

Minijets and the two-body parton correlation

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A large number of double parton scatterings have been recently observed by the CDF collaboration. The double parton scattering process measures σ_{eff} , a nonperturbative quantity related to the hadronic transverse size and with the dimensions of a cross section. The actual value measured by the CDF collaboration is considerably smaller as compared with the naive expectation, namely, the value of the inelastic nondiffractive cross section. The small value of σ_{eff} may be an effect of the hadron structure in the transverse plane. We discuss the problem by taking into account, at all orders, the two-body parton correlations in the many-body parton distributions. [S0556-2821(98)00601-8]

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I. INTRODUCTION

The Collider Detector at Fermilab (CDF) collaboration has recently shown evidence of a large number of double parton collisions [1], namely, events where, in the same inelastic interaction, two different pairs of partons scatter independently with large momentum transfer. The inclusive cross section for a double parton scattering is written as [2]

$$\sigma_D = \int_{p_i^c} D_2(x_A, x'_A; \mathbf{b}) \hat{\sigma}(x_A, x_B) \hat{\sigma}(x'_A, x'_B) \times D_2(x_B, x'_B; \mathbf{b}) d\mathbf{b} dx_A dx_B dx'_A dx'_B, \quad (1.1)$$

where $\hat{\sigma}(x_A, x_B)$ is the parton-parton cross section integrated with the cutoff p_i^c , which is the lower threshold to observe final state partons as minijets, x is the momentum fraction, and A and B are labels to identify the two interacting hadrons. σ_D is a function of the product $\hat{\sigma}(x_A, x_B) \hat{\sigma}(x'_A, x'_B)$. Actually the two different partonic interactions are localized in two regions in transverse space with a size of order $(1/p_i^c)^2$ and at a relative distance of the order of the hadronic radius r , in such a way that the two partonic interactions add incoherently in the double scattering cross section. The nonperturbative input in Eq. (1.1) is the two-body parton distribution $D_2(x, x'; \mathbf{b})$, which depends on the fractional momenta of the two partons taking part to the interaction and on their relative transverse distance \mathbf{b} . The transverse distance \mathbf{b} has to be the same for the two partons of hadron A and the two partons of hadron B , in order to have the alignment which is needed for a double collision to occur. D_2 is a dimensional quantity and therefore the process introduces a non perturbative scale factor which is related to the hadronic transverse size.

The simplest possibility to consider is the one where the dependence of D_2 on the different variables is factorized:

$$D_2(x, x'; \mathbf{b}) = f_{\text{eff}}(x) f_{\text{eff}}(x') F(\mathbf{b}). \quad (1.2)$$

f_{eff} is the effective parton distribution, namely, the gluon plus $\frac{4}{9}$ of the quark and antiquark distributions and $F(\mathbf{b})$ is normalized to one. Multiparton distributions are then uncorrelated and D_2 does not contain further information with

respect to the one-body parton distribution (actually f_{eff}) apart from the dependence on \mathbf{b} , whose origin is the dimensionality of D_2 and which gives rise to the scale factor σ_{eff} . In fact in this case one may write

$$\sigma_D = \frac{\sigma_S^2}{\sigma_{\text{eff}}} \quad (1.3)$$

with

$$\frac{1}{\sigma_{\text{eff}}} = \int F^2(\mathbf{b}) d^2b \quad (1.4)$$

and

$$\sigma_S = \int_{p_i^c} f_{\text{eff}}(x_A) f_{\text{eff}}(x_B) \hat{\sigma}(x_A, x_B) dx_A dx_B, \quad (1.5)$$

the single scattering expression of the perturbative QCD parton model.

Equation (1.2) is the basic hypothesis underlying the signature of a double parton collision which one has been looking for in the experimental search [1,3]. The expected characteristic feature of a double collision is in fact that it should produce a final state analogous to the one obtained by superposing two single scattering processes. CDF measures

$$\sigma_{\text{eff}} = 14.5 \pm 1.7_{-2.3}^{+1.7} \text{ mb.}$$

By looking at the dependence of σ_{eff} on x CDF has been able to verify the correctness of the factorization hypothesis in Eq. (1.2). The range of values of x available is limited to $x \leq 0.2$, for the interaction producing a pair of minijets, and to $x \leq 0.4$ for the interaction giving rise to a minijet and a photon. In the limited range of values of x available, the factorization hypothesis has shown to be consistent with the experimental evidence.

Since the uncorrelation hypothesis, as expressed in Eq. (1.2), is not inconsistent with experiment, one can work out the case where all multiparton distributions are uncorrelated and one may look for the sum of all multiparton interactions to the hadronic inelastic cross section. The subset where all multiple parton collisions are disconnected can be easily

summed up in the uncorrelated case [4]. The result is the semihard hadronic cross section σ_H , which represents the contribution to the hadronic inelastic cross section from events with at least one semihard partonic interaction. The actual expression is

$$\begin{aligned}\sigma_H &= \int d^2\beta [1 - e^{-\sigma_S F(\beta)}] \\ &= \sum_{n=1}^{\infty} \int d^2\beta \frac{[\sigma_S F(\beta)]^n}{n!} e^{-\sigma_S F(\beta)}.\end{aligned}\quad (1.6)$$

The integration on the impact parameter of the hadronic collision β gives the dimensionality to the cross section. The argument of the integral has the meaning of a Poissonian distribution of multiple semihard partonic interactions with average number depending on the impact parameter. The expression in Eq. (1.6) can be also obtained by introducing the semi-hard component in the conventional eikonalization of multiple scatterings [5].

