Exact solutions in the Yang-Mills-Wong theory

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This paper discusses in a systematical way exact retarded solutions to the classical SU(N) Yang-Mills equations with the source composed of several colored point particles. A new method of finding such solutions is reviewed. Relying on features of the solutions, a toy model of quark binding is suggested. According to this model, quarks forming a hadron are influenced by no confining force in spite of the presence of a linearly rising term of the potential. The large-N dynamics of quarks conforms well with Witten's phenomenology. On the semiclassical level, hadrons are color neutral in the Gauss law sense. Nevertheless, a specific multiplet structure is observable in the form of the Regge sequences related to infinite-dimensional unitary representations of SL(4,R) which is shown to be the color gauge group of the background field generated by any hadron. The simultaneous consideration of SU(N), SO(N), and Sp(N) as gauge groups offers a plausible explanation of the fact that clusters containing two or three quarks are more stable than multiquark clusters. [S0556-2821(98)05908-6]

I. INTRODUCTION

In this paper, we look into the Yang-Mills-Wong (YMW) theory, a classical non-Abelian gauge model describing closed systems of several spinless colored point particles interacting with a gluon field. Emphasis is put on features of exact solutions, some of which were obtained previously [1,2], and others are discussed for the first time.

A solvable nonlinear model is of interest by itself. Further still, one may expect that exact solutions of the YMW theory will be useful in studying nonperturbative vacua of quantum chromodynamics (QCD). We get, at the least, a toy model of bound states in QCD.

It may appear at first glance that classical particles have nothing to do with real quarks, but closer inspection casts doubt on this belief. Indeed, a medium of the normal nuclear density offers a fertile ground for creations and annihilations of quark pairs. Nevertheless, the quark-antiquark sea is largely suppressed in hadrons. According to Zweig's rule, a quark and an antiquark with opposite quantum numbers defy their annihilation. Such persistence of particles is typical for the classical picture.

The classicality of constituent quarks is difficult to understand, but one can describe it explicitly. The bulk of the hadron phenomenology is grasped by planar diagrams, which implies in particular that world lines of valence quarks are subjected to neither bifurcation nor termination in the Feynman path integral, unless hadrons collide or decay. Thus, to a good approximation, the number of bound quarks remains fixed.

Starting from the QCD Lagrangian, Wong was able to show [3] that the behavior of a quark in the limit $\hbar \to 0$ is governed by the classical equation of motion of a spinless colored point particle.[1] Alternative methods of "dequantization" [4] lead to classical actions of the spinning particle with Wong's action as a constituent. Thus the classical limit of the quark dynamics remains to be completed. Wong's dynamics may provisionally be regarded as the simplest reasonable approximation to the limiting theory. In addition, it would be appropriate to use spinless particles as the starting point of the bound quark description, for measurements of the polarized proton structure in deep inelastic leptoproduction indicate that quarks carry only a small fraction of the spin of the nucleon [5]. The Wong particles will hereafter be called quarks though this name is rather conventional and should not be confused with the standard QCD term, taking into account that the Wong color charges are in the adjoint while quark fields are in the fundamental representation of the gauge group.

As is well known, the classical limit of QCD is related to the limit of large number of colors [6]. Substituting SU(3)$_c$ by SU(N) and going to the limit

$$N \to \infty, \quad g^2 N = \text{const},$$

't Hooft established that the planar diagrams are dominating in this limit [7]. Witten found [8] that the real hadronic world is qualitatively displayed even in the zeroth approximation of the $1/N$ expansion. In the limit $N \to \infty$, the vacuum expectation value of the product of gauge invariant operators obeys the so-called factorization relation, and quantum fluctuations disappear [8]. Thus QCD becomes a classical theory as $N \to \infty$. We suggest that the large-N YMW theory is intimately related to the classical limit of QCD.

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[1]The present model is not quite that obtained in the original Wong approach. We deal with arbitrary number of color particles while Wong's procedure is matched with a single particle; the situations are apparently different if one keeps in mind the nonlinear field dynamics.

Note, however, that the confinement problem is out of the question now. Indeed, it is conceivable that quarks constituting a hadron experience an attractive constant force originating from a term of the potential A_μ which linearly rises with distance between the quarks, and such a behavior of A_μ is to provide the area law for the Wilson loop functional [9]. Are we correct in interpreting the area law as the evidence of the constant attractive force? As will be shown, an exact classical solution A_μ with the linearly rising term actually exists. Although this term contributes to the field strength, it produces no force. The general reason for such a surprising result is the conformal invariance. The linearly rising term violates the scale invariance. While such a violation being allowable for the gauge quantities A_μ and $F_{\mu\nu}$, it cannot be tolerated for observables. One may expect a dimensional parameter, measuring a gap in the energy spectrum and violating the scale symmetry, to emerge only upon quantization leading to anomalies. Meanwhile exact classical solutions are crucial in learning the symmetry of the vacuum.

One believes two phases of the strong interacting matter to exist, hot and cold, which must be distinguished by their symmetry. At high temperatures, the asymptotical freedom dominates, hence the conventional $SU(3)_c$ symmetry is inherent in the hot phase. On the other hand, Ne'eman and Šijački [10] developed an exhaustive phenomenological classification of hadrons on the basis of infinite-dimensional unitary representations of $SL(4,R)$, which hints that $SL(4,R)$ is the cold phase symmetry. Where does this $SL(4,R)$ come from?

Coleman [11] argued that the symmetry of the vacuum is the symmetry of world. Given the vacuum invariant under $SL(4,R)$, excitations about it possess the same symmetry. Since the symmetry of the gluon vacuum is nothing but the symmetry of the background field, the responsibility for $SL(4,R)$ rests with the background described by a certain solution of the QCD equations in the classical limit. It is the background generated by quarks in hadrons that provides the $SL(4,R)$ relief for gluon excitations.

We will find two classes of exact retarded solutions to the classical Yang-Mills (YM) equations. Solutions of the first class, invariant under $SU(N)$, appear to be related to the background in the hot phase. Solutions of the second class might be treated as the background generated by bound quarks in the cold phase. These solutions are complex valued with respect to the Lie algebra su(N), but one can convert them to the real form to yield the invariance under $SL(N,R)$ or its subgroups. In particular, the background generated by any three-quark cluster is invariant under $SL(4,R)$, and that generated by any two-quark cluster is invariant under $SL(3,R)$.

Notice that $SL(4,R)$ of Ne'eman and Šijački operates in spacetime while the present $SL(4,R)$ acts in the color space. However, we attempt to interweave two arenas by reference to that color degrees of freedom may be convertible into spin degrees of freedom, the fact discovered by Jackiw and Rebbi, and Hasenfratz and 't Hooft [12].

The paper is organized as follows. Section II outlines the general formalism of the YMW theory. The next section is devoted to a justification of the Ansatz whereby we seek exact retarded solutions of the YM equations with the source composed of several arbitrarily moving quarks. Finding such

solutions is traced by the simplest example of the single-quark source, Sec. IV. Properties of the background generated by two-quark sources are reviewed in Sec. V. Fields generated by arbitrary number of color particles, and their features are considered in Sec. VI. We show that bound quarks are affected by no external force as $N\rightarrow\infty$. The equation of motion of a dressed quark is discussed, and its exact solution in the absence of external forces is given. The large-N dynamics of quarks is shown to conform with Witten's phenomenology. Moreover, a step forward can be made by the simultaneous consideration of $SU(N)$, $SO(N)$, and $Sp(N)$ as gauge groups, which offers a plausible explanation of the fact that clusters composed of two or three quarks are more stable than multiquark clusters. The stability of the solutions is the subject of Sec. VII. We conclude that free quarks are ruled out by a consistency reasoning. Issues concerning the semiclassical quantization and the resulting picture are considered in Sec. VIII. The fulfilment of the Wilson criterion has been confirmed. We show that any hadron is color neutral in the sense of the Gauss law. Nevertheless, a certain multiplet structure is observable. These multiplets are described by infinite-dimensional unitary representations of $SL(4,R)$, the gauge group of the background field generated by any hadron.

II. GENERAL FORMALISM

We work in Minkowski space with the metric $\eta_{\mu\nu} = \mathrm{diag}(+,-,-,-)$. Let us consider classical point particles interacting with the $SU(N)$ Yang-Mills field. The particles will be called quarks and labeled by index $I, I = 1, \ldots, K$. Each quark is assigned a color charge Q_I^a [transforming as the adjoint representation of $SU(N)$, the color index a runs from 1 to N^2-1], and a bare mass m_0^I. Any other specification is omitted, so that quarks and antiquarks are indistinguishable in the present context. Let every quark be moving along a timelike world line $z_I^\mu(\tau_I)$ parametrized by the proper time τ_I. This gives rise to the current

$$j_\mu(x) = \sum_{I=1}^K \int d\tau_I Q_I(\tau_I) v_\mu^I(\tau_I) \delta^4[x - z_I(\tau_I)], \quad (1)$$

where $Q_I = Q_I^a T_a$, T_a are generators of $SU(N)$, $v_\mu^I \equiv \dot{z}_\mu^I \equiv dz_\mu^I/d\tau_I$ is the four-velocity of Ith quark. The action is written [13] as

$$S = -\sum_{I=1}^K \int d\tau_I (m_0^I \sqrt{v_\mu^I v_I^\mu} + \mathrm{tr}\, Z_I \lambda_I^{-1} \dot{\lambda}_I)$$
$$- \int d^4x\, \mathrm{tr}\left(j_\mu A^\mu + \frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu}\right). \quad (2)$$

Here, $\lambda_I = \lambda_I(\tau_I)$ are time-dependent elements of $SU(N)$, $Z_I = e_I^a T_a$, e_I^a being some constants whereby the color charge is specified, $Q_I = \lambda_I Z_I \lambda_I^{-1}$. The field strength is

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu],$$

with g being the coupling constant. The middle two terms of Eq. (2) can be combined into $-\Sigma \mathrm{tr}\, Z_I \lambda_I^{-1} D_\tau \lambda_I$ with the covariant derivative $D_\tau \equiv d/d\tau_I + v_I^\mu A_\mu(z_I)$. Since λ_I re-

sponds to a local gauge transformation by $\lambda_I \to \lambda_I' = \Omega^{-1}\lambda_I$, the gauge invariance of the action is quite clear. Elements T_a of the Lie algebra su(N) satisfy the commutation relations

$$[T_a, T_b] = i f_{abc} T^c \tag{3}$$

with the structure constants f_{abc} of SU(N), and the orthonormalization condition

$$\text{tr}(T_a T_b) = \delta_{ab}. \tag{4}$$

Note that the invariance of the action under SU(N) automatically entails the invariance under SL(N,C) unless a constraint is imposed so as to preserve the real valueness of the gauge field variables. If we have no prior knowledge of the symmetry, it can be identified by the structure constants which are present in the action. The specific values of f_{abc} entering into Eq. (3) imply that S is invariant under SU(N). However, for any simple complex Lie algebra, there exists a basis, referred to as the Cartan basis, such that the structure constants are found to be real, antisymmetric, and identical to the structure constants of the real compact form of this Lie algebra [14]. The basis of su(N) is simultaneously the Cartan basis of its complexification sl(N,C). Thus the presence of the structure constants of SU(N) in Eq. (3) need not be the evidence for that the symmetry of S is SU(N); allowing for the complex-valued field variables, we enlarge the symmetry up to SL(N,C).

