Self-dual Maxwell Chern-Simons solitons in 1+1 dimensions

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We study the domain wall soliton solutions in the relativistic self-dual Maxwell Chern-Simons model in $1+1$ dimensions obtained by the dimensional reduction of the $2+1$ model. Both topological and nontopological self-dual solutions are found in this case. In the manner of BPS dyons here the Bogomol'nyi bound on the energy is expressed in terms of two conserved quantities. We discuss the underlying supersymmetry. The nonrelativistic limit of this model is also considered and static, nonrelativistic self-dual soliton solutions are obtained. [S0556-2821(98)03408-0]

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I. INTRODUCTION

The $(2+1)$-dimensional Maxwell Chern-Simons (MCS) system has already been studied, and the existence of charged vortices of finite energy has been shown [1]. Self-dual topological and nontopological soliton solutions can be obtained in this case provided one also adds a neutral scalar field to the theory [2]. The nonrelativistic limit of this model was also considered and self-dual soliton solutions have been obtained [3].

Some time ago the $(1+1)$-dimensional nonlinear sigma model [4] was obtained by dimensional reduction of certain $(2+1)$-dimensional nonlinear sigma models and soliton solutions in the $(1+1)$-dimensional case were shown to be similar to the Bogomol'nyi-Prasad-Sommerfield (BPS) dyons of $(3+1)$-dimensional Yang-Mills Higgs theory. This work was extended further by the inclusion of the Chern-Simons term [5] and self-dual soliton solutions were again obtained. Recently the $(1+1)$-dimensional reduction of the Abelian Higgs model with a pure Chern-Simons term was considered and explicit topological and nontopological domain wall solutions were obtained.

The purpose of this paper is to consider the dimensional reduction of the Abelian Higgs model with a Chern-Simons (as well as Maxwell) term (and a neutral scalar field). We show the existence of self-dual topological as well as nontopological domain wall solutions in this dimensionally reduced theory. The nonrelativistic limit of this model is also considered and soliton solutions are again obtained. In Sec. II we obtain the model by dimensional reduction of the $2+1$ MCS model. Here we study the invariance of the Lagrangian and the corresponding two conserved charges. In Sec. III we obtain the BPS-type bound [6] on the energy and show that the bound is saturated when the self-dual equations are satisfied by the fields. In Sec. IV we study the self-dual equations and obtain both topological and nontopological domain wall solutions for the system. In Sec. V we consider the underlying $N=2$ supersymmetry. In Sec. VI we consider the nonrelativistic limit of the model and again obtain self-dual soliton solutions and study their properties. Finally in Sec. VII we conclude the results.

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II. MODEL

The Lagrangian for the Maxwell-Chern Simons system is given by [2,3]

$$\mathcal{L}_{2+1}^{MCS}=-\frac{1}{4e^2}F_{\rho\nu}F^{\rho\nu}+\frac{\mu}{2e^2}\epsilon^{\eta\nu\rho}A_\eta\partial_\nu A_\rho+(D_\rho\phi)^*(D^\rho\phi)+\frac{1}{2e^2}\partial_\rho N\partial^\rho N-\frac{1}{c^2}|\phi|^2\left(N-\frac{e^2v^2}{\mu c}\right)^2-\frac{e^2}{2c^2}\left(|\phi|^2-\frac{\mu c}{e^2}N\right)^2. \quad (1)$$

Here N is a real scalar field, ϕ is complex scalar field, c is the velocity of light, and A_μs are gauge fields. This model has two degenerate vacua. The symmetric phase, having a vacuum expection value $\langle\phi\rangle=0$, $\langle N\rangle=0$ and the asymmetric phase having $\langle\phi\rangle=v$, $\langle N\rangle=e^2v^2/\mu c$. In the symmetric phase the complex scalar field ϕ has mass $e^2v^2/\mu c^2$. The neutral scalar field N and the gauge fields have mass μ. In the asymmetric phase there are two massive gauge degrees of freedom with masses given by [7]

$$m_\pm^2=\frac{2e^2v^2}{c^2}+\frac{\mu^2}{2}\pm\frac{\mu}{2}\sqrt{\mu^2+\frac{8e^2v^2}{c^2}}. \quad (2)$$

The scalar fields also combine into two massive modes with masses $m_\pm$. This model (in $2+1$ dimensions) possesses a Bogomol'nyi-type bound [8] and has static self-dual soliton solutions.