σ_{eff} can be obtained from σ_H by taking twice the opposite of the second term of the expansion of σ_H in powers of multiple collisions. The actual value of σ_H is related to the value of σ_{eff} through Eq. (1.6). The relation depends on the functional form of $F(\beta)$. In the simplest case where $F(\beta) = \exp(-\beta^2/R^2)/\pi R^2$ one obtains a closed analytic expression for σ_H :

$$\sigma_H = 2\pi R^2 [\gamma + \ln \kappa + E_1(\kappa)], \quad (1.7)$$

where $\gamma = 0.5772 \dots$ is Euler's constant, $\kappa = \sigma_S / (2\pi R^2)$, and $E_1(x)$ is the exponential integral. In this example the relation with the hadronic radius r is $R = r\sqrt{2}$. For small κ one obtains $\sigma_H \rightarrow 2\pi R^2 \kappa = \sigma_S$, for large κ , namely, $\sigma_S \rightarrow \infty$, one obtains $\sigma_H \rightarrow 2\pi R^2 (\gamma + \ln \kappa)$. In this simplest example $\sigma_{\text{eff}} = 2\pi R^2$. The value of σ_H is therefore proportional to the measured value of σ_{eff} , the proportionality factor is slightly dependent on energy and on the cutoff. Sensible values of the hadron-hadron c.m. energy and of the cutoff give values for σ_H which are some 30–40 % larger with respect to the value of σ_{eff} . Different analytic forms for $F(\beta)$ give qualitatively similar results. Although σ_H has no reason in principle to be close to σ_{inel} , a value of σ_H as small as $\sigma_{\text{inel}}/2$, which would result from such considerations, seems nevertheless too small. Indeed σ_H , as given in Eq. (1.6), has a smooth dependence on the infrared cutoff. If one could really have the interaction under control also at small values of p_i^c one would say that σ_H has to be the same as the inelastic non diffractive cross section σ_{inel} . It would therefore be rather natural to expect for σ_H a value not much smaller as compared to the value of σ_{inel} . The large difference between σ_H and σ_{inel} could therefore be an indication that the simplest assumptions underlying the derivation of the expression in Eq. (1.6) have to be revised.

The main hypothesis which has been done to obtain the expression for σ_H in Eq. (1.6) is the Poissonian multiparton distribution. On the other hand, one has to expect correlations between partons as a consequence of the binding force. While most probably correlations will affect the x dependence of the multiparton distribution only for finite values of x , and therefore at large rapidities, correlations in the trans-

verse parton coordinates are present in every kinematical regime. Indeed the main reason of interest in multiple parton collisions, besides the identification of the process itself, is precisely the measure of the many-body parton correlations, which is an information on the hadron structure independent on the one-body parton distributions usually considered in hard processes.

In the present paper we work out the most general expression for the semihard cross section σ_H , which one obtains by assuming that only two-body parton correlations are present in the many-body parton distributions and by summing all disconnected multiple parton interactions. The two body parton correlation in transverse plane is then worked out in great detail in a few cases by considering different explicit shapes.

II. GENERAL FORMALISM AND SEMIHARD CROSS SECTION

At a given resolution, provided by the cutoff p_i^{min} that defines the lower threshold for the production of minijets, one can find the hadron in various partonic configurations. The probability of an exclusive n -parton distribution, namely, the probability to find the hadron in a configuration with n -partons, is denoted by $W_n(u_1 \dots u_n)$. $u_i \equiv (\mathbf{b}_i, x_i)$ represents the transverse partonic coordinate \mathbf{b}_i and longitudinal fractional momentum x_i while color and flavor variables are not considered explicitly. The distributions are symmetric in the variables u_i . One defines the generating functional of the multiparton distributions as

$$\mathcal{Z}[J] = \sum_n \frac{1}{n!} \int J(u_1) \dots J(u_n) W_n(u_1 \dots u_n) du_1 \dots du_n, \quad (2.1)$$

where the dependence on the infrared cutoff p_i^{min} is implicitly understood, and one may introduce also the logarithm of the generating functional $\mathcal{F}[J] = \ln(\mathcal{Z}[J])$. The conservation of the probability yields the overall normalization condition

$$\mathcal{Z}[1] = 1. \quad (2.2)$$

One may use the generating functional to derive the many body densities, i.e., the inclusive distributions $D_n(u_1 \dots u_n)$:

$$\begin{aligned}D_1(u) &= W_1(u) + \int W_2(u, u') du' \\ &+ \frac{1}{2} \int W_3(u, u', u'') du' du'' + \dots \\ &= \left. \frac{\delta \mathcal{Z}}{\delta J(u)} \right|_{J=1} = \left. \frac{\delta \mathcal{F}}{\delta J(u)} \right|_{J=1},\end{aligned}$$

$$\begin{aligned}
D_2(u_1, u_2) &= W_2(u_1, u_2) + \int W_3(u_1, u_2, u') du' \\
&+ \frac{1}{2} \int W_4(u_1, u_2, u', u'') du' du'' \dots \\
&= \frac{\delta^2 \mathcal{Z}}{\delta J(u_1) \delta J(u_2)} \Big|_{J=1} = \frac{\delta^2 \mathcal{F}}{\delta J(u_1) \delta J(u_2)} \Big|_{J=1} \\
&+ \frac{\delta \mathcal{F}}{\delta J(u_1)} \frac{\delta \mathcal{F}}{\delta J(u_2)} \Big|_{J=1} \dots \quad (2.3)
\end{aligned}$$

The many-body parton correlations are defined by expanding $\mathcal{F}[J]$ in the vicinity of $J=1$:

$$\begin{aligned}
\mathcal{F}[J] &= \int D(u) [J(u) - 1] du + \sum_{n=2}^{\infty} \frac{1}{n!} \int C_n(u_1 \dots u_n) \\
&\times [J(u_1) - 1] \dots [J(u_n) - 1] du_1 \dots du_n. \quad (2.4)
\end{aligned}$$

Here $D=D_1$ and the correlations C_n describe how much the distribution deviates from a Poisson distribution, which corresponds in fact to $C_n \equiv 0$, $n \geq 2$.