The Euler-Lagrange equations for the action (2) are the Yang-Mills equations

$$\mathcal{E}_\mu \equiv D^\nu F_{\mu\nu} + 4\pi j_\mu = 0 \tag{5}$$

with $D^\nu = \partial^\nu - ig[A^\nu,]$, the equation of motion of Ith bare quark

$$\varepsilon_I^\lambda \equiv m_0^I a_I^\lambda - v_\mu^I \, \text{tr}[Q_I F^{\lambda\mu}(z_I)] = 0, \tag{6}$$

where $a_I^\lambda \equiv \dot{v}_I^\lambda$ is the four-acceleration of this quark, and the Wong equation

$$\dot{Q}_I = -ig[Q_I, v_\mu^I A^\mu(z_I)] \tag{7}$$

describing the evolution of the color charge of Ith quark.

It follows from Eq. (7) that

$$\frac{d}{d\tau_I} \, \text{tr} \, Q_I^2 = -2ig \, \text{tr}(Q_I[Q_I, v_\mu^I A^\mu]) = 0,$$

i.e., the magnitude of Q_I remains unchanged, specifically, Q_I may be constant.

The total color charge of a K-quark system is defined by

$$Q = \int_\Sigma d\sigma^\mu j_\mu, \tag{8}$$

where the integral is taken over an arbitrary spacelike hypersurface Σ, and the domain of integration covers all K points of intersection of Σ with the world lines. However, it would be more convenient to do with somewhat narrow class of hypersurfaces with a rigidly fixed mutual arrangement of any hypersurface and the worldlines in the vicinities of their in-

tersections. Consider, for example, the intersection of a hyperplane by a timelike curve at a right angle. Such an arrangement can be achieved for any hypersurface Σ by replacing a small fragment of Σ in the vicinity of every intersection point z_μ^I by a fragment of a hyperplane orthogonal to the worldline at z_μ^I and smoothing off this piecewise hypersurface. The resulting hypersurface will be called *locally adjusted* and denoted by $\widetilde{\Sigma}$.

Since j_μ is not a conserved current, the total color charge Q is in general hypersurface dependent. But Q ceases to depend on $\widetilde{\Sigma}$ if the color charge of each quark is constant,

$$\dot{Q}_I = 0, \tag{9}$$

which imposes certain restrictions on the form of A_μ. We will discuss just this case.

In view of Eq. (5), the definition of Q can also be rewritten in terms of the field variables:

$$Q = \frac{1}{4\pi} \int_{\widetilde{\Sigma}} d\sigma_\nu D_\mu F^{\mu\nu}. \tag{10}$$

Under the local gauge transformations

$$A_\mu \to \Omega^{-1} A_\mu \Omega - \frac{i}{g} \Omega^{-1} \partial_\mu \Omega,$$

the covariant derivatives of the field strength transform as

$$D_\mu F^{\mu\nu} \to \Omega^{-1} D_\mu F^{\mu\nu} \Omega,$$

so that Eq. (5) is covariant providing j_μ transforms as

$$j_\mu \to \Omega^{-1} j_\mu \Omega.$$

One could always find such unitary matrix Ω as to diagonalize Hermitean matrix j_μ. Since the Lie algebra su(N) is of rank $N-1$, there exist $N-1$ diagonal elements H_i. Thus, without loss of generality, one can set

$$Q_I(\tau_I) = \sum_{i=1}^{N-1} e_I^i(\tau_I) H_i. \tag{11}$$

We will find e_I^i to be constants fixed exactly by the solution itself.

Picking Q_I in the form (11) reduces the gauge freedom of A_μ. The color charges Q_I may thereon be rotated within the Cartan subgroup, in particular, through discrete angles associated with permutations of H_i. We will see that the diagonalization of Q_I leads to

$$[Q_I, v_\mu^I A^\mu] = 0 \tag{12}$$

which can be treated as a gauge fixing condition.

The symmetric energy-momentum tensor is

$$T_{\mu\nu} = \Theta_{\mu\nu} + t_{\mu\nu},$$

where

$$\Theta_{\mu\nu} = \frac{1}{4\pi} \, \text{tr}\left(F_{\mu\alpha} F_\nu^\alpha + \frac{1}{4} \eta_{\mu\nu} F_{\alpha\beta} F^{\alpha\beta}\right), \tag{13}$$

$$t_{\mu\nu}(x) = \sum_{I=1}^{K} m_0^I \int d\tau_I v_\mu^I(\tau_I) v_\nu^I(\tau_I) \delta^4[x - z_I(\tau_I)].$$

(14)

One can readily verify the Noether identity

$$\partial_\mu T^{\lambda\mu} = \frac{1}{4\pi} \mathrm{tr}(\mathcal{E}_\mu F^{\lambda\mu}) + \sum_{I=1}^{K} \int d\tau_I \varepsilon_I^\lambda \delta^4[x - z_I(\tau_I)],$$

(15)

where $\mathcal{E}_\mu$ and ε_I^λ are the left-hand sides (LHS's) of Eqs. (5) and (6), respectively.

As is well known (see, e.g., [15]), the conformal invariance is ensured by the condition

$$T_\mu^\mu = 0.$$

Leaving aside sophistications regarding the modification of the energy-momentum tensor on the quantum level [16], we merely remark that the YM sector is conformally invariant for

$$\Theta_\mu^\mu = 0.$$

Equation (13) shows that this condition is met for $D=4$. Thus, in the four-dimensional (4D) case, the spacetime symmetry of the YM equations is enlarged to include the conformal transformations.

The regularized total four-momentum can be defined as

$$P_\mu = \int_{\widetilde{\Sigma}} d\sigma^\nu T_{\mu\nu},$$

(16)

where the integration of $\Theta_{\mu\nu}$ is taken over an adjusted hypersurface $\widetilde{\Sigma}$ with small invariant holes cut out by the future light cones drawn from points on the worldlines slightly below their intersections with $\widetilde{\Sigma}$.

To gain insight into the YMW dynamics, one should find simultaneous solutions of Eqs. (5), (6), and (7). At first, one solves Eq. (5), i.e., A_μ is expressed in terms of z_I^μ. Since the resulting field is singular on the worldlines, its insertion into Eqs. (6) and (7) brings to ultraviolet divergences. There are two means of tackling this difficulty, the mass renormalization and the restriction to such situations that Eq. (7) can be put in its trivial form $\dot{Q}_I = 0$. Upon the mass renormalization, one derives an equation of motion of the dressed quark allowing for finite self-action. Finally, if one succeeds in solving this equation, then the set of dynamical equations is entirely integrated.

A more refined approach is to use the Noether identity (15), implying that the equation of motion of the dressed quark is due to substituting a solution of the YM equations into the equation of motion of the bare quark, accompanied by the mass renormalization. On the other hand, Eq. (15) expresses the local energy-momentum balance of the whole system.

III. ANSATZ

It is easily seen that the coefficients for highest derivatives in Eq. (5) coincide with those in Maxwell equations.

Thus the characteristic cones are identical in both theories.

The retarded signal is of primary importance for every 4D classical field theory because it is associated with the idea of causality. Let us turn first to the single-quark case. With a given point of observation x_μ, such a signal carries information on a single point of the world line, z_μ^{ret}. Indeed, the support of the retarded function of the wave equation

$$D_{\mathrm{ret}}(x) = 2\,\theta(x_0)\,\delta(x^2)$$

(17)

is localized on the boundary of the future light cone, with expression (17) containing no derivatives. The advanced function $D_{\mathrm{adv}}(x)$ reveals similar properties but its physical meaning is less clear. Other Green functions carry signals from several points or even from some region of the world line.

Thus the retarded YM potential A_μ generated by a single quark may depend on two kinematic quantities, the four-velocity v_μ at the retarded instant τ_{ret} and the lightlike vector $R_\mu = x_\mu - z_\mu^{\mathrm{ret}}$ drawn from the point of emission, z_μ^{ret} to the point of observation x_μ.

We recall elements of technique of covariant retarded quantities [17–19]. Consider a plane built out of R^μ and v^μ. A normalized vector u^μ orthogonal to v^μ and the lightlike vector

$$c^\mu \equiv v^\mu + u^\mu$$

(18)

can be drawn here. All this is expressible analytically as

$$v^2 = -u^2 = 1, \quad v\cdot u = 0, \quad c^2 = 0, \quad c\cdot v = -c\cdot u = 1,$$

$$R^\mu = \rho c^\mu,$$

where the scalar

$$\rho = -u\cdot R = v\cdot R$$

(19)

represents the distance between z_μ^{ret} and x_μ in the reference frame with the arrow of time v^μ.

From the condition $R^2 = 0$, one readily derives the following rules of differentiation:

$$\partial_\mu \tau = c_\mu,$$

(20)

$$\partial_\mu \rho = v_\mu + [\rho(a\cdot u) - 1]c_\mu.$$

(21)

This enables us to find derivatives of any kinematic quantities, for example, $\partial_\mu v_\nu = a_\nu c_\mu$.

Let us further turn to the K-quark case. Define the retarded invariants

$$\rho_I \equiv R^I \cdot v^I, \quad \beta_{IJ} \equiv v^I(R^I - R^J), \quad \gamma_{IJ} \equiv v^I \cdot v^J,$$

$$\Delta_{IJ} \equiv (R^I - R^J)^2 = -2R^I \cdot R^J,$$

(22)

where $I, J = 1, \ldots, K$, and v_μ^I is taken at τ_I^{ret}. We have

$$\partial_\mu \beta_{IJ} = [a^I \cdot (R^I - R^J) - 1]c_\mu^I + \gamma_{IJ} c_\mu^J,$$

$$\partial_\mu \gamma_{IJ} = (a^I \cdot v^J)c_\mu^I + (a^J \cdot v^I)c_\mu^J,$$

$$\partial_\mu \Delta_{IJ} = -2(\beta_{IJ} c_\mu^I + \beta_{JI} c_\mu^J).$$

(23)

Thereafter the generic retarded solution to Eq. (5) is

$$A_\mu(x) = \sum_{I=1}^{K} \sum_{a=1}^{N^2-1} T_a(v_\mu^I f^{aI} + R_\mu^I h^{aI}), \qquad (24)$$

where the sought functions f^{aI} and h^{aI} may depend on ρ_I, β_{IJ}, γ_{IJ}, and Δ_{IJ} [2].

The expression (24) is inserted in Eq. (5) and the differentiations are made by means of Eqs. (20), (21), and (23). One gets the expressions in which it is necessary to equate to zero the coefficients for the linearly independent vectors c_μ^I, v_μ^I, and a_μ^I, as well as for each color basis element T_a. Recall that we search for solutions of the YM equations off the quark world lines where the differentiation formulas (20), (21), and (23) are just valid. If the procedure is to be self-consistent, we must separately equate to zero coefficients for every scalar kinematic quantity of which f^{aI} and h^{aI} are independent, e.g., scalars containing a_μ^I.

A distinctive feature of this procedure is that any supplementary condition on A_μ is unnecessary. We thus arrive at a class of equivalence of solutions A_μ related by gauge transformations rather than a particular potential.

One should emphasize that the ansatz (24) rests crucially on the following points: the field is massless; the dynamics is gauge invariant; the signals are retarded; the dimension of spacetime is four; the world lines are timelike.

Given a massive field, the support of the retarded Green function is the interior of the past light cone. Therefore the expression (24) is no longer solution of the field equation. It is clear that this scheme is unsuited for the dynamics without gauge invariance; the case is typified by replacing $\Box \eta_{\mu\nu} - \partial_\mu \partial_\nu$ by $\Box \eta_{\mu\nu}$.

We recall also that, in $2n$-dimensional spacetimes with $n>2$, the retarded function is built out of derivatives of the δ function, therefore the retarded signal carries information on $v_\mu^{\rm ret}$ as well as $a_\mu^{\rm ret}$, and the like. It would be necessary to supplement the expression (24) by appropriate kinematical terms. As to $(2n+1)$-dimensional spacetimes, the retarded signal carries information on the entire history of the source preceeding the point $z_\mu^{\rm ret}$, and the Huygens principle underlying our approach turns out to be invalid.