After compactification of the y direction we get the following Lagrangian in $1+1$ dimensions:

$$\mathcal{L}_{1+1}^{MCS}=\frac{1}{2e^2}F_{01}^2+\frac{\mu}{e^2}R(x)F_{01}(x)+(D_\rho\phi)^*(D^\rho\phi)+\frac{1}{2e^2}\partial_\rho N\partial^\rho N+\frac{1}{2e^2}\partial_\rho R\partial^\rho R-U(R,\phi,N), \quad (3)$$

where

$$D_\rho=\partial_\rho-iA_\rho,$$

$$U(R,\phi,N)=R^2(x)|\phi|^2+|\phi|^2\left(N-\frac{e^2}{\mu}v^2\right)^2+\frac{e^2}{2}\left(|\phi|^2-\frac{\mu}{e^2}N\right)^2. \quad (4)$$

Here for simplicity we put $c=1$. We identify the y-independent component of $A_y(t,x,y)$ as $R(x)$. The symmetric phase is again given by $\langle\phi\rangle=0$, $\langle N\rangle=0$, but now $\langle R\rangle=R_0$, where R_0 is arbitrary. In this case the gauge fields as well as the R field are massless while the N field has mass μ and the Higgs field has mass $\sqrt{R_0^2+e^4v^4/\mu^2}$. In the broken phase $\langle N\rangle=e^2v^2/\mu$, $\langle\phi\rangle=v$, $\langle R\rangle=0$. In this case the gauge fields and the scalar field R become massive, having masses equal to $\sqrt{2e^2v^2}$, while the scalar field N and Higgs field combine to give two massive modes with masses $m_\pm$ given by Eq. (2) with $c=1$.

We express the Lagrangian in terms of dimensionless fields $n(\tilde{x}),r(\tilde{x}),f_{01}(\tilde{x})$, and $\varphi(\tilde{x})$ where

$$N(x)=\frac{e^2v^2}{\mu}n(\tilde{x}),\quad R(x)=\frac{e^2v^2}{\mu}r(\tilde{x}),$$

$$\phi(x)=v\varphi(\tilde{x}),\quad A_\rho(x)=\frac{e^2v^2}{\mu}a_\rho(\tilde{x}), \quad (5)$$

$$x^\mu=\frac{\mu}{e^2v^2}\tilde{x}^\mu. \quad (6)$$

From now on we do the calculations with the new field variables which are functions of $\tilde{x}^\mu$. We shall omit the tilde from x and now we will take x for $\tilde{x}$ unless otherwise specified. Then the dimensionless Lagrangian becomes

$$\frac{\mu^2}{e^4v^6}\mathcal{L}_{1+1}^{MCS}=L=\frac{1}{2k}f_{01}^2(x)+r(x)f_{01}(x)+(d_\rho\varphi)^*(d^\rho\varphi)+\frac{1}{2k}\partial_\rho n\partial^\rho n+\frac{1}{2k}\partial_\rho r\partial^\rho r-V(f,n,r), \quad (7)$$

where

$$V(f,n,r)=f(x)r^2(x)+f(x)[n(x)-1]^2+\frac{k}{2}[f(x)-n(x)]^2,$$

$$f(x)=|\varphi|^2,\quad k=\frac{\mu^2}{e^2v^2},\quad d_\rho=\partial_\rho-ia_\rho(x). \quad (8)$$

In the limit $k\to\infty$ the Lagrangian reduces to [9]

$$L=r(x)f_{01}(x)+(d_\rho\varphi)^*(d^\rho\varphi)-f(x)r^2(x)-f(x)[f(x)-1]^2, \quad (9)$$

which is the pure Chern-Simons Higgs Lagrangian. This system admits known topological and nontopological soliton solutions. The theory is invariant under $U(1)$ gauge transformations and the corresponding current is

$$j^\rho=i\{(d^\rho\varphi)^*\varphi-\varphi^*d^\rho\varphi\}. \quad (10)$$

The theory has another invariance. In particular there exists another set of transformations of fields,

$$\delta\varphi(x)=i\varphi\left(r(x)+\frac{1}{k}f_{01}(x)\right),\quad \delta a_1(x)=j^0+\frac{(n^2+r^2)'}{2\left(r(x)+\frac{1}{k}f_{01}(x)\right)},$$

$$\delta a_0(x)=j^1+\frac{C_0}{-r(x)+\frac{f_{01}(x)}{k}},\quad \delta n(x)=0,\quad \delta r(x)=0, \quad (11)$$

where a prime denotes a derivative with respect to space coordinates. Here C_0 is some arbitrary constant. The above transformations leave the theory invariant only after using the equations of motion. The corresponding charge is

$$\left(\frac{\mu}{e^2v^4}\right)Y=\int dx\left\{(d^0\varphi)^*\delta\varphi+\delta\varphi^*d^0\varphi+\left(r(x)+\frac{f_{01}(x)}{k}\right)\delta a_1(x)\right\}. \quad (12)$$