In the case of hadron-nucleus and nucleus-nucleus collisions a systematic use of the Abramovskii-Gribov-Kancheli (AGK) cutting rules [6] allows one to express the total inelastic cross section as a probabilistic superposition of nucleon-nucleon interaction probabilities [7]. The same feature holds for the self-shadowing cross sections [8]. When considering hadron-hadron collisions as interactions between objects composed with partons, one can make the assumption that similar relations hold with nucleons replacing nuclei and partons replacing nucleons. In this respect semihard parton-parton interactions have to be regarded as a particular case of self-shadowing interactions [9]. The semihard nucleon-nucleon cross section is then expressed as the sum of all the probabilities of multiple parton collisions:

$$\sigma_H = \int d^2\beta \sigma_H(\beta)$$

with

$$\begin{aligned}
\sigma_H(\beta) &= \int \sum_n \frac{1}{n!} \frac{\delta}{\delta J(u_1)} \dots \frac{\delta}{\delta J(u_n)} \mathcal{Z}_A[J] \\
&\times \sum_m \frac{1}{m!} \frac{\delta}{\delta J'(u'_1 - \beta)} \dots \frac{\delta}{\delta J'(u'_m - \beta)} \mathcal{Z}_B[J'] \\
&\times \left\{ 1 - \prod_{i=1}^n \prod_{j=1}^m [1 - \hat{\sigma}_{i,j}(u, u')] \right\} \\
&\times \prod dudu' \Big|_{J=J'=0}, \quad (2.5)
\end{aligned}$$

where β is the impact parameter between the two interacting hadrons A and B and $\hat{\sigma}_{i,j}$ is the elementary probability for parton i (of A) to have a hard interaction with parton j (of B). The semihard cross section is constructed summing over all possible partonic configurations of the two interacting hadrons (the sums over n and m) and, for each configuration

with n partons from A and m partons from B , summing over all possible multiple partonic interactions. This last sum is constructed asking for the probability of no interaction between the two configurations (actually $\prod_{i=1}^n \prod_{j=1}^m [1 - \hat{\sigma}_{i,j}]$). One minus the probability of no interaction is equal to sum over all semihard interaction probabilities.

The presence of multiple parton interactions is induced by the large flux of partons which is effective at large energies. The most important contribution to the semihard cross section, as a consequence, is the contribution of the disconnected partonic collisions, namely, the interactions where each parton undergoes at most one semihard collision. These are, in fact, those multiple partonic interactions that, at a given number of partonic collisions, maximize the parton flux. Indeed the search and the observation of the first evidence of multiple semihard parton interactions has been focused to the case of double disconnected parton interactions [1,3]. We simplify therefore the problem by expanding the interaction probability (the factor in curly brackets) as sums and by removing all the addenda containing repeated indices:

$$\left\{ 1 - \prod_{i,j}^{n,m} [1 - \hat{\sigma}_{ij}] \right\} \Rightarrow \sum_{ij} \hat{\sigma}_{ij} - \frac{1}{2!} \sum_{ij} \sum_{k \neq i, l \neq j} \hat{\sigma}_{ij} \hat{\sigma}_{kl} + \dots \quad (2.6)$$

as a result the semihard cross section is constructed with multiple disconnected parton collisions only, where disconnected refers to the perturbative component of the interaction. Some features of semihard parton rescatterings are presented in Appendix A. Because of the symmetry of the derivative operators in Eq. (2.5) one can replace the expression in Eq. (2.6) with

$$nm \hat{\sigma}_{11} - \frac{1}{2!} n(n-1)m(m-1) \hat{\sigma}_{11} \hat{\sigma}_{22} + \dots$$

in such a way that the sums over m and n can be performed explicitly. As a consequence the cross section at fixed impact parameter $\sigma_H(\beta)$ can be expressed by the operatorial form

$$\sigma_H(\beta) = [1 - \exp(-\delta \cdot \hat{\sigma} \cdot \delta')] \mathcal{Z}_A[J+1] \mathcal{Z}_B[J'+1] \Big|_{J=J'=0}. \quad (2.7)$$

We have avoided writing explicitly the variables u and u' and the functional derivative $\delta/\delta J(u_i)$ has been simply indicated as δ_i .

The form of $\sigma_H(\beta)$ given by Eq. (2.6) is still too complicated to be worked out in its general form, since all possible multiparton correlations are present in \mathcal{Z} . Therefore we further simplify the problem by taking into account two-body parton correlations only. Our explicit expression for \mathcal{F} is, therefore,

$$\begin{aligned}
\mathcal{F}_{A,B}[J+1] &= \int D_{A,B}(u) J(u) du \\
&+ \frac{1}{2} \int C_{A,B}(u, v) J(u) J(v) dudv, \quad (2.8)
\end{aligned}$$

where $D(u)$ is the average number of partons and $C(u, v)$ is the two-body parton correlation.

