathbf{X}$. Then, using this with Eqs. (3.20) and (3.21) [with the gauge choice $\widetilde{\alpha}^b_0(\mathbf{r}')=\omega a_i\overline{A^b_i}(\mathbf{r}')$, made for the consistency of our ansatz], the expressions for $\widetilde{\Pi}^a(\mathbf{r}')$ and $\widetilde{\alpha}^a_i(\mathbf{r}')$ follow. In this way, long-range fields in the present process have been identified as

$$\mathbf{B}^{\rm em}(\mathbf{r},t)\sim\mp i\omega^2\mathbf{a}fe^{i\mathbf{k}\cdot\mathbf{r}-iwt}\mp i\omega^2[\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times\mathbf{a})]\frac{e^{i\omega r-iwt}}{er},$$
$$\mathbf{E}^{\rm em}(\mathbf{r},t)\sim\pm i\omega^2(\hat{\mathbf{k}}\times\mathbf{a})fe^{i\mathbf{k}\cdot\mathbf{r}-iwt}\mp i\omega^2(\hat{\mathbf{r}}\times\mathbf{a})\frac{e^{i\omega r-iwt}}{er},\tag{3.23}$$
$$\mathbf{H}(\mathbf{r},t)\sim i\omega^2\frac{\mathbf{a}\cdot\hat{\mathbf{r}}}{er}e^{i\omega r-iwt}\hat{\mathbf{r}},$$
$$H^0(\mathbf{r},t)\sim i\omega^2\frac{\mathbf{a}\cdot\hat{\mathbf{r}}}{er}e^{i\omega r-iwt}$$

where only the real parts are relevant. Notice the appearance of outgoing spherical waves, describing electromagnetic and Higgs scalar radiations. Based on these results, the related differential cross sections are determined as

$$\left(\frac{d\sigma}{d\Omega}\right)_{\rm em}=\frac{(\omega^4/2e^2)|\hat{\mathbf{r}}\times\mathbf{a}|^2}{\frac{1}{2}\omega^4f^2a^2}=\left(\frac{g^2}{4\pi M}\right)^2\sin^2\Theta,\tag{3.24}$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{\rm Higgs}=\frac{(\omega^4/2e^2)|\mathbf{a}\cdot\hat{\mathbf{r}}|^2}{\frac{1}{2}\omega^4f^2a^2}=\left(\frac{g_s^2}{4\pi M}\right)^2\cos^2\Theta,\tag{3.25}$$

where Θ is the angle between $\hat{\mathbf{r}}$ (i.e., the observation direction) and the incident $\mathbf{B}^{\rm em}$ field and we used the relation $g^2=g_s^2=(4\pi/e)^2$ here. Notice a close similarity between these results for a BPS monopole and the corresponding formulas (1.8) for an electrically charged W particle.

B. Accelerating dyon solution in weak uniform asymptotic fields

A BPS dyon, having nonzero dilaton charge, will have a nontrivial coupling to the Higgs field. To deduce the corresponding force law from the field equations in a convincing way, it is necessary to consider a more general, uniform, asymptotic field than in Eq. (3.1). In this section, we therefore suppose that there exists also a weak, uniform, Higgs field strength asymptotically, viz.,

$$\frac{\phi^a}{|\phi|}(D_i\phi)^a \rightarrow -(\mathbf{H}_0)_i \quad \text{as} \quad r\rightarrow\infty \tag{3.26}$$

in addition to the electromagnetic field strengths $(\mathbf{E}_0,\mathbf{B}_0)$ specified as in Eq. (3.1). Of course, the imposition of Eq. (3.26) would make the asymptotic condition (2.6), required for any field configuration with finite total energy, obsolete. This is not a problem; our interest here is in studying the time-dependent flow of energy from one spatial region to another, as predicted by field equations. For sufficiently small $(\mathbf{E}_0,\mathbf{B}_0,\mathbf{H}_0)$, we may again seek the appropriate perturbative solution to the field equations on the basis of the ansatz given in Eqs. (3.2) and (3.3). This will lead to Eqs. (3.4) and (3.5) and also to the relation (3.6) for E_i^a. However, the solution of our present interest is, unlike that given in Ref. [11], the one satisfying Eqs. (3.4) and (3.5) for nonzero $\mathbf{H}_0$.

Our first task is to determine the dyon acceleration $\mathbf{a}$ under this generalized asymptotic condition. For this purpose, we assume the asymptotic form of the function α_0^a to be given as

$$\alpha_0^a(\mathbf{r}';\beta)\rightarrow \cos\beta\mathbf{C}\cdot\mathbf{r}'\hat{\mathbf{r}}'_a \quad \text{for} \quad r\rightarrow\infty \tag{3.27}$$

($\mathbf{C}$ is some constant vector) and then we have

$$\frac{\phi^a}{|\phi|}(\tilde{D}_i\alpha_0)^a\rightarrow \pm\cos\beta C_i \qquad \text{for} \quad r\rightarrow\infty. \tag{3.28}$$

Now we use this information and the given asymptotic conditions with Eqs. (3.4) and (3.6) to deduce two linear relations involving $\mathbf{B}_0$, $\mathbf{E}_0$, $\mathbf{H}_0$, $\mathbf{a}$, and $\mathbf{C}$. Solving the latter for $\mathbf{a}$ and $\mathbf{C}$, we immediately obtain

$$\mathbf{a}=\mp\frac{1}{f}[\cos\beta\mathbf{B}_0+\sin\beta\mathbf{E}_0\mp\mathbf{H}_0] \tag{3.29}$$

and

$$\mathbf{C}=\mp\frac{1}{f}[\sin\beta\mathbf{B}_0-\cos\beta\mathbf{E}_0]. \tag{3.30}$$

Notice that $\mathbf{C}$ does not depend on $\mathbf{H}_0$. If Eq. (3.29) is rewritten using $g=\mp4\pi/e$, $q=\mp4\pi\tan\beta/e$, and $g_s=4\pi/(e\cos\beta)=M/f$, it assumes the form

$$M\mathbf{a}=g\mathbf{B}_0+q\mathbf{E}_0+g_s\mathbf{H}_0. \tag{3.31}$$

This is the desired equation of motion for a dyon in its instantaneous rest frame. We remark here that, by considering the Lorentz boost of the above solution, Eq. (3.31) may be generalized to the form (see Appendix B)

$$\frac{d}{dt}\left(\frac{(M-g_sX_\mu H^\mu)\mathbf{V}}{\sqrt{1-\mathbf{V}^2}}\right)=g(\mathbf{B}_0-\mathbf{V}\times\mathbf{E}_0)+q(\mathbf{E}_0+\mathbf{V}\times\mathbf{B}_0)+g_s\mathbf{H}_0\sqrt{1-\mathbf{V}^2}. \tag{3.32}$$

This should be compared with the force law for a W particle, given in Eq. (1.5).

If the strengths of the asymptotic fields are such that

$$\mathbf{H}_0=\pm(\cos\beta\mathbf{B}_0+\sin\beta\mathbf{E}_0), \tag{3.33}$$

we see from Eq. (3.29) that $\mathbf{a}=0$, i.e., the dyon does not "feel" any force (at least to the first order in the applied fields). In view of Eq. (3.2), the corresponding solution is necessarily static. Here one has the special case where the applied fields are consistent with the original Bogomol'nyi equations (2.16). This happens if $\alpha_0^a=0$ (and hence $\mathbf{C}=\mathbf{0}$) and

$$\mathbf{B}_0=\pm\cos\beta\mathbf{H}_0, \quad \mathbf{E}_0=\pm\sin\beta\mathbf{H}_0. \tag{3.34}$$

We are now talking about a static BPS dyon solution in the presence of *self-dual uniform fields*. After some calculation we have found that the appropriate static solution for $\beta=0$ (i.e., the case of a neutral monopole) and to $O(H_0)$ is given by

$$\phi_a(\mathbf{r})=\pm\hat{r}_af\left(\coth m_vr-\frac{1}{m_vr}\right)\pm\frac{1}{2}\mathbf{H}_0\cdot\mathbf{r}\hat{\mathbf{r}}_a\frac{m_vr}{\sinh^2 m_vr} \mp\frac{1}{2}[(\mathbf{H}_0)_a-\mathbf{H}_0\cdot\hat{\mathbf{r}}\hat{\mathbf{r}}_a]\frac{r}{\sinh m_vr}\mp\coth m_vr\mathbf{H}_0\cdot\mathbf{r}\hat{\mathbf{r}}_a, \tag{3.35}$$

$$A_a^i(\mathbf{r})=\epsilon_{aij}\frac{\hat{\mathbf{r}}_j}{er}\left(1-\frac{m_vr}{\sinh m_vr}\right)+\frac{1}{2}\epsilon_{aij}\hat{\mathbf{r}}_j\frac{\partial}{\partial r}\left(\frac{r^2}{\sinh m_vr}\right)\mathbf{H}_0\cdot\hat{\mathbf{r}}+\epsilon_{aij}[(\mathbf{H}_0)_j-\mathbf{H}_0\cdot\hat{\mathbf{r}}\hat{\mathbf{r}}_j]\frac{r}{2\sinh m_vr}+\frac{r-r\cosh m_vr}{2\sinh m_vr}\hat{\mathbf{r}}_a\epsilon_{ilm}\hat{\mathbf{r}}_l(\mathbf{H}_0)_m, \tag{3.36}$$

$$A_0^a(\mathbf{r})=0. \tag{3.37}$$

The corresponding solution for $\beta\neq0$ (i.e., the BPS dyon case) then also follows once the trick in Eq. (2.18) is used.[6]

We now consider the solution to Eqs. (3.4) and (3.5) when $\mathbf{a}$ is nonzero; this will lead to a time-dependent solu-

[6]While Eq. (3.35) is only an approximate solution [i.e., valid to $O(H_0)$] of the Bogomol'nyi equation, we remark that its $m_v\rightarrow\infty$ limit, namely, $\phi_a(\mathbf{r})=\pm\hat{r}_a(-\mathbf{H}_0\cdot\mathbf{r}+f-1/er)$ and $A_a^i(\mathbf{r})=\epsilon_{aij}(\hat{r}_j/er)-\frac{1}{2}\hat{r}_a\epsilon_{ilm}x_l(\mathbf{H}_0)_m$, corresponds to an exact but singular solution of $B_i^a=\mp(D_i\phi)^a$.

tion, accompanied by suitable radiation fields. Following Ref. [11], we introduce rescaled quantities

$$\mathbf{y}=\mathbf{r}' \cos\beta, \quad \mathcal{A}_i^a(\mathbf{y})=\frac{1}{\cos\beta}\widetilde{A}_i^a\left(\mathbf{r}'=\frac{\mathbf{y}}{\cos\beta};\beta\right) \tag{3.38}$$

and recast Eqs. (3.4) and (3.5) as

$$\mathcal{B}_i^a=\mp\left(\mathcal{D}_i^{(y)}+\frac{a_i}{\cos\beta}\right)^{ab}\left(\widetilde{\phi}^b\pm\frac{\sin\beta}{\cos^2\beta}\alpha_0^b\right), \tag{3.39}$$

$$(\overline{D_i^{(y)}}\,\overline{D_i^{(y)}}\alpha_0)^a=-e^2\epsilon_{abc}\epsilon_{bdf}\overline{\phi^c}(\mathbf{y};\beta=0)\,\overline{\phi^f}(\mathbf{y};\beta=0)\,\alpha_0^d, \tag{3.40}$$

where $\overline{D_i^{(y^i)ac}}\equiv(\partial/\partial y^i)\,\delta_{ac}+e\epsilon_{abc}\overline{A_i^b}(\mathbf{y};\beta=0)$, $\mathcal{D}_i^{(y)ab}\equiv(\partial/\partial y)\,\delta_{ac}+e\epsilon_{abc}\mathcal{A}_i^b(\mathbf{y})$, and $\mathcal{B}_i^a(\mathbf{y})$ denotes the magnetic field strength obtained from the vector potential $\mathcal{A}_i^a(\mathbf{y})$. The solution to Eq. (3.40) that fulfills the condition (3.27) is given by

$$\alpha_0^a=\coth m_v y(\mathbf{C}\cdot\mathbf{y})\hat{y}_a+\frac{y}{\sinh m_v y}[(\mathbf{C})_a-(\mathbf{C}\cdot\hat{\mathbf{y}})\hat{y}_a], \tag{3.41}$$

where $y\equiv|\mathbf{y}|=r'\cos\beta$ and the vector $\mathbf{C}$ is given by Eq. (3.30). We have also the solution to Eq. (3.39) expressed as

$$\begin{aligned}\widetilde{\phi}_a=&\pm\hat{y}_a\left[f\left(\coth m_v y-\frac{1}{m_v y}\right)+\frac{\hat{\mathbf{y}}\cdot\mathbf{a}}{2e\cos\beta}\right.\\&\left.\times\left(1-\frac{m_v y}{\sinh m_v y}\right)\right]\pm\frac{a_a f y}{2\cos\beta\sinh m_v y}\mp\frac{\sin\beta}{\cos^2\beta}\alpha_0^a\\&\mp\hat{y}_a\frac{\partial}{\partial y}(y\coth m_v y V)\mp\left(\frac{\partial}{\partial y^a}-\hat{y}_a\frac{\partial}{\partial y}\right)\left(\frac{y}{\sinh m_v y}V\right),\end{aligned} \tag{3.42}$$

$$\begin{aligned}\mathcal{A}_a^i=&\epsilon_{aij}\frac{\hat{y}_j}{ey}\left(1-\frac{m_v y}{\sinh m_v y}\right)+\frac{fy}{2\cos\beta}\left(\coth m_v y\right.\\&\left.-\frac{1}{m_v y}\right)\epsilon_{ijk}\hat{y}_j a_k\hat{y}_a+\epsilon_{aij}\frac{\partial}{\partial y^j}\left(\frac{y}{2\sinh m_v y}V\right)\\&+(1-\cosh m_v y)\hat{y}_a\epsilon_{ilm}\hat{y}_l\frac{\partial}{\partial y^m}\left(\frac{y}{\sinh m_v y}V\right),\end{aligned} \tag{3.43}$$

where the function V, which is adjustable, must satisfy the Laplace equation $\nabla^2 V=0$. All asymptotic boundary conditions, including Eq. (3.26), are satisfied if we here choose

$$V=-\frac{1}{2\cos^2\beta}(\sin\beta\mathbf{C}\cdot\mathbf{y}-\cos\beta\mathbf{H}_0\cdot\mathbf{y}). \tag{3.44}$$