III. EQUATION OF MOTION AND SELF DUALITY

Now varying the Lagrangian (7) with respect to the fields we get the equations of motion

$$\frac{1}{k}f_{01}'(x)+r'(x)-i[(d_0\varphi)^*\varphi-\varphi^*d^0\varphi]=0, \quad (13)$$

$$\frac{1}{k}\partial_\rho\partial^\rho r+2r(x)f(x)-f_{01}(x)=0, \quad (14)$$

$$-\frac{1}{k^2}\partial_\rho\partial^\rho n+[f(x)-n(x)]-\frac{2}{k}f(x)[n(x)-1]=0, \quad (15)$$

$$d_\rho d^\rho\varphi+\{r^2(x)+[n(x)-1]^2+k[f(x)-n(x)]\}\varphi=0, \quad (16)$$

where Eq. (13) is the Gauss law equation. Using the Gauss law equation (and putting appropriate dimensions), Q and Y as given by Eqs. (10) and (12) take the form

$$Q=ev^2\int_{-\infty}^{+\infty}j_0(x)dx, \tag{17}$$

$$Y=\left(\frac{e^2v^4}{\mu}\right)\int_{-\infty}^{+\infty}dx\frac{1}{2}\{n^2(x)+r^2(x)\}'. \tag{18}$$

The dimensionless energy density $\mathcal{E}(x)$ is given by

$$\frac{\mu^2}{v^6e^4}\tilde{\mathcal{E}}(x)=\mathcal{E}(x)=\frac{1}{2k}[f_{01}^2(x)+r'^2(x)+n'^2(x)+\dot{r}^2(x)+\dot{n}^2(x)]+|d_0\varphi|^2+|d_1\varphi|^2+V(f,n,r). \tag{19}$$

Here an overdot denotes a derivative with respect to time. We study the system in the static ansatz where we have $\dot{n}(x)=0=\dot{r}(x)$. Integrating by parts and using the Gauss law equation the energy density can be expressed as

$$\mathcal{E}(x)=|d_0\varphi+i(n(x)-1)\varphi(x)\cos\alpha-ir(x)\varphi(x)\sin\alpha|^2+|d_1\varphi+[n(x)-1]\varphi(x)\sin\alpha+r(x)\varphi(x)\cos\alpha|^2+\frac{1}{2k}\{r'(x)+k[f(x)-n(x)]\cos\alpha-f_{01}(x)\sin\alpha\}^2+\frac{1}{2k}\{n'(x)+k[f(x)-n(x)]\sin\alpha+f_{01}\cos\alpha\}^2+\frac{1}{2}[n^2(x)+r^2(x)]'\sin\alpha+\{r(x)[1-f(x)]\}'\cos\alpha+\left(\frac{1}{k}f_{01}(x)\{r(x)\sin\alpha+[1-n(x)]\cos\alpha\}+f(x)[1-n(x)]\sin\alpha\right)'. \tag{20}$$

Here α is an arbitrary angle variable. The last term in the above expression vanishes after integration as $f_{01}(x)\to 0$ asymptotically. Then we can write the following inequality:

$$\mathcal{E}(x)\geqslant\frac{1}{2}[n^2(x)+r^2(x)]'\sin\alpha+\{r(x)[1-f(x)]\}'\cos\alpha. \tag{21}$$

This is a Bogomol'nyi-type bound. The lower bound on energy is saturated when the following self-dual equations hold:

$$d_0\varphi+i[n(x)-1]\varphi(x)\cos\alpha-ir(x)\varphi(x)\sin\alpha=0,$$

$$d_1\varphi+[n(x)-1]\varphi(x)\sin\alpha+r(x)\varphi(x)\cos\alpha=0,$$

$$\frac{1}{k}[r'(x)-f_{01}(x)\sin\alpha]+[f(x)-n(x)]\cos\alpha=0,$$

$$\frac{1}{k}[n'(x)+f_{01}(x)\cos\alpha]+[f(x)-n(x)]\sin\alpha=0. \tag{22}$$

These along with the Gauss law, Eq. (13), are consistent with the second order static equations of motion as given by Eqs. (14)–(16). We can rewrite the above self-dual equations as

