Applying the linear δ expansion to the $i\phi^3$ interaction

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The linear δ expansion (LDE) is applied to the Hamiltonian $H=\frac{1}{2}(p^2+m^2x^2)+igx^3$, which arises in the study of Lee-Yang zeros in statistical mechanics. Despite being non-Hermitian, this Hamiltonian appears to possess a real, positive spectrum. In the LDE, as in perturbation theory, the eigenvalues are naturally real, so a proof of this property devolves on the convergence of the expansion. A proof of convergence of a modified version of the LDE is provided for the ix^3 potential in zero dimensions. The methods developed in zero dimensions are then extended to quantum mechanics, where we provide numerical evidence for convergence. [S0556-2821(98)05208-4]

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I. INTRODUCTION

In a recent paper [1], Bender and Milton carried out an investigation of the following non-Hermitian Hamiltonian:

$$H=\frac{1}{2}(p^2+m^2x^2)+igx^3. \tag{1}$$

Among other results, they provided analytical evidence for the remarkable property that its eigenvalues are real and positive. This Hamiltonian and its field theory counterparts arise in the study of the zeros of the Ising model partition function as a function of complex magnetic field, the so-called Lee-Yang zeros (see, e.g., Ref. [2] and references therein). The distribution of zeros in the complex parameter plane of the partition function of a given system can yield useful information concerning its phase transitions. For the Ising model, the zeros lie on the imaginary magnetic field axis and, for imaginary field, the effective theory is an interacting scalar field theory with dominant interaction $i\phi^3$ for small fluctuations [3].

In order to give a proper analytical demonstration of the reality of the eigenvalues of Hamiltonian (1), as well as investigate the higher dimensional quantum field theory analogues, we require an expansion method which converges for finite, but otherwise arbitrary values of the coupling g and mass m. The method must therefore be necessarily nonperturbative in these parameters. Bender and Milton employed a variant of their previous δ expansion method [4], in which the interaction $i\phi^3$ is replaced by $(i\phi)^{2+\delta}$. A Taylor expansion in δ of the desired quantity (N-point Green function, nth energy eigenvalue, etc.) is obtained and then δ is set equal to one. The terms of the resulting series are nonperturbative in m and g. However, in this method it is extremely difficult to go beyond first order in δ and hence test for convergence.

In the present paper we employ an alternative expansion method, the so-called linear δ expansion (LDE). The LDE has been employed as a non-perturbative approximation method to study problems in, for example, ϕ^4 theory [5–7], quantum chromodynamics [8,9], relativistic nuclear models [10], and electron dynamics in disordered systems [11]. The method involves constructing a modified action which involves the original action of the theory, S, and a soluble trial action, S_0, containing one or more variational parameters λ_i:

$$S_\delta=S_0+\delta(S-S_0). \tag{2}$$

The Green function of interest is evaluated as a power series in δ, which is then set equal to 1. For the success of the method the trial action S_0 needs to be simple, so that one can perform high order calculations, but also as close to the true action as possible, so that expanding around S_0 is a reasonable procedure with a good chance of converging. Although $S_{\delta=1}=S$ is independent of the λ_i, there is nonetheless a residual dependence on λ_i in the truncated series evaluated at $\delta=1$, and it is therefore necessary to choose these parameters according to some well-defined criterion. Perhaps the most commonly used criterion is the principle of minimal sensitivity (PMS), according to which the λ_i are chosen to be stationary points of the truncated expansion, where the dependence is minimal. Whatever criterion is adopted, it is to be applied at each order N in the expansion, so that the λ_i become N dependent. This feature of the method is crucial for the convergence of the LDE, which can be characterized as an order-dependent split between the bare and the interaction terms in the action. In many cases where the fixed split of conventional perturbation theory leads to a divergent series, the LDE can be proved to lead to a convergent sequence of approximants.

In this paper we apply different variants of the LDE to the ix^3 potential in zero dimensions, where we give a proof of convergence of the expansion for the analogue of the vacuum persistence amplitude Z. We then go on to consider the one-dimensional problem and provide numerical evidence of convergence of the LDE for the finite-temperature partition function and the ground-state energy.

In Sec. II, which deals with the zero-dimensional problem, we first describe how the conventional LDE converges, but to the wrong answer. One possible resolution is to split the integral up for positive and negative x and apply the PMS separately to each integral [12]. Numerically, this gives a sequence of approximants converging to the correct answer, and we provide a proof of convergence of this procedure. This LDE variant can be generalized to the path integral

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expression of the 1D finite-temperature partition function. However, difficulties are encountered when applied to the ground-state energy, owing to the non-analyticity of the splitting procedure. We resolve this problem by using a modified δ expansion which involves a shift parameter. This retains the essential features of the integration splitting procedure, but has the advantage of being analytic, thus making the calculations straightforward to carry out and allowing the possibility of generalization to higher dimensions. In the last part of Sec. II, a numerical study of the shift method is carried out, and a proof of convergence provided using saddle-point techniques.

In Sec. III, where we deal with the quantum mechanical problem, we show that the conventional LDE again fails. Numerical evidence of convergence is given for the path integral expression of the partition function (using both the splitting and shift techniques) and especially for the ground state energy using the shift technique.

In the conclusion we outline further directions, including proving convergence for the 1D problem and generalizing the shift method to the higher dimensional field theory analogues.