Either by using techniques of functional integration or by means of a suitable diagrammatic expansion [10] one is able to obtain in this case a closed expression for $\sigma_H(\beta)$:

$$\sigma_H(\beta) = 1 - \exp\left[-\frac{1}{2} \sum_n a_n - \sum_n \frac{1}{2n} b_n\right], \quad (2.9)$$

where a_n and b_n are functions of the impact parameter β and are given by

$$a_n = \int D_A(u_1) \hat{\sigma}(u_1, u'_1) C_B(u'_1 - \beta, u'_2 - \beta) \hat{\sigma}(u'_2, u_2) \\ \times C_A(u_2, u_3) \cdots D_B(u'_n - \beta) \prod du_i du'_i, \quad (2.10)$$

$$b_n = \int C_A(u_n, u_1) \hat{\sigma}(u_1, u'_1) C_B(u'_1 - \beta, u'_2 - \beta) \\ \times \hat{\sigma}(u'_2, u_2) \cdots C_B(u'_{n-1} - \beta, u'_n - \beta) \\ \times \hat{\sigma}(u'_n, u_n) \prod du_i du'_i. \quad (2.11)$$

The actual expression for a_n (which will be referred to as ‘‘open chain’’) holds for n odd. When n is odd one may also have the symmetric case, where the expression begins with D_B and ends with D_A . When n is even the initial and final distribution are either both D_A or both D_B . In the definition of b_n , which will be referred to as ‘‘closed chain’’ n is always even, so that one of the ends is A and the other is B . Sometimes the expression $n=2m$ will be used. One may notice that, at a given order in the number of partonic interactions, one can obtain a term of kind a from a term of kind b by replacing one C with a pair of D 's. The operation can be done in n ways. The combinatorial meaning of the $1/n$ factor multiplying each term of kind b in Eq. (2.9) is then understood. The factor $\frac{1}{2}$ in Eq. (2.9) is the consequence of the symmetry between A and B .

The cross section is given by an integral on the impact parameter of the interaction probability $\sigma_H(\beta)$ that is expressed as one minus the probability of no interaction. The probability of no interaction is given by the negative exponential of the sum over all possible different connected structures, namely, all structures of kind a_n and of kind b_n . With our approximations, Eq. (2.6) and Eq. (2.8), these are in fact all possible connected structures which can be built with the average numbers $D_{A,B}$, the two-body correlations $C_{A,B}$, and the interaction $\hat{\sigma}$. Expanding the exponential, the cross section can then be expressed as the sum over all possible structures, both connected and disconnected.

One will notice that, when no correlations are present, all terms of kind b disappear and only the first of the terms of kind a , namely, $D_A \hat{\sigma} D_B$, is left. In that limit the cross section is given simply by

$$\sigma_H = \int d^2\beta \left\{ 1 - \exp\left[-\int D_A(u - \beta) \right. \right. \\ \left. \left. \times D_B(u') \hat{\sigma}(u, u') dud u'\right] \right\}, \quad (2.12)$$

which corresponds to the Poissonian distribution discussed in the Introduction.

III. AN EXPLICIT CASE: GAUSSIAN CORRELATION

The bulk of the semihard cross section originates in the region of small x values. The experimental observation on the independence of σ_{eff} as a function of x , at small x values, on the other hand, indicates that correlations in x , at small x , are not a strong effect in the two-body parton distribution. A sensible case to consider is therefore the one where there are no correlations in fractional momenta, nor between fractional momenta and transverse coordinates, while only transverse coordinates are correlated:

$$D(u) = f(x)D(\mathbf{b}), \quad C(u, u') = f(x)f(x')C(\mathbf{b}, \mathbf{b}'). \quad (3.1)$$

Given the localization of the partonic collisions in transverse space, the dependence of the parton-parton interaction probability on \mathbf{b} and \mathbf{b}' is represented as a δ function:

$$\hat{\sigma}(u, u') = \sigma_{x,x'} \delta(\mathbf{b} - \mathbf{b}'). \quad (3.2)$$

All integrations on the fractional momenta x and x' are then factorized from the integrations on the transverse coordinates and result in the single scattering term cross section σ_S , whose expression is given in Eq. (1.5). a_n and b_n are therefore considerably simplified:

$$a_n = \sigma_S^n \int D_A(\mathbf{b}_1) C_B(\mathbf{b}_1 - \beta, \mathbf{b}_2 - \beta) C_A(\mathbf{b}_2, \mathbf{b}_3) \cdots \\ \times D_B(\mathbf{b}_n - \beta) \prod d\mathbf{b}_i, \quad (3.3)$$

$$b_n = \sigma_S^n \int C_A(\mathbf{b}_n, \mathbf{b}_1) C_B(\mathbf{b}_1 - \beta, \mathbf{b}_2 - \beta) \cdots \\ \times C_B(\mathbf{b}_{n-1} - \beta, \mathbf{b}_n - \beta) \prod d\mathbf{b}_i \cdots. \quad (3.4)$$

To proceed further one needs to consider explicit functional forms for $C(\mathbf{b}, \mathbf{b}')$. A simplest extension of the Gaussian model discussed in the introduction corresponds to the following Gaussian expression of the correlation term:

$$D(\mathbf{b}) = h \exp[-F\mathbf{b}^2], \\ C(\mathbf{b}, \mathbf{b}') = k \exp\left[-\frac{1}{2} F(\mathbf{b} + \mathbf{b}')^2 - \frac{1}{2} G(\mathbf{b} - \mathbf{b}')^2\right]. \quad (3.5)$$

After defining $\mathbf{b} = \mathbf{y} + \frac{1}{2}\beta$, b_n is explicitly written as

$$b_n = \sigma_S^n k^n \int \exp\left[-\frac{1}{2} nF\beta^2\right] \exp\left[-(F+G) \sum \mathbf{y}_i^2\right] \\ \times \exp\left[-(F-G) \sum \mathbf{y}_i \cdot \mathbf{y}_{i+1}\right] \prod d\mathbf{y}_i,$$

with the convention that the variable \mathbf{y}_{n+1} coincides with \mathbf{y}_1 . b_n can then be worked out through the substitution

$$\mathbf{v}_i = \mu \mathbf{y}_i - \nu \mathbf{y}_{i+1}; \quad \mu - \nu = \sqrt{2F} \mu + \nu = \sqrt{2G},$$

which leads to

$$b_n = J \sigma_S^n k^n \exp\left[-\frac{1}{2} n F \beta^2\right] \int \exp\left[-\sum \mathbf{v}_i^2\right] \prod d\mathbf{v}_i, \quad (3.6a)$$

where the Jacobian is $J = (\mu^n - \nu^n)^{-2}$, as discussed in Appendix B. b_n is therefore finally written as

$$b_n = \sigma_S^n k^n \pi^n (\mu^n - \nu^n)^{-2} \exp\left[-\frac{1}{2} n F \beta^2\right]. \quad (3.6b)$$