$$\frac{1}{k}f_{01}'(x)+r'(x)+2f(x)\{[n(x)-1]\cos\alpha-r(x)\sin\alpha\}=0,$$

$$f'(x)+2f(x)\{[n(x)-1]\sin\alpha+r(x)\cos\alpha\}=0,$$

$$\frac{1}{k}[r'(x)-f_{01}(x)\sin\alpha]+[f(x)-n(x)]\cos\alpha=0,$$

$$\frac{1}{k}[n'(x)+f_{01}(x)\cos\alpha]+[f(x)-n(x)]\sin\alpha=0. \tag{23}$$

Eliminating the fields r, f_{01}, and n from the above equations we find the fourth order equation

$$-\frac{1}{k^2}(\ln f)''''+\left(1+\frac{2f(x)}{k}\right)(\ln f)''+\frac{2}{k}f''(x)-4f(x)[f(x)-1]=0. \tag{24}$$

In the self-dual limit the energy

$$E=\frac{e^2v^4}{\mu}\int_{-\infty}^{+\infty}dx\mathcal{E}(x) \tag{25}$$

can be written as

$$E=\left\{Y\sin\alpha+\left(\frac{ev^2}{\mu}\right)Q\cos\alpha\right\}, \tag{26}$$

where we have used the fact that (rf) vanishes at both $\pm\infty$.

IV. ASYMPTOTIC PROPERTIES

The finiteness of energy requires that the fields have to satisfy some boundary conditions. From the expression of the energy density we find that the fields have to take one of the following two set of values as $x\rightarrow\pm\infty$. They are

$$f(x)=1,\quad n(x)=1,\quad f_{01}(x)=0,\quad r(x)=0 \tag{27}$$

or

$$f(x)=0,\quad n(x)=0,\quad f_{01}(x)=0,\quad r(x)=r_0, \tag{28}$$

where r_0 is an arbitrary real constant. For the asymmetric solutions the fields at $+\infty$ take one of the above sets of values and at $-\infty$ the other. But for symmetric solutions the fields takes the second set of values both at $+\infty$ and at $-\infty$.

A. Asymmetric solution

It is difficult to solve the self-dual equations analytically. However, we can obtain the asymptotic form of the various fields. Consider first the asymmetric solution for which the fields take the values (27) at $-\infty$ and (28) at ∞. Then from the fourth order equation we find that for $x\rightarrow-\infty$, the field f behaves as

$$f(x)\rightarrow(1-qe^{lx})+\cdots, \tag{29}$$

where

$$l=-\frac{k}{2}+\frac{1}{2}\sqrt{k^2+8k}. \tag{30}$$

Here q is some arbitrary constant.

Using the above expression for f in the self-dual equations we find that the behavior of the fields n, r, and f_{01} as $x\rightarrow-\infty$ are

$$n(x)\rightarrow\left(1-q\frac{l}{2}e^{lx}\right)+\cdots,$$

$$r(x)\rightarrow q\frac{l\cos\alpha}{2(1-\sin\alpha)}e^{lx}+\cdots,$$

$$f_{01}(x)\rightarrow q\frac{l^2\cos\alpha}{2(1-\sin\alpha)}e^{lx}+\cdots. \tag{31}$$

Similarly for $x\rightarrow\infty$ we find, from the fourth order equation,

$$f(x)\rightarrow a_0e^{-mx}(1+\tilde{a}_0e^{-mx})+\cdots, \tag{32}$$

where

$$-\frac{m^4}{k^2}\frac{\tilde{a}_0}{a_0}+m^2\left(\frac{\tilde{a}_0}{a_0}+\frac{2}{k}\right)+4=0. \tag{33}$$

Then from the self-dual equations we find

$$n(x)\rightarrow b_0e^{-mx}+\cdots,$$

$$r(x)\rightarrow\tilde{d}_0+d_0e^{-mx}+\cdots,$$

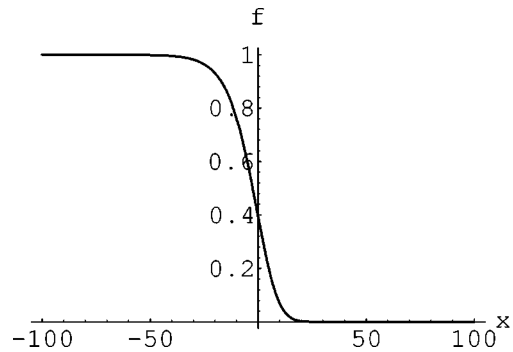

FIG. 1. Asymmetric solution for $k=0.01$.