II. ZERO DIMENSIONS

The zero-dimensional analogue of the finite-temperature partition function in quantum mechanics, or the vacuum persistence amplitude in field theory, is the ordinary integral

$$Z=\int_{-\infty}^{\infty} dx e^{-m^2x^2+igx^3}. \tag{3}$$

For simplicity we will take $m=0$. In that case $Z\propto g^{-1/3}$ and we can set $g=1$ without loss of generality. In spite of the absence of the convergence factor the integral is still well defined, and can be calculated by splitting up the integration range into $x<0$ and $x>0$, and then rotating the contour by $\pm\pi/6$. The exact result, obtained in this way, is $Z=\Gamma(1/3)/\sqrt{3}=1.54668588415598$, to 15 significant figures.

A. Naive application of the LDE

In evaluating Z using the LDE, the standard approach is to modify the exponent simply by adding and subtracting a quadratic term, to give

$$Z(\delta)=\int_{-\infty}^{\infty} dx e^{-\lambda x^2+\delta(\lambda x^2+ix^3)}. \tag{4}$$

The procedure is then to expand Z_δ to order δ^N, set $\delta=1$ and then choose $\lambda=\lambda_N$ by some criterion or other. In the absence of any additional information the most reasonable criterion is the principle of minimal sensitivity (PMS), namely to choose λ_N as a stationary point of Z_N.

The truncated series is $\Sigma_{n=0}^{N} c_n$, with

$$c_n=\frac{1}{\sqrt{\lambda}}\sum_{r=0}^{[n/2]}\frac{\Gamma(n+r+1/2)}{(2r)!(n-2r)!}\left(-\frac{1}{\lambda^3}\right)^r. \tag{5}$$

Using (5), Z_N can be calculated to high order. However, the residual dependence on λ turns out to be radically different from the x^4 case discussed in [5], where, for odd N there was a single maximum, and the value of Z at that maximum steadily approached the exact value as N increased. As shown in Fig. 1, where $Z_N(\lambda)$ is plotted against λ for $N=30$, the situation here is that there are extremely violent oscillations for small λ, which gradually decrease in amplitude as λ increases, with a final, very broad maximum lying above the exact value of Z.

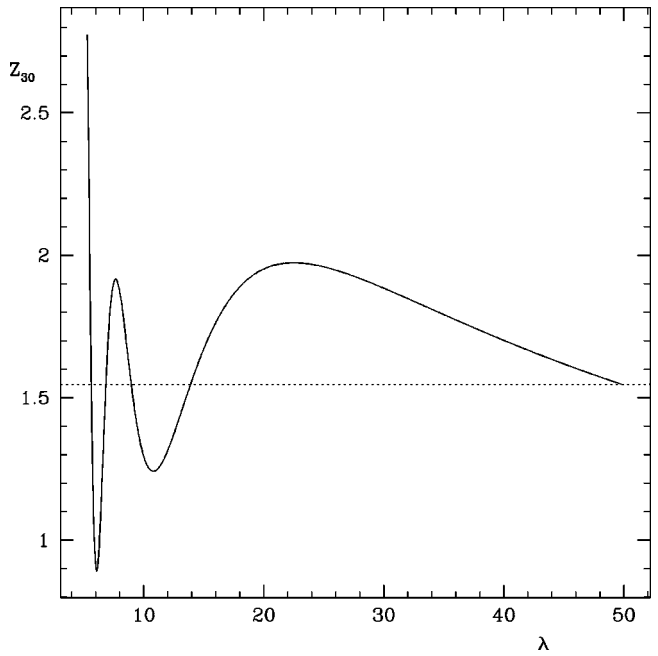

FIG. 1. Z_N for $N=30$ as a function of λ in the naive LDE in zero dimensions. The dotted line shows the exact answer.

One's natural choice of a PMS point for λ would be the position of this last maximum: the residual dependence of Z_N on λ around this point is much less than anywhere else on the graph. However, it turns out that the sequence $Z_N(\lambda_N)$, with λ_N so chosen, indeed converges, but to the wrong answer. The same is true of the previous minimum, and indeed none of the stationary points converges to the correct answer. This is a stark warning, in this admittedly somewhat pathological case, of the shortcomings of the PMS. It is worth mentioning that the scaling behavior with N of these λ_N is anomalous: for large N they grow like N, rather than the $N^{1/3}$ which would be expected from a saddle-point analysis and which indeed is obtained in the variants of the LDE discussed in the following two subsections. In an estimation of the error $Z-Z_N$ on the lines of the double saddle-point procedure of Ref. [6], the interaction term ix^3 would then be sub-dominant, a clear signal that one is on the wrong track.