IV. YANG-MILLS FIELD GENERATED BY A SINGLE QUARK

If the source is a single quark, then it is sufficient to consider the gauge group SU(2). The extension to SU(N) offers no significant changes in the final results.

We specify a moving basis of the color space spanned by a triplet Γ_1^a, Γ_2^a, $\Gamma_3^a \equiv Q^a / \sqrt{Q^2}$ (with Q^a precessing around $v^\mu A_\mu^a$ in the color space) obeying the condition of orientability

$$\varepsilon_{abc} \Gamma_i^a(\tau) \Gamma_j^b(\tau) = \varepsilon_{ijk} \Gamma_c^k(\tau), \qquad (25)$$

where ε^{abc} are the structure constants of SU(2), and the condition of orthonormalization

$$\delta_{ab} \Gamma_i^a(\tau) \Gamma_j^b(\tau) = \frac{1}{2} \delta_{ij}. \qquad (26)$$

These relations are equivalent to Eqs. (3) and (4).

A retarded solution of Eq. (5) is written [1] as

$$\Gamma_j^a(\tau) = {\rm const,}$$

$$A_\mu^a = \mp \frac{2i}{g} \Gamma_3^a \frac{v_\mu}{\rho} + \kappa(\Gamma_1^a \pm i\Gamma_2^a) R_\mu, \qquad (27)$$

where κ is an arbitrary nonzero integration constant with the dimensionality of (length)$^{-2}$.

The first term of A_μ^a is a generalized Liénard-Wiechert part of the potential. The coefficient for Γ_3^a is an imaginary integration constant e_3 exactly fixed by the condition

$$g^2 e_3^2 = -4 \qquad (28)$$

assuring the compatibility of an overdetermined system of nonlinear equations for $h_j(\rho)$.

From Eq. (27), one obtains the field strength

$$F = c \wedge W, \qquad (29)$$

$$W_\mu^a = \mp \frac{2i}{g} \Gamma_3^a \frac{V_\mu}{\rho^2} + \kappa(\Gamma_1^a \pm i\Gamma_2^a) v_\mu. \qquad (30)$$

Here, the symbol $\wedge$ signifies the exterior product of two four-vectors, and

$$V_\mu = v_\mu + \rho[a_\mu + (a \cdot u) u_\mu]. \qquad (31)$$

Notice that the linearly rising term of A_μ^a contributes to the field strength, hence it cannot be purely gauge.

Now, the Gauss law can be represented in its familiar form: The flux of the generalized Liénard-Wiechert part of the field strength through any two-dimensional surface, surrounding the source with the color charge Q^a, equals $4\pi Q^a$, other terms cancel out.[2] In combination with Eqs. (27) and (29)–(31), this yields the color charge of the quark

$$Q^a = \pm \frac{2i}{g} \Gamma_3^a. \qquad (32)$$

We draw attention to the non-analytical dependence of A_μ^a on the coupling g. It follows that A_μ^a involves a nonperturbative information, and this is a good reason for sampling it as a nontrivial background in the semiclassical description.

For $\kappa = 0$, the condition (28) does not appear, and the retarded solution is

$$A_\mu^a = q \Gamma_3^a \frac{v_\mu}{\rho}, \qquad (33)$$

with q being an arbitrary constant. Considering the field invariants

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[2]Although this result was established in the single-quark case [19], it can be extended to the general K-quark case proceeding from Eq. (10) and taking advantage of a locally adjusted hypersurface $\bar{\Sigma}$.

$$F^a_{\mu\nu}{}^*F^{\mu\nu}_a = 0, \quad F^a_{\mu\nu}F^{\mu\nu}_a = \frac{4}{g^2\rho^4},$$

one recognizes the configuration (27) to be neither self-dual nor anti-self-dual. Equation (27) describes the field of the magnetic type while Eq. (33) describes the field of the electric type.

If the retarded condition is replaced by the advanced one, then we arrive at similar expressions for the potential and the field strength, the only modification is the change of sign for $i\Gamma^a_2$ in Eqs. (27) and (30) as well as for the term in the square brackets of Eq. (31). This implies that the spacetime and color arguments of A^a_μ are correlated in a specific fashion; under time reversal, associated with replacing the retarded condition by the advanced one, the isotropic directions in the color space $\Gamma^a_1 + i\Gamma^a_2$ and $\Gamma^a_1 - i\Gamma^a_2$ interchange.

Setting a new basis of the color space

$$\mathcal{T}_1 \equiv i\Gamma_1, \quad \mathcal{T}_2 \equiv \Gamma_2, \quad \mathcal{T}_3 \equiv i\Gamma_3,$$

and considering the parameter κ to be imaginary, one rearranges Eq. (27) to the form

$$A_\mu = \mathcal{A}^a_\mu \mathcal{T}_a$$

with real-valued $\mathcal{A}^a_\mu$. Elements of the new basis can be represented by traceless imaginary-valued 2×2 matrices satisfying the commutation relations of the Lie algebra sl(2,R), as becomes clear upon specifying the abstract basis by Pauli matrices $\Gamma_j = \sigma_j/2$.

It may appear that the doubling of colored degrees of freedom is attributable to opposite color charges, as Eq. (32) suggests. However, the complex-conjugate potentials (27), being represented in the matrix form, are interconvertible by the gauge transformation

$$\bar{A}_\mu = \Omega^{-1} A_\mu \Omega$$

with

$$\Omega = \Omega^{-1} = \sigma_1.$$

Thus the availability of opposite color charges is deceptive in the single quark case.

To sum up, we have the retarded solutions (27) and (33) describing the YM field of two different phases. The first phase is specified by the noncompact gauge group SL(2,R) while the second by the compact group SU(2).

Let us verify that Eqs. (27) and (33) give an exhaustive collection of retarded solutions to Eq. (5) in that there are no other functions $f_j(\rho)$ and $h_j(\rho)$ representing solutions. The potential generated by a single quark is

$$A^a_\mu = \sum_{j=1}^{3} \Gamma^a_j(\tau)[f_j(\rho)v_\mu + h_j(\rho)R_\mu]. \quad (34)$$

Insert it into Eq. (5). From the requirement of vanishing the coefficient for a_μ, one obtains

$$\rho f'_j + f_j = 0$$

which is readily integrated to yield

$$f_j(\rho) = \frac{e_j}{\rho}, \quad e_j = \text{const}. \quad (35)$$

Substituting Eqs. (34) and (35) in Eq. (7) results in divergent terms which cannot be removed by the standard renormalization procedure since there is no physical parameter of an appropriate dimensionality at our disposal. This difficulty can be circumvented, if one takes all constants of integration in Eq. (35) to be zero, with one exception, say, e_3. Thus,

$$\dot{\Gamma}^a_3 = 0.$$

There remains the possibility of arbitrary rotations of Γ^a_1 and Γ^a_2 around Γ^a_3. Having regard to the Gauss law applied to small distances, we impose an additional restriction on the class of sought functions: The behavior of $h_j(\rho)$ should be less singular than that of $f_j(\rho)$, namely, $\rho h_j(\rho) \to 0$ as $\rho \to 0$. Then, due to lack of finite parameters whereby these rotations can be specified, we conclude that

$$\Gamma^a_1(\tau) = \Gamma^a_2(\tau) = \text{const}.$$

Let us define two isotropic color vectors

$$\Gamma^a_\pm \equiv \Gamma^a_1 \pm i\Gamma^a_2$$

which together with Γ^a_3 span a new time-independent color basis. Equation (34) takes the form

$$A^a_\mu = e_3\Gamma^a_3\frac{v_\mu}{\rho} + (\Gamma^a_3 h_3 + \Gamma^a_+ h_+ + \Gamma^a_- h_-)R_\mu. \quad (36)$$

Substitute Eq. (36) in Eq. (5) and equate to zero the coefficients for v_ν and R_ν. In the latter case, we separately equate to zero the coefficient for $a \cdot u$ and the sum of remaining terms. Introducing $\xi \equiv \ln\rho$ and denoting the derivative with respect to ξ by a prime, we get

$$h''_3 + 3h'_3 + 2h_3 = 0, \quad (37)$$

$$h''_+ + (3 - 2ige_3)h'_+ + (2 - 3ige_3 - g^2e_3^2)h_+ = 0, \quad (38)$$

$$h''_3 + h'_3 = 0, \quad (39)$$

$$h''_+ + (1 - ige_3)h'_+ = 0, \quad (40)$$

$$h_+ h'_- - h_- h'_+ + 2ige_3 h_+ h_- = 0, \quad (41)$$

$$e_3(h'_+ + 2h_+) + \exp(2\xi)(h'_+ h_3 - h'_3 h_+)$$
$$- ige_3[e_3 + \exp(2\xi)h_3]h_+ = 0, \quad (42)$$

and in addition three equations resulting from Eqs. (38), (40), and (42) by the complex conjugation and the replacement of h_+ by h_-.

We have arrived at an apparently overdetermined set of equations: Nine equations are used to determine three sought functions. It can be resolved if some compatibility conditions are satisfied, and this is accomplished if the constants of integration take certain fixed values.

Since Eqs. (37)–(40) and their complex-conjugate are linear, we are seeking the simultaneous solution of these equations in the form

$$h_3 \propto \exp(\lambda_3 \xi), \quad h_+ \propto \exp(\lambda_+ \xi), \quad h_- = \bar{h}_+ .$$

We get $\lambda_3 = -2$ or $\lambda_3 = -1$ from Eq. (37), and $\lambda_3 = 0$ or $\lambda_3 = -1$ from Eq. (39). Thus, Eqs. (37) and (39) are compatible if

$$h_3 = \kappa_3 \exp(-\xi). \tag{43}$$

One can find next $\lambda_+ = -2 + ige_3$ or $\lambda_+ = -1 + ige_3$ from Eq. (38), and $\lambda_+ = 0$ or $\lambda_+ = -1 + ige_3$ from Eq. (40). Thus Eqs. (38) and (40) are compatible if $\lambda_+ = -1 + ige_3$. The compatibility can also be established in the case $\lambda_+ = 0$ which is achievable for $e_3 = -2i/g$ or $e_3 = -i/g$.

Let us examine further the compatibility of Eqs. (38), (40) with Eq. (41). Assuming $h_+ h_- \neq 0$, we conclude from Eq. (41) that

$$\lambda_- - \lambda_+ + 2ige_3 = 0.$$

This equation is satisfied identically for $\lambda_+ = \bar{\lambda}_- = -1 + ige_3$, but it has no solution when $\lambda_+ = \lambda_- = 0$. The compatibility of Eqs. (38), (40), and (41) is also established for $\lambda_+ = \lambda_- = 0$ if

$$h_+ h_- = 0. \tag{44}$$

We consider lastly the compatibility of Eq. (42) with Eqs. (37)–(41). Taking $\lambda_+ = -1 + ige_3$, in combination with Eq. (43), we obtain $e_3 h_+ = 0$, while the complex-conjugate equation yields $e_3 h_- = 0$. This implies either $e_3 = 0$, with reducing the potential (36) to the form

$$A_\mu^a = \gamma^a \frac{R_\mu}{\rho}, \tag{45}$$

where γ^a is an arbitrarily color vector or $h_+ = h_- = 0$ resulting in

$$A_\mu^a = \Gamma_3^a \frac{e_3 v_\mu + \kappa_3 R_\mu}{\rho}. \tag{46}$$

Recall that $R_\mu/\rho = \partial_\mu \tau$, so that the potential (45) is purely gauge while the expression (46) differs from Eq. (33) by a gauge term.