$$f_{01}(x)\rightarrow c_0e^{-mx}+\cdots, \tag{34}$$

where the coefficients a_0,b_0,c_0,d_0, and $\tilde{d}_0$ satisfy the following relations:

$$\frac{b_0}{a_0}=\frac{k+2}{k^2-m^2},$$

$$\frac{c_0}{a_0}=\frac{m(2+m\sin\alpha)+k(m+2\sin\alpha)}{(k^2-m^2)\cos\alpha},$$

$$\frac{d_0}{a_0}=\frac{k(2+m\sin\alpha)+m(m+2\sin\alpha)}{m(m^2-k^2)\cos\alpha},$$

$$\tilde{d}_0=\frac{m+2\sin\alpha}{2\cos\alpha}. \tag{35}$$

The charges Q and Y are given in this case by (putting appropriate dimensions)

$$Y=\left(\frac{e^2v^4}{\mu}\right)\left(\frac{\tilde{d}_0^2-1}{2}\right), \tag{36}$$

$$Q=ev^2\tilde{d}_0, \tag{37}$$

while E is given by Eq. (26). In case one assumes the same exponent at $\pm\infty$, i.e., $m=l$, then Y, Q, and E are solely determined by k and the angle α. Further, as $k\rightarrow\infty$, Y, Q, and E reduce to their expressions in the pure Chern-Simons case [9].

We have also found a numerical solution to the fourth order equation (24) in the asymmetric case with the boundary values as given by Eqs. (29) and (32). The profile of the Higgs field in the case $k=0.01$ is given in Fig. 1.

B. Symmetric solution

For the symmetric case, we assume that the profile of the field f is symmetric around $x=0$. To know the behavior of f around $x=0$, we expand the field f as

$$f(x)=a+bx^2+cx^4+\cdots. \tag{38}$$

On substituting the above expression in Eq. (24) we get

$$-6c+\left(\frac{k}{2}+2a\right)kb-a^2k^2(a-1)=0. \tag{39}$$

For $x\to\infty$ we find from the fourth order equation that $f(x)$ is similar to that given in Eq. (32), and hence the fields n, r, and f_{01} are similar to those given in Eq. (34).

The expressions for Y,Z, and E for the symmetric domain wall soliton are found to be (putting appropriate dimensions)

$$Y=\frac{1}{2}(\tilde{d}_2^{\,2}-\tilde{d}_1^{\,2})=\frac{e^2v^4}{\mu}\left(\frac{m\ \sin\ \alpha}{\cos^2\alpha}\right), \tag{40}$$

$$Q=ev^2(\tilde{d}_2-\tilde{d}_1)=\frac{mev^2}{\cos\ \alpha}, \tag{41}$$

$$E=\frac{e^2v^4}{\mu}\left(\frac{m}{\cos^2\alpha}\right), \tag{42}$$

where $\tilde{d}_1$ and $\tilde{d}_2$ are the values of $r(x)$ at $-\infty$ and at $+\infty$, respectively. From the above expressions it can be seen that

$$E=\sqrt{Y^2+\left(\frac{ev^2}{\mu}Q\right)^2} \tag{43}$$

and $Y=E\ \sin\alpha$, $(ev^2/\mu)Q=E\ \cos\ \alpha$. Then the ratio of energy to charge is given by

$$\frac{E}{Q}=\frac{ev^2}{\mu\cos\ \alpha} \tag{44}$$

$$=\frac{1}{e}\sqrt{\overline{R^2}+\frac{e^4v^4}{\mu^2}}, \tag{45}$$

where $\overline{R}=\tan\alpha$ is the average value of R. The mass of the elementary excitation in the unbroken phase is $m_e=\sqrt{R_0^2+e^4v^4/\mu^2}$. Taking $R_0=\overline{R}$ we find

$$\frac{E}{Q}=\frac{m_e}{e}, \tag{46}$$

which means that the symmetric solution is at the threshold of stability.

We have also found numerical solutions to the fourth order equation (24) in the symmetric case with the boundary value as given by Eqs. (40) and (32). The profile of the Higgs field $f(x)$ in the case $k=0.01$ is given in Fig. 2 [we have chosen $a=0.8,b=0.08$, where a,b are given by Eq. (39)].