B. The splitting procedure

A possible resolution of this problem is to split up the integration range into positive and negative x. This gives $Z_N(\lambda)=Z_N^+(\lambda)+Z_N^-(\lambda)$, where

$$Z_N^+(\lambda)=\int_0^{\infty} dx e^{-\lambda x^2}\{e^{\lambda x^2+ix^3}\}_N, \tag{6}$$

and similarly for Z_N^-, where $\{f(z)\}_N$ denotes the Taylor ex-

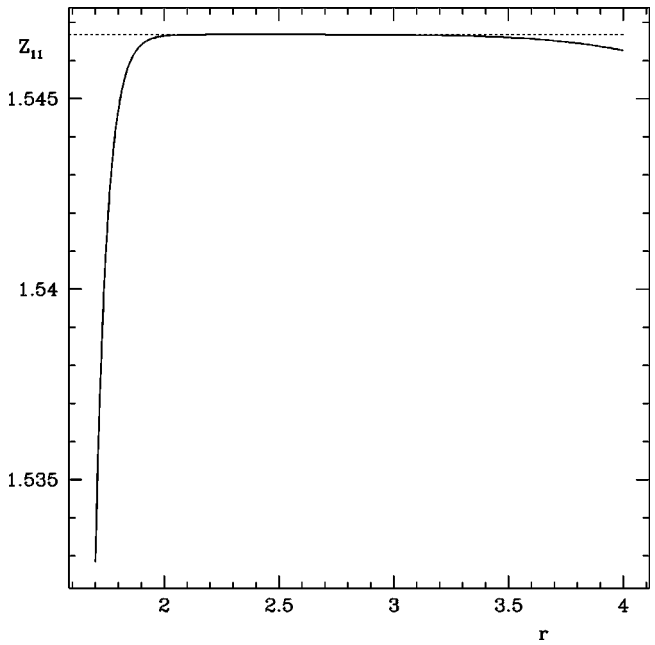

FIG. 2. Z_N for $N=11$ as a function of r using the splitting method in zero dimensions. The dotted line shows the exact answer.

pansion of f to order z^N. If λ is taken as $\lambda = re^{-i\pi/3}$, where r is real, and the integration contour rotated by $x \to xe^{i\pi/6}$, then $Z_N^+(\lambda) = e^{i\pi/6}\mathcal{Z}_N(r)$ where

$$\mathcal{Z}_N(r) = \int_0^\infty dx e^{-rx^2}\{e^{rx^2-x^3}\}_N. \tag{7}$$

For Z_N^- we need to take $\lambda = re^{i\pi/3}$ and rotate the contour in the opposite direction, to obtain $Z_N^- = e^{-i\pi/6}\mathcal{Z}_N(r)$.

The analysis of $\mathcal{Z}_N$ is very similar to the x^4 case discussed in Ref. [5]. For odd N there is a single maximum in r, which steadily converges to the exact result. For $N=11$ the situation is depicted in Fig. 2, where we have multiplied $\mathcal{Z}_N$ by the appropriate factor of $\sqrt{3}$. The contrast with Fig. 1 could hardly be more striking. The crucial difference between this and the naive application of the LDE is that by splitting the integral we are in fact using two different (complex conjugate) values of λ_N for the two integrals rather than the same value for both.

The proof of convergence of this procedure is similar to that for the x^4 case, though different in detail. The remainder R_N is essentially given by the integral for the Nth coefficient in the expansion of $\mathcal{Z}$, namely

$$c_N = \frac{1}{N!}\int_0^\infty dx e^{-rx^2}(rx^2-x^3)^N, \tag{8}$$

which can be estimated by saddle-point methods for large N. In this case the PMS value of r_N scales as $N^{1/3}$. Writing $r_N = \alpha N^{1/3}$ and $x = zN^{1/3}$ and using Stirling's approximation for $N!$, the leading behavior of the integral is

$$c_N = \int_0^\infty dz e^{N\varphi}, \tag{9}$$

where

$$\varphi = -\alpha z^2 + \ln(\alpha z^2 - z^3) + 1. \tag{10}$$

The saddle-point equation $d\varphi/dz = 0$ is

$$2\alpha z^3 - 2\alpha^2 z^2 - 3z + 2\alpha = 0, \tag{11}$$

which has two positive roots $z_\pm$. What is required for convergence is that both Re $\varphi(z_+)$ and Re $\varphi(z_-)$ be negative, and then the rate of convergence is governed by the larger of the two. They become equal at the PMS point, when α is such that Re $\varphi(z_+)$=Re $\varphi(z_-)$. Numerically the solution of this equation is $\alpha = 1.0272$, when Re $\varphi(z_\pm) = -1.2398$. Thus the sequence of approximants converges like $\exp(-1.2398N)$.

C. The shift method

If we were only concerned with zero dimensions this would be an entirely satisfactory resolution of the problem. However, in higher dimensions one is dealing with a *functional* integral, and the generalization of such a splitting procedure is fraught with difficulties, even though some progress can be made along these lines, as we show in Sec. III B. Fortunately, consideration of the quantum mechanical problem suggests another solution, which is immediately generalizable to higher dimensions, namely to incorporate a linear term, or shift, into the δ-modified action.

The motivation for introducing such a term is discussed briefly in Sec. III C. In the context of the zero-dimensional model it is worth noting that one of the features of the successful stratagem of splitting the integration range and treating each half separately is that terms odd in x survive the integration, whereas they cancel in the naive application of the δ expansion. Introducing a linear term into the δ-modified action also avoids such a cancellation, but in a simple algebraic way.

Thus, in zero dimensions the relevant modification of Eq. (4) is

$$Z(\delta) = \int_{-\infty}^\infty dx e^{-\lambda x^2 + \delta[\lambda x^2 + ig(x+ia)^3]}, \tag{12}$$

where we have introduced a shift ia, which will indeed turn out to be pure imaginary. Notice that when $\delta=1$ this does reduce to the original integration, as the shift is then immaterial.