The case of the open chain a_n is less symmetric and requires a slightly less straightforward treatment. Using the \mathbf{y} variables, defined previously, the expression of a_n takes the form

$$a_n = \sigma_S^n h^2 k^{n-1} \exp\left[-\frac{1}{2} n F \beta^2\right] \times \int \exp\left[-(F+G)\mathcal{Y}^T \cdot \mathcal{M} \cdot \mathcal{Y}\right] \prod d\mathbf{y}_i, \quad (3.7a)$$

where

$$\mathcal{Y}^T = (\mathbf{y}_1, \mathbf{y}_2, \dots, \mathbf{y}_n) \quad (3.8)$$

and

$$\mathcal{M} = \begin{pmatrix} 1+r & r & 0 & \dots & 0 & 0 \\ r & 1 & r & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & r & 1+r \end{pmatrix}, \quad (3.9)$$

$$r = \frac{1}{2}(F-G)/(F+G). \quad (3.10)$$

The Gaussian integral can be evaluated (details of the calculations are reported in Appendix B) leading to the relation

$$a_n = \sigma_S^n h^2 k^{n-1} \pi^n r^{n-1} [(2r+1)U_{n-1}(1/2r)]^{-1} \times \exp\left[-\frac{1}{2} n F \beta^2\right], \quad (3.7b)$$

where U_n is the Chebyshev polynomial of second kind [11].

While the terms a_n and b_n are computed exactly, the sum of the series in Eq. (2.9) can be performed only in limiting cases. We discuss the case where the correlation length is much smaller with respect to the hadronic radius. The parameter G characterizes the correlation while F is related to the hadronic transverse size. Small correlation lengths correspond to $F \ll G$. We work out therefore the leading order term in F while keeping the full structure in G both in a_n and in b_n . In order to find the limiting expression of a_n when $F \ll G$, one needs to work out the limit of $U_n(1/2r)$ for $1/2r \rightarrow -1 + \epsilon$. One obtains

$$U_n(-1 + \epsilon) \approx (-)^n (n+1).$$

By using this expression for U_n and with similar, but simpler, manipulations one obtains the following limiting form for a_n :

$$a_n = \frac{G h^2}{4 n k F} Z^n, \quad (3.7c)$$

with

$$Z = \left(\frac{2\pi k}{G}\right) \sigma_S \exp\left[-\frac{1}{2} F \beta^2\right]. \quad (3.11)$$

The whole series of Eq. (3.1) can be then be summed yielding as a result

$$S_a = \frac{G h^2}{4 k F} [-\ln(1-Z)]. \quad (3.12)$$

To work out the limiting case for b_n we keep the exact value of the difference $(\mu - \nu)^2 = 2F$, while setting $\mu \approx \nu \approx \sqrt{G/2}$ everywhere else. Keeping, moreover, into account the condition $n = 2m$, we obtain

$$b_{2m} = \frac{G}{16 F m^2} Z^{2m}. \quad (3.6c)$$

The sum of the series appearing in Eq. (2.9) is therefore given by

$$S_b = \frac{G}{32 F} \mathcal{L}_3(Z^2), \quad (3.13)$$

where $\mathcal{L}_3(x) = \sum x^n/n^3$ is the trilogarithm function [12].

The final limiting form for the cross section, when $F \ll G$, is therefore

$$\sigma_H = \int d^2\beta \sigma_H(\beta) = \int d^2\beta \left\{ 1 - \exp\left[-\frac{1}{2} S_a - \frac{1}{2} S_b\right] \right\}. \quad (3.14)$$

In the limit $G \rightarrow \infty$ the correlation goes to zero. In this case one obtains $Z \rightarrow 0$, $S_a \rightarrow (\pi \sigma_S/2kF) \exp[-\frac{1}{2} F \beta^2]$ and $S_b \rightarrow 0$. The expression of σ_H in Eq. (3.14) is then reduced to the Poissonian distribution discussed in the Introduction.

IV. TWO QUALITATIVELY DIFFERENT FEATURES OF THE CORRELATION TERM

The uncorrelated multiparton distributions are characterized by two independent features. One feature is that the n -body parton distribution is factorized in the product of n times the one-body parton distribution. The second feature is that the distribution in the number of partons (namely, after integrating over all other degrees of freedom) is Poissonian. Both features are affected by the introduction of the two-body correlations and it is therefore interesting to study the two effects separately.

One may modify the number distribution, without introducing nonfactorized two-body correlations in \mathbf{b} , by using the factorized expression

$$C(u, u') = -\lambda D(u) D(u'). \quad (4.1)$$

The terms appearing in Eqs. (2.10) and (2.11) are very much simplified in this case. After integrating over the longitudinal variables, the transverse integrations always appears in the unique form

$$T \equiv T(\beta) = \sigma_S \int D_A(\mathbf{b}) D_B(\mathbf{b} - \beta) d\mathbf{b}. \quad (4.2)$$

One therefore obtains

$$\frac{1}{2} \sum_n a_n = T/(1 + \lambda T),$$

$$\frac{1}{2} \sum_m b_{2m}/2m = \ln(1 - \lambda^2 T^2).$$

and the cross section, at a fixed value of the impact parameter, is expressed as

$$\sigma_H(\beta) = 1 - [1 - \lambda^2 T^2]^{-1/2} \exp[-T/(1 + \lambda T)]. \quad (4.3)$$

The parameter λ represents a measure of the deviation from the Poissonian distribution, both for the initial states and for the elementary collisions. In Appendix C some further elaboration on the factorized case are presented.