Using Eqs. (3.42) and (3.43), we find completely regular expressions for the functions $\widetilde{\phi}(\mathbf{r}';\beta)$, $\widetilde{A}_i^a(\mathbf{r}';\beta)=\cos\beta\mathcal{A}_i^a(\mathbf{y}=\mathbf{r}'\cos\beta)$, and $\widetilde{A}_0^a(\mathbf{r}';\beta)=\mp\sin\beta\widetilde{\phi}(\mathbf{r}';\beta)+\alpha_0^a(\mathbf{r}';\beta)$ [see Eq. (3.3)], immediately. If those are inserted into Eq. (3.2), we have the explicit perturbative solution appropriate to a BPS dyon in the presence of uniform electromagnetic and Higgs field strengths asymptotically. Note that only elementary functions enter our solution (but in a rather complicated way) and the result for $\mathbf{H}_0=\mathbf{0}$ of course coincides with that already given in Ref. [11]. Long-range electromagnetic and Higgs fields, which are easily extracted from this time-dependent solution to the field equations, again take simple forms. As for the $\mathbf{B}^{\text{em}}$, $\mathbf{E}^{\text{em}}$, and H^0, the expressions (3.10), (3.12), and (3.13) are still valid under the condition that the acceleration parameter $\mathbf{a}$ is now specified by Eq. (3.29). On the other hand, the expression of $\mathbf{H}$ now contains also a uniform-field term over the result (3.12), viz.,

$$\mathbf{H}(\mathbf{r},t)\sim\mathbf{H}_0+\frac{g_s}{4\pi}\frac{\hat{\mathbf{R}}-\mathbf{v}_{\text{ret}}}{(1-\hat{\mathbf{R}}\cdot\mathbf{v}_{\text{ret}})^3R^2}+\left\{\frac{g_s}{4\pi}\frac{(\hat{\mathbf{R}}\cdot\mathbf{a})\hat{\mathbf{R}}}{R}\right\}. \tag{3.45}$$

This in turn implies that one may continue to use formula (3.14), with $\mathbf{a}$ specified by Eq. (3.29), to find the radiated energy flux in the form of electromagnetic and Higgs waves.

C. Electromagnetic and Higgs waves incident on dyons

In Sec. III A, the light scattering off a neutral BPS monopole was described in the long-wavelength limit. Since the theory admits also a massless Higgs boson, one might also consider a Higgs wave scattering by a BPS monopole or dyon, which would reveal tree-level interactions between a massless Higgs boson and a BPS dyon. Therefore, to make our analysis complete, we will here analyze light and Higgs wave scattering by a BPS dyon with the help of an analogous perturbative scheme.

In the presence of incident electromagnetic and Higgs plane waves, the dyon is expected to undergo a motion of the form (3.16) with the vector $\mathbf{a}$ describing the oscillating direction and amplitude of the dyon in response to the incident waves. The vector $\mathbf{a}$ is taken to be real; this amounts to choosing the initial condition $\mathbf{X}(0)=\mathbf{0}$. Here $\mathbf{X}(t)$ describes the position (i.e., the center) of the dyon that is defined as the zero of the Higgs field $\phi(\mathbf{r},t)$. We shall again construct a solution to the field equations (2.4) and (2.5) corresponding to this oscillating dyon with incident electromagnetic and Higgs plane waves. Due to the oscillatory motion, it must radiate electromagnetic and Higgs waves as in the case of a neutral monopole. Hence the solution describes the scattering of light and Higgs particle by a dyon.

One may begin the analysis with an ansatz for the solution

$$\phi^a(\mathbf{r},t)=\overline{\phi^a}(\mathbf{r}-\mathbf{X};\beta)+\text{Re}[\widetilde{\Pi}^a(\mathbf{r}-\mathbf{X};\beta)e^{-i\omega t}], \tag{3.46}$$

$$A_\mu^a(\mathbf{r},t)=\overline{A_\mu^a}(\mathbf{r}-\mathbf{X};\beta)+\text{Re}[\widetilde{\alpha}_\mu^a(\mathbf{r}-\mathbf{X};\beta)e^{-i\omega t}]+O(a^2), \tag{3.47}$$

where $(\overline{\phi^a}(\mathbf{r};\beta),\overline{A_\mu^a}(\mathbf{r};\beta))$ is the static dyon solution characterized by magnetic and electric charges $(g=\mp4\pi/e,\ q=\mp4\pi\tan\beta/e)$. The functions $(\widetilde{\Pi}^a(\mathbf{r}-\mathbf{X};\beta),\widetilde{\alpha}_\mu^a(\mathbf{r}-\mathbf{X};\beta))$ represent excitations from the undeformed but moving dyon with the center at $\mathbf{X}(t)$ and in particular contain the asymptotic fields required for the motion and the radiations emitted by the dyon. In spite of the clarity in their interpretation, we shall not work with these functions due to the complexity in resulting equations. Instead, we define new functions

$$\tilde{\Pi}'^{a}=\tilde{\Pi}^{a}(\mathbf{r}-\mathbf{X};\beta)-\tilde{\mathbf{X}}\cdot\nabla\bar{\phi}^{a}(\mathbf{r}-\mathbf{X};\beta), \tag{3.48}$$

$$\tilde{\alpha}'^{a}_{\mu}=\tilde{\alpha}^{a}_{\mu}(\mathbf{r}-\mathbf{X};\beta)-\tilde{\mathbf{X}}\cdot\nabla\bar{A}^{a}_{\mu}(\mathbf{r}-\mathbf{X};\beta), \tag{3.49}$$

where $\tilde{\mathbf{X}}$ is implicitly defined by the relation $\mathbf{X}(t)=\mathrm{Re}[\tilde{\mathbf{X}}e^{-i\omega t}]$. These functions in fact represent the entire time-dependent corrections to the static configurations. As in the case of a monopole, the functions $(\tilde{\Pi}^{a},\tilde{\alpha}^{a}_{\mu})$ and $(\tilde{\Pi}'^{a},\tilde{\alpha}'^{a}_{\mu})$ are assumed to be $O(a)$ and we will solve the field equations to the first order in a. The field equation (2.4) now reads

$$(D_iD_iA_0)^a-i\omega(\bar{D}_i\tilde{\alpha}_0')^a=ie\omega\epsilon^{abc}\bar{\phi}^b\tilde{\Pi}'^c-e^2\epsilon^{abc}\epsilon^{cde}\phi^bA_0^d\phi^e, \tag{3.50}$$

$$(D_jG^{ij})^a-\omega^2\tilde{\alpha}'^{a}_i+i\omega(\bar{D}_i\tilde{\alpha}_0')^a+2ie\omega\epsilon^{abc}\tilde{\alpha}'^{b}_i\bar{A}^c_0=e\epsilon^{abc}A^b_0(D_iA_0)^c-e\epsilon^{abc}\phi^b(D_i\phi)^c, \tag{3.51}$$

while the other field equation (2.5) becomes

$$(D_iD_i\phi)^a+\omega^2\tilde{\Pi}'^a+2ie\omega\epsilon^{abc}\bar{A}^b_0\tilde{\Pi}'^c+ie\omega\epsilon^{abc}\tilde{\alpha}'^{b}_0\bar{\phi}^c=e^2\epsilon^{abc}\epsilon^{cde}A^b_0A^d_0\phi^e, \tag{3.52}$$

where only part of the relevant quantities are expressed in terms of $(\tilde{\Pi}'^a,\tilde{\alpha}'^a_{\mu})$.

To proceed further, we find it convenient to introduce the functions $\tilde{b}^a_i(\mathbf{r}-\mathbf{X})$ by

$$B^a_i(\mathbf{r},t)=\mp\frac{(D_i\phi)^a(\mathbf{r},t)}{\cos\beta}-\tan\beta E^a_i\pm\frac{\tilde{b}^a_i(\mathbf{r}-\mathbf{X})e^{-i\omega t}}{\cos\beta}. \tag{3.53}$$

Note that $\tilde{b}^a_i$ effectively describes dynamical excitations from the BPS saturated state satisfying the combined Bogomol'nyi equation [see Eq. (2.16)]

$$B_k(\mathbf{r},t)=\mp\frac{(D_k\phi)^a(\mathbf{r},t)}{\cos\beta}-\tan\beta E^a_k. \tag{3.54}$$

If we use the relation (3.53) to eliminate $D_i\phi$ from Eq. (3.52) and the Bianchi identity $(D_iB_i)^a=0$, we obtain

$$\omega^2\tilde{\Pi}'_a=(\bar{D}_i\tilde{b}_i)_a-ie\omega\epsilon_{abc}(\tilde{\alpha}'^{b}_0\bar{\phi}^c+\bar{A}^b_0\tilde{\Pi}'^c), \tag{3.55}$$

while direct insertion of Eq. (3.53) into Eq. (3.51) yields

$$\omega^2\tilde{\alpha}'^{a}_i+i\omega(\bar{D}_i\tilde{\alpha}'_0)^a=\mp\frac{1}{\cos\beta}\epsilon_{ijk}(\bar{D}_jb_k)^a+e\epsilon_{abc}\tilde{b}^b_i\bar{\phi}^c-i\omega e\tan\beta\epsilon_{ijk}(\bar{D}_j\tilde{\alpha}'_k)^a-i\omega e\epsilon_{abc}\tilde{\alpha}'^{b}_i\bar{A}^c_0. \tag{3.56}$$

One may also reexpress relation (3.53) in terms of $\tilde{\Pi}'^a$ and $\tilde{\alpha}'^a_{\mu}$ as

$$\epsilon_{ijk}(\bar{D}_j\tilde{\alpha}'_k)^a=\mp\frac{(\bar{D}_k\tilde{\Pi}')^a}{\cos\beta}\mp\cos\beta\epsilon_{abc}\tilde{\alpha}'^{b}_i\bar{\phi}^c+\tan\beta[(\bar{D}_i\tilde{\alpha}'_0)^a-i\omega\tilde{\alpha}'^{a}_i]\pm\frac{\tilde{b}^a_i}{\cos\beta}. \tag{3.57}$$

It is then not difficult to verify that Eq. (3.50) is identically satisfied when Eqs. (3.57), (3.55), and (3.56) are used.

Taking an appropriate combination of Eqs. (3.57) and (3.56) to eliminate the $\tilde{\alpha}'^{a}_i$ dependence and using relation (3.55), we can derive a second-order equation for $\tilde{b}^a_i$, which reads

$$[(\bar{D}_k\bar{D}_k+\omega^2)\tilde{b}_i]_a+e^2\cos^2\beta\epsilon_{abc}\epsilon_{bde}\tilde{b}^d_i\bar{\phi}^e\bar{\phi}^c=0. \tag{3.58}$$

On the other hand, eliminating the $\epsilon_{ijk}(\bar{D}_j\tilde{\alpha}'_k)^a$ terms from Eqs. (3.57) and (3.56) leads to

$$\omega^2\tilde{\alpha}'^{a}_i=\mp\cos\beta\epsilon_{ijk}(\bar{D}_j\tilde{b}_k)^a+e\cos^2\beta\epsilon_{abc}\tilde{b}^b_i\bar{\phi}^c-i\omega(\bar{D}_i\tilde{\alpha}'_0)^a\pm i\omega\sin\beta[(\bar{D}_i\tilde{\Pi}')^a+\tilde{b}^a_i]. \tag{3.59}$$