Truncating the expansion at order δ^N and setting $\delta=1$ we now have the series $\Sigma_{n=0}^N c_n$, where

$$c_n = \sum_{r=0}^n \sum_{s=0}^{[3r/2]} \binom{3r}{2s} \frac{\Gamma(n-r+s+1/2)}{r!(n-r)!\lambda^{s+1/2}}(-1)^s g^r a^{3r-2s}. \tag{13}$$

One can now search for double PMS points in the two parameters a, λ. For $N>2$ there are several such points. For example, at order 16 there are ten solutions, with λ lying in the range 4–6 and a close to 1. There is no very convincing

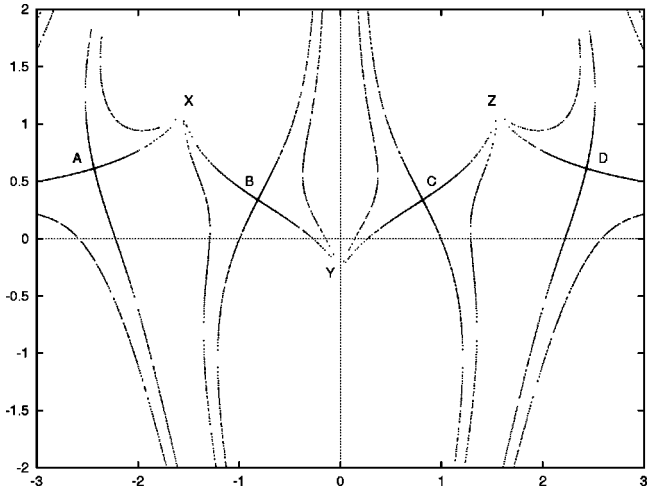

FIG. 3. The location of the saddle points and sinks in the integrand of Eq. (15). The required stationary-phase path, passing through all four saddle points, is $AXBYCZD$.

criterion for choosing between these multiple PMS solutions, but they all agree with the exact answer up to the 7th decimal place.

However, the PMS was only a means to an end, namely to obtain a sequence of approximants converging to the exact answer, and in the present case we can instead adopt this latter requirement directly as a criterion for choosing a and λ. This can be done via a saddle-point analysis of the error R_N.

As in the previous subsection, the error is essentially given by the expression for the Nth coefficient c_N, which in the present case reads

$$c_N=\frac{1}{N!}\int_{-\infty}^{\infty}dx e^{-\lambda x^2}[\lambda x^2+ig(x+ia)^3]^N. \qquad (14)$$

The appropriate scaling for a saddle-point approximation in which both parameters play a non-trivial role is $\lambda=\alpha N^{1/3}$, $a=\beta N^{1/3}$. Then the large-N behavior of c_N is

$$c_N=\int_{-\infty}^{\infty}dz e^{N\varphi}, \qquad (15)$$

where now (again setting $g=1$)

$$\varphi=-\alpha z^2+\ln[\alpha z^2+i(z+i\beta)^3]+1. \qquad (16)$$

The saddle-point equation in this case is a quartic:

$$2\alpha z^4+2i\alpha(3\beta-\alpha)z^3-3(2\beta^2\alpha+1)z^2 -2i(\beta^3\alpha+3\beta-\alpha)z+3\beta^2=0, \qquad (17)$$

with four complex roots.

The location of these roots is shown in Fig. 3 for typical values of α and β, when they occur as two pairs of the form $\pm u+iv$. Also shown in that diagram are the paths of stationary phase. It is necessary to know the geometry of these paths in order to determine which saddle points are encountered, and in what direction, when the contour is distorted from its original location along the real axis from $-\infty$ to ∞. In fact the required path goes through all four saddle points

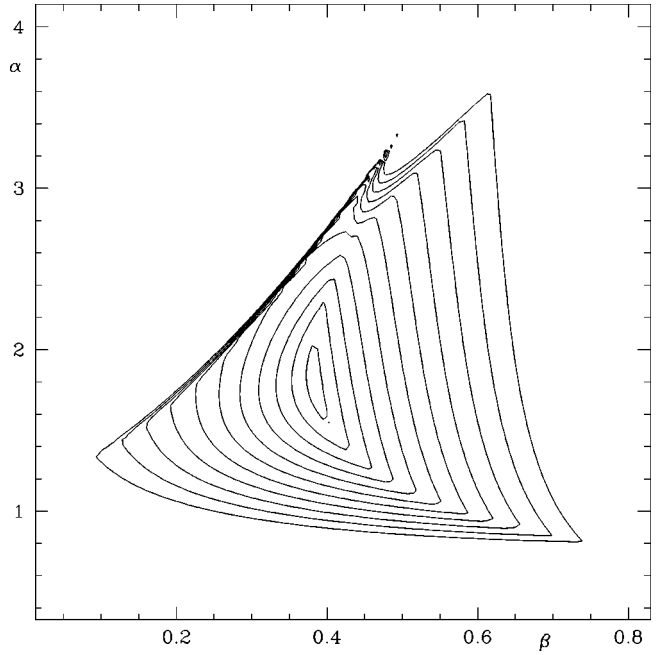

FIG. 4. Contours of Re φ in the parameter space of α and β. Convergence with the shift method is assured for all points within the outer contour. For points within the inner contour the convergence is faster than e^{-N}.