Perhaps more interesting is the case where the number distribution of the incoming parton is Poissonian, because the correlation integrates to zero, while the resulting distribution in the number of parton collisions at fixed impact parameter is non-Poissonian, as a consequence of the presence of the correlation term. One may, e.g., consider the expressions

$$D(\mathbf{b}) = h \exp[-F\mathbf{b}^2],$$

$$C(\mathbf{b}, \mathbf{b}') = -k' \Delta_2 \exp[-\frac{1}{2} F(\mathbf{b} + \mathbf{b}')^2 - \frac{1}{2} G(\mathbf{b} - \mathbf{b}')^2], \quad (4.4)$$

where Δ_2 indicates the two-dimensional Laplace operator acting on the transverse difference $\mathbf{b} - \mathbf{b}'$ or $\mathbf{y} - \mathbf{y}'$. By integrating C over the transverse variables one obtains zero.

The number distribution depends only on D and it is Poissonian by construction. The parameter k' is obviously related to the parameter k in the previous paragraph. Since k was the strength of the correlation at $\mathbf{b} = \mathbf{b}' = 0$ in order to keep the same normalization we make the position $k' = k/G$.

In order to study this case a different way of dealing with the approximation introduced in obtaining Eqs. (3.6c) and (3.7c) is useful. This alternative procedure is the following: The variables \mathbf{y}_i are substituted by the mean value $\mathbf{Y} = \Sigma \mathbf{y}/n$ and by the differences $\mathbf{x}_i = \mathbf{y}_{i+1} - \mathbf{y}_i$, with the constraint $\Sigma \mathbf{x}_i = 0$. The relevant Jacobian is $J = 1$ and so the integration volume is transformed according to $\Pi d\mathbf{y}_i = d\mathbf{Y} \Pi d\mathbf{x}_i \delta(\Sigma \mathbf{x}_i)$. Then in the terms containing F , which simply defines the size of the hadron, one performs the substitutions $\mathbf{y}_i \approx \mathbf{Y}/n$, whereas the terms containing G and the differences \mathbf{x}_i are not modified. For the calculation of a_n , since only $n - 1$ differences appear, the integrations in $d\mathbf{Y}$ and $d\mathbf{x}_i$ decouple in the Gaussian integration and the result of Eqs. (3.7c) and (3.11) is recovered; for the calculation of b_n the constraint on the \mathbf{x}_i is essential, it may be implemented in

the standard exponential form $\delta(\Sigma \mathbf{x}_i) = (2\pi)^{-2} \int \exp(i\Sigma \mathbf{q} \cdot \mathbf{x}_i) d\mathbf{q}$ and carrying out all the integrations the result of Eq. (3.6c) is reproduced.

All open chains a_n with $n > 1$ contain at least one integration over the C and give therefore zero since, as already observed, in this case the differences have no constraint. Every independent integration over a transverse difference corresponds therefore to an integration of a single correlation term, which gives zero. Small nonzero effects have been possibly lost in the approximate substitution $\mathbf{y}_i \approx \mathbf{Y}/n$ in the terms containing F in Eqs. (3.6b) and (3.7b).

When one computes the contribution of the closed chain b_n , the constraint is effective. The calculation is again performed by means of the exponential representation of the δ function and the resulting expression can be brought to the following form:

$$\tilde{b}_{2m} = \frac{G}{8Fm^2} (Z/m)^{2m} (2m)!. \quad (4.5)$$

The final result is obtained by summing all the terms above, with the weight $(1/2m)$. The cross section (at β fixed) is therefore

$$\sigma_H(\beta) = 1 - \exp[-T - \tilde{S}_b], \quad \tilde{S}_b = \sum_{m=1}^{\infty} \frac{\tilde{b}_{2m}}{2m}. \quad (4.6)$$

While a closed analytic form for the sum \tilde{S}_b is not easily written in the general case, in the limiting case where the number of partonic collisions is large one may use the Stirling approximation for the factorial $m^{-m} \approx \sqrt{2\pi m} e^{-m}/m!$. With the help of the decomposition $(2m)! = 2^{2m} m! (\frac{1}{2})_m$ the sum can be expressed as

$$\tilde{S}_b = \frac{\pi G}{8F} \psi(w), \quad (4.7)$$

where

$$w = (2Z/e)^2, \quad \psi(w) = \sum_1^{\infty} \frac{(1/2)_m}{m^2 m!} w^m.$$

$\psi(w)$ can be evaluated by computing, as an intermediate step,

$$(w \partial_w)(w \partial_w) \psi(w) = \sum_1^{\infty} \frac{(1/2)_m}{m!} w^m,$$

and by performing the integrations afterwards. The resulting expression is

$$\psi(w) = 2\mathcal{L}_2(\frac{1}{2} [1 - \sqrt{1-w}]) - \ln^2(\frac{1}{2} [1 + \sqrt{1+w}]), \quad (4.8)$$

where $\mathcal{L}_2(x) = \Sigma x^n/n^2$ is the dilogarithm function [12].

V. CONCLUDING DISCUSSION

The small value of σ_{eff} , the dimensional parameter characterizing double parton scatterings, which has been measured recently by CDF, is an indication that two-body parton

correlations, in the many-body parton distribution of the proton, are likely to be sizable. In the case of an uncorrelated many-body parton distribution, the value of σ_{eff} puts a constraint on the range of possible values of σ_H , the semihard contribution to the hadronic inelastic cross section. The actual measured value of σ_{eff} would give rise to values of σ_H of the order of $\sigma_{\text{inel}}/2$ also at very large c.m. energies, where one would rather expect $\sigma_H \approx \sigma_{\text{inel}}$. The experimental evidence is also that, in the x region accessible experimentally, namely, at small x values, the correlation in fractional momenta is not a large effect.