Once $\tilde{b}^a_i$ are obtained from Eq. (3.58), we may use Eqs. (3.55) and (3.59) to fix $(\tilde{\Pi}'^a,\ \tilde{\alpha}'^a_i)$ up to unknown functions $\tilde{\alpha}'^a_0$. Again note that there is no equation for $\tilde{\alpha}'^a_0$, which merely reflects that the choice of $\tilde{\alpha}'^a_0$ is related to pure gauge degrees of freedom. Equation (3.58) is the same as Eq. (3.19) when we scale $\mathbf{r}$ to $\mathbf{r}/\cos\beta$ and ω to $\omega\cos\beta$. Thus the scattering solution immediately follows if we use the results of Sec. III A:

$$\tilde{b}^a_i=\pm i\omega^2a_if\coth m_v r'\cos\beta e^{i\mathbf{k}\cdot\mathbf{r}}\hat{\mathbf{r}}'^a\mp i\omega^2a_i\frac{e^{i\omega r}}{er'\cos\beta}\hat{\mathbf{r}}'^a+O(a\omega^3), \tag{3.60}$$

where $\mathbf{r}'\equiv\mathbf{r}-\mathbf{X}$. (We will see below that this particular homogeneous solution in fact describes the oscillating dyon by incident electromagnetic and Higgs planewaves. Of course, the solution is not the most general solutions of the above equation.) Upon making the gauge choice

$$\tilde{\alpha}'^a_0=\mp\sin\beta\tilde{\Pi}'^a+\omega a_i\bar{A}^a_i \tag{3.61}$$

and using Eqs. (3.55) and (3.55), we find the expressions

$$\tilde{\Pi}'^a\sim\mp\omega\left[\mathbf{a}\cdot\hat{\mathbf{k}}fe^{i\mathbf{k}\cdot\mathbf{r}}-\frac{\mathbf{a}\cdot\hat{\mathbf{r}}}{er\cos\beta}e^{i\omega r}\right]\hat{r}^a, \tag{3.62}$$

$$\widetilde{\alpha}_i'^a \sim \omega f[(\hat{\mathbf{k}}\times\mathbf{a})_i \cos\beta - a_i \sin\beta] e^{i\mathbf{k}\cdot\mathbf{r}}\hat{r}^a - \omega[(\hat{\mathbf{r}}\times\mathbf{a})_i \cos\beta - a_i \sin\beta] e^{i\omega r}\hat{r}^a \quad (3.63)$$

in the scattering region where the terms of $O(r^{-2})$ are ignored. Consequently, the electromagnetic and Higgs fields in the asymptotic region are given as

$$\mathbf{B}^{\mathrm{em}} = \mp i\omega^2[\cos\beta\hat{\mathbf{k}}\times(\hat{\mathbf{k}}\times\mathbf{a}) - \sin\beta(\hat{\mathbf{k}}\times\mathbf{a})] f e^{i\mathbf{k}\cdot\mathbf{r}-iwt} \mp i\omega^2[\cos\beta\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times\mathbf{a}) - \sin\beta(\hat{\mathbf{r}}\times\mathbf{a})]\frac{e^{i\mathbf{k}\cdot\mathbf{r}-iwt}}{er\cos\beta}, \quad (3.64)$$

$$\mathbf{E}^{\mathrm{em}} = \mp i\omega^2[\cos\beta(\hat{\mathbf{k}}\times\mathbf{a}) + \sin\beta\hat{\mathbf{k}}\times(\hat{\mathbf{k}}\times\mathbf{a})] f e^{i\mathbf{k}\cdot\mathbf{r}-iwt} \mp i\omega^2[\cos\beta(\hat{\mathbf{r}}\times\mathbf{a}) + \sin\beta\hat{\mathbf{r}}\times(\hat{\mathbf{r}}\times\mathbf{a})]\frac{e^{i\mathbf{k}\cdot\mathbf{r}-iwt}}{er\cos\beta}, \quad (3.65)$$

$$\mathbf{H} = -i\omega^2(\mathbf{a}\cdot\hat{\mathbf{k}})\hat{\mathbf{k}} f e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t} + i\omega^2(\mathbf{a}\cdot\hat{\mathbf{r}})\hat{\mathbf{r}}\frac{e^{i\omega r-i\omega t}}{er\cos\beta}, \quad (3.66)$$

$$\mathbf{H}_0 = -i\omega^2(\mathbf{a}\cdot\hat{\mathbf{k}}) f e^{i\mathbf{k}\cdot\mathbf{r}-i\omega t} + i\omega^2(\mathbf{a}\cdot\hat{\mathbf{r}})\frac{e^{i\omega r-i\omega t}}{er\cos\beta}, \quad (3.67)$$

where only the real parts are relevant. From those expressions, one may clearly see the presence of incident plane-waves as well as the electromagnetic and Higgs radiation fields emitted by the dyon. As expected, the force law can be verified explicitly by finding zero of $\phi(\mathbf{r},t)$:

$$M\ddot{\mathbf{X}} = M\frac{d^2}{dt^2}\mathrm{Re}[-i\mathbf{a}e^{-i\omega t}] = \mathrm{Re}[g\mathbf{B}^{\mathrm{em}}_{\mathrm{inc}} + q\mathbf{E}^{\mathrm{em}}_{\mathrm{inc}} + g_s\mathbf{H}^{\mathrm{em}}_{\mathrm{inc}}]_{\mathbf{r}=\mathbf{X}}. \quad (3.68)$$

Here the subscript inc indicates that it refers only to the incident part of the given field. The results (3.64)–(3.67) can be used to calculate the related scattering cross sections. With the energy momentum tensor (2.11), the time-averaged incident flux densities in electromagnetic and Higgs sectors are, respectively,

$$(T^{0i})^{\mathrm{em}}_{\mathrm{inc}} = \frac{1}{2}\omega^4 f^2|\mathbf{a}\times\hat{\mathbf{k}}|^2\hat{\mathbf{k}}_i, \quad (3.69)$$

$$(T^{0i})^{\mathrm{Higgs}}_{\mathrm{inc}} = \frac{1}{2}\omega^4 f^2|\mathbf{a}\cdot\hat{\mathbf{k}}|^2\hat{\mathbf{k}}_i, \quad (3.70)$$

while the time-averaged radiated energy flux densities are

$$(T^{0i})^{\mathrm{em}}_{\mathrm{rad}} = \frac{\omega^4}{2e^2r^2\cos^2\beta}|\mathbf{a}\times\hat{\mathbf{r}}|^2\hat{\mathbf{r}}_i, \quad (3.71)$$

$$(T^{0i})^{\mathrm{Higgs}}_{\mathrm{rad}} = \frac{\omega^4}{2e^2r^2\cos^2\beta}|\mathbf{a}\cdot\hat{\mathbf{r}}|^2\hat{\mathbf{r}}_i. \quad (3.72)$$

Based on these, we find that, when a light is incident upon the dyon, i.e., $\mathbf{a}\cdot\mathbf{k}=0$, the related differential cross sections are[7]

$$\left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{em}\to\mathrm{em}} = \left(\frac{g^2+q^2}{4\pi M}\right)^2 \sin^2\Theta, \quad (3.73)$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{em}\to\mathrm{Higgs}} = \left(\frac{g^2+q^2}{4\pi M}\right)\left(\frac{g_s^2}{4\pi M}\right)\cos^2\Theta, \quad (3.74)$$

where Θ is the angle between $\hat{\mathbf{r}}$ and the combination $g\mathbf{B}^{\mathrm{em}}_{\mathrm{inc}} + q\mathbf{E}^{\mathrm{em}}_{\mathrm{inc}}$. On the other hand, for an incident Higgs wave, we find

$$\left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{Higgs}\to\mathrm{em}} = \left(\frac{g_s^2}{4\pi M}\right)\left(\frac{g^2+q^2}{4\pi M}\right)\sin^2\theta, \quad (3.75)$$

$$\left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{Higgs}\to\mathrm{Higgs}} = \left(\frac{g_s^2}{4\pi M}\right)^2\cos^2\theta, \quad (3.76)$$

where θ is the angle between $\hat{\mathbf{r}}$ and $\hat{\mathbf{k}}$.

As should be the case, the cross sections (3.73) and (3.74) for vanishing β agree with those of light scattering by a monopole in Eqs. (3.24) and (3.25). (However, the case of the Higgs and dyon or monopole scattering was not considered before.) Also it should be stressed that the cross sections found above are manifestly duality symmetric (i.e., involve the combination g^2+q^2 only) and have the same form as the corresponding cross sections for a W particle [see Eq. (1.8)]. In fact, formulas (3.73)–(3.76) apply to solitons and elementary quanta alike only if appropriate values for the mass and various charges are used.

IV. EFFECTIVE THEORY FOR ELECTROMAGNETIC AND HIGGS SCALAR INTERACTIONS OF BPS DYONS

A. Duality-invariant Maxwell theory

According to the results of the preceding section, the behaviors of BPS dyons in low-energy processes are not very different from those of W particles; that is, solitons and elementary field quanta behave alike. This in turn suggests that there should exist a simple effective field theory for low-energy BPS dyons interacting with long-range fields. However, unlike W particles, dyons carry both electric and magnetic charges and so their electromagnetic interactions cannot be accounted for by the usual Maxwell theory: We need a duality-symmetric generalization of the latter. Even from 1960s, Schwinger considered such a duality-symmetric Maxwell theory seriously [19] and then several different versions were developed by him and others [20] since. For our discussion we find the simple first-order action approach,

[7]In view of the relation $g^2+q^2=g_s^2$, the multiplicative factors appearing on the right hand sides of Eqs. (3.73) and (3.74) are actually the same; here [and also in Eqs. (3.75) and (3.76)] we have just written the expression in such a way that the vertices involved may be seen clearly.

given by Schwinger [21] in 1975, adequate. Its basic idea will be recalled briefly in this subsection.

The goal is to find a simple Lagrangian description for the generalized Maxwell system

$$\partial_\nu F^{\nu\mu}=J_e^\mu(x), \tag{4.1a}$$

$$\partial_\nu\, {}^*F^{\nu\mu}=J_g^\mu(x), \tag{4.1b}$$

where ${}^*F^{\mu\nu}=\frac{1}{2}\epsilon^{\mu\nu\lambda\delta}F_{\lambda\delta}$ and J_e and J_g denote conserved electric and magnetic sources, respectively. This system is marked by the duality symmetry

$$J_e'^\mu(x)=\cos\,\alpha J_e^\mu(x)+\sin\,\alpha J_g^\mu(x),$$

$$J_g'^\mu(x)=-\sin\,\alpha J_e^\mu(x)+\cos\,\alpha J_g^\mu(x),$$

$$F'^{\mu\nu}(x)=\cos\,\alpha F^{\mu\nu}(x)+\sin\,\alpha\,{}^*F^{\mu\nu}(x). \tag{4.2}$$

For a given distribution of J_e^μ and J_g^μ, the field strengths $F^{\mu\nu}$ (satisfying suitable asymptotic conditions) can be determined using Eqs. (4.1). However, for a Lagrangian, vector potentials are needed. Based on Eq. (4.1b), we may here introduce the vector potential $A^\mu(x)$ by

$$F^{\mu\nu}(x)=\partial^\mu A^\nu(x)-\partial^\nu A^\mu(x) - \int d^4x'(n\cdot\partial)^{-1}(x,x')\frac{1}{2}\epsilon^{\mu\nu\lambda\delta}[n_\lambda J_{g\delta}(x') - n_\delta J_{g\lambda}(x')]. \tag{4.3}$$

Here n^μ may be any fixed, spacelike, unit vector and the Green's function $(n\cdot\partial)^{-1}$ is realized by

$$\begin{aligned}(n\cdot\partial)^{-1}(x,x')&=\int_0^\infty d\xi[a\,\delta^4(x-x'-n\xi) -(1-a)\,\delta^4(x-x'+n\xi)]\\ &=\{a\Theta[n\cdot(x-x')] -(1-a)\Theta[-n\cdot(x-x')]\}\delta_n(x-x'),\end{aligned} \tag{4.4}$$

where one can choose either $a=0,1$ (semi-infinite string) or $a=1/2$ (symmetric infinite string) and $\delta_n(x-x')$ denotes a three-dimensional δ function with a support on the hypersurface orthogonal to n^μ. Similarly, Eq. (4.1a) informs us that we may also write

$$F^{\mu\nu}(x)=-\frac{1}{2}\epsilon^{\mu\nu\lambda\delta}[\partial_\lambda C_\delta(x)-\partial_\delta C_\lambda(x)] +\int d^4x'(n\cdot\partial)^{-1}(x,x')[n^\mu J_e^\nu(x')-n^\nu J_e^\mu(x')], \tag{4.5}$$

$C^\mu(x)$ being another vector potential that is unrestricted by Eq. (4.1a) alone.

The two potentials A^μ and C^μ cannot be completely independent, since they are connected through Eqs. (4.3) and (4.5). In fact, the latter relations allow one to determine the potentials in terms of $F^{\mu\nu}$ up to a gauge transformations separately for A^μ and C^μ. Explicitly, we have

$$A^\mu(x)=-\int d^4x'(n\cdot\partial)^{-1}(x,x')n_\nu F^{\mu\nu}(x')+\partial^\mu\Lambda_e(x), \tag{4.6}$$

$$C^\mu(x)=-\int d^4x'(n\cdot\partial)^{-1}(x,x')n_\nu^* F^{\mu\nu}(x')+\partial^\mu\Lambda_g(x), \tag{4.7}$$

where $\Lambda_e(x)$ and $\Lambda_g(x)$ are arbitrary gauge functions [which may be set to zero in the gauge $n_\mu A^\mu(x)=n_\mu C^\mu(x)=0$]. Because of these, we can regard the potential C^μ to represent the field-strength-dependent function $C_\mu(F)$ as specified by Eq. (4.7), while the field strengths $F^{\mu\nu}$ are expressed in terms of the potential A^μ through Eq. (4.3).[8] We also remark that, with the choice $n^\mu=(0,\hat{\mathbf{n}})$ [see Eq. (4.4)], using formula (4.6) [for $\Lambda_e(x)=0$] with the magnetic Coulomb field of a point monopole leads to the famous Dirac vector potential with a semi-infinite string along the direction $\hat{\mathbf{n}}$ if the values $a=0,1$ are assumed in the Green's function realization (4.4). Varying the direction of $\hat{\mathbf{n}}$ just leads to gauge equivalent potentials if the magnetic charge carried by the monopole satisfies the well-known quantization condition [22]. On the other hand, if one adopts the Schwinger value [19,21] $a=1/2$ in Eq. (4.4), the resulting monopole vector potential will contain a symmetrically located infinite string singularity along the direction $\pm\hat{\mathbf{n}}$. In the latter case, the vector potentials corresponding to different choices of $\hat{\mathbf{n}}$ can be shown to be gauge equivalent [19,21] if the magnetic charge is quantized by twice the Dirac unit. As for the magnetic monopoles of the Yang-Mills-Higgs system, either value of a may be adopted to define $(n\cdot\partial)^{-1}$; however, if one wishes to have a manifestly duality-symmetric action formulation, the Schwinger value $a=1/2$ may be chosen (see below).