A, B, C, D, in each case at a maximum of the integrand because of the intervening ''sinks'' X, Y, Z where the argument of the logarithm vanishes.

The values of φ at the mirror points A and B are the complex conjugates of those at D and C respectively. The criterion for convergence therefore reduces to ensuring that both Re $\varphi(z_C)$ and Re $\varphi(z_D)$ are negative. Optimal convergence occurs when the two are equal.

Thus in the parameter space of α and β we solve the equation Re $\varphi(z_C)$=Re $\varphi(z_D)$ and then look for points where this common value is negative. We display the results as a contour plot in Fig. 4, where the outer contour corresponds to Re $\varphi(z_C)=0$ and the inner one to Re $\varphi(z_C)=-1$. Any point within the outer contour will give convergence, while points within the inner contour will give convergence with an error smaller than e^{-N}. Notice that there is quite a wide range of acceptable values of $\beta\equiv aN^{-1/3}$, but of course this does not include $\beta=0$, which would correspond to the naive LDE with no shift term.

In Fig. 5 we demonstrate the nature of the numerical convergence up to $N=30$ for the optimal values $\alpha=1.79055$, $\beta=0.38378$. The abolute value of the error is shown, on a logarithmic scale. Although this decrease lies within the predicted envelope, the error is periodically significantly smaller than the general trend. A similar pattern of convergence has been seen [15] in the application of variational perturbation theory to the calculation of the strong-coupling coefficients of the quartic oscillator.

In conclusion, the shift method described in this subsection involves a simple algebraic modification of the original integrand and an unambiguous choice of the two variational parameters according to the requirement of convergence. The method may be straightforwardly generalized to higher dimensions, where the calculations involved in the δ expansion

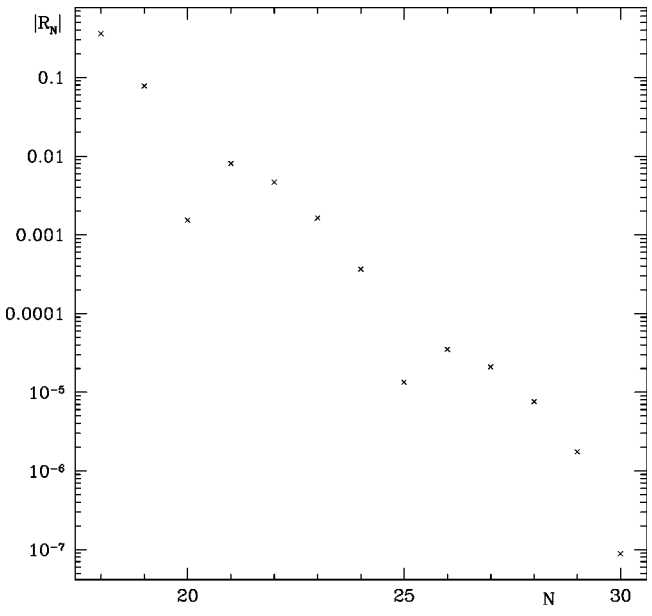

FIG. 5. Convergence of the LDE for the shift method in zero dimensions for the optimal values of α and β. The absolute value of the error is plotted on a logarithmic scale against N.

are not significantly more complicated than in ordinary perturbation theory. In the case of quantum mechanics the analytic nature of the δ-modified Hamiltonian allows one to go to very high orders, as we show in Sec. III C.

III. ONE DIMENSION

We now return to the quantum mechanical problem armed with the insights gained from the zero-dimensional analogue.

The initial evidence for the reality and positivity of the spectrum comes from a numerical analysis using the matrix method, whereby the Hamiltonian is rewritten in terms of the raising and lowering operators of the bare Hamiltonian and then regarded as a matrix H_{mn} in the infinite-dimensional space labelled by the occupation numbers n. If this matrix is truncated at some finite order N the resulting spectrum of eigenvalues can easily be calculated numerically, up to N of the order of 100. The pattern which emerges from such a calculation is that as N increases more and more of the lower eigenvalues emerge as real numbers from the amalgamation of complex conjugate pairs. After emergence they each tend to a definite positive limit. The ground-state energy is always present from $N=3$ onwards, and for $m=g=1$ is stable at $E_0=0.797342612$ to the 9th decimal place beyond $N=40$.

Since the E_i appear to be real, so also will be the finite-temperature partition function $Z(\beta)=\Sigma_i \exp(-\beta E_i)$, which can be calculated to the desired accuracy by including a sufficient number of energy levels in the sum.

To our knowledge there is no proof of the convergence of the matrix method, and our ultimate goal will be to provide such a proof for a suitable generalization of the LDE. In this section we discuss in turn the three variants of the δ expansion which were used in the zero-dimensional analogue.

A. Naive application of the LDE

The standard δ modification of Eq. (1) amounts basically to expanding around a SHO of a different, order-dependent frequency according to

$$H=\frac{1}{2}[p^2+(m^2+2\lambda)x^2]+\delta(igx^3-\lambda x^2). \tag{18}$$

The new effective interaction is $igx^3-\lambda x^2$, a polynomial. For a polynomial $V(x)$ one has the possibility of expanding to very high orders via the use of recursion relations rather than the much more cumbersome Rayleigh-Schrödinger perturbation theory.