For $\lambda_+ = \lambda_- = 0$, Eq. (42) becomes

$$[e_3(2 - ige_3) + \exp(2\xi)h_3]h_+ = 0. \tag{47}$$

If $e_3 = -2i/g$, then Eq. (47) reduces to $h_+ h_3 = 0$, and the complex-conjugate equation is $h_- h_3 = 0$. This provides two possibilities. First, $h_+ = h_- = 0$, resulting in a purely gauge potential, Eq. (45). Second, $h_3 = 0$, which is allowable for $\kappa_3 = 0$, and taking into account Eq. (44), one arrives at the expression (27).

In the case $e_3 = -i/g$, Eq. (47) is satisfied only for $h_+ = 0$. With the corresponding result for the complex-conjugate equation, namely, $h_- = 0$, we return to the potential (45).

So, the compatibility of all the equations is established if relation (44) holds. In the case $h_+ = h_- = 0$, there is no constraint on the parameter e_3 which yields Eq. (33). On the assumption of vanishing only h_- (or only h_+), one should equate e_3 to $-2i/g$ (or $2i/g$), and this results in Eq. (27). This completes our justification of the uniqueness of the retarded solutions (27) and (33).

V. YANG-MILLS FIELD GENERATED BY TWO QUARKS

A detailed procedure of obtaining exact retarded solutions of Eq. (5) with the source composed of two quarks, starting from the ansatz (24), was given in [2]. We thus dwell on analytical and geometrical features of these solutions.

We adopt SU(3), the minimal group whereby the retarded field generated by two bound quarks is constructed. The point is that the field of a bound quark occupies individually some SL(2,R) cell of the color space while SL(3,C) contains two such cells.

One usually realizes su(3) with the aid of the Gell-Mann matrices $T_a = \lambda_a/2$. It is more convenient, however, for our purposes to use an overcomplete color basis spanned by the nonet of 3×3 matrices including three diagonal matrices

$$H_1 \equiv \frac{1}{2}\left(\lambda_3 + \frac{\lambda_8}{\sqrt{3}}\right) = \frac{1}{3}\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{48}$$

$$H_2 \equiv -\frac{1}{2}\left(\lambda_3 - \frac{\lambda_8}{\sqrt{3}}\right) = \frac{1}{3}\begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \tag{49}$$

$$H_3 \equiv -\frac{\lambda_8}{\sqrt{3}} = \frac{1}{3}\begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}, \tag{50}$$

which are related by

$$\sum_{n=1}^{3} H_n = 0,$$

and six raising and lowering matrices

$$E_{12}^+ \equiv \frac{1}{2}(\lambda_1 + i\lambda_2) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$E_{12}^- \equiv E_{21}^+ \equiv \frac{1}{2}(\lambda_1 - i\lambda_2) = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \tag{51}$$

$$E_{13}^+ \equiv \frac{1}{2}(\lambda_4 + i\lambda_5) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$E^-_{13} \equiv E^+_{31} \equiv \frac{1}{2}(\lambda_4 - i\lambda_5) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \qquad (52)$$

$$E^+_{23} \equiv \frac{1}{2}(\lambda_6 + i\lambda_7) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$E^-_{23} \equiv E^+_{32} \equiv \frac{1}{2}(\lambda_6 - i\lambda_7) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}. \qquad (53)$$

Given this color basis, three retarded solutions are

$$A^{(1)}_\mu = \mp \frac{2i}{g}\left(H_1 \frac{v^1_\mu}{\rho_1} + H_2 \frac{v^2_\mu}{\rho_2} \right) + \kappa(E^\pm_{13}R^1_\mu + E^\pm_{23}R^2_\mu)\,\delta(R^1 \cdot R^2), \qquad (54)$$

$$A^{(2)}_\mu = \mp \frac{2i}{g}\left(H_3 \frac{v^1_\mu}{\rho_1} + H_1 \frac{v^2_\mu}{\rho_2} \right) + \kappa(E^\pm_{32}R^1_\mu + E^\pm_{12}R^2_\mu)\,\delta(R^1 \cdot R^2), \qquad (55)$$

$$A^{(3)}_\mu = \mp \frac{2i}{g}\left(H_2 \frac{v^1_\mu}{\rho_1} + H_3 \frac{v^2_\mu}{\rho_2} \right) + \kappa(E^\pm_{21}R^1_\mu + E^\pm_{31}R^2_\mu)\,\delta(R^1 \cdot R^2). \qquad (56)$$

They represent actually the same YM field related by the gauge transformations

$$A^{(j)}_\mu = \Omega^{-1}_{1j} A^{(1)}_\mu \Omega_{1j}$$

with

$$\Omega_{12} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \qquad \Omega_{13} = \Omega^{-1}_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}.$$

Therefore, only one of Eqs. (54)–(56), say, Eq. (54), will thereafter be referred to.

Taking into account the Gauss law, one finds the color charge of Ith quark

$$Q_I = \mp \frac{2i}{g} H_I. \qquad (57)$$

Thus the solutions suggest the existence of two-quark systems with the total color charges

$$Q_{(+)} = \frac{2i}{g}(H_1 + H_2) \qquad (58a)$$

and

$$Q_{(-)} = -\frac{2i}{g}(H_1 + H_2). \qquad (58b)$$

Therein lies the most outstanding distinction from the single-quark case where the complex-conjugate potentials, being interconvertible by gauge transformations, originate from the

same source. As for the present case, it is impossible to convert the complex-conjugate solutions to one another since there exists a nonzero field invariant

$$C_3 = \mathrm{tr}(F_{\lambda\mu}F^\mu_\nu F^{\nu\lambda})$$

which is of different signs for the complex-conjugate solutions. So, we have two different field configurations generated by two sources with the total color charges $Q_{(+)}$ and $Q_{(-)}$. It follows from Eq. (58) that no colorless two-quark cluster is feasible on the classical level.

Let us turn to the spacetime dependence of the solutions. It was found in [2] that f^{iI} and h^{iI} are independent of β_{12}, and γ_{12}, whereas Δ_{12} is shown by the factor $2\delta(\Delta_{12}) = \delta(R_1 \cdot R_2)$.

Since R_1 and R_2 are lightlike, the equality $R_1 \cdot R_2 = 0$ is ensured only for collinear R_1 and R_2. In view of the factor $\delta(R^1 \cdot R^2)$, this means that the linearly rising term of A_μ is localized on the enveloping surface of two families of rays

$$x_\mu = z^1_\mu(\tau) + \theta(\sigma)n_\mu\sigma,$$

$$n_\mu = z^1_\mu - z^2_\mu, \quad n^2 = 0, \quad n_0 > 0,$$

and

$$x_\mu = z^2_\mu(\tau) + \theta(\sigma)m_\mu\sigma,$$

$$m_\mu = z^2_\mu - z^1_\mu, \quad m^2 = 0, \quad m_0 > 0,$$

parametrized by τ and σ. At the intersection of a spacelike hyperplane with this surface, two fragments of a curve arise. Thus the force lines of the YM field corresponding to the linearly rising term of A_μ are squeezed to a string. This is not a finite string joining two quarks. What we have now are two half-infinite strings which begin at the positions of the quarks and go outward at different sides to spatial infinity.

By the construction of the enveloping surfaces, they are ruled surfaces. Having lightlike rulings and timelike directrices (the world lines), we find warped world sheets of the strings.

There exists also an alternative solution identical to Eq. (54) in every respect except the linearly rising term that is stripped of the factor $\delta(R^1 \cdot R^2)$,

$$A_\mu = \mp \frac{2i}{g}\left(H_1 \frac{v^1_\mu}{\rho_1} + g\kappa E^\pm_{13}R^1_\mu \right) \mp \frac{2i}{g}\left(H_2 \frac{v^2_\mu}{\rho_2} + g\kappa E^\pm_{23}R^2_\mu \right). \qquad (59)$$

Now the linearly rising terms describe the force lines distributed over all directions of space.

The solution (59) is truly non-Abelian because

$$[A_\mu, A_\nu] \neq 0, \quad [A_\mu, F^{\mu\nu}] \neq 0.$$

How can the nonlinearity of the YM equations be compatible with that A_μ is the sum of two single-quark potentials? Equation (59) does combine two such terms, yet making no hint about the plausibility to represent the solution as an arbitrary superposition of them. If either of terms with some coefficient different from 1 is added to another, no new solution arises.

It should be realized that Eq. (59) describes the YM fields generated by a system of two *bound* quarks. Would the field be generated by two free quarks, the sign of the color charge of any quark might be subjected to variation regardless of the sign of the color charge of other quark. In the present case, however, changing the signs for the first one-quark term and leaving intact the signs for the second one-quark term, we arrive at the expression that is no longer solution as it is clear from that $E_{13}^{\pm}$ and $E_{23}^{\mp}$ do not commute. One may change the signs only simultaneously for both terms. This correlation of signs is precisely the evidence that we are dealing with bound quarks.

For $\kappa = 0$, a retarded solution is a superposition of two single-quark potentials (33),

$$A_\mu = \sum_{I=1}^{2} \sum_{n=1}^{3} e_I^n H_n \frac{v_\mu^I}{\rho_I}. \tag{60}$$

The solutions (54)–(56) and (59) become real-valued with respect to the color basis

$$\mathcal{T}_1 \equiv i\frac{\lambda_1}{2}, \quad \mathcal{T}_2 \equiv \frac{\lambda_2}{2}, \quad \mathcal{T}_3 \equiv i\frac{\lambda_3}{2}, \quad \mathcal{T}_4 \equiv i\frac{\lambda_4}{2},$$

$$\mathcal{T}_5 \equiv \frac{\lambda_5}{2}, \quad \mathcal{T}_6 \equiv i\frac{\lambda_6}{2}, \quad \mathcal{T}_7 \equiv \frac{\lambda_7}{2}, \quad \mathcal{T}_8 \equiv i\frac{\lambda_8}{2},$$

or else

$$\mathcal{H}_n \equiv iH_n, \quad \mathcal{E}_{mn}^{\pm} \equiv iE_{mn}^{\pm}.$$

With reference to the explicit form of H_n and $E_{mn}^{\pm}$, Eqs. (48)–(53), one finds that $\mathcal{T}_n$ are traceless imaginary 3×3 matrices satisfying the commutation relations of the Lie algebra sl(3,R), thus the gauge symmetry of the solutions (54)–(56) and (59) is SL(3,R).

When either of two quarks, say, the first, is elimitated, then Eq. (55) acquires the form

$$A_\mu = A_\mu' + A_\mu'',$$

$$A_\mu' = \mp \frac{i}{g}\lambda_3 \frac{v_\mu}{\rho} + \kappa E_{\pm}^{12} R_\mu, \quad A_\mu'' = \mp \frac{i}{g}\frac{\lambda_8}{\sqrt{3}}\frac{v_\mu}{\rho}. \tag{61}$$

A_μ' is the single-quark solution (27) while A_μ'' is an Abelian term, decoupled from A_μ' since λ_8 commutes with λ_3 and $E_{12}^{\pm}$. The adequacy of the gauge group SU(2) in the single-quark case is thus confirmed; the non-Abelian piece of the solution is built out of color vectors forming the Lie algebra su(2). The field invariant C_3 is zero for both Eqs. (27) and (61).

It is no great surprise that SL(2,C) stands out against SL(N,C),N>2, in the single-quark case. The metrical structure of the base embodied in the Lorentz group SL(2,C) is all that should be mapped by the future light cone into the fiber, so that the color space SL(2,C) is the only exact image. Based on SO(N) or Sp(N) as the starting point, one reaches the same solution since su(2)∼so(3)∼sp(1), and sl(2,R)∼su(1,1)∼so(2,1)∼sp(1,R) [14].