We are now ready to present Schwinger's first-order action approach. It is based on the action

$$S=\int d^4x\left\{\frac{1}{4}F^{\mu\nu}F_{\mu\nu}-\frac{1}{2}F^{\mu\nu}(\partial_\mu A_\nu-\partial_\nu A_\mu)-J_e^\mu A_\mu -J_g^\mu C_\mu(F)\right\}, \tag{4.8}$$

where A_μ and $F^{\mu\nu}$ are taken to be independent fields and $C_\mu(F)$ are specified as above, i.e., through Eq. (4.7). Obviously, the first Maxwell equation $\partial_\nu F^{\nu\mu}=J_e^\mu(x)$ is the consequence of $\delta S/\delta A_\mu(x)=0$. On the other hand, from $\delta S/\delta F_{\mu\nu}(x)=0$ we obtain

$$F^{\mu\nu}(x)=\partial^\mu A^\nu(x)-\partial^\nu A^\mu(x)+\int d^4x'\frac{1}{2}\epsilon^{\mu\nu\lambda\delta}[n_\lambda J_{g\delta}(x') -n_\delta J_{g\lambda}(x')](n\cdot\partial)^{-1}(x',x), \tag{4.9}$$

[8]Alternatively, utilizing the relations (4.6) and (4.5), one may assign a primary role on the *dual* potential C_μ (rather than A_μ).

or taking the dual,

$$*F^{\mu\nu}(x)=\frac{1}{2}\epsilon^{\mu\nu\lambda\delta}[\partial_\lambda A_\delta(x)-\partial_\delta A_\lambda(x)]$$

$$-\int d^4x'[n^\mu J_g^\nu(x')-n^\nu J_g^\mu(x')](n\cdot\partial)^{-1}(x',x). \tag{4.10}$$

Then, based on Eq. (4.10), it is easy to derive the second Maxwell equation $\partial_\nu *F^{\nu\mu}=J_g^\mu(x)$ also. Therefore, the action (4.8) can be used to describe the system (4.1). Here notice another consequence of Eq. (4.9): Multiplying Eq. (4.9) by n_ν and picking the gauge $n_\mu A^\mu=0$ yields

$$n_\nu F^{\mu\nu}(x)=-(n\cdot\partial)A^\mu(x) \tag{4.11}$$

and hence relation (4.6) follows. Moreover, our definition of $C^\mu(F)$ and the first Maxwell equation $\partial_\nu F^{\nu\mu}=J_e^\mu(x)$ may be used to confirm the representation (4.5).

Astute readers should have noticed that Eq. (4.9) is not quite our earlier equation (4.3), unless our Green's function $(n\cdot\partial)^{-1}(x',x)$ satisfies the symmetry property

$$(n\cdot\partial)^{-1}(x',x)=-(n\cdot\partial)^{-1}(x,x'). \tag{4.12}$$

Actually, this odd character of the Green's function is also necessary for the action (4.8) to be invariant under the duality transformation (4.2) [now generalized to include the duality rotation between A_μ and $C_\mu(F)$ in an obvious way] [21]. The condition (4.12) is met if the Schwinger value $a=\frac{1}{2}$ is chosen with our representation (4.4).

B. Low-energy effective theory of BPS dyons

Our detailed analysis of nonlinear field equations (given in Sec. III) revealed that BPS dyons behave just like point-like objects carrying electric, magnetic and dilaton charges. (This does not mean that the core region of the dyon profile remains rigid; rather, the core profile gets deformed suitably to accommodate any change in the long-range tail part.) This observation applies to our force law (3.32), to the near-zone and radiation-zone fields given in Eqs. (3.10)–(3.13) and (3.45), and to the scattered waves of electromagnetic and Higgs particles found in Eqs. (3.64)–(3.67). As a matter of fact, these results are exact parallels of the corresponding formulas for the *W* particles, aside from the ubiquitous sign of duality-invariant electromagnetic coupling in all of our formulas derived for BPS dyons. Therefore, we should be able to account for the entire low-energy dynamics involving *N* BPS dyons and massless fields by a simple effective field theory, described by an action corresponding to a duality-symmetric generalization of the low-energy *W* particle action (1.2). We shall make this statement more precise below.

What we ask for our effective field theory is that it should be able to describe to a good approximation the dynamical development of a configuration of *N* well-separated BPS dyons (i.e., at any given instant, the Higgs field has *N* zeros at various locations), while allowing incoming and outgoing radiations (with moderate frequency) of massless fields. For this purpose, we must first specify appropriate dynamical variables that may enter our effective theory. We shall here keep the position coordinates of BPS dyons (or the location of zeros in the Higgs field), $\mathbf{X}_n(t)(n=1,\cdots,N)$, the electromagnetic fields $[A_\mu(x)]$, and the Higgs field $[\varphi(x)]$. Each BPS dyon has three kinds of charges, that is, q_n, g_n $(=\mp 4\pi/e)$, and $(g_s)_n$ $(=\sqrt{g_n^2+q_n^2})$ for the nth dyon; these charges are made local sources for the electromgnetic and Higgs fields. The electromagnetic field strength $F^{\mu\nu}$ may be defined so that Eq. (4.3) may hold, and the Higgs field strength by $H_\mu=-\partial_\mu\varphi$. In our perturbative solutions given in Sec.III, how should one identify the contributions that may duly be associated with the fields A^μ and φ (or the field strengths $F^{\mu\nu}$ or H_μ)? Actually, for all of our explicit solutions, the field strengths $G_a^{\mu\nu}$ and $(D_\mu\phi)^a$ in the region away from the dyon core (i.e., for $m_v r\gg 1$) have nonvanishing components only in the *isospin* direction $\hat{\phi}^a$. Only the fields in this region are relevant for the present discussion and here one may identify A^μ and φ unambiguously by going to the unitary gauge,[9] that is, $\phi^a(x)=[f+\varphi(x)]\delta_{a3}$ and $A^\mu(x)=A_3^\mu(x)$ away from the core region. Fields within the dyon core and charged vector fields correspond to the freedoms to be integrated out.

We are now ready to write down the action, which incorporates all of our findings on low-energy processes involving BPS dyons. Noting that the results of our analysis for the dyons differ from those for *W* particles only by the presence of the electromagnetic duality symmetry, the desired low-energy action is given by the form

$$S_{\rm eff}=\int d^4x\left\{\frac{1}{4}F^{\mu\nu}F_{\mu\nu}-\frac{1}{2}F^{\mu\nu}(\partial_\mu A_\nu-\partial_\nu A_\mu)\right.$$

$$\left.-\frac{1}{2}\partial_\mu\varphi\partial^\mu\varphi\right\}+\int dt\sum_{n=1}^{N}\{-[M_n+(g_s)_n\varphi(\mathbf{X}_n,t)]$$

$$\times\sqrt{1-\dot{\mathbf{X}}_n^2}-q_n[A^0(\mathbf{X}_n,t)-\dot{\mathbf{X}}_n\cdot\mathbf{A}(\mathbf{X}_n,t)]$$

$$-g_n[C^0(\mathbf{X}_n,t)-\dot{\mathbf{X}}_n\cdot\mathbf{C}(\mathbf{X}_n,t)]\}, \tag{4.13}$$

where $C^\mu=(C^0,\mathbf{C})$, as a function of $F^{\mu\nu}$, are defined by Eq. (4.7) with the Green's functions $(n\cdot\partial)^{-1}$ satisfying the symmetry property (4.12). As one can easily verify, the above action is still invariant under the scale transformation of the form (1.4). Considering the variations of $F^{\mu\nu}$ and A^μ, we then obtain Eq. (4.3) and the generalized Maxwell equations (4.1) with the source term given by

$$J_g^0(x)=\sum_{n=1}^{N}g_n\delta^3(\mathbf{x}-\mathbf{X}_n(t)),$$

[9] A gauge-invariant identification can also be given. Clearly, in the region away from the core, we may set $\varphi\simeq|\phi|-f$, which in turn leads to $H_\mu\simeq-\hat{\phi}^a(D_\mu\phi)^a$. Also note that $\hat{\phi}^a G_a^{\mu\nu}$ in this region is essentially the same as the gauge-invariant 't Hooft tensor [1], $F^{\mu\nu}=\hat{\phi}^a G_a^{\mu\nu}-(1/e)\,\epsilon^{abc}\hat{\phi}^a(D^\mu\hat{\phi})^b(D^\nu\hat{\phi})^c$, which is known to satisfy the generalized Maxwell equation (4.1). Using this 't Hooft tensor, one may then simply define the electromagnetic field A^μ, say, by relation (4.3).

$$\mathbf{J}_g(x)=\sum_{n=1}^{N} g_n\dot{\mathbf{X}}_n(t)\,\delta^3(\mathbf{x}-\mathbf{X}_n(t)),$$
$$J_e^0(x)=\sum_{n=1}^{N} q_n\delta^3(\mathbf{x}-\mathbf{X}_n(t)), \tag{4.14}$$
$$\mathbf{J}_e(x)=\sum_{n=1}^{N} q_n\dot{\mathbf{X}}_n(t)\,\delta^3(\mathbf{x}-\mathbf{X}_n(t)).$$

The corresponding equation of motion for the field φ reads

$$\partial_\mu\partial^\mu\varphi(x)=\sum_{n=1}^{N}(g_s)_n\sqrt{1-\dot{\mathbf{X}}_n}\delta^3(\mathbf{x}-\mathbf{X}_n(t))\equiv J_s(x). \tag{4.15}$$

On the other hand, the $\mathbf{X}_n$ variation with our action leads to the equation of motion

$$\frac{d}{dt}\left[\{m_v+(g_s)_n\varphi(\mathbf{X}_n,t)\}\frac{\mathbf{V}_n}{\sqrt{1-\mathbf{V}_n^2}}\right]$$
$$=q_n[\mathcal{F}^{0i}(\mathbf{X}_n,t)+V_n^j\mathcal{F}^{ij}(\mathbf{X}_n,t)]$$
$$+g_n[\overline{\mathcal{F}}^{0i}(\mathbf{X}_n,t)+V_n^j\overline{\mathcal{F}}^{ij}(\mathbf{X}_n,t)]$$
$$+(g_s)_n\sqrt{1-\mathbf{V}_n^2}\nabla\varphi(\mathbf{X}_n,t), \tag{4.16}$$

where we have defined $\mathcal{F}^{\mu\nu}\equiv\partial^\mu A^\nu-\partial^\nu A^\mu$ and $\overline{\mathcal{F}}^{\mu\nu}\equiv\partial^\mu C^\nu-\partial^\nu C^\mu$. Here, because of Eqs. (4.3) and (4.5), we have $\mathcal{F}^{\mu\nu}=F^{\mu\nu}$ and $\overline{\mathcal{F}}^{\mu\nu}={}^*F^{\mu\nu}$ almost everywhere, that is, away from the string singularity; in this way, the force law (3.32) is also incorporated in our action. The effective theory defined by the above action, by its very construction, will reproduce all the consequences in Sec. III in the proper kinematical regime.

When BPS dyons in the system are sufficiently slowly moving so that only negligible radiations are produced, the above effective field theory may be turned into the effective particle Lagrangian analogous to Eq. (1.9). For this, it suffices to integrate out the fields $A^\mu(x)$ and $\varphi(x)$ using the near-zone solutions to the respective equations of motion [for a given distribution of sources $J_g^\mu(x)$, $J_e^\mu(x)$, and $J_s(x)$]; this is the same procedure to obtain the slow motion Lagrangian (1.9) for W particles (see also Appendix A). Then the Higgs field is expressed as [see Eq. (A6)]

$$\varphi(\mathbf{x},t)=-\frac{1}{4\pi}\sum_n\frac{(g_s)_n\sqrt{1-\dot{\mathbf{X}}_n^2}}{|\mathbf{x}-\mathbf{X}_n|}$$
$$+\frac{1}{4\pi}\frac{\partial}{\partial t}\left(\sum_n(g_s)_n\sqrt{1-\dot{\mathbf{X}}_n^2}\right)$$
$$-\frac{1}{8\pi}\frac{\partial^2}{\partial t^2}\left(\sum_n(g_s)_n\sqrt{1-\dot{\mathbf{X}}_n^2},|\mathbf{x}-\mathbf{X}_n|\right)+\cdots. \tag{4.17}$$

To obtain the corresponding expression for $A^\mu(x)$, one may use the formula (4.4) with the help of the following expression for $F^{\mu\nu}$ [describing the near-zone solution to the generalized Maxwell equation (4.1)]:

$$F^{0i}(\mathbf{x},t)=\frac{1}{4\pi}\sum_n\frac{q_n(x^i-X_n^i)}{|\mathbf{x}-\mathbf{X}_n|^{3/2}}\left[1-\frac{3}{2}\frac{(\mathbf{x}-\mathbf{X}_n)\cdot\dot{\mathbf{X}}_n}{|\mathbf{x}-\mathbf{X}_n|}+\frac{1}{2}\dot{\mathbf{X}}_n^2\right]$$
$$-\frac{1}{4\pi}\sum_n\frac{g_n\epsilon^{ijk}\dot{X}_n^j(t)(x^k-X_n^k)}{|\mathbf{x}-\mathbf{X}_n|^{3/2}}+O(\dot{X}^3), \tag{4.18}$$

$$\frac{1}{2}\epsilon^{ijk}F_{jk}(\mathbf{x},t)=\frac{1}{4\pi}\sum_n\frac{q_n\epsilon^{ijk}\dot{X}_n^j(x^k-X_n^k)}{|\mathbf{x}-\mathbf{X}_n|^{3/2}}$$
$$+\frac{1}{4\pi}\sum_n\frac{g_n(x^i-X_n^i)}{|\mathbf{x}-\mathbf{X}_n|^{\frac{3}{2}}}+O(\dot{X}^2). \tag{4.19}$$