These recursion relations are derived by first factoring off the asymptotic behavior of the wave function according to $\psi=e^{-(1/2)y^2}\chi$, where $y=x\sqrt{\omega}$ and $\omega=(m^2+2\lambda)^{1/2}$. Then χ and the ground-state energy are both expanded as power series in δ, with $\chi=1+\Sigma_{n=1}^{\infty}\delta^n\chi_n$ and $E=1/2\omega(1+\Sigma_{n=1}^{\infty}\delta^n\epsilon_n)$. The crucial observation is that with a polynomial interaction of order 3, the functions χ_n are polynomials of order $3n$, so that we can write $\chi_n=\Sigma_{p=1}^{3n}A_{n,p}(iy)^p$. Substituting these expressions into the Schrödinger equation gives coupled recursion relations for the ϵ_n and $A_{n,p}$, with $\epsilon_n=2A_{n,2}$.

Using MACSYMA we have evaluated E up to order $N=20$. The results are similar to those obtained for Z in zero dimensions, namely there are very large oscillations for small λ, while for larger λ there are a number of PMS points, with λ scaling like N, where the values of E tend to a constant different from the exact value.

B. The splitting procedure

In Sec. II B, we saw how applying LDE separately to Z^+ and Z^- gave a sequence which converged to the correct answer. A natural generalization of the splitting procedure to the path integral formulation of the quantum-mechanical partition function is given by $Z_\delta=Z_\delta^+ + Z_\delta^-$, where

$$Z_\delta^{\pm}=\int [dx]\theta\left(\pm\int_{-\beta/2}^{\beta/2}x\,dt\right)\exp\left(-\int_{-\beta/2}^{\beta/2}dt\left\{\frac{1}{2}[\dot{x}^2+(m^2+2\lambda_{\pm})x^2]+\delta(igx^3-\lambda_{\pm}x^2)\right\}\right). \tag{19}$$

This expression is rather formal as it stands, since it is not possible to evaluate the partition function directly in the presence of the step function. A possible resolution [11] is to use the integral representation of the step function:

$$\theta(z)=\frac{1}{2\pi i}\int_{-\infty}^{\infty}dq\,\frac{e^{iqz}}{q-i\epsilon} \tag{20}$$

which leads to

$$Z_\delta^{\pm}=\frac{1}{2\pi i}\int_{-\infty}^{\infty}dq\,\frac{Z_{\mp\delta}^{\pm}(q)}{q-i\epsilon}, \tag{21}$$

where

$$Z_\delta^\pm(q)=\int [dx]\exp\left(-\int_{-\beta/2}^{\beta/2}dt\left\{\frac{1}{2}[\dot{x}^2+(m^2+2\lambda_\pm)x^2]\mp iqx+\delta(igx^3-\lambda_\pm x^2)\right\}\right). \tag{22}$$

We have checked that the zero-dimensional analogue of (21) gives the same result as splitting up the integral directly.

To first order in δ, Eq. (21) gives (cf. [14])

$$Z^\pm=\frac{1}{2}\,\overline{Z}^\pm\left\{1+\beta\left[\lambda_\pm\Delta_\pm\mp\frac{2ig}{\sqrt{(2\pi)}\omega_\pm}(3\Delta_\pm-1/\omega_\pm^2)\right]\right\} \tag{23}$$

where $\omega_\pm^2=m^2+2\lambda_\pm$, $\Delta_\pm=\coth(1/2\,\omega_\pm\beta)/2\omega_\pm$, and $\overline{Z}^\pm=1/[2\sinh(1/2\,\omega_\pm\beta)]$ is the partition function for the simple harmonic oscillator.

For each of $Z^\pm$ there is a single complex PMS point, with $\lambda_-=\lambda_+^*$ and $Z^-=(Z^+)^*$. A similar calculation can be carried out to order δ^2, where in each case we find two complex PMS points. In Fig. 6 we plot the order δ and δ^2 approximations to the finite-temperature partition function as a function of the coupling g for $\beta=2$ and $m^2=1$, and compare them with the partition function calculated via $Z=\Sigma_i e^{-\beta E_i}$, with the energy eigenvalues[1] obtained by the matrix method. Both possible solutions at $O(\delta^2)$ are given, and it will be seen that they do not differ appreciably from each other. The graph provides some evidence for the convergence of this variant of the LDE method, but it is obviously not conclusive. However, it is difficult to go to significantly higher orders in this approach.