On the other hand, given SO(N) or Sp(N) in the two-quark case, we have other results as opposed to SU(N). Both the so(4,C) and so(5,C) color spaces are suitable for an accommodation of two ''elementary'' color cells so(3,C) ∼sl(2,C). But so(4,C) is not semisimple, and the Cartan-Killing metric is singular here. As for so(5,C), it is isomorphic to sp(2,C), and we envisage two alternatives in the description of the color space in the two-quark case, either sl(3,C) or so(5,C)∼sp(2,C).

VI. YANG-MILLS FIELD GENERATED BY SEVERAL QUARKS

The discussion of solutions of the YM equations with the source composed of K quarks echoes in many respects that in the two-quark case. We adopt now the gauge group SU(N) with sufficiently large N, at least $N \geq K+1$, to allow an accommodation of all K quarks.

A. Solutions

We use the Cartan-Weyl basis of the Lie algebra su(N) spanned by the set of N^2 matrices which includes N diagonal elements H_n, the Cartan subalgebra,

$$(H_n)_{AB} \equiv \delta_{An}\delta_{Bn} - N^{-1}\delta_{AB},$$

satisfying

$$\sum_{n=1}^{N} H_n = 0,$$

and $N^2 - N$ raising and lowering elements E_{mn}^+ and E_{mn}^-, with $n > m$,

$$(E_{mn}^+)_{AB} \equiv \delta_{Am}\delta_{Bn}, \quad (E_{mn}^-)_{AB} \equiv \delta_{Bm}\delta_{An}.$$

Here, $m,n,A,B = 1,\ldots,N$. The nontrivial commutators are as follows:

$$[H_m, E_{mn}^{\pm}] = \pm E_{mn}^{\pm}, \tag{62}$$

$$[E_{mn}^+, E_{mn}^-] = H_m - H_n, \tag{63}$$

$$[E_{kl}^{\pm}, E_{lm}^{\pm}] = \pm E_{km}^{\pm}. \tag{64}$$

With these commutation relations, one can ascertain that Eq. (5) is satisfied by

$$A_\mu = \mp \frac{2i}{g}\sum_{I=1}^{K}\left[H_I \frac{v_\mu^I}{\rho_I} + g\kappa E_{IK+1}^{\pm} R_\mu^I \prod_{I=1}^{K-1} \delta(R^K \cdot R^I) \right]. \tag{65}$$

There exist C_N^K solutions of this type. Consider the transformation $A_\mu \rightarrow \Omega^{-1} A_\mu \Omega$ with

$$\Omega = E_{1N}^- + \sum_{I=1}^{N-1} E_{II+1}^+ = \begin{pmatrix} 0 & . & . & . & & 0 & 1 \\ 1 & 0 & & & & & \\ & 1 & 0 & & & & \\ & & & . & . & & \\ & & & & . & . & \\ & & & & & . & . \end{pmatrix}$$

and

$$\Omega^{-1} = E_{1N}^+ + \sum_{I=1}^{N-1} E_{II+1}^- = \begin{pmatrix} 0 & 1 & & & & \\ & 0 & 1 & & & \\ & & . & . & & \\ & & & . & . & \\ & & & & . & . \\ & & & & 0 & 1 \\ 1 & 0 & . & . & . & 0 \end{pmatrix}.$$

It increases each index of H_I and $E_{IK+1}^\pm$ by one, and the transformed A_μ turns out to be a new solution. Other solutions can be obtained by repetitions of this gauge transformation. [A similar situation already encountered in the two-quark case where the solutions (54)–(56) were shown to convert to each other by transformations of this sort.]

The solution (65) describes a background field generated by K quarks which form some K-quark cluster. The color charge of Ith quark [cf. Eq. (57)] is

$$Q_I = \mp \frac{2i}{g} H_I.$$

The total color charge of such a cluster [cf. Eq. (58)] is either

$$Q_{(+)} = \frac{2i}{g} \sum_{I=1}^K H_I$$

or

$$Q_{(-)} = -\frac{2i}{g} \sum_{I=1}^K H_I.$$

There are also solutions describing background fields generated by several clusters. Each cluster is defined by the condition that the signs of the color charges are simultaneously either $+$ or $-$ for every quark of the cluster whereas relative signs of the total color charges of the clusters are arbitrary. For example, the potential generated by two two-quark clusters is $A_\mu = A_\mu^1 \pm A_\mu^2$, where A_μ^j is the potential generated by the jth cluster,

$$A_\mu^1 = \pm \frac{2i}{g} \sum_{I=2}^3 \left[H_I \frac{v_\mu^I}{\rho_I} + g\kappa E_{1I}^\pm R_\mu^I \delta(R^2 \cdot R^3) \right],$$

$$A_\mu^2 = \pm \frac{2i}{g} \sum_{I=5}^6 \left[H_I \frac{v_\mu^I}{\rho_I} + g\kappa E_{4I}^\pm R_\mu^I \delta(R^5 \cdot R^6) \right]. \tag{66}$$

Omitting $\delta(R^I \cdot R^J)$ in Eqs. (65) and (66) gives alternative solutions [cf. Eq. (59)].

At last, there exist solutions describing YM fields of free quarks with the color charges

$$Q_I = \pm \frac{i}{g}(H_{I+1} - H_I).$$

For example, the YM field of two free quarks (labeled by numbers 1 and 3) is

$$A_\mu = \pm \left[\frac{i}{g}(H_2 - H_1)\frac{v_\mu^1}{\rho_1} + \kappa E_{12}^\pm R_\mu^1 \right]$$
$$\pm \left[\frac{i}{g}(H_4 - H_3)\frac{v_\mu^3}{\rho_3} + \kappa E_{34}^\pm R_\mu^3 \right]. \tag{67}$$

The gauge transformation $A_\mu \to \Omega^{-1} A_\mu \Omega$ with

$$\Omega = \Omega^{-1} = \frac{N-2}{N}\mathbf{1} + \sum_{I=3}^N H_I + E_{12}^+ + E_{12}^-$$

$$= \begin{pmatrix} 0 & 1 & & & & \\ 1 & 0 & & & & \\ & & 1 & & & \\ & & & 1 & & \\ & & & & . & \\ & & & & & . \\ & & & & & & . \end{pmatrix}$$

changes the $\pm$ signs of the first square bracket of Eq. (67) while the signs of the second remains invariable. It is easy to recognize this gauge transformation as that rendering the potential complex-conjugate in the single-quark case. Thus the color charge of the free quark is determined modulo $\exp(i\pi n)$.

There is a number of ways to separate a given K-quark system into groups of clusters of a certain quark content and free quarks. We interpret them as *scenarios of hadronization*.

We now look at the symmetry of these solutions. One can define $(K+1)^2$ traceless imaginary matrices $\mathcal{H}_n$ and $\mathcal{E}_{mn}^\pm$ as follows:

$$\mathcal{H}_n \equiv iH_n, \quad \mathcal{E}_{mn}^\pm \equiv iE_{mn}^\pm$$

which are elements of the Lie algebra $\mathrm{sl}(K+1,R)$. Thereafter, every solution above becomes real valued with respect to this basis. The solutions constructed from M^2 such elements obeying the closed set of commutation relations are invariant under $\mathrm{SL}(M,R)$, $M \leq K+1$.

In particular, the YM field generated by a two-quark cluster (meson) is invariant under $\mathrm{SL}(3,R)$ and that of a three-quark cluster (barion) is invariant under $\mathrm{SL}(4,R)$. Since $\mathrm{SL}(3,R)$ is a subgroup of $\mathrm{SL}(4,R)$, the YM field of every hadron is specified by the gauge group $\mathrm{SL}(4,R)$. This symmetry is independent of N and is retained in the limit $N \to \infty$.

For $\kappa = 0$, the YM equations linearize, and one gets an Abelian solution

$$A_\mu = \sum_{I=1}^{K} \sum_{n=1}^{N} e_I^n H_n \frac{v_\mu^I}{\rho_I}, \qquad (68)$$

where e_I^n are arbitrary parameters. The gauge symmetry of this solution is SU(N).

We have obtained two types of solutions corresponding to two phases of matter. The YM background of the first phase is invariant under the noncompact gauge group SL($K+1,R$) while that of the second phase is invariant under the compact gauge group SU(N).

B. Energetical considerations

It follows from the trace relations

$$\mathrm{tr}(H_l E_{mn}^\pm) = 0 \qquad (69)$$

that the linearly rising term of A_μ does not contribute to the color force

$$f_I^\mu = v_\nu^I \, \mathrm{tr}[Q_I F^{\mu\nu}(z_I)]. \qquad (70)$$

Thus, the well-known mechanism of quark binding by a constant force lacks support from the exact solutions of the classical YM equations.

A surprising thing is that the linearly rising term of A_μ, while ensuring nonzero contribution to the field strength $F_{\mu\nu}$, results in no force. An explanation is simple. Equation (70) includes the scalar product of two color vectors $F_{\mu\nu}$ and Q_I which are not arbitrary; they come from the exact solutions and turn out to be orthogonal to each other.

One can find from Eq. (69) and

$$\mathrm{tr}(E_{mn}^\pm E_{mn}^\pm) = 0 \qquad (71)$$

that the linearly rising term of A_μ does not contribute to color singlets altogether. This is because it depends on either E_{mn}^+ or E_{mn}^-, not both. A more fundamental reason is the conformal invariance of the 4D classical YM theory which implies that the parameter κ violating the scale symmetry cannot manifest itself in observables.

The energy-momentum tensor can be split into

$$\Theta_{\mu\nu} = \sum_{I=1}^{K} \Theta_{\mu\nu}^I + \Theta_{\mu\nu}^{\mathrm{int}},$$

where $\Theta_{\mu\nu}^I$ is the self-action term containing the contribution of the YM field generated by Ith quark and $\Theta_{\mu\nu}^{\mathrm{int}}$ is the interaction part comprised of mixed contributions.

The four-momentum P_μ defined by Eq. (16) contains divergent terms due to $\Theta_{\mu\nu}^I$'s. If the solution is invariant under SL($K+1,R$), the self-energy of each quark is negative definite. This suggests that such backgrounds are most favorable at zero temperature. It is the energetical advantage that enables attributing them to the gluon vacuum in the cold world.

The self-energy is positive-definite for the solutions invariant under SU(N). These solutions seem to be related to the hot phase. (One should mention also the configurations [20] invariant under SU(N) with an energy lower than that of the Coulomb solution.)

However, it is beyond the scope of this work to review a temperature-dependent version of the YMW theory. We merely drew a parallel to the Yang-Mills-Higgs (YMH) theory where two classical solutions with different symmetries also exist. The solution with broken symmetry being stable and energetically favorable corresponds to the cold phase. Although the solution with unbroken symmetry is unstable (thus bearing no relation to physical world), its availability on the fundamental level motivates the quest of a phase with such a symmetry.

The present situation contrasts with that of the YMH theory in three aspects. First, there is no spontaneous symmetry breakdown. We deal with solutions invariant under two different real forms of the complex group SL(N,C). The occurrence of the solution invariant under a noncompact group different from the initial one is a new field-theoretic phenomenon referred to as the *spontaneous symmetry deformation* [2]. The epithet spontaneous emphasizes that the scenario of hadronization is accomplished quite accidentally.

Second, both solutions are now stable against small disturbances (see Sec. VII).

Third, the critical point $\kappa = 0$ is independent of parameters appearing in the action, as opposed to the YMH theory where the spontaneous symmetry breakdown is directly related to parameters controlling the convexity of the Higgs potential.