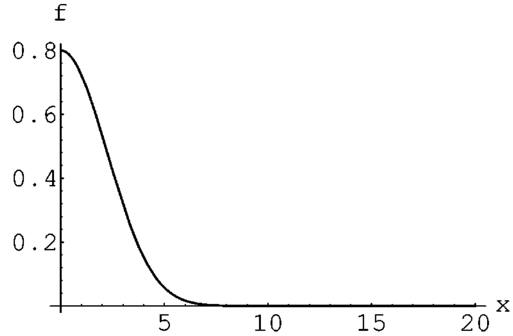

FIG. 2. Symmetric solution for $k=0.01$.

V. SUPERSYMMETRY

We have obtained the supersymmetric model by dimensional reduction of the three-dimensional supersymmetric Maxwell Chern-Simons model [10,11]. The Lagrangian in two dimensions is found to be

$$\begin{aligned}L=&\frac{1}{2e^2}F_{01}^2+\frac{\mu}{e^2}RF_{01}+\frac{1}{2e^2}\partial_\rho R\partial^\rho R+\frac{1}{2e^2}\partial_\rho N\partial^\rho N\\&+(D_\rho\phi)^*(D^\rho\phi)-R^2|\phi|^2-\frac{e^2}{2}\left(|\phi|^2+\frac{\mu}{e^2}N-v^2\right)^2\\&-N^2|\phi|^2+i\bar{\psi}\gamma^\rho D_\rho\psi-i\bar{\psi}\gamma^5R\psi+\frac{i}{e^2}\bar{\chi}\gamma^\rho\partial_\rho\chi-\frac{\mu}{e^2}\bar{\chi}\chi\\&+i\sqrt{2}(\bar{\psi}\chi\phi-\bar{\chi}\psi\phi^*)-N\bar{\psi}\psi.\end{aligned} \tag{47}$$

The bosonic part of this Lagrangian is equal to Eq. (3) after a redefinition of the field N by $(-N+e^2v^2/\mu)$. The Lagrangian has the following $N=2$ supersymmetry:

$$\begin{aligned}&\delta_\eta A_\rho=i(\bar{\eta}\gamma_\rho\chi-\bar{\chi}\gamma_\rho\eta),\\&\delta_\eta R=(\bar{\chi}\gamma^5\eta-\bar{\eta}\gamma^5\chi),\\&\delta_\eta\phi=\sqrt{2}\bar{\eta}\chi,\\&\delta_\eta N=i(\bar{\chi}\eta-\bar{\eta}\chi),\\&\delta_\eta\psi=\sqrt{2}(-i\gamma^\rho\eta D_\rho\phi+i\gamma^5\eta R\phi-\eta N\phi),\\&\delta_\eta\chi=-i(\gamma^0\partial_1R+\gamma^1\partial_0R)\eta+\gamma^5F_{01}\eta+\gamma^\rho\partial_\rho N\eta\\&\qquad-i\eta e^2\left(|\phi|^2+\frac{\mu}{e^2}N-v^2\right).\end{aligned} \tag{48}$$

Here we have chosen $\gamma^0=\sigma_2$, $\gamma^1=i\sigma_1$, $\gamma^5=\gamma^0\gamma^1=\sigma_3$, $\eta_{\rho\sigma}=\mathrm{diag}(1,-1)$. The supercharge which generates this transformation is given by

$$\begin{aligned}Q=\int dx\Bigg[&\sqrt{2}(D_\rho\phi)^*\gamma^\rho\gamma^0\psi-\sqrt{2}R\phi^*\gamma^1\psi+\sqrt{2}iN\phi^*\gamma^0\psi\\&-\frac{i}{e^2}\partial_\rho N\gamma^\rho\gamma^0\chi-\frac{i}{e^2}F_{01}\gamma^1\chi-\frac{1}{e^2}\gamma^5\partial_0R\chi+\frac{1}{e^2}\partial_1R\chi\\&+\left(|\phi|^2+\frac{\mu}{e^2}N-v^2\right)\gamma^0\chi\Bigg].\end{aligned} \tag{49}$$

The superalgebra is found to be

$$\frac{1}{2}\{Q_\alpha,\overline{Q}_\beta\}=(\gamma^\rho)_{\alpha\beta}P_\rho+\delta_{\alpha\beta}Z+i(\gamma^5)_{\alpha\beta}Y, \tag{50}$$

where the central charges Y and Z are given by

$$Y=\int dx\frac{\mu}{2e^2}\left[R^2+\left(N-\frac{e^2v^2}{\mu}\right)^2\right]',$$

$$Z=\int dx[R(|\phi|^2-v^2)]'. \tag{51}$$

It can be easily verified that the central charges Y and Z are equal to the Noether and topological charges after a field redefinition.