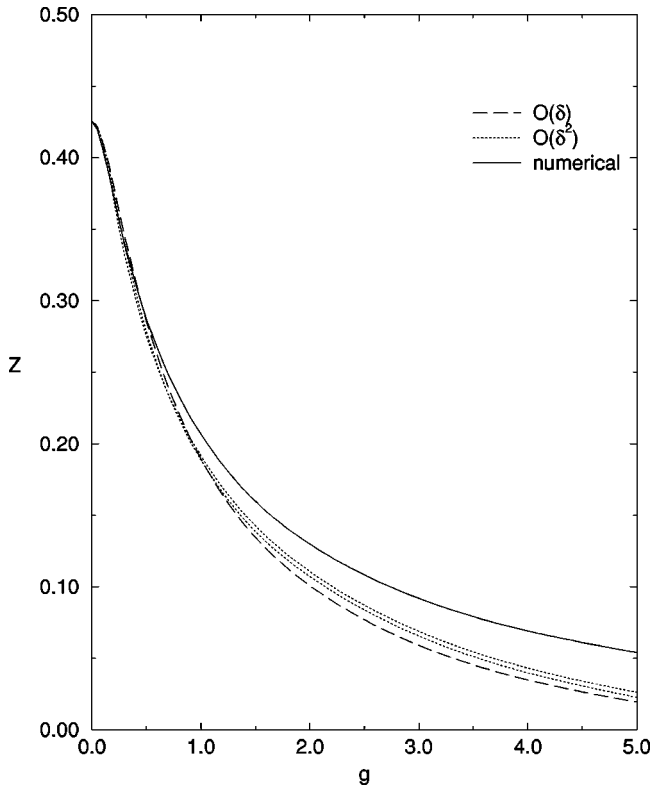

FIG. 6. The finite temperature partition function versus g, with fixed $m^2=1$ and $\beta=2$. The solid line is the numerical matrix method approximation, the dashed line the order δ approximation and the dotted lines the order δ^2 approximation using the splitting procedure.

C. The shift method

While we have seen that the splitting method can be implemented in the calculation of Z via the integral representation of the theta function, calculations to higher orders become progressively more complicated. Moreover, there is no obvious way to implement the method for the calculation of the energy levels themselves. The shift method, in contrast, can be applied to both Z and the energy levels, and in the latter case recursion relations allow one easily to go to high orders in δ.

In the context of quantum mechanics, the motivation for introducing a shift arises from a modification of the variational approach for the calculation of the ground-state energy using a trial Gaussian-like wave function

$$\psi=N[\theta(x)e^{-\lambda_1 x^2}+\theta(-x)e^{-\lambda_2 x^2}] \tag{24}$$

which distinguishes between $x<0$ and $x>0$. Because of the symmetry of the Hamiltonian under $x\to -x$ combined with complex conjugation it is clear that $\lambda_1=\lambda_2^*$.

In the approach of [13] a delta-modified Hamiltonian is constructed which coincides with the Gaussian variational approximation to lowest order in Rayleigh-Schrödinger perturbation theory, but then allows one systematically to improve on the variational approximation. The difficulty with this approach in the present problem is the non-analyticity of the Hamiltonian, which considerably complicates the calculation of the higher order terms.

The shift method can be regarded as a modification of this approach which retains its essential features but involves an analytic wave function and Hamiltonian. Thus with $\lambda_1=b+ia$ the wave function of Eq. (24) can be written as $\psi=Ne^{-(bx^2+iax|x|)}$, with the non-analyticity residing in the $|x|$ factor in the imaginary part of the exponent, which is, as a consequence, odd in x. The simplest alternative construction which retains this feature, while at the same time being analytic, is $\psi\sim e^{-(bx^2+iax)}$. Thus, a (pure imaginary) shift parameter ia has been introduced. A similar shift procedure is in fact described in [13], and a method given to δ modify a Hamiltonian incorporating a shift. Following this method we obtain the Hamiltonian

$$H_\delta=\frac{1}{2}p^2+\lambda x^2-\frac{1}{2}m^2a^2+ga^3+\delta\left[\frac{1}{2}m^2(x+ia)^2-\lambda x^2+\frac{1}{2}m^2a^2-ga^3+ig(x+ia)^3\right]. \tag{25}$$

The corresponding Lagrangian can be used for the calculation of the finite-temperature partition function as a func-

[1]For $\beta=2$ sufficient accuracy was attained by including at most 10 energy levels in the sum.

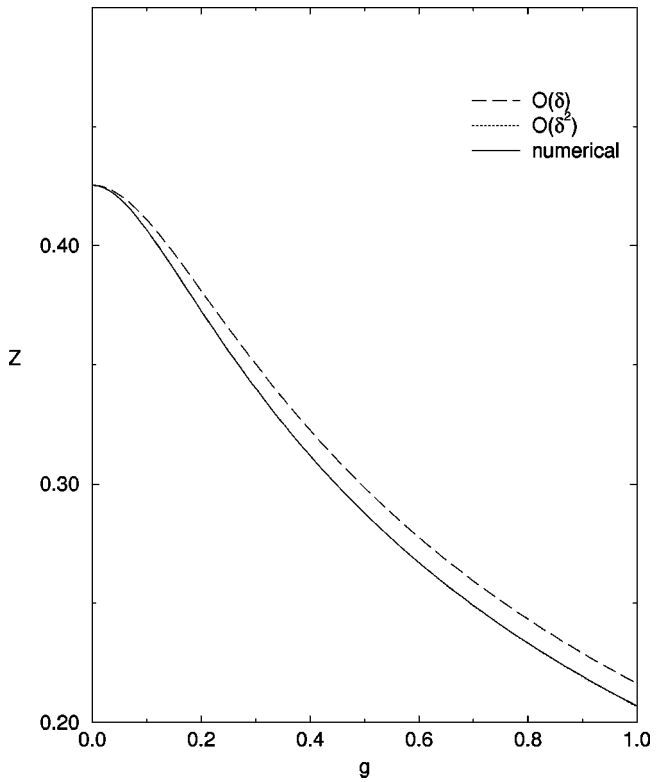

FIG. 7. The finite temperature partition function versus g, with fixed $m^2=1$ and $\beta=2$. The solid line is the numerical matrix method approximation, the dashed line the order δ approximation and the dotted line the order δ^2 approximation using the shift procedure.