Let us turn to the self-action problem. We follow the basic Teitelboim pattern [21] developed in the Maxwell-Lorentz theory. The SU(N) phase is treated with no noticeable distinctions from electrodynamics; one needs simply to substitute everywhere e^2 by Q_I^2.

A completely different situation arises in the cold phase. After the mass renormalization, the negative definiteness of the field energy does not disappear without leaving a trace. It reveals itself in the "wrong" sign of the radiation energy of accelerated quarks [1,19],

$$\frac{dE_I}{d\tau_I} = \frac{2}{3} |\mathrm{tr}\, Q_I^2| a_I^2 < 0. \qquad (72)$$

Thus the self-action of a quark in the cold phase is such that the flux of energy is directed inward the source. This feature of the self-action is unrelated to the boundary condition; replacing the retarded condition by the advanced one leaves the direction of the flux intact.

To gain better understanding of this point, one should derive the equation of motion of a dressed quark. Taking into account Eqs. (13)–(15), one can find [1,19]

$$m_I[a_\mu^I + \tau_0(\dot{a}_\mu^I + v_\mu^I a_I^2)] = v_I^\nu \, \mathrm{tr}[Q_I F_{\mu\nu}(z_I)]. \qquad (73)$$

Here, m_I is the renormalized mass of Ith quark, $F_{\mu\nu}$ is the field of all other quarks at the position of Ith quark z_I^μ, and

$$\tau_0 \equiv \frac{2}{3m_I} |\mathrm{tr}\, Q_I^2|.$$

A similar parameter in electrodynamics $\tau_0 = 2e^2/3mc^2 \approx 10^{-13}$ cm is related to the classical radius of electron. Every effect of the scale τ_0 is neglected there, keeping in mind that quantum phenomena come into play already at the range

of the Compton wavelength of electron $\lambda = \hbar/mc \approx 10^{-11}$ cm. On the contrary, the classical radius of quark in the cold phase far exceeds its wavelength, $\tau_0 \gg \lambda$, since, in the semiclassical treatment, the coupling g is held to be much less than 1, and therefore

$$|\text{tr } Q_I^2| = \frac{4}{g^2}\left(1 - \frac{1}{N}\right) \gg 1. \qquad (74)$$

Equation (73) can be rewritten as

$$\overset{v}{\perp}(\dot{p} - f) = 0, \qquad (75)$$

where the operator

$$\overset{v}{\perp} \equiv \mathbf{1} - \frac{v \otimes v}{v^2} \qquad (76)$$

projects vectors on a hyperplane normal to the four-velocity v^μ, and

$$p^\mu = m(v^\mu + \tau_0 a^\mu). \qquad (77)$$

Note that Eq. (75) is Newton's second law governing the behavior of an object specified by its point of location z^μ and four-momentum p^μ defined by Eq. (77). We call this object the dressed quark, or the *color complex* [19], keeping in mind the field-mechanical origin of p^μ.

The invariance of the action (2) under the local reparametrizations

$$\delta\tau = \epsilon, \qquad \delta z^\mu = v^\mu \epsilon$$

endows the equation of motion of a bare quark with the factor $\perp$. The regularization of P_μ indicated in Sec. II ensures that such an invariance remains valid after the mass renormalization. Thus the form of the dynamical equations of bare and dressed quarks is the same, only their four-momenta p^μ are distinct in the dependence on kinematical variables.

The complex is unaffected by the *radiation reaction*; the Newtonian behavior of this object implies that only external force acts on it. The hallmark of the complex is not the evolution law, which is not uncommon, but the indefiniteness of p^0, as Eq. (77) suggests.

On the other hand, Eq. (73) expresses the local four-momentum balance: The increment of the complex four-momentum dp_μ^I originates from the total effect of all other quarks $v_I^\nu \text{tr}[Q_I F_{\mu\nu}(z_I)]d\tau_I$ and the absorbed four-momentum $\tau_0 v_\mu^I a_I^2 d\tau_I$ [19]. The greater the acceleration (determined by the total effect of other quarks) the greater the absorption.

A more familiar viewpoint is that Eq. (73) describes the evolution of an object with the four-momentum $p^\mu = mv^\mu$, visualized as a point particle. The behavior of such a particle is beyond the control of the Newton second law. Departures from the Newtonian behavior are commonly attributed to the radiation reaction. This is a source of many paradoxes [19].

For a vanishing right-hand side (RHS) of Eq. (73), one gets the solution

$$v^\mu = \{\cosh(C + De^{-\tau/\tau_0}), \mathbf{n}\sinh(C + De^{-\tau/\tau_0})\},$$

where C and D are constants, and $\mathbf{n}$ is a fixed unit vector. Thus, in the absence of the external force, the absorption of energy is exponentially decreasing in time, and the motion of the color complex asymptotically approaches Galileo's inertial regime. The solution describes a straight world line when the asymptotical condition $a_\mu \to 0$, $\tau \to -\infty$ is imposed.

We further turn to the interquark forces. In the cold phase, like color charges attract and unlike ones repel. However, it follows from the trace relations

$$\text{tr}(H_{J+1} - H_J)^2 = 2, \quad \text{tr}(H_{J+1} - H_J)H_I = 0 \qquad (78)$$

that a free quark, while experiencing the self-action, does not act on other quarks.

We have seen that, in the cold phase, each quark individually occupies some sl$(2,R)$ cell. Neither of two backgrounds generated by different quarks may be contained in the same sl$(2,R)$. This is similar to the Pauli blocking principle. Just as a cell of volume h^3 in the phase space might be occupied by at most one fermion with a definite spin polarization, so any sl$(2,R)$ cell is intended for a background of only one quark. Choosing SO(N) or Sp(N), instead of SU(N), one singles out the same color cell so$(2,1) \sim$ sp$(1,R) \sim$ sl$(2,R)$.

By contrast, in the hot phase, assuming the total color charge of quarks in a given plasma lump to be zero, the parameters e_I^n in Eq. (68) are to be appropriately fitted. Then the most energetically advantageous field configuration is such that the color charges of quarks are lined up into a fixed color direction, thereby reducing SU(N) to SU(2). This bears some resemblance to the Bose-Einstein condensation in the color space.

Thus the "color Pauli principle" preventing a body of $K+1$ color cells against shrinkages is an evidence of that the large-N limit is adequate to the cold phase description, whereas the "color Bose-Einstein condensation" suggests the sufficiency of SU(2) for the hot phase.

Consider $N \to \infty$ limit in the cold phase, assuming the coupling g to be fixed. (Note that the factorization condition is therewith assured.) The relations

$$\text{tr}(H_I)^2 = 1 - N^{-1}, \quad \text{tr}(H_I H_J) = -N^{-1} \qquad (79)$$

show that the color repulsion between bound quarks vanishes in this limit, unless the number of quarks at the given cluster is of order N. Thus K-quark clusters with $K = O(N)$ are unstable while any cluster of finite number of quarks survives as $N \to \infty$.

This is in agreement with Witten's phenomenology [8], where mesons made out of quark-antiquark pairs are stable and noninteracting (their decay and scattering amplitudes are suppressed respectively as $1/\sqrt{N}$ and $1/N$), and barions, imagined as N-quark clusters, are unstable (the barion-barion and barion-meson vertices are, respectively, of order N and 1).

In the present context, however, barions being considered as three-quark clusters turn out to be stable. The consistency with Witten's phenomenology is true for both mesons and multiquark clusters with the number of quarks of order N.

Thus, the classical YMW quarks do not interact as $N \rightarrow \infty$. The quark binding is characterized only by the correlation of signs of the color charges of quarks comprising a cluster. Since the color self-action prevents the motion from the runaway regime, the quarks could be conceived as moving along parallel straight world lines. This accords with an intuitive idea of the ground state of a cluster with zero orbital momentum.[3]

Switching on electromagnetic forces violates this equilibrium. It is possible, however, to introduce the electrodynamical terms of the action in conjunction with choosing the YM coupling g such that

$$|\text{tr } Q_I^2| = \frac{4}{g^2}\left(1 - \frac{1}{N}\right) = e^2. \tag{80}$$

This enables a consistent treatment of orbital motions. It follows from Eqs. (79) and (80) that the centrifugal force is finite while the absorbed YM energy exactly compensates the radiated electromagnetic energy, and helical world lines are no longer infrared troublesome. Moreover, the resulting picture is free of ultraviolet divergences.

Unfortunately, this attractive possibility may pretend only to a toy status. With the actual value of the elementary electric charge $e^2 \approx 1/137$, Eq. (80) results in $g \approx 24$ invalidating the semiclassical tratment. Furthermore, the electric charge of real quarks takes two values $-e/3$ and $2e/3$ so that the picture of accelerated quarks emitting no energy can be matched with only either type of quarks (and with electroneutral two-quark clusters).

VII. STABILITY PROBLEM

We have found that the YMW bound quarks are balanced being in the state of indifferent equilibrium. One should then examine the spectrum of excitations about the classical background. In order that a given cluster be stable, the energy of every mode (the translation mode apart) must be positive; if the balance is upset by some external influence, then excitations responsible for increasing the energy should occur.

For our prime interest is in the ground state of clusters where the quarks rest relative to each other, we consider the static background field generated by such quarks. Set $B_\mu = A_\mu + b_\mu$ where A_μ is a static configuration, and b_μ is a small disturbance about A_μ. As can be readily shown (see, e.g., [23]), the positivity of the excitation energy about a given static background A_μ is tantamount to that the equation of excitations

$$\frac{\delta^2 S}{\delta A_\mu^a(x) \delta A_\nu^b(y)} b_\mu^a(x) = 0 \tag{81}$$

[3]A similar situation occurs in the monopole dynamics. As was shown by Manton [22], the monopoles forming a static multimonopole are influenced by no intermonopole forces; they are balanced due to exact cancellation of the repulsive magnetic YM force and the attractive Higgs force.

has no solution exponentially increasing with time. Any oscillatory solution $\exp(i\omega_k t)$ determines a positive mode $\varepsilon_k \propto \omega_k^2$. (For a more extended discussion see, e.g., [24].)

Let us show that the single-quark solution (27) is stable against small field disturbances [25]. In the static case, when $v^\mu = (1,0,0,0)$, $z^\mu(\tau) = z^\mu(0) + v^\mu \tau$, the proper time τ is identified with the laboratory time t, and the retarded distance ρ with the usual radius r.

As is well known [23], the classical limit $\hbar \rightarrow 0$ is equivalent to that of the weak coupling $g \rightarrow 0$. Taking into account that the expression (27) depends on g as g^{-1}, we must retain only quantities of order g^0 in Eq. (81).