Given this expressions and the choice $n^\mu=(0,\hat{n})$, the integral on the right-hand side of Eq. (4.4) may be performed to discover, modulo gauge transformation, the following (near-zone) expression for the field $A^\mu(x)$:

$$A^0(\mathbf{x},t)=\sum_n\left[\frac{q_n}{4\pi|\mathbf{x}-\mathbf{X}_n|}+\frac{q_n}{8\pi}\frac{\partial^2}{\partial t^2}|\mathbf{x}-\mathbf{X}_n|\right.$$
$$\left.+g_n\dot{\mathbf{X}}_n\cdot\boldsymbol{\omega}(\mathbf{x};\mathbf{X}_n)\right]+O(\dot{X}^3), \tag{4.20}$$

$$A^i(\mathbf{x},t)=\sum_n\left[\frac{q_n\dot{\mathbf{X}}_n^i}{4\pi|\mathbf{x}-\mathbf{X}_n(t)|}+g_n\omega^i(\mathbf{x};\mathbf{X}_n)\right]+O(\dot{X}^2), \tag{4.21}$$

where $\omega^i(\mathbf{x};\mathbf{X}_n)$ denotes the unit-monopole static vector potential (with a symmetrically located infinite string), given by

$$\omega(\mathbf{x};\mathbf{X}_n)=-\frac{1}{8\pi}\left[\frac{\hat{n}\times(\mathbf{x}-\mathbf{X}_n)/|\mathbf{x}-\mathbf{X}_n|}{|\mathbf{x}-\mathbf{X}_n|-\hat{n}\cdot(\mathbf{x}-\mathbf{X}_n)}\right.$$
$$\left.-\frac{\hat{n}\times(\mathbf{x}-\mathbf{X}_n)/|\mathbf{x}-\mathbf{X}_n|}{|\mathbf{x}-\mathbf{X}_n|+\hat{n}\cdot(\mathbf{x}-\mathbf{X}_n)}\right]. \tag{4.22}$$

Note that the electric charge contributions in Eqs. (4.20) and (4.21) are identical to those in Eqs. (A4) and (A5). Also required is the expression for the magnetic potential C^μ. Using Eqs. (4.18) and (4.19) in Eq. (4.7) and making appropriate gauge transformation, one has an expression dual to Eqs. (4.20) and (4.21):

$$C^0(\mathbf{x},t)=\sum_n\left[\frac{g_n}{4\pi|\mathbf{x}-\mathbf{X}_n|}+\frac{g_n}{8\pi}\frac{\partial^2}{\partial t^2}|\mathbf{x}-\mathbf{X}_n|\right.$$
$$\left.-q_n\dot{\mathbf{X}}_n\cdot\boldsymbol{\omega}(\mathbf{x};\mathbf{X}_n)\right]+O(\dot{X}^3), \tag{4.23}$$

$$C^i(\mathbf{x},t)=\sum_n\left[\frac{g_n\dot{\mathbf{X}}_n^i}{4\pi|\mathbf{x}-\mathbf{X}_n|}-q_n\omega^i(\mathbf{x};\mathbf{X}_n)\right]+O(\dot{X}^2). \tag{4.24}$$

The desired effective Lagrangian will result if the fields $A^\mu(x)$ and $\varphi(x)$ are eliminated from the action (4.13) by using the above effective solutions. Here it is useful to notice that, thanks to the field equations satisfied by A^μ and φ, contributions from the massless field action in Eq. (4.13) are equal to one-half of those from the interaction terms with matter. In particular, for the action given in Eq. (4.8), the use of Eqs. (4.1), (4.3), and (4.5) allows us to replace its field action (i.e., the part not involving matter current explicitly) by

$$\int d^4x\left\{-\frac{1}{4}F^{\mu\nu}(\partial_\mu A_\nu-\partial_\nu A_\mu)-\frac{1}{4}{}^*F^{\mu\nu}\int d^4x'(n\cdot\partial)^{-1}(x,x')[n_\mu J_{g\nu}(x')-n_\nu J_{g\mu}(x')]\right\}$$
$$\sim\int d^4x\left\{-\frac{1}{4}F^{\mu\nu}(\partial_\mu A_\nu-\partial_\nu A_\mu)-\frac{1}{2}\partial^\mu C^\nu\int d^4x'(n\cdot\partial)^{-1}(x,x')[n_\mu J_{g\nu}(x')-n_\nu J_{g\mu}(x')]\right\}$$
$$\sim\int d^4x\left\{\frac{1}{2}J_e^\mu A_\mu+\frac{1}{2}J_g^\mu C_\mu(F)\right\}, \tag{4.25}$$

where, on the second line, we have dropped the contribution apparently describing the string-string interaction. As analogous reduction holds for the Higgs field action of Eq. (4.13) also. Based on this observation, using the solutions (4.17), (4.20), (4.21), (4.23), and (4.24) in the action (4.13) leads to the effective Lagrangian of the form

$$\int dtL=\int dt\left\{-\sum_n M_n\sqrt{1-\dot{\mathbf{X}}_n^2}+\frac{1}{2}\sum_{n,m\,(\neq n)}(q_ng_m-g_nq_m)(\dot{\mathbf{X}}_n-\dot{\mathbf{X}}_m)\cdot\omega(\mathbf{X}_n,\mathbf{X}_m)-\frac{1}{8\pi}\sum_{n,m\,(\neq n)}(q_nq_m+g_ng_m)\left(\frac{1}{|\mathbf{X}_n-\mathbf{X}_m|}\right.\right.$$
$$\left.+\frac{1}{2}\left[\frac{\partial^2}{\partial t^2}|\mathbf{x}-\mathbf{X}_m|\right]_{\mathbf{x}=\mathbf{X}_n}-\frac{\dot{\mathbf{X}}_n\cdot\dot{\mathbf{X}}_m}{|\mathbf{X}_n-\mathbf{X}_m|}\right)+\sum_{n,m\,(\neq n)}\frac{(g_s)_n(g_s)_m}{8\pi}\left(\frac{\sqrt{1-\dot{\mathbf{X}}_n^2}\sqrt{1-\dot{\mathbf{X}}_m^2}}{|\mathbf{X}_n-\mathbf{X}_m|}\right.$$
$$\left.\left.+\frac{1}{2}\left[\frac{\partial^2}{\partial t^2}|\mathbf{x}-\mathbf{X}_m|\right]_{\mathbf{x}=\mathbf{X}_n}\right)\right\}, \tag{4.26}$$

with irrelevant self-interaction terms dropped. Ignoring terms beyond $O(\dot{X}^2)$, this Lagrangian may then be changed to the form (cf. Appendix A)

$$L=-\sum_n M_n+\frac{1}{2}\sum_n M_n\dot{\mathbf{X}}_n^2-\frac{1}{16\pi}\sum_{n,m\,(\neq n)}(g_s)_n(g_s)_m\frac{|\dot{\mathbf{X}}_n-\dot{\mathbf{X}}_m|^2}{|\mathbf{X}_n-\mathbf{X}_m|}+\frac{1}{2}\sum_{n,m\,(\neq n)}(q_ng_m-g_nq_m)(\dot{\mathbf{X}}_n-\dot{\mathbf{X}}_m)\cdot\omega(\mathbf{X}_n,\mathbf{X}_m)$$
$$-\frac{1}{16\pi}\sum_{n,m\,(\neq n)}[(g_s)_n(g_s)_m-q_nq_m-g_ng_m]\left\{\frac{\dot{\mathbf{X}}_n\cdot\dot{\mathbf{X}}_m}{|\mathbf{X}_n-\mathbf{X}_m|}+\frac{(\mathbf{X}_n-\mathbf{X}_m)\cdot\dot{\mathbf{X}}_n(\mathbf{X}_n-\mathbf{X}_m)\cdot\dot{\mathbf{X}}_m}{|\mathbf{X}_n-\mathbf{X}_m|^3}\right\}$$
$$+\frac{1}{8\pi}\sum_{n,m\,(\neq n)}\frac{(g_s)_n(g_s)_m-q_nq_m-g_ng_m}{|\mathbf{X}_n-\mathbf{X}_m|}. \tag{4.27}$$

Some comments are in order regarding the slow-motion effective Lagrangian derived above. If the given system consists of BPS dyons with the same values of charges only [i.e., $q_n=q$, $g_n=g$, and $(g_s)_n=\sqrt{g^2+q^2}$ for all n], all the terms in Eq. (4.27) that are not quadratic in velocities cancel. This is the case in which *static* multimonopole solutions are possible, and for some given initial velocities the dynamics is governed solely by the kinetic Lagrangian of the same form as found for slowly moving equal-charge W particles (see Sec. I). Another case of interest follows if we let the magnetic charge of all BPS dyons to be equal (i.e., $g_n=g$ for all n) and keep in Eq. (4.27) only terms that are at most quadratic in velocity or electric charge. Then $(g_s)_n\approx g+q_n^2/2g$ and the Lagrangian (4.27) reduces to (here $M=gf$)

$$L=\frac{1}{2}\sum_n M(\dot{\mathbf{X}}_n^2-q_n^2/g^2)-\frac{g^2}{16\pi}\sum_{n,m\,(\neq n)}\frac{|\dot{\mathbf{X}}_n-\dot{\mathbf{X}}_m|^2}{|\mathbf{X}_n-\mathbf{X}_m|}$$
$$+\frac{1}{16\pi}\sum_{n,m(\neq n)}\frac{(q_n-q_m)^2}{|\mathbf{X}_n-\mathbf{X}_m|}+\frac{g}{2}\sum_{n,m(\neq n)}(q_n-q_m)$$
$$\times(\dot{\mathbf{X}}_n-\dot{\mathbf{X}}_m)\cdot\omega(\mathbf{X}_n,\mathbf{X}_m). \tag{4.28}$$

Using precisely this form, Gibbons and Manton [13] showed that one can derive the Lagrangian appropriate to geodesic motion of n well-separated monopoles on the corresponding multiple-monopole moduli space; this generalizes the earlier work by Manton [13] on the nature of two-monopole moduli space, where the relevant asymptotic metric was known as the self-dual Euclidean Taub–Newman-Unti-Tambeurino metric [23] with a negative mass parameter. Without repeating this analysis mention here only that the electric charge variables q_n in Eq. (4.28) may be interpreted as conserved momenta conjugate to the collective coordinates representing U(1) phase angles of individual monopoles. In conclusion, our low-energy action (4.13) predicts the same physics as the moduli-space geodesic approach (for well-separated BPS monopoles of the same magnetic charges), when the effect of radiation can be ignored. Our action (4.13) can be used to describe low-energy processes involving radiation of the A^μ or φ explicitly also.

V. EXTENSION TO MORE GENERAL GAUGE MODELS

A. Preliminaries

Up to this point our discussion was exclusively in the context of SU(2) Yang-Mills-Higgs model. We now want to generalize our discussion to the case of BPS dyons appearing in a gauge theory with an arbitrary compact simple gauge group G that is maximally broken to $U(1)^k$ (k is the rank of G). As we shall see, much of the structure derived in the $G=SU(2)$ model will find a direct generalization to this case.

Using the matrix notations $A_\mu \equiv A_\mu^p T_p$ and $\phi \equiv \phi^p T_p (p=1,\ldots,d=\dim G)$ with Hermitian generators T_p normalized by $\mathrm{Tr}(T_p T_q)=\kappa\delta_{pq}$, the Lagrange density reads

$$\mathcal{L}=-\frac{1}{4\kappa}\mathrm{Tr}G^{\mu\nu}G_{\mu\nu}-\frac{1}{2\kappa}\mathrm{Tr}D_\mu\phi D^\mu\phi, \tag{5.1}$$

where $G^{\mu\nu}\equiv\partial^\mu A^\nu-\partial^\nu A^\mu-ie[A^\mu,A^\nu]$ and $(D_\mu\phi)\equiv\partial_\mu\phi-ie[A_\mu,\phi]$. As is well known, generators may be decomposed into k mutually commuting operators T_r that span the Cartan subalgebra and lowering and raising operators $E_{\vec\alpha}$ obeying $[T_r,E_{\vec\alpha}]=\alpha_r E_{\vec\alpha}$ and $[E_{\vec\alpha},E_{-\vec\alpha}]=\Sigma_{r=1}^k\alpha_r T_r$ $(\equiv\vec\alpha\cdot\vec T)$. The nature of the symmetry breaking is determined by the asymptotic value of the Higgs field in some fixed direction, say, on the positive z axis. It may be taken to lie in the Cartan subalgebra; this then defines a unit vector $\hat h$ by

$$\langle\phi\rangle_v=\sum_{r=1}^k f\hat h_r T_r \equiv f\hat h\cdot\vec T, \tag{5.2}$$

where f is some positive number. We have a maximal symmetry breaking, i.e., $G\rightarrow U(1)^k$, if $\hat h$ is orthogonal to none of the root vectors. In the latter case, there is a unique set of so-called simple roots $\vec\beta_r$ ($r=1,\ldots,k$) that satisfies the conditions $\hat h\cdot\vec\beta_r>0$ for all r and all other roots can be expressed as linear combinations of these simple roots with integer coefficients all of the same sign. Only this case will be considered in this paper.