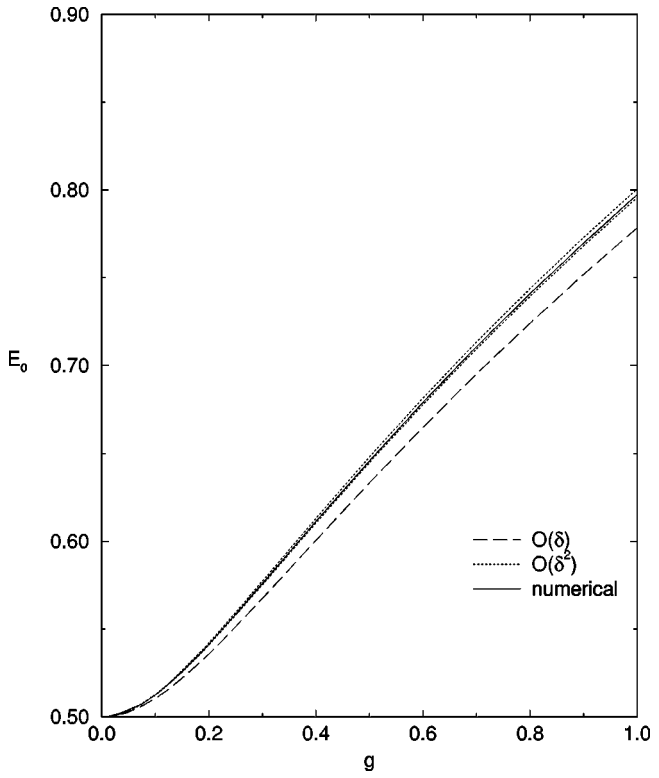

FIG. 8. The ground state energy versus g, with fixed $m^2=1$. The solid line is the numerical matrix method approximation, the dashed line the order δ approximation and the two dotted lines the order δ^2 approximation with two different PMS values for λ, using the shift method.

tional integral rather than as $\Sigma_i e^{-\beta E_i}$. The first–order approximation to Z obtained in this way [14] is

$$Z=\overline{Z}e^{\beta(1/2\,m^2a^2-ga^3)}\left[1+\beta\Delta_0\left(3ga+\lambda-\frac{1}{2}m^2\right)\right], \quad (26)$$

where $\Delta_0=\coth(\sqrt{2\lambda}\beta/2)/2\sqrt{2\lambda}$ and $\overline{Z}=1/[2\sinh(\sqrt{2\lambda}\beta)/2]$. We have also calculated the second-order approximation. In Fig. 7 we plot both the order δ and δ^2 approximations to Z as a function of the coupling g, where we have again taken $m^2=1$ and $\beta=2$. On the scale of the diagram the $O(\delta^2)$ approximation is almost indistinguishable from that obtained from the matrix method, which provides strong evidence for the rapid convergence of this variant of the LDE for the partition function.

However, the most convincing numerical evidence for the numerical convergence of the shift method comes from a calculation of the ground state energy. This can be done by generating recursion relations for Eq. (25) in a similar manner to that described in Sec. III A.

We have applied the recursion relation method to eighth order in δ, taking $m=1$ for definiteness. In Fig. 8 we show the results of the $O(\delta)$ and $O(\delta^2)$ calculations for a range of g, while in Table I we illustrate the nature of the numerical convergence up to $O(\delta^8)$ for $g=1$. In the latter case we consistently chose the largest PMS value of λ. The numerical evidence is strong and encourages us to seek a proof of convergence for the one-dimensional case.

IV. CONCLUSION

We have applied the LDE approximation method to the ix^3 potential in zero and in one dimension. A proof of convergence was given for the massless case in zero dimensions. The methods employed in zero dimensions were then extended to one dimension where numerical evidence of convergence for the partition function and the ground state energy was obtained. This evidence was particularly strong for the ground-state energy within the shift method.

The next step is to extend the proof of convergence from zero to one dimension. For Z one would need to apply techniques similar to those used in [6], while for the energy levels an extension of the methods of Guida *et al.* [16] would be necessary.

TABLE I. Results of the shift method for E_0 (to be compared with 0.797342612...).

N	λ	$-a$	E_0
1	1.651	0.2847	0.7785
2	2.352	0.3018	0.7957
3	2.648	0.3038	0.7967
4	2.942	0.3077	0.79720
5	3.188	0.3102	0.79729
6	3.4079	0.3121	0.79732
7	3.6088	0.3133	0.797334
8	3.79431	0.31418	0.797339

In higher dimensions one has to tackle the problems of regularization and renormalization within the LDE. A possible way of doing this has been outlined by the last-quoted authors in the context of the calculation of the critical exponents of ϕ_3^4 using the order-dependent mapping method, which is equivalent to the conventional LDE without a shift. It would be interesting to generalize their methods to trial actions including a shift, and to apply these techniques to the study of Lee-Yang zeros in statistical mechanics.

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