Let us take the gauge condition $v^\mu b_\mu^a = 0$ for any quark world line. Then the color charge of the quark remains constant $\dot{Q}^a = 0$ even with the presence of the excitations b_μ^a. In the static case, this condition is reduced to

$$b_0^a = 0.$$

Among the spatial components of $\mathbf{b}^a$, we must separate only those which are orthogonal to the gauge modes. This is guaranteed by the condition [26]

$$\nabla \mathbf{b}^a + g f^{abc} \mathbf{B}^b \cdot \mathbf{b}^c = 0,$$

which, in the weak coupling limit, becomes

$$\nabla \mathbf{b}^a = 0. \tag{82}$$

Putting

$$\mathbf{b} = \mathbf{b}^3 \Gamma_3 + \mathbf{b}^+ \Gamma_+ + \mathbf{b}^- \Gamma_-,$$

and taking into account Eq. (82), we obtain [25]

$$\Box \mathbf{b}^3 = 0, \tag{83}$$

$$\left(\Box \mp \frac{4}{r}\frac{\partial}{\partial t} + \frac{4}{r^2}\right)\mathbf{b}^\pm = 0, \tag{84}$$

$$\mathbf{r} \cdot \mathbf{b}^\pm = 0. \tag{85}$$

It is clear from Eq. (83) that $\mathbf{b}^3$ does not violate the stability of the background A_μ^a. The function $\mathbf{b}^-$, satisfying Eqs. (82), (84), and (85), with oscillatory behavior in time is

$$\mathbf{b}^-(t,\mathbf{r}) = \int_0^\Lambda d\omega \sum_{l,m} \{\alpha_{lm}(\omega)e^{-i\omega t}\mathbf{Y}_{lm}(\theta,\phi)K_j(\omega r)$$

$$+ \beta_{lm}(\omega)e^{i\omega t}[\mathbf{Y}_{lm}(\theta,\phi)K_j(\omega r)]^*\}. \tag{86}$$

Here, Λ is a frequency cutoff parameter that characterizes a boundary of the infrared region, $\mathbf{Y}_{lm}(\theta,\phi)$ is a spherical vector harmonic, $K_j(s)$ is expressed in terms of the confluent hypergeometric function

$$K_j(s) = s^j e^{-is} F(j-1, 2j+2, 2is), \tag{87}$$

and j runs through values which are positive roots of the equation

$$j(j+1) = l(l+1) + 4, \quad l = 1, 2, \ldots. \tag{88}$$

We are looking for solutions in the class of functions with an appropriate behavior at spatial infinity and singular points of the background. Every solution corresponding to a negative root of Eq. (88) is more singular than the background (27) at $r=0$, and should be excluded.

The solution $\mathbf{b}^+$ is obtained from Eq. (86) by replacing t by $-t$. $K_j(s)$ is regular at $s=0$ while, in the limit $s\rightarrow\infty$, it has the asymptotic

$$K_j(\omega r)=\{c_j\omega r+d_j+O[(\omega r)^{-1}]\}\exp(i\omega r), \quad (89)$$

where c_j and d_j are certain known constants.

Note that the simultaneous presence of $\mathbf{b}^+$ and $\mathbf{b}^-$ ensures a nonvanishing contribution to the energy-momentum tensor. From the behavior of $\mathbf{b}^\pm$ at spatial infinity, Eq. (89), it follows that the four-momentum P_μ is infrared divergent, and the semiclassical treatment of the cold phase turns out to be inconsistent. Therefore, scenarios of hadronization with the presence of free quarks must be ruled out.

Let us turn to the problem of stability of the solution (33). We restrict ourselves to the static case which though is hard justified now if one remembers the runaway problem. Thus we consider only necessary (not sufficient) conditions of stability.

Equation (33) is independent of g, hence $\pm(2i/g)$ must be replaced by q in each foregoing relation. Equation (84) converts to the form

$$\left(\Box\mp\frac{2igq}{r}\frac{\partial}{\partial t}-\frac{g^2q^2}{r^2}\right)\mathbf{b}^\pm=0. \quad (90)$$

Equations (87)–(89) are modified appropriately,

$$K_j(s)=s^je^{-is}F(-igq+j+1,2j+2,2is), \quad (91)$$

$$j(j+1)=l(l+1)-g^2q^2, \quad l=1,2,\ldots, \quad (92)$$

$$K_j(s)=O(s^{igq-1}e^{is}), \quad s\rightarrow\infty. \quad (93)$$

It is clear from Eq. (93) that q must be *real* for $\mathbf{b}^\pm$ to decrease as $1/r$ at spatial infinity, similar to the background (33). Let us compare their behaviors at $r=0$. From Eq. (91) it follows that $K_j(s)$ is regular at $s=0$ if $j\geq0$. Write the positive solution of Eq. (92),

$$j=\frac{1}{2}[\sqrt{(2l+1)^2-4g^2q^2}-1]$$

and take $l=1$, the minimal allowable value, then one finds that j is positive for

$$g^2q^2\leq2. \quad (94)$$

A similar result was obtained by Mandula [27]. Thus the solution (27) is stable whereas the solution (33) would be so provided that q is a real quantity less than $\sqrt{2}/g$.

We next go to the two-quark case. Decompose b_μ into vectors of the color basis (48)–(53),

$$b_\mu=\sum_{n=1}^{3}\left[b_\mu^nH_n+\sum_{k=1}^{3}(b_\mu^{kn-}E_{nk}^-+b_\mu^{kn+}E_{kn}^+)\right].$$

We restrict ourselves to the situation of static quarks. We adopt the gauge condition

$$b_0=0,$$

which ensures the constancy of the color charges of both quarks. Let us consider the background potential (54). Now, by repeating what was done in the single-quark case, we find that $\mathbf{b}^n$ satisfy Eqs. (82) and (83) while $\mathbf{b}^{23\pm}$ and $\mathbf{b}^{13\pm}$ satisfy Eqs. (82), (84), and (85) with r playing the role of ρ_1 for $\mathbf{b}^{23\pm}$ and ρ_2 for $\mathbf{b}^{13\pm}$. From this identification, one checks the stability of these components. Note that $\mathbf{b}^{23\pm}$ and $\mathbf{b}^{13\pm}$ are associated with the position of corresponding quarks while $\mathbf{b}^n$ does not relate to either quark specifically.

As for $\mathbf{b}^{12\pm}$, it obeys Eq. (82) and

$$\left[\Box\mp4\left(\frac{1}{r_2}-\frac{1}{r_1}\right)\frac{\partial}{\partial t}+4\left(\frac{1}{r_2}-\frac{1}{r_1}\right)^2\right]\mathbf{b}^{12\pm}=0, \quad (95)$$

where the operator $\Box$ acts on the variables t and $\mathbf{x}$, and $r_I=|\mathbf{x}-\mathbf{z}_I|$. One can see that $\mathbf{b}^{12\pm}$ fluctuates with respect to both quarks, hence quark binding is ensured by just this component of excitations. It is essential to gain insight into the behavior of solutions of Eq. (95) at spatial infinity.

If the quarks are separated by distance d, then, for $r_I\gg d$, Eq. (95) is reduced to the wave equation, and its asymptotical solutions are either

$$\mathbf{b}^{12\pm}\sim\text{const} \quad (96)$$

or

$$\mathbf{b}^{12\pm}\sim\sum_{k,l,m}j_l(kr)[\mathbf{c}_{lm}^\pm(k)Y_{lm}(\theta,\phi)e^{-ikt}$$
$$+\mathbf{d}_{lm}^\pm(k)Y_{lm}(\theta,\phi)^*e^{ikt}], \quad (97)$$

where $j_l(kr)$ are the spherical Bessel functions

$$j_l(kr)\sim\frac{1}{kr}\sin\left(kr-\frac{\pi l}{2}\right), \quad kr\gg l. \quad (98)$$

The solution (97) poses no infrared problem. By contrast, the solution (96) gives rise to the infrared divergence of P_μ and should therefore be considered as *redundant*.

The analysis of stability of the background (65) generated by K-quark clusters is identical to that in the two-quark case, with $\mathbf{b}_\pm^{IK+1}$, $I=1,\ldots,K$ playing the role of $\mathbf{b}^{23\pm}$ and $\mathbf{b}^{13\pm}$, while $\mathbf{b}^{12\pm}$ being represented by $\mathbf{b}_\pm^{JL}$, $J,L=1,\ldots,K$. The last field fluctuates with respect to the pair of quarks labeled by the numbers J and L, which ensures their binding.

The existence of excitations $\mathbf{b}_\pm^{IK+1}$, $I=1,\ldots,K$ entails the infrared divergences of P_μ due to their asymptotical behavior, Eq. (89), and the situation cannot be remedied by mere selecting scenarios of hadronization. How to exclude such excitations, is not yet understood. A possible direction in which one might search is studying a nonlinear problem of stability with the requirement that every excitation becomes purely gauge at spatial infinity.

VIII. SEMICLASSICAL TREATMENT

We have seen that the linearly rising term of A_μ produces no confining force. This brings up a question: Is Wilson's criterion fulfilled? From the trace relations

$$\text{tr } E_{mn}^\pm = \text{tr } H_n = \text{tr}(E_{mn}^\pm E_{mn}^\pm) = \text{tr}(H_n E_{mn}^\pm) = 0,$$

it follows that the loop operator

$$W(C) = \text{tr } P \exp\left[ig \oint_C dz^\mu A_\mu(z)\right]$$

with the background A_μ develops the perimeter law for both phases.

Consider the effect of gluon excitations b_μ about A_μ in the cold phase. Substitution of $B_\mu = A_\mu + b_\mu$ to the YM action gives

$$S(B) \approx S(A) + \frac{1}{2}\int d^4x \, d^4y \, b_\mu^a(x)\frac{\delta^2 S(A)}{\delta A_\mu^a(x)\delta A_\nu^b(y)}b_\nu^b(y).$$

Although $S(A)$ is divergent it should be discarded, just as in electrodynamics. Due to the gauge invariance, the differential operator $\delta^2 S/\delta A_\mu^a \delta A_\nu^b$ is irreversible, and a gauge-fixing term of the Lagrangian is called for. (It would be reasonable for the present purposes to use some linear gauge-fixing condition to avoid complications associated with the Faddeev-Popov ghosts.) Thereafter a certain nonsingular Lagrangian of gluon excitations results,

$$\mathcal{L} = b_\mu^a(x)\Lambda_{ab}^{\mu\nu}(A;\partial)b_\nu^b(x),$$

where $\Lambda_{ab}^{\mu\nu}(A;\partial)$ is a background-dependent reversible differential operator.

We average $W(C)$ over either of two complex-conjugate backgrounds A_μ^+ or A_μ^-,

$$\int \mathcal{D}b_\mu^c \exp\left[-\int d^4x \, b_\mu^a(x)\Lambda_{ab}^{\mu\nu}(A;\partial)b_\nu^b(x)\right.$$
$$\left. + i \oint_C dz^\mu b_\mu^b(z)\right].$$

This integral can be readily worked out to yield

$$\exp\left[-\oint_C dy^\mu \oint_C dz^\nu \mathcal{G}_{\mu\nu}^{ab}(y,z)\right], \tag{99}$$

where $\mathcal{G}_{\mu\nu}^{ab}(y,z)$ is the gluon propagator obeying the equation

$$\Lambda_{ab}^{\lambda\mu}(A;\partial)(y)\mathcal{G}_{\mu\nu}^{bc}(y,z) = -\delta_a^c \delta_\nu^\lambda \delta^4(z).$$

The area law for Eq. (99) would be the case if $\mathcal{G}_{\mu\nu}^{ab}(y,z)$ tends to a constant as $(y-z)^2 \to -\infty$. Since the behavior of the propagator at spacelike infinity is the same as that of excitations b_μ^a which obey the corresponding homogeneous equations, the responsibility for the area law rests with the excitations $\mathbf{b}_\pm^{JL}$ approaching asymptotically a constant as $r \to \infty$, the redundant solution of Eq. (95). Thus Wilson's criterion is fulfilled, though the area law cannot already be in-

terpreted as an evidence of the classical attractive constant force between quarks composing a cluster.

Note also that the area law stems from the excitations described by the redundant solution rendering P_μ infrared divergent. This resembles the situation with the energetical criterion valid for the prerenormalization stage when P_μ is still ultraviolet divergent.

We learned from the exact YM solutions that every classical cluster has a certain nonzero color charge. The situation reverses on the semiclassical level where the color neutralness of clusters is attained on the average of the gluon vacuum.