Let us briefly summarize known properties of monopoles or dyons in this model [24]. In the asymptotic region, the magnetic field $B_i\equiv B_i^p T_p$ must commute with the Higgs field and therefore, in the spatial direction chosen to define $\langle\phi\rangle_v$, must assume the form

$$B_i(\mathbf{r})\sim\frac{x_i}{4\pi r^3}\vec g\cdot\vec T. \tag{5.3}$$

Topological arguments lead to the quantization condition

$$\vec g=\frac{4\pi}{e}\sum_{r=1}^k n_r\vec\beta_r^* \quad (\vec\beta_r^*\equiv\vec\beta_r/\vec\beta_r^2), \tag{5.4}$$

the non-negative integer n_r being the topologically conserved charges related to the homotopy class of the Higgs field at spatial infinity. We may now define the special U(1) electric and magnetic charges using the asymptotic Higgs field direction as

$$Q_E=\frac{1}{f}\oint_{r=\infty}dS_i\frac{1}{\kappa}\mathrm{Tr}(\phi E_i),$$
$$Q_M=\frac{1}{f}\oint_{r=\infty}dS_i\frac{1}{\kappa}\mathrm{Tr}(\phi B_i)\quad(=\vec g\cdot\hat h) \tag{5.5}$$

and similarly the dilaton charge as

$$Q_S=\frac{1}{f}\oint_{r=\infty}dS_i\frac{1}{\kappa}\mathrm{Tr}(\phi D_i\phi). \tag{5.6}$$

Then, just as in the $G=SU(2)$ model discussed in Sec. II, one can show that the mass of a static soliton, which is always equal to fQ_S, satisfies the Bogomol'nyi bound $M\geqslant f\sqrt{Q_M^2+Q_E^2}$. Hence, for given values of Q_E and Q_M, one may obtain static solutions to field equations with the lowest possible energy, $M=f\sqrt{Q_E^2+Q_M^2}$, by solving again the Bogomol'nyi equations which have the same structure as the corresponding equations of the SU(2) model, viz., (2.16). Especially, with $Q_E=0$, these lowest energy configurations will have the mass

$$M=f\vec g\cdot\hat h=\sum_{r=1}^k n_r\left(\frac{4\pi}{e}f\hat h\cdot\vec\beta_r\right). \tag{5.7}$$

On the other hand, Weinberg [24] showed that the dimension of the corresponding moduli space is equal to $4\Sigma_{r=1}^k n_r$. This suggests that, in analogy to the SU(2) case, all static solutions might be viewed as being composed of a number of *fundamental* BPS monopoles, each with a single unit of topological charge (i.e., $n_r=\delta_{rr'}$, for the r' type).

The fundamental static BPS monopole solutions can be obtained by simple embeddings [25] of the spherically symmetric SU(2) solution given in Eq. (2.19). Note that, with each root $\vec\alpha$, we can always define an SU(2) subalgebra with generators

$$t^1_{(\vec\alpha)}=\frac{1}{\sqrt{2\vec\alpha^2}}(E_{\vec\alpha}+E_{-\vec\alpha}),\quad t^2_{(\vec\alpha)}=\frac{i}{\sqrt{2\vec\alpha^2}}(-E_{\vec\alpha}+E_{-\vec\alpha}),$$
$$t^3_{(\vec\alpha)}=\frac{\vec\alpha\cdot\vec T}{\vec\alpha^2}. \tag{5.8}$$

Now, if $\overline{A_i^a}(\mathbf{r},f)$ and $\overline{\phi_i^a}(\mathbf{r},f)$ denotes the static SU(2) BPS monopole solution corresponding to a Higgs expectation value f [see Eq. (2.19)], then

$$A_i(\mathbf{r})=\sum_{i=1}^{3}\overline{A_i^a}(\mathbf{r},f\hat{h}\cdot\vec{\beta}_r)t^a_{(\vec{\beta}_r)},$$
$$\phi_i(\mathbf{r})=\sum_{i=1}^{3}\overline{\phi_i^a}(\mathbf{r},f\hat{h}\cdot\vec{\beta}_r)t^a_{(\vec{\beta}_r)}+f[\hat{h}-(\hat{h}\cdot\vec{\beta}_r^*)\vec{\beta}_r]\cdot\vec{T} \tag{5.9}$$

is the fundamental monopole solution with $\vec{g}=-(4\pi/e)\vec{\beta}_r^*$ and mass $M_r=(4\pi/e)f\hat{h}\cdot\vec{\beta}_r$. As in the SU(2) case, we can also obtain the dyon solution corresponding to these fundamental monopoles by applying the trick (2.18). Here, to push the SU(2) analogy further, it will be useful to write the corresponding asymptotic field strengths as[10]

$$B_i\sim g_r\frac{x_i}{4\pi r^3}\hat{r}_a t^a_{(\vec{\beta})},\quad E_i\sim q_r\frac{x_i}{4\pi r^3}\hat{r}_a t^a_{(\vec{\beta})},$$
$$D_i\phi\sim-(g_s)_r\frac{x_i}{4\pi r^3}\hat{r}_a t^a_{(\vec{\beta})}, \tag{5.10}$$

which means, on the positive z axis (i.e., the direction chosen to define $\overline{\phi_0}$), the behaviors

$$B_i\sim g_r\frac{x_i}{4\pi r^3}(\vec{\beta}_r^*\cdot\vec{T}),\quad E_i\sim q_r\frac{x_i}{4\pi r^3}(\vec{\beta}_r^*\cdot\vec{T}),$$
$$D_i\phi\sim-(g_s)_r\frac{x_i}{4\pi r^3}(\vec{\beta}_r^*\cdot\vec{T}). \tag{5.11}$$

For the r-type fundamental dyon, we then have the values $g_r=-4\pi/e$, $q_r=g_r\tan\beta$, and $(g_s)_r=\sqrt{g_r^2+q_r^2}$; the mass of this dyon is equal to $M_r=(g_s)_r f\hat{h}\cdot\vec{\beta}_r^*$.

B. Low-energy effective theory

What sort of low-energy dynamics for fundamental BPS dyons follows from the field equations of the theory? As in the SU(2) case, some of the most direct information on this problem can be obtained by considering the fundamental dyons in the presence of some weak asymptotic uniform fields. Only the asymptotic, gauge, or Higgs field strengths that commute with the Higgs field ϕ may be allowed here [i.e., the uniform Higgs field belonging to the unbroken $U(1)^k$ subgroup only]. We may specify the nature of these applied field strengths by their values on the z axis where the Higgs field originally there is $\overline{\phi_0}=f\hat{h}\cdot\vec{T}$. (This way of specifying the applied field strengths will have a clear physical meaning if one works in a unitary gauge where the Higgs field is everywhere aligned in the direction of $\langle\phi\rangle_v$.) Now the problem is to find the solution to the field equations, describing the motion of the r-type fundamental dyon in a nonzero asymptotic field as specified through the conditions

$$B_i(\mathbf{r},t)\to(\vec{B}_0)_i\cdot\vec{T},\quad E_i(\mathbf{r},t)\to(\vec{E}_0)_i\cdot\vec{T},$$
$$D_i\phi(\mathbf{r},t)\to-(\vec{H}_0)_i\cdot\vec{T} \tag{5.12}$$

along the z axis and as $r\to\infty$. Here note that $(\vec{B}_0)_i\cdot\vec{T}\equiv\Sigma_{r=1}^k(B_0)_i^rT_r$, etc., and the constant vectors $\vec{B}_0$, $\vec{E}_0$, and $\vec{H}_0$ are assumed to be of sufficiently small magnitude.

Remarkably, the desired solution can be given using the corresponding solution of the SU(2) model, which we discussed in Sec. III. This is the generalization of the embedding procedure described in Eq. (5.9). Let $\overline{A_\mu^a}(x;f,\mathbf{B}_0,\mathbf{E}_0,\mathbf{H}_0)$ denote the (in general time dependent) SU(2) BPS dyon solution in the presence of the asymptotic field $(\mathbf{B}_0,\mathbf{E}_0,\mathbf{H}_0)$. Then it may directly be verified that

$$A_\mu(x)=\sum_{i=1}^{3}\overline{A_i^a}(x;f\hat{h}\cdot\vec{\beta}_r,\vec{\mathbf{B}}_0\cdot\vec{\beta}_r,\vec{\mathbf{E}}_0\cdot\vec{\beta}_r,\vec{\mathbf{H}}_0\cdot\vec{\beta}_r)t^a_{(\vec{\beta})}+x^\lambda\{(\vec{G}_0)_{\lambda\mu}-[(\vec{G}_0)_{\lambda\mu}\cdot\vec{\beta}_r^*]\vec{\beta}_r\}\cdot\vec{T}, \tag{5.13}$$

$$\phi_i(x)=\sum_{i=1}^{3}\overline{\phi_i^a}(x;f\hat{h}\cdot\vec{\beta}_r,\vec{\mathbf{B}}_0\cdot\vec{\beta}_r,\vec{\mathbf{E}}_0\cdot\vec{\beta}_r,\vec{\mathbf{H}}_0\cdot\vec{\beta}_r)t^a_{(\vec{\beta})}+f[\hat{h}-(\hat{h}\cdot\vec{\beta}_r^*)\vec{\beta}_r]\cdot\vec{T}-x^i\{(\vec{H}_0)_i-[(\vec{H}_0)_i\cdot\vec{\beta}_r^*]\vec{\beta}_r\}\cdot\vec{T} \tag{5.14}$$

[here $(\vec{G}_0)_{ij}\equiv\epsilon_{ijk}(\vec{B}_0)_k$ and $(\vec{G}_0)^{0i}\equiv(\vec{E}_0)_k$] is a solution describing the r-type dyon in the nonzero asymptotic field as specified by Eq. (5.12). Then, based on our SU(2) solution, we may immediately conclude that the r-type dyon in its instantaneous rest frame should accelerate according to the formula [see Eq. (3.29)]

$$a_i=-\frac{1}{f\hat{h}\cdot\vec{\beta}_r}[\cos\beta(\vec{B}_0)_i\cdot\vec{\beta}_r+\sin\beta(\vec{E}_0)_i\cdot\vec{\beta}_r-(\vec{H}_0)_i\cdot\vec{\beta}_r] \tag{5.15}$$

which may be rewritten, using the charges defined by Eq. (5.11), as

$$M_ra_i=g_r\vec{\beta}^*\cdot(\vec{B}_0)_i+q_r\vec{\beta}^*\cdot(\vec{E}_0)_i+(g_s)_r\vec{\beta}^*\cdot(\vec{H}_0)_i. \tag{5.16}$$

To find the associated long-distance fields (including radiation), recall that, for the SU(2) case, the relevant field strengths have nonvanishing components only in the direction of $\hat{r}^a$ (or the Higgs field) and have the amplitude described by $\mathbf{B}^{\rm em}$, $\mathbf{E}^{\rm em}$, $\mathbf{H}$ and H^0 through Eqs. (3.10)–(3.13) and (3.45). This term implies that, for our solution given by Eqs. (5.13) and (5.14), the corresponding field strengths would have the following large-distance behaviors on the z axis:

[10]In a quantized theory, the electric charge q_r defined by Eq. (5.10) will be required to be an integer multiple of $e\vec{\beta}_j^2$.

$$\mathbf{B}_i(\mathbf{r},t)\sim(\vec{B}_0)_i\cdot\vec{T}+\frac{g_r\vec{\beta}_r^*\cdot\vec{T}}{4\pi}\left\{\frac{(\hat{\mathbf{R}}-\mathbf{v}_{\rm ret})_i}{(1-\hat{\mathbf{R}}\cdot\mathbf{v}_{\rm ret})^3R^2}+\frac{[\hat{\mathbf{R}}\times(\hat{\mathbf{R}}\times\mathbf{a})]_i}{R}\right\}-\frac{q_r\vec{\beta}_r^*\cdot\vec{T}}{4\pi}\frac{(\hat{\mathbf{R}}\times\mathbf{v}_{\rm ret})_i}{R^2}+\frac{(\hat{\mathbf{R}}\times\mathbf{a})_i}{R}, \tag{5.17}$$

$$\mathbf{E}_i(\mathbf{r},t)\sim(\vec{E}_0)_i\cdot\vec{T}+\frac{q_r\vec{\beta}_r^*\cdot\vec{T}}{4\pi}\left\{\frac{(\hat{\mathbf{R}}-\mathbf{v}_{\rm ret})_i}{(1-\hat{\mathbf{R}}\cdot\mathbf{v}_{\rm ret})^3R^2}+\frac{[\hat{\mathbf{R}}\times(\hat{\mathbf{R}}\times\mathbf{a})]_i}{R}\right\}+\frac{g_r\vec{\beta}_r^*\cdot\vec{T}}{4\pi}\left\{\frac{(\hat{\mathbf{R}}\times\mathbf{v}_{\rm ret})_i}{R^2}+\frac{(\hat{\mathbf{R}}\times\mathbf{a})_i}{R}\right\}, \tag{5.18}$$

$$-D_i\phi(\mathbf{r},t)\sim(\vec{H}_0)_i\cdot\vec{T}+\frac{(g_s)_r\vec{\beta}_r^*\cdot\vec{T}}{4\pi}\left\{\frac{(\hat{\mathbf{R}}-\mathbf{v}_{\rm ret})_i}{(1-\hat{\mathbf{R}}\cdot\mathbf{v}_{\rm ret})^3R^2}+\frac{(\hat{\mathbf{R}}\cdot\mathbf{a})\hat{R}_i}{R}\right\}, \tag{5.19}$$

$$-D^0\phi(\mathbf{r},t)\sim\frac{(g_s)_r\vec{\beta}_r^*\cdot\vec{T}}{4\pi}\left\{\frac{\hat{\mathbf{R}}\cdot\mathbf{v}_{\rm ret}}{(1-\hat{\mathbf{R}}\cdot\mathbf{v}_{\rm ret})^3R^2}+\frac{\hat{\mathbf{R}}\cdot\mathbf{a}}{R}\right\}. \tag{5.20}$$