VI. NONRELATIVISTIC MODEL

To get the nonrelativistic Lagrangian we substitute

$$\phi=\sqrt{\frac{\mu c^3}{2e^2v^2}}(e^{-imc^2t}\psi+e^{imc^2t}\widetilde{\psi}) \tag{52}$$

in Lagrangian (1). Neglecting higher order terms of $1/c$ we have the nonrelativistic Lagrangian [3]

$$\mathcal{L}^{NR}_{2+1}=-\frac{1}{4e^2}F_{\rho\nu}F^{\rho\nu}+\frac{\mu}{2e^2}\epsilon^{\eta\nu\rho}A_\eta\partial_\nu A_\rho+\frac{1}{2e^2}\partial_\rho N\partial^\rho N$$
$$-\frac{\mu^2}{2e^2}N^2+ic\psi^*D_0\psi+ic\widetilde{\psi}^*(\partial_0+iA_0)\widetilde{\psi}$$
$$-\frac{\mu c^3}{2e^2v^2}(D_i\psi)^*(D_i\psi)-\frac{\mu c^3}{2e^2v^2}(D_i\widetilde{\psi})^*(D_i\widetilde{\psi})$$
$$-\frac{\mu^2c^4}{8e^2v^4}(|\psi|^2+|\widetilde{\psi}|^2)^2+\left(1+\frac{\mu^2c^2}{2e^2v^2}\right)$$
$$\times(|\psi|^2+|\widetilde{\psi}|^2)N. \tag{53}$$

(Here all the fields are functions of x^μ and not of $\widetilde{x}^\mu$.) After dimensional reduction and restricting ourselves only to the zero antiparticle sector we get the following nonrelativistic Lagrangian in $1+1$ dimensions:

$$\mathcal{L}^{NR}_{1+1}=\frac{1}{2e^2}F^2_{01}+\frac{\mu}{e^2}RF_{01}+\frac{1}{2e^2}\partial_\rho R\partial^\rho R+\frac{1}{2e^2}\partial_\rho N\partial^\rho N$$
$$-\frac{\mu^2}{2e^2}N^2(x)+i\psi^*(x)D_0\psi-\frac{\mu}{2e^2v^2}(D_x\psi)^*(D_x\psi)$$
$$+\left(1+\frac{\mu^2}{2e^2v^2}\right)|\psi|^2N(x)-\frac{\mu^2}{8e^2v^4}|\psi|^4$$
$$-\frac{\mu}{2e^2v^2}R^2(x)|\psi|^2. \tag{54}$$

This is same as the nonrelativistic limit of the dimensionally reduced relativistic $(1+1)$-dimensional Lagrangian. As before, let us express the Lagrangian in terms of the dimensionless fields which are functions of $\widetilde{x}$, and omit the tilde:

$$\frac{\mu^2}{e^4v^6}\mathcal{L}^{NR}_{1+1}=L^{NR}=\frac{1}{2k}f^2_{01}+r(x)f_{01}(x)+\frac{1}{2k}\partial_\rho r\partial^\rho r$$
$$+\frac{1}{2k}\partial_\rho n\partial^\rho n+2i\chi^*d_0\chi-(d_x\chi)^*(d_x\chi)$$
$$+2n(x)f(x)-\frac{k}{2}[f(x)-n(x)]^2-r^2(x)f(x), \tag{55}$$

where $\chi=(1/ev)\sqrt{\mu/2}\psi$ is dimensionless. The dimensionless energy density $\mathcal{E}(x)$ is

$$\frac{\mu^2}{e^6v^4}\widetilde{\mathcal{E}}(x)=\mathcal{E}(x)=\frac{1}{2k}\{f^2_{01}(x)+[r'(x)]^2+[n'(x)]^2+\dot{r}^2(x)$$
$$+\dot{n}^2(x)\}+\frac{k}{2}[n(x)-f(x)]^2+(d_x\chi)^*(d_x\chi)$$
$$+r^2(x)f(x)-2n(x)f(x). \tag{56}$$