Given a K-quark cluster, one may define the gluon vacuum as a state with no excitation about the background generated by this cluster. This state is represented by a vector of Hilbert space Ψ such that the expectation value of every color-invariant quantity coincides with its classical value. However, there are invariants which are finite and of different signs for the complex-conjugate potentials A_μ^+ and A_μ^-, as exemplified by $C_3 = \text{tr}(F_{\lambda\mu}F_\nu^\mu F^{\nu\lambda})$ with the complex-conjugate solutions (65). Requiring the uniqueness of the vacuum expectation value of C_3, one has inevitably to assign $(\Psi, C_3\Psi) = 0$. Let Ψ_+ and Ψ_- be vectors of Hilbert space associated with $+$ and $-$ terms of Eq. (65). Being eigenvectors of the total color charge operator $\hat{Q}$ [defined by Eq. (10)],

$$\hat{Q}\Psi_\pm = Q_{(\pm)}\Psi_\pm,$$

they are mutually orthogonal,

$$(\Psi_-, \Psi_+) = 0.$$

If the gluon vacuum is defined as

$$\Psi = \frac{1}{2}(\Psi_+ + \eta\Psi_-), \tag{100}$$

where $\eta = \exp i\delta$ is an arbitrary phase factor, then one gets

$$(\Psi, \hat{Q}\Psi) = 0.$$

Thus the condition of the color-neutralness of K-quark clusters is met on the average of the gluon vacuum; the cluster finds itself partly at the state Ψ_+ with the color charge $Q_{(+)} = 2i/g\Sigma_{I=1}^K H_I$ and partly at the state Ψ_- with the color charge $Q_{(-)} = -2i/g\Sigma_{I=1}^K H_I$.

In the case of several clusters, the gluon vacuum is defined in a similar way. For example, given two two-quark clusters, the construction

$$\Psi = \frac{1}{4}[\Psi(1_+,2_+) + \eta_{-+}\Psi(1_-,2_+) + \eta_{+-}\Psi(1_+,2_-)$$
$$+ \eta_{--}\Psi(1_-,2_-)] \tag{101}$$

ensures the color neutralness of both clusters. Here, $\Psi(1_{\sigma1},2_{\sigma2})$ are vectors of Hilbert space associated with four solutions A_μ given by Eq. (66), σ_J the sign of the color charge of the Jth quark, and $\eta_{\sigma1\sigma2}$ are arbitrary phase factors.

Were A_μ^- to be convertible to A_μ^+ by a gauge transformation, any superposition of Ψ_+ and Ψ_- such as that of Eqs. (100) or (101) would be forbidden from realization as a physical state due to the availability of some superselection rule [28]. Only the potential generated by free quarks can be converted to the complex conjugate by a gauge transformation. There is no such transformation for A_μ generated by bound quarks, hence the superselection rule does not occur, and every cluster remains color neutral.

We regard every K-quark cluster on the equal footing since the existence of multiquark clusters was revealed experimentally [29]. It is well known, however, that hadrons are much more stable than multiquark clusters. One may wonder what a plausible explanation of this fact may be.

We can envision consecutive constructions of the YMW systems with the color spaces $SL(N,C)$, $SO(N,C)$, and $Sp(N,C)$. There is nothing to decide between these alternatives, hence all should persist and interfere. Is there the largest color cell outside of which three pictures become quite different? Such a cell does correspond to the three-quark case. For $n > 4$, there are no isomorphisms between members of the series $sl(n,C)$, $so(2n-1,C)$, $sp(n,C)$, and $so(2n,C)$. The interference of distinct color backgrounds is responsible for the splitting of energetical levels, which leads to the decay of clusters. No interference occurs in the single-quark case because $sl(2,C) \sim so(3,C) \sim sp(1,C)$. In the two-quark and three-quark cases, two alternatives interfere, respectively, $sl(3,R)$ and $so(3,2) \sim sp(2,R)$, and $sp(3,R)$ and $sl(4,R) \sim so(3,3)$. Thus clusters with two or three quarks are moderately stable. The interference of three alternatives keeps multiquark clusters away from stability.

The color neutralness of hadrons in the Gauss law sense may well be compatible with the observability of some specific color multiplet structure which reveals itself by infinite-dimensional unitary multiplets of $SL(4,R)$. Dothan, Gell-Mann, and Ne'eman [30] suggested that unitary multiplets of $SL(3,R)$ are related to the Regge trajectories of mesons. This group is generated by the angular momentum operators L_i and the quadrupole operators T_{ij} with the commutation relations

$$[L_i, L_j] = i\epsilon_{ijk}L_k, \qquad (102)$$

$$[L_i, T_{jk}] = i\epsilon_{ijl}T_{lk} + i\epsilon_{ikl}T_{jl}, \qquad (103)$$

$$[T_{ij}, T_{kl}] = -i(\delta_{ik}\epsilon_{jlm} + \delta_{il}\epsilon_{jkm} + \delta_{jl}\epsilon_{ikm})L_m. \quad (104)$$

The algebra $sl(3,R)$ represents the minimal scheme capable to explain two features of Regge trajectories: The $\Delta J = 2$ rule and the apparently infinite sequence of hadronic states.

It was found in [30] that two infinite unitary representations belonging to the ladder series

$$D_{SL(3,R)}^{\text{ladd}}(0;R): \quad \{J\} = \{0,2,4,\ldots\},$$

$$D_{SL(3,R)}^{\text{ladd}}(1;R): \quad \{J\} = \{1,3,5,\ldots\},$$

are associated with the π and ρ trajectories. Besides, there exists [31] a unique spinorial ladder representation related to the N trajectory

$$D_{SL(3,R)}^{\text{ladd}}\left(\frac{1}{2};R\right): \quad \{J\} = \left\{\frac{1}{2}, \frac{5}{2}, \frac{9}{2}, \ldots\right\},$$

while the spinorial representation starting with $J = \frac{3}{2}$ belongs to the discrete series [32]

$$D_{SL(3,R)}^{\text{disc}}\left(\frac{3}{2};R\right): \quad \{J\} = \left\{\frac{3}{2}, \frac{5}{2}, \frac{7^2}{2}, \frac{9^2}{2}, \frac{11^2}{2}, \ldots\right\}.$$

Thus the $SL(3,R)$ scheme of Dothan, Gell-Mann, and Ne'eman, being usefully applied to the Regge trajectories of mesons, turns out to be inadequate to account for those of baryons.

Ne'eman and Šijački [10] assumed that matters can be improved by a simultaneous application of $sl(3,R)$ and $so(1,3)$. The commutation relations can be closed by embedding two algebras in $sl(4,R)$, a relativistic generalization of $sl(3,R)$.

With adopting this $SL(4,R)$, one can classify the $SU(3)_f$ octet states acording to the $D_{SL(4,R)}^{\text{disc}}(\frac{1}{2},0) \oplus D_{SL(4,R)}^{\text{disc}}(0,\frac{1}{2})$ representation while the symmetrized product of this reducible representation and the finite-dimensional $SL(4,R)$ representation $(\frac{1}{2},\frac{1}{2})$ is used for the decuplet states. Although this scheme is quite restrictive, it is in a good agreement with known data of hadronic spectroscopy and predicts several new states [10]. The present exact solutions show that the gauge symmetry of the background generated by clusters composed of two or three quarks is just $SL(4,R)$.[4]

A basis of $sl(4,R)$ contains six antisymmetric elements $M_{\mu\nu}$ and nine symmetric elements $T_{\mu\nu}$, $\mu,\nu = 1 \ldots 3$, which can be regrouped in the subsets $L_i = \frac{1}{2}\epsilon_{ijk}M_{jk}$, $K_i = M_{0i}$, T_{ij}, $N_i = T_{0i}$, T_{00}, satisfying the commutation relations (102)–(104) together with

$$[K_i, K_j] = -i\epsilon_{ijk}K_k, \quad [N_i, N_j] = i\epsilon_{ijk}N_k; \quad (105)$$

$$[L_i, K_j] = i\epsilon_{ijk}K_k, \quad [L_i, N_j] = i\epsilon_{ijk}N_k, \quad (106)$$

$$[K_i, N_j] = -i(T_{ij} + \delta_{ij}T_{00}); \quad (107)$$

$$[K_i, T_{jk}] = -i(\delta_{ij}N_k + \delta_{ik}N_j), \quad (108)$$

$$[N_i, T_{jk}] = -i(\delta_{ij}K_k + \delta_{ik}K_j), \quad (109)$$

$$[L_i, T_{00}] = [T_{ij}, T_{00}] = 0, \quad [K_i, T_{00}] = -2iN_i,$$

$$[N_i, T_{00}] = -2iK_i. \quad (110)$$

$SL(4,R)$ is thus split into several subgroups: $SO(4) = SO(3) \times SO(3)$ the maximal compact subgroup generated by L_i and N_i, $SO(1,3)$ the Lorentz group generated by L_i

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[4]The isomorphism $sl(4,C) \sim so(6,C)$ renders selected this gauge symmetry. Indeed, if the conformal extension of Minkowski space $M^\#$ is to be mapped in a topologically nontrivial way into the color space, then the color space $SL(4,C)$ is favored over other images since it has a real form $SL(4,R) \sim SO(3,3)$ isomorphic to the conformal group of the pseudo-Euclidean space $E_{2,2}$.

and K_i, SL(3,R) the ''three-volume-preserving'' group generated by L_i and T_{ij}, R^+ the noncompact Abelian group generated by T_{00}.

The subgroup SO(4) is utilized as a basis with J^P content of some (j_1,j_2) representation:

$$J^P=(j_1+j_2)^P,(j_1+j_2-1)^{-P},\ldots,(|j_1-j_2|)^{\pm P}.$$

The operator T_{ij} shifts SO(4) multiplets in (j_1,j_2) by $\Delta j_{1,2}=2$ [see Eq. (103)], and the structure of Regge sequences is reproduced by such shifting. A remarkable fact is that we have arrived at hadrons with different total angular momenta J, including the half-integer.

However, in the present context, quarks have neither spin nor orbital momentum. In the limit $N\to\infty$, we deal with bound quarks moving along straight world lines. Where did these higher angular momenta come from? We suppose them to be built out of gluon degrees of freedom. Indeed, $M_{\mu\nu}$ and $T_{\mu\nu}$ are related to our color basis as follows:

$$M_{ij}\equiv-i(E^+_{ij}-E^-_{ij}),\quad K_j\equiv i(E^+_{0j}-E^-_{0j}),\quad(111)$$

$$T_{ij}\equiv-i(E^+_{ij}+E^-_{ij}),\quad N_j\equiv i(E^+_{0j}+E^-_{0j}),\quad(112)$$

$$T_{00}\equiv2iH_0,\quad T_{jj}\equiv-2iH_j.\quad(113)$$

It is conceivable that gluon excitations about the background with the SL(4,R) color symmetry can manifest themselves as if their color degrees of freedom were converted into spin degrees of freedom described by irreducible unitary representations of SO(1,3).

A conversion of isospin into spin in gauge theories discovered in [12] seems to be of the direct relevance to our discussion. This phenomenon has its origin in a combination of some singular gauge field of magnetic type, such as the magnetic field generated by a monopole, and an isospin-degenerate field which is the source of a Coulomb-like electric field. The Liénard-Wiechert term of the background is analogous to the field of the magnetic monopole while the components of excitations $\mathbf{b}^{IJ\pm}$, $I,J=1,\ldots,3$ with the asymptotical behavior (96) play the role of the long-range electric field. The rotation generators M_{ij} related to $E^{\pm}_{ij}$ by Eq. (111) occur in the term of the angular momentum independent of the radius of gyration.

Unfortunately, this similarity is not quite complete. An external color field with an appropriate SL(4,R) degeneracy generating a long-range counterpart of the initial background field can hardly be conceived in the present context. On the other hand, restricting the consideration to a pure YM system, one faces infrared divergences due to the excitations $\mathbf{b}^{I4\pm}$.

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