Also considering the Lorentz-boosted solution would change the force law (5.16) into the corresponding covariant form [cf. Eq. (3.32)]

$$\frac{d}{dt}\left(\frac{(M_r-(g_s)_r\vec{\beta}_r^*\cdot X_\mu\vec{H}^\mu)V_i}{\sqrt{1-\mathbf{V}^2}}\right)=g_r\vec{\beta}_r^*\cdot[(\vec{B}_0)_i-\epsilon_{ijk}V_j(\vec{E}_0)_k]+q_r\vec{\beta}_r^*\cdot[(\vec{E}_0)_i+\epsilon_{ijk}V_j(\vec{B}_0)_k]+(g_s)_r\vec{\beta}_r^*\cdot(\vec{H}_0)_i\sqrt{1-\mathbf{V}^2}. \tag{5.21}$$

Without any further analysis, it is clear from the above discussion that the differences from the SU(2) dyon case are mainly in prolification of various charges as we have more massless fields. In detail we are just seeing that, given the r-type fundamental dyon associated with the root $\vec{\beta}_r$, it interacts with k different pairs of massless photon and Higgs field [all in a identical manner to the SU(2) case], with the strength of its coupling with the r'th photon or Higgs field set by the magnetic charge $g^{rr'}=g_r(\vec{\beta}_r^*)_{r'}$, electric charge $q^{rr'}=q_r(\vec{\beta}_r^*)_{r'}$, and dilaton charge $(g_s)^{rr'}=(g_s)_r(\vec{\beta}_r^*)_{r'}$. The massless fields here are precisely the ones one easily identifies by going to the unitary gauge where the Higgs field is everywhere in the direction of $\overline{\phi}_0$; the components lying in the Cartan subalgebra from the gauge and Higgs fields correspond to nonmassive physical excitations. The low-energy effective action may now be written down on the basis of this observation and the corresponding result in the SU(2) case. The effective theory would involve a set of position coordinates $\mathbf{X}_n$ ($n=1,\ldots,N$) for fundamental dyons (the type of which may also be indicated by the index n), U(1) gauge fields $A_\mu^{(r)}(x)$ ($r=1,\ldots,k$) and Higgs fields $\varphi^{(r)}(x)$ ($r=1,\ldots,k$), while the massive vector boson modes are to be integrated out. We then have the action [cf. Eq. (4.13)]

$$S_{\rm eff}=\int d^4x\left\{\frac{1}{4}F^{(r)\mu\nu}F_{(r)\mu\nu}-\frac{1}{2}F^{(r)\mu\nu}(\partial_\mu A_\nu^{(r)}-\partial_\nu A_\mu^{(r)})-\frac{1}{2}\partial_\mu\varphi^{(r)}\partial^\mu\varphi^{(r)}\right\}+\int dt\sum_{n=1}^{N}\times\left\{-\left(M_n+\sum_r(g_s)^{nr}\varphi^{(r)}(\mathbf{X}_n,t)\right)\sqrt{1-\dot{\mathbf{X}}_n^2}-\sum_r q^{nr}[A^{(r)0}(\mathbf{X}_n,t)-\dot{\mathbf{X}}_n\cdot\mathbf{A}^{(r)}(\mathbf{X}_n,t)]-\sum_r g^{nr}[C^{(r)0}(\mathbf{X}_n,t)-\dot{\mathbf{X}}_n\cdot\mathbf{C}^{(r)}(\mathbf{X}_n,t)]\right\} \tag{5.22}$$

with $C^{(r)\mu}$, as functions of $F^{(r)\mu\nu}$, defined in the same way as Eq. (4.7). [In Eq. (5.22), the index n in q^{nr}, g^{nr}, and $(g_s)^{nr}$ is actually r' if the nth dyon in question is of the r' type, viz., $q^{nr}=q_{r'}(\vec{\beta}_{r'}^*)_r$, $g^{nr}=g_{r'}(\vec{\beta}_{r'}^*)_r$, etc.]

The action (5.22) captures low-energy dynamics of any number of fundamental BPS dyons (corresponding to various type) and massless fields in the system. This includes scattering physics involving dyons and on-shell photons or Higgs particles. Also, for a slowly moving system of BPS dyons, one may ignore radiation effects and go on to eliminate all massless fields from this action by using the near-zone solutions to the respective field equations. This procedure, which parallels verbatim our discussion in the SU(2) case, leads to the effective particle Lagrangian, which has the same structure as the SU(2)-case Lagrangian (4.26). Changes appear just in the interaction strengths, i.e., the second, third, and fourth terms on the right-hand side of Eq. (4.26) now come with the strengths

$$\sum_{r=1}^{k}(g_s)^{nr}(g_s)^{mr}=(g_s)_n(g_s)_m\vec{\beta}_n^*\cdot\vec{\beta}_m^*,$$

$$\sum_{r=1}^{k}(q^{nr}q^{mr}+g^{nr}g^{mr})=(q^nq^m+g^ng^m)\vec{\beta}_n^*\cdot\vec{\beta}_m^*, \tag{5.23}$$

$$\sum_{r=1}^{k}(q^{nr}g^{mr}-g^{nr}q^{mr})=(q^ng^m-g^nq^m)\vec{\beta}_n^*\cdot\vec{\beta}_m^*$$

instead of having the values $(g_s)_n(g_s)_m$, $(q_nq_m+g_ng_m)$, and $(q_ng_m-g_nq_m)$. Similarly, when terms beyond $O(\dot{X}^2)$ are ignored, the Lagrangian (4.27) is valid for the present case also only if we insert the multiplicative factor $\vec{\beta}_n^*\cdot\vec{\beta}_m^*$ inside the summation symbol of every term on the right-hand side of Eq. (4.27) except for the first two purely kinematical ones. If one sets $g_n = g = -4\pi/e$ and further makes the expansion $(g_s)_n = |g| + q_n^2/2|g|$ with this quadratic particle Lagrangian, one obtains the slow-motion Lagrangian of Lee, Weinberg, and Yi [12], which is quadratic not only in velocities but also in electric charges. Then, as was shown in Ref. [12], a simple Legendre transform may be performed to change the latter into the Lagrangian appropriate to geodesic motion in the corresponding multiple-monopole moduli space. [11]

VI. DISCUSSION

In this paper an effective field theory approach for BPS dyons and massless fields has been developed, starting from the analysis of nonlinear field equations of the Yang-Mills-Higgs system. Our approach, while being consistent with the moduli-space dynamics of Manton, can describe the low-energy interaction of oppositely charged BPS dyon and also the process involving radiation of various massless quanta explicitly. Our discussion was entirely at the classical level, but, for an appropriately supersymmetrized system, our effective theory might be generalized to have a quantum significance. The electromagnetic duality and (spontaneously broken) scale invariance, which are manifest in our approach, may play a useful role in such an endeavor. It would also be desirable to make some contact with the results of Seiberg and Witten [2].

There are some interesting related problems that require further study. We mention a few of them.

(i) Our effective action is correct when all monopoles are separated a large distance compared to the core size. If two identical monopoles overlap, the individual coordinates are not meaningful anymore. We can describe the low-energy dynamics by the geodesic motion on the Atiyah-Hitchin moduli space. However, radiation, however weak it may be, should come out from this motion in the moduli space, including the exchange of the relative charge between two identical monopoles. Our point particle approximation does not capture this physics. It would be interesting to couple the full moduli space dynamics to the weak radiation.

(ii) The present effective field theory approach should be generalized to the case of full, $N=2$ or $N=4$, super–Yang-Mills system. In particular, the spin effect including the electric and magnetic dipole moments would appear. See Ref. [27] for the corresponding moduli-space description.

(iii) For larger gauge groups, we have only considered the cases where the given simple gauge group is maximally broken. If a non-Abelian subgroup remains unbroken, there are fundamental monopoles carrying non-Abelian magnetic charges and their low-energy dynamics would be richer. (For a recent investigation on this subject, see Ref. [28].) An extension of our analysis in this direction would be most desirable; for instance, one might consider here following the behavior of the effective theory as one varies the asymptotic Higgs field from a value giving a purely Abelian symmetry breaking to one that leaves a non-Abelian subgroup unbroken. Finally, we should mention the recent work by one of the authors and Min [29] where some interesting observation was made regarding to the radiation reaction and the finite-size effect in the dynamics of the BPS monopole and the duality of these effects against those of the W particles.

[11] It has been shown recently [12,26] that the moduli space metric obtained by this procedure for distinct fundamental monopoles is in fact the exact metric over the whole moduli space, i.e., for all values of intermediate distances. This may imply that our effective action is correct even when two distinct monopoles overlap each other.

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APPENDIX A: EFFECTIVE LAGRANGIAN FOR A SYSTEM OF *W* PARTICLES

From the low-energy effective action (1.2) we can derive the effective Lagrangian for a system of slowly moving W particles, given in Eq. (1.9), in the following way. The field equations for the massless fields $A^\mu(x)$ and $\varphi(x)$ read

$$\partial_\nu(\partial^\mu A^\nu - \partial^\nu A^\mu) = J^\mu(x)$$

$$J_0(x) = \sum_n q_n \delta^3(\mathbf{x} - \mathbf{X}_n(t)),$$

$$\mathbf{J}(x) = \sum_n q_n \dot{\mathbf{X}}_n(t)\delta^3(\mathbf{x} - \mathbf{X}_n(t)), \tag{A1}$$

$$\partial_\nu \partial^\nu \varphi = \sum_n g_s \sqrt{1 - \dot{\mathbf{X}}_n^2(t)}\,\delta^3(\mathbf{x} - \mathbf{X}_n(t)) \equiv J_s(x). \tag{A2}$$

Assuming slowly varying sources, we may then express the fields $A^\mu(x), \varphi(x)$ by their usual retarded solutions considered in the near-zone approximation. This gives the electromagnetic potential

$$A^\mu(\mathbf{x},t) = \frac{1}{4\pi}\int \frac{J^\mu(\mathbf{x}', t - |\mathbf{x} - \mathbf{x}'|)}{|\mathbf{x} - \mathbf{x}'|} d^3\mathbf{x}'$$

$$= \frac{1}{4\pi}\int \frac{J^\mu(\mathbf{x}',t)}{|\mathbf{x} - \mathbf{x}'|} d^3\mathbf{x}' - \frac{1}{4\pi}\frac{\partial}{\partial t}\left[\int J^\mu(\mathbf{x}',t)d^3\mathbf{x}'\right]$$

$$+ \frac{1}{8\pi}\frac{\partial^2}{\partial t^2}\left[\int |\mathbf{x} - \mathbf{x}'| J^\mu(\mathbf{x}',t) d^3\mathbf{x}'\right] + \cdots \tag{A3}$$

and so for the point sources

$$A^0(\mathbf{x},t)=\frac{1}{4\pi}\sum_n \frac{q_n}{|\mathbf{x}-\mathbf{X}_n(t)|}+\frac{1}{8\pi}\frac{\partial^2}{\partial t^2}\left(\sum_n q_n|\mathbf{x}-\mathbf{X}_n(t)|\right)+\cdots, \tag{A4}$$

$$\mathbf{A}(\mathbf{x},t)=\frac{1}{4\pi}\sum_n \frac{q_n\dot{\mathbf{X}}_n}{|\mathbf{x}-\mathbf{X}_n(t)|}+\cdots . \tag{A5}$$

Similarly, for the Higgs field, we have

$$\varphi(\mathbf{x},t)=-\frac{1}{4\pi}\int \frac{J_s(\mathbf{x}',t-|\mathbf{x}-\mathbf{x}'|)}{|\mathbf{x}-\mathbf{x}'|}d^3\mathbf{x}'$$

$$=-\frac{1}{4\pi}\sum_n \frac{g_s\sqrt{1-\dot{\mathbf{X}}_n^2}}{|\mathbf{x}-\mathbf{X}_n(t)|}+\frac{1}{4\pi}\frac{\partial}{\partial t}\left(\sum_n g_s\sqrt{1-\dot{\mathbf{X}}_n^2}\right)$$

$$-\frac{1}{8\pi}\frac{\partial^2}{\partial t^2}\left(\sum_n g_s\sqrt{1-\dot{\mathbf{X}}_n^2}|\mathbf{x}-\mathbf{X}_n(t)|\right)+\cdots . \tag{A6}$$

These expressions may also be obtained by considering the small-velocity expansion of the known Liénard-Wiechert–type potentials.