The equations of motion are

$$2id_0\chi+d_x^2\chi+(2+k)n(x)\chi(x)-[kf(x)+r^2(x)]\chi(x)=0,$$

$$\frac{1}{k}\partial_\rho\partial^\rho n+k[n(x)-f(x)]-2f(x)=0,$$

$$\frac{1}{k}\partial_\rho\partial^\rho r-f_{01}(x)+2r(x)f(x)=0, \tag{57}$$

$$\frac{1}{k}f'_{01}(x)+r'(x)+2f(x)=0. \tag{58}$$

For static fields we have $\dot{r}(x)=0=\dot{n}(x)$. Using the Gauss law equation (55) we express the energy density as

$$\mathcal{E}(x)=|d_x\chi-r(x)\chi(x)|^2+\frac{k}{2}\left(\frac{1}{k}r'(x)+n(x)-f(x)\right)^2$$
$$+\frac{1}{2k}\{n'(x)-f_{01}(x)\}^2+\{r(x)f(x)\}'$$
$$+\frac{1}{k}\{n(x)f_{01}(x)\}'. \tag{59}$$

In the case of static fields we have the following coupled first order self-dual equations which can be shown to be consistent with the above field equations:

$$\frac{1}{k}r'(x)=[f(x)-n(x)],$$

$$n'(x)=f_{01}(x),$$

$$f'(x)=2r(x)f(x),$$

$$\frac{1}{k}f'_{01}(x)+r'(x)+2f(x)=0. \tag{60}$$

The energy has a lower bound of zero which is saturated when the fields satisfy the above self-dual equations. Eliminating the fields n,r,f_{01} from the above equations we get the uncoupled fourth order equation for the field f,

$$(-\partial_x^2+k^2)\partial_x^2\ln f=-4k\left(\frac{1}{2}\partial_x^2+k\right)f(x), \tag{61}$$

which is analogous to the Liouville equation. The finiteness of energy requires that the field f vanish at both $\pm\infty$. It can be easily shown that the falloff is not a power law but an exponential and as $x\rightarrow-\infty$ we have

$$f(x)\rightarrow Ae^{Bx}(1+Ce^{Bx})+\cdots, \tag{62}$$

where A, B, and C satisfy the relation

$$-\frac{B^4}{k^2}\frac{C}{A}+B^2\left(\frac{C}{A}+\frac{2}{k}\right)+4=0. \tag{63}$$

Using this in the self-dual equations (57) and (55) we find

$$n(x)\rightarrow A_0e^{Bx}+\cdots,$$

$$r(x)\rightarrow\frac{B}{2}+A_1e^{Bx}+\cdots,$$

$$f_{01}(x)\rightarrow A_2e^{Bx}+\cdots, \tag{64}$$

where the coefficients A_0, A_1, A_2, B, and A are related as

$$A_0=-\frac{A}{\left(1+\dfrac{2k+B^2}{k(k+2)}\right)},$$

$$A_1=\frac{kA}{B\left(1+\dfrac{k(k+2)}{2k+B^2}\right)},$$

$$A_2=-\frac{AB}{\left(1+\dfrac{2k+B^2}{k(k+2)}\right)}. \tag{65}$$

This solution is expected to be symmetric around $x=0$. To know the behavior around $x=0$ we can expand the fields around it as

$$f(x)=\widetilde{A}+\widetilde{B}x^2+\widetilde{C}x^4+\cdots. \tag{66}$$

Then from Eq. (58) we find

$$k^2(\widetilde{B}+2\widetilde{A}^2)+2k\widetilde{A}\widetilde{B}-12\widetilde{C}=0. \tag{67}$$

The charge in this case is found to be

$$Q=-\int_{-\infty}^{\infty}dx f(x)=\frac{r(\infty)-r(-\infty)}{2}=\frac{B}{2}, \tag{68}$$

in case one assumes the same behavior of the fields at $\pm\infty$.

VII. CONCLUSION

In this paper we have studied domain wall soliton solutions in the self-dual Maxwell Chern-Simons systems in $1+1$ dimensions. Here we found a BPS-type bound which is saturated when the self-dual equations hold. Numerical solutions for both the topological and the nontopological soliton equations were obtained. Further the asymptotic properties of the fields are also studied. We considered the underlying $N=2$ supersymmetry of this model by dimensional reduction of the $(2+1)$-dimensional model. Finally we have studied the nonrelativistic limit of the model. This work raises a few questions which need to be looked into. For example, can one generalize the model to the non-Abelian case? There may be a gauged version of the nonlinear sigma models with Maxwell and Chern-Simons terms with a higher gauge group which could lead to a richer variety of two-dimensional models with a BPS-type energy bound. Further, in three dimensions, the maximal supersymmetry for the Maxwell Chern-Simons system is $N=3$ [12]. In two dimensions it may be worth enquiring as to what maximal supersymmetry is allowed. Also it would be interesting to study the fermionic and bosonic zero modes and to check whether the energy bound is saturated at the quantum level or if there are quantum corrections.

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