In the present paper we have worked out the semihard cross section σ_H in the case where initial state partons are correlated in the transverse parton coordinates. In fact σ_H can be worked out rather explicitly when only two-body parton correlations are included in the many-body parton distributions and when each parton can have at most one semihard interaction. There are two qualitatively different features in the two-body parton correlation, and both change the relation between σ_H and σ_{eff} with respect to the uncorrelated case: (1) The distribution in the number of partons is not any more Poissonian, although the dependence on the kinematical variables of the different partons is factorized; (2) the overall distribution in the number of partons, namely after integrating on the partonic kinematical variables, is Poissonian but the dependence on the partonic kinematical variables is not factorized, in which case the two-body parton correlation integrates to zero. The general case is obviously a combination of the two possibilities. We point out, however, that both cases separately can give rise to a small value of σ_{eff} while keeping the value of σ_H close to σ_{inel} .

In the first case σ_H is obtained by integrating the expression in Eq. (4.3). In the second by integrating the expression in Eq. (4.6). The critical value of the impact parameter β_c , which gives the size to the cross section σ_H , is the value which makes small the argument of the exponential in the expression of $\sigma_H(\beta)$. The detailed dependence of the argument of the exponential at $\beta < \beta_c$ is not of great importance for the determination of σ_H when, for $\beta < \beta_c$, the argument of the exponential is already large: σ_H is obtained by integrating the probability of having at least one semihard interaction. When the probability is close to one the contribution to the integral is very similar for events with the same impact parameter and with different but large average number of partonic collisions.

The critical value of the impact parameter which gives the size to σ_H is therefore determined by the argument of the exponential at the edge of the interaction region. In case (1) the behavior of the argument of the exponential when going at the edge of the interaction region corresponds to the limiting behavior of the expression below at small T values:

$$\frac{T}{1+\lambda T} + \frac{1}{2} \ln(1 - \lambda^2 T^2) \rightarrow T = \sigma_S \int D_A(\mathbf{b}) D_B(\mathbf{b} - \beta) d^2 b,$$

while in case (2) it corresponds to neglect \tilde{S}_b as compared with T in Eq. (4.6):

$$T + \tilde{S}_b \rightarrow T = \sigma_S \int D_A(\mathbf{b}) D_B(\mathbf{b} - \beta) d^2 b.$$

In both cases the critical value β_c has the same value which one finds for the uncorrelated distribution and therefore σ_H , as in the uncorrelated case, is roughly equal to $4\pi/F$, where F is the parameter giving the extension of the one-body parton distribution in the transverse plane.

The value of the semihard cross section is therefore determined by the behavior at large distances and it does not depend on correlations. σ_{eff} , on the contrary, is related directly to the correlation term, as it is seen by working out the double scattering cross section from the expansion of σ_H . One obtains in fact the relation

$$\sigma_{\text{eff}} = \frac{4\pi}{F(1+\lambda)^2}$$

in case (1), and

$$\frac{1}{\sigma_{\text{eff}}} = \frac{F}{4\pi} + \int C(\mathbf{b}, \mathbf{b}') C(\mathbf{b} - \beta, \mathbf{b}' - \beta) d^2 b d^2 b' d^2 \beta$$

in case (2).

A qualitative feature is that in both cases one obtains a value of σ_{eff} which may be sizably smaller with respect to $4\pi/F \approx \sigma_H$. While, on the other hand, nothing prevents the value of σ_H to be close to the value of σ_{inel} . The smaller value of σ_{eff} , with respect to the expectation of the uncorrelated case, is rather generally associated with the increased dispersion of the distribution in the number of partonic collisions: In the case of no correlations the distribution is strictly Poissonian when the impact parameter is fixed. When correlations are introduced the distribution in the number of parton collisions, at fixed β , is not Poissonian any more and the natural consequence is that the dispersion in the number of collisions is increased.

The indication from the measure of the rate of double parton scatterings is therefore that two-body parton correlations are likely to be important while, unfortunately, one cannot say much about dynamical quantities, such as the correlation length. Useful observables to be measured, in order to get some more insight into the problem, would be the semihard cross section σ_H and the triple parton scattering cross section. Our present analysis shows that σ_H can be reliably discussed in perturbation theory. The measure of σ_H , in association with σ_{eff} , would help considerably in clarifying the size of the effect induced by the presence of the two-body parton correlations: All the considerations of the present paper are based on the prejudice that σ_H should have a value rather close to the value of σ_{inel} .

The measure of triple parton scattering would allow one to test the possibility discussed in case (1), since in that case the rate of triple scatterings would be strictly fixed by the measured rate of double scatterings. In the other cases presently discussed the knowledge of the rate of triple parton collisions would allow one to obtain the actual values of the parameters of the correlation term.

A lot of effort has been put on the study of the proton structure as a function of the momentum fraction x . The distribution of partons however depends on three degrees of freedom, the momentum fraction x and the transverse parton coordinate \mathbf{b} . The measure of the rate of triple and of higher

order partonic collisions is the essential tool to learn on the parton structure of the proton in transverse plane.