The desired effective Lagrangian for slowly moving W particles is obtained if we eliminate (or integrate out) the massless fields $A^\mu(x)$ and $\varphi(x)$ from the action (1.2) by using the above (approximate) solutions to the field equations[12]. Here note that, because of Eqs. (A1) and (A2), the contribution from the massless field action in Eq. (1.2) can be written in the same form as the interaction terms appearing in the matter action $\int dt\, L_{\rm eff}$. So, to our approximation, the result of using Eqs. (A4)–(A6) in the action (with irrelevant self-interactions dropped) is

$$\int dt\, L=\int dt\Bigg\{-\sum_n m_v\sqrt{1-\dot{\mathbf{X}}_n^2}+\frac{g_s^2}{8\pi}\sum_{n,m\,(\neq n)}\left(\frac{\sqrt{1-\dot{\mathbf{X}}_n^2}\sqrt{1-\dot{\mathbf{X}}_m^2}}{|\mathbf{X}_n-\mathbf{X}_m|}+\frac{1}{2}\left[\frac{\partial^2}{\partial t^2}|\mathbf{x}-\mathbf{X}_m(t)|\right]_{\mathbf{x}=\mathbf{X}_n}\right)$$
$$-\frac{1}{8\pi}\sum_{n,m\,(\neq n)} q_nq_m\left(\frac{1}{|\mathbf{X}_n-\mathbf{X}_m|}+\frac{1}{2}\left[\frac{\partial^2}{\partial t^2}|\mathbf{x}-\mathbf{X}_m(t)|\right]_{\mathbf{x}=\mathbf{X}_n}-\frac{\dot{\mathbf{X}}_n\cdot\dot{\mathbf{X}}_m}{|\mathbf{X}_n-\mathbf{X}_m|}\right)\Bigg\}; \tag{A7}$$

Here notice that

$$\left[\frac{\partial^2}{\partial t^2}|\mathbf{x}-\mathbf{X}_m|\right]_{\mathbf{x}=\mathbf{X}_n}=\left[\frac{\partial}{\partial t}\frac{[\mathbf{x}-\mathbf{X}_m(t)]\cdot\dot{\mathbf{X}}_m(t)}{|\mathbf{x}-\mathbf{X}_m|}\right]_{\mathbf{x}=\mathbf{X}_n}=\frac{\dot{\mathbf{X}}_n\cdot\dot{\mathbf{X}}_m}{|\mathbf{X}_n-\mathbf{X}_m|}-\frac{(\mathbf{X}_n-\mathbf{X}_m)\cdot\dot{\mathbf{X}}_n(\mathbf{X}_n-\mathbf{X}_m)\cdot\dot{\mathbf{X}}_m}{|\mathbf{X}_n-\mathbf{X}_m|^3}-\frac{d}{dt}\left[\frac{(\mathbf{X}_n-\mathbf{X}_m)\cdot\dot{\mathbf{X}}_m}{|\mathbf{X}_n-\mathbf{X}_m|}\right] \tag{A8}$$

and so if we ignore terms beyond $O(\dot{\mathbf{X}}^2)$ and also total time derivative terms from L, we obtain the Lagrangian of the form

$$L=\frac{1}{2}\sum_n m_v\dot{\mathbf{X}}_n^2-\frac{g_s^2}{16\pi}\sum_{n,m\,(\neq n)}\frac{|\dot{\mathbf{X}}_n-\dot{\mathbf{X}}_m|^2}{|\mathbf{X}_n-\mathbf{X}_m|}-\frac{1}{16\pi}\sum_{n,m\,(\neq n)}(g_s^2-q_nq_m)\left\{\frac{\dot{\mathbf{X}}_n\cdot\dot{\mathbf{X}}_m}{|\mathbf{X}_n-\mathbf{X}_m|}+\frac{(\mathbf{X}_n-\mathbf{X}_m)\cdot\dot{\mathbf{X}}_n(\mathbf{X}_n-\mathbf{X}_m)\cdot\dot{\mathbf{X}}_m}{|\mathbf{X}_n-\mathbf{X}_m|^3}\right\}+\frac{1}{8\pi}\sum_{n,m\,(\neq n)}\frac{g_s^2-q_nq_m}{|\mathbf{X}_n-\mathbf{X}_m|}. \tag{A9}$$

As one can easily verify, this can readily be rewritten in the form in Eqs. (1.9) and (1.10).

APPENDIX B: DERIVATION OF THE FORCE LAW IN LORENTZ BOOSTED FRAME

The system in Eq. (2.1) is invariant against the Lorentz (boost) transformation

$$t\rightarrow t^*=\frac{t+\mathbf{u}\cdot\mathbf{r}}{\sqrt{1-\mathbf{u}^2}},$$

$$\mathbf{r}\rightarrow\mathbf{r}^*=\mathbf{r}-(\hat{\mathbf{u}}\cdot\mathbf{r})\hat{\mathbf{u}}+\frac{1}{\sqrt{1-\mathbf{u}^2}}[(\hat{\mathbf{u}}\cdot\mathbf{r})\hat{\mathbf{u}}+\mathbf{u}t], \tag{B1}$$

under which (A_μ,ϕ) transform as

$$A_\mu(x)\rightarrow A_\mu^*(x^*)=\frac{dx^\nu}{dx^{*\mu}}A_\nu(x),$$

$$\phi(x)\rightarrow\phi^*(x^*)=\phi(x). \tag{B2}$$

This of course implies that the fields $(A_\mu^*(\mathbf{r},t),\phi^*(\mathbf{r},t))$ obtained by the Lorentz boost of an initially given solution $(A_\mu(\mathbf{r},t),\phi(\mathbf{r},t))$ should also satisfy the field equations. Here we use this simple observation in order to show that the moving dyon seen in a different inertial frame obeys the covariant equation of motion.

Let $(A_\mu(\mathbf{r},t),\phi(\mathbf{r},t))$ be a dyon solution of the field equations (2.4) and (2.5), subject to the constant asymptotic fields $(\mathbf{B},\mathbf{E},\mathbf{H})$ with zero initial (center) velocity. The trajectory of the dyon will be governed by the equation of motion

[12]This is equivalent to the more traditional approach described, for instance, in the textbook by Landau and Lifshitz [30].

$$M\frac{d^2}{dt^2}\mathbf{X}=g\mathbf{B}+q\mathbf{E}+g_s\mathbf{H}, \tag{B3}$$

as was shown in Eq. (3.31). In this reference frame the asymptotic value of $H^0[\equiv -(\phi^a/|\phi|)(D^0\phi)^a]$ may be taken to be $O(a^2)$ at most. Then a new solution $(A^*_\mu(\mathbf{r},t),\phi^*(\mathbf{r},t))$ generated by the Lorentz boost in Eq. (B1) is associated with the asymptotic fields $(\mathbf{B}^*,\mathbf{E}^*)$ specified by

$$\mathbf{E}=(\hat{\mathbf{u}}\cdot\mathbf{E}^*)\hat{\mathbf{u}}+\frac{1}{\sqrt{1-\mathbf{u}^2}}[\mathbf{E}^*-(\hat{\mathbf{u}}\cdot\mathbf{E}^*)\hat{\mathbf{u}}+\mathbf{u}\times\mathbf{B}], \tag{B4}$$

$$\mathbf{B}=(\hat{\mathbf{u}}\cdot\mathbf{B}^*)\hat{\mathbf{u}}+\frac{1}{\sqrt{1-\mathbf{u}^2}}[\mathbf{B}^*-(\hat{\mathbf{u}}\cdot\mathbf{B}^*)\hat{\mathbf{u}}-\mathbf{u}\times\mathbf{E}^*]$$

and $(H^{*0},\mathbf{H}^*)$ by

$$\mathbf{H}=\frac{1}{\sqrt{1-\mathbf{u}^2}}[(\hat{\mathbf{u}}\cdot\mathbf{H}^*)\hat{\mathbf{u}}-\mathbf{u}H^{*0}]+\mathbf{H}^*-(\hat{\mathbf{u}}\cdot\mathbf{H}^*)\hat{\mathbf{u}}, \tag{B5}$$

$$H^0=\frac{1}{\sqrt{1-\mathbf{u}^2}}[H^{*0}-\mathbf{u}\cdot\mathbf{H}^*].$$

Let $X^\mu\equiv(t,\mathbf{X}(t))$ denotes the dyon trajectory seen in the original frame and $X^{*\mu}\equiv(t^*,\mathbf{X}^*(t^*))$ the trajectory in the boosted frame. Then they should be related by [cf. Eq. (B1)]

$$t=\frac{t^*-\mathbf{u}\cdot\mathbf{X}^*}{\sqrt{1-\mathbf{u}^2}},$$

$$\mathbf{X}=\mathbf{X}^*-(\hat{\mathbf{u}}\cdot\mathbf{X}^*)\hat{\mathbf{u}}+\frac{1}{\sqrt{1-\mathbf{u}^2}}[(\hat{\mathbf{u}}\cdot\mathbf{X}^*)\hat{\mathbf{u}}-\mathbf{u}t^*]. \tag{B6}$$

We may now reexpress each side of Eq. (B3) using the variables in the boosted frame. The left-hand side is rewritten, to $O(a)$, as

$$\begin{aligned}M\frac{d^2}{dt^2}\mathbf{X}(t)&=M\frac{dt^*}{dt}\frac{d}{dt^*}\left(\frac{dt^*}{dt}\frac{d}{dt^*}\mathbf{X}(t)\right)\\&=M\left(\frac{\left(\mathbf{u}\cdot\frac{d\mathbf{V}^*}{dt^*}\right)\mathbf{u}}{(1-\mathbf{u}^2)^{3/2}}+\frac{\left(\hat{\mathbf{u}}\cdot\frac{d\mathbf{V}^*}{dt^*}\right)\hat{\mathbf{u}}}{(1-\mathbf{u}^2)^{1/2}}\right.\\&\qquad\left.+\frac{\frac{d\mathbf{V}^*}{dt^*}-\left(\hat{\mathbf{u}}\cdot\frac{d\mathbf{V}^*}{dt^*}\right)\hat{\mathbf{u}}}{(1-\mathbf{u}^2)}\right),\end{aligned} \tag{B7}$$

where $\mathbf{V}^*=(d/dt^*)\,\mathbf{X}^*$. On the other hand, inserting Eqs. (B4) and (B5) into Eq. (B3), we find that the right-hand side can be expressed as

$$\begin{aligned}g\mathbf{B}+q\mathbf{E}+g_s\mathbf{H}&=g\{(\hat{\mathbf{u}}\cdot\mathbf{B}^*)\hat{\mathbf{u}}\\&\quad+1\sqrt{1-\mathbf{u}^2}[\mathbf{B}^*-(\hat{\mathbf{u}}\cdot\mathbf{B}^*)\hat{\mathbf{u}}-\mathbf{u}\times\mathbf{E}^*]\}\\&\quad+q\left\{(\hat{\mathbf{u}}\cdot\mathbf{E}^*)\hat{\mathbf{u}}+\frac{1}{\sqrt{1-\mathbf{u}^2}}[\mathbf{E}^*-(\hat{\mathbf{u}}\cdot\mathbf{E}^*)\hat{\mathbf{u}}\right.\\&\quad\left.+\mathbf{u}\times\mathbf{B}]\right\}+g_s\left\{\frac{1}{\sqrt{1-\mathbf{u}^2}}[(\hat{\mathbf{u}}\cdot\mathbf{H}^*)\hat{\mathbf{u}}\right.\\&\quad\left.-\mathbf{u}H^{*0}]+\mathbf{H}^*-(\hat{\mathbf{u}}\cdot\mathbf{H}^*)\hat{\mathbf{u}}\right\}.\end{aligned} \tag{B8}$$

The equation of motion (B3) implies that the last line of Eq. (B7) should be equal to the right-hand side of Eq. (B8). Since this is a vector equality, the components parallel to $\mathbf{u}$ on each side should agree and so should the components perpendicular to $\mathbf{u}$ on each side. We multiply each perpendicular component by the factor $\sqrt{1-\mathbf{u}^2}$ and then add the resulting perpendicular parts on each side to the parallel parts on the corresponding side. These operations lead to the relation

$$\begin{aligned}&M\left(\frac{\left(\mathbf{u}\cdot\frac{d\mathbf{V}^*}{dt^*}\right)\mathbf{u}}{(1-\mathbf{u}^2)^{3/2}}+\frac{1}{(1-\mathbf{u}^2)^{1/2}}\frac{d\mathbf{V}^*}{dt^*}\right)\\&\quad=g(\mathbf{B}^*-\mathbf{u}\times\mathbf{E}^*)+q(\mathbf{E}^*+\mathbf{u}\times\mathbf{B})\\&\qquad+g_s\sqrt{1-\mathbf{u}^2}\mathbf{H}^*-\check{\mathbf{F}},\end{aligned} \tag{B9}$$

where $\check{\mathbf{F}}$ is given by

$$\begin{aligned}\check{\mathbf{F}}&=g_s\frac{\mathbf{u}}{\sqrt{1-\mathbf{u}^2}}(\mathbf{u}\cdot\mathbf{H}^*-H^*_0),\\&=g_s\frac{\mathbf{u}}{\sqrt{1-\mathbf{u}^2}}\frac{d}{dt^*}(\mathbf{X}^*\cdot\mathbf{H}^*-t^*H^{*0}).\end{aligned} \tag{B10}$$

Ignoring $O(a^2)$ terms, we may replace $\mathbf{u}$ in Eq. (B9) by $\mathbf{V}^*$ since $\mathbf{V}^*$ is $\mathbf{u}+O(a)$. Thus it is now straightforward to find the desired covariant equation

$$\begin{aligned}\frac{d}{dt^*}\left(\frac{[M-g_sX^*_\mu H^{*\mu}]\mathbf{V}^*}{\sqrt{1-\mathbf{V}^{*2}}}\right)&=g(\mathbf{B}^*-\mathbf{V}^*\times\mathbf{E}^*)+q(\mathbf{E}^*+\mathbf{V}^*\\&\quad\times\mathbf{B})+g_s\mathbf{H}^*\sqrt{1-\mathbf{V}^{*2}}.\end{aligned} \tag{B11}$$

A few comments are in order. First we assumed the acceleration, $d\mathbf{V}/dt$ or $d\mathbf{V}^*/dt^*$, to be small as before and so the above covariant equation of the dyon motion is of course valid to first order in the acceleration. The term $\check{\mathbf{F}}$ in Eq. (B10) is of second order in the acceleration, but it has been included in the above covariant equation. The reason comes from the following observation. Let us consider the case where the Higgs field has the constant asymptotic value f' ($\neq f$). If we carried out the same analyses to find the dyon motion with this choice, the mass parameter that enters into the dyon equation of motion is $g_s f'$ instead of $g_s f$. Hence the change in the asymptotic value of the Higgs field should be reflected in the mass appearing in the dyon equation of motion [cf. Eq. (1.5)]. This reasoning can be properly taken into account if we add the second-order contribution.

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