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APPENDIX A

In this appendix some short observations about the effects of the rescattering processes will be displayed. The general form of the hadron-hadron scattering cross section Eq. (2.5) has been reduced to a more manageable form by throwing away all the rescattering process by means of the position:

$$\left\{ 1 - \prod_{i,j}^{n,m} [1 - \hat{\sigma}_{ij}] \right\} \Rightarrow \sum_{ij} \hat{\sigma}_{ij} - \frac{1}{2!} \sum_{ij} \sum_{k \neq i, l \neq j} \hat{\sigma}_{ij} \hat{\sigma}_{kl} + \dots \quad (\text{A1})$$

It is possible to deal with some rescattering process without too much effort if we look at processes where only one of the two colliding partons has already suffered some collision. In this case instead of Eq. (A1) we write

$$\begin{aligned} \sigma_H = \int d\beta \left\{ 1 - \exp \left[- \int D_A(u) \hat{\sigma}(u, u') D_B(u') du du' \right] \left(\exp \left[\frac{1}{2} \int D_A(u) \hat{\sigma}(u, u') \hat{\sigma}(u, u'') D_B(u') D_B(u'') du du' du'' \right] \right. \right. \\ \left. \left. + \exp \left[\frac{1}{2} \int D_A(u) D_A(u'') \hat{\sigma}(u, u') \hat{\sigma}(u'', u') D_B(u') du du' du'' \right] - 1 \right) \right\}. \quad (\text{A4}) \end{aligned}$$

This form is not unexpected, it could have been written by hand, Eq. (A3) shows a more systematic way of deriving it and possible further corrections. The final distribution of the collision is *not* Poissonian and this deviation is purely due to the hard dynamics. It can be observed that precisely this origin in the hard collision may also offer a way to distinguish these effects from the effects of correlations in the incoming two-body distributions, in fact in the case here sketched there should be an unbalance among the kinematical variables of the pair of jets originating from the scattered partons.

APPENDIX B

The Jacobian appearing in Eq. (3.6), which arises from the transformation from the variables \mathbf{y} to the variables \mathbf{v} , is expressed as $J = [\det \mathcal{J}]^{-2}$. The actual form of \mathcal{J} is

$$\mathcal{J} = \begin{pmatrix} \mu & -\nu & 0 & \dots & 0 & 0 \\ 0 & \mu & -\nu & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -\nu & 0 & 0 & \dots & 0 & \mu \end{pmatrix},$$

$$\begin{aligned} \left\{ 1 - \prod_{i,j}^{n,m} [1 - \hat{\sigma}_{ij}] \right\} \Rightarrow \sum_{ij} \hat{\sigma}_{ij} - \frac{1}{2!} \sum_{ij} \left\{ \sum_{k \neq i, l} \hat{\sigma}_{ij} \hat{\sigma}_{kl} \right. \\ \left. + \sum_{k, l \neq j} \hat{\sigma}_{ij} \hat{\sigma}_{kl} - \sum_{k \neq i, l \neq j} \hat{\sigma}_{ij} \hat{\sigma}_{kl} \right\} + \dots \quad (\text{A2}) \end{aligned}$$

In this expression the first term shows that the partons belonging to the first hadron suffer only one collision, the partons belonging to the second hadron undergo any number of collisions, the second term describes the symmetrical situation, the third term eliminates the double counting of the single-collision processes.¹

The operatorial form of the cross section at fixed impact parameter $\sigma_H(\beta)$ acquires now the more complicated shape

$$\begin{aligned} \sigma_H(\beta) = \{ 1 - \exp(\delta \cdot [e^{\hat{\sigma} \cdot \delta'} - 1]) - \exp(\delta' \cdot [e^{\hat{\sigma} \cdot \delta} - 1]) \\ + \exp(-\delta \cdot \hat{\sigma} \cdot \delta') \} \mathcal{Z}_A[J+1] \mathcal{Z}_B[J'+1] |_{J=J'=0}. \quad (\text{A3}) \end{aligned}$$

This expression seems of uneasy interpretation but it gives some information when reduced to simpler particular cases. The simplest, but nontrivial result is produced when the incoming parton distribution is purely Poissonian and only the double scattering is actually considered. In these situations it is easily seen that instead of Eqs. (2.9), (2.10), (2.11) one obtains the form

and the result $\det \mathcal{J} = \mu^n - \nu^n$ is easily obtained by expansion according to the last row. The exponent -2 appears because the matrix describes the inverse transformation and there are two transverse dimensions.

In order to calculate the determinant of the matrix \mathcal{M} in Eq. (3.9) an auxiliary matrix \mathcal{A} is introduced so that we have

$$\mathcal{M} = \begin{pmatrix} 1+r & r & 0 & \dots & 0 & 0 \\ r & 1 & r & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & r & 1+r \end{pmatrix},$$

$$\mathcal{A} = \begin{pmatrix} 1 & r & 0 & \dots & 0 & 0 \\ r & 1 & r & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & r & 1 \end{pmatrix}.$$

¹This possibility was already considered in Ref. [10], for an unsymmetrical situation.

By means of the standard rules for the computation of the determinants two recurrence relations are obtained:

$$\det \mathcal{M}_n = \det \mathcal{A}_n + 2r \det \mathcal{A}_{n-1} + r^2 \det \mathcal{A}_{n-2},$$

$$\det \mathcal{A}_n = \det \mathcal{A}_{n-1} - r^2 \det \mathcal{A}_{n-2}$$

which imply the simpler relation

$$\det \mathcal{M}_n = (2r+1) \det \mathcal{A}_{n-1}.$$

The recurrence relations for the determinants of \mathcal{A}_n are very similar to the recurrence relation for the Chebyshev polynomials U_n [11] and in this way it is possible to get the final expression for them

$$\det \mathcal{A}_n = r^n U_n(1/2r).$$

For completeness we give also the explicit expression of the U polynomials:

$$U_n(\cos \theta) = \frac{\sin(n+1)\theta}{\sin \theta}.$$

APPENDIX C

In this appendix we shall briefly discuss some non-Poissonian one-body densities, the essential point were already shown in Ref. [10], but it may be useful to state again them in order to have a comparison of the previous treatment. The starting point of Eq. (2.7) is specialized to the case of pure one-body densities in the form

$$\sigma_H = [1 - \exp(-T\partial\partial')] X_A(\mathcal{Y}) X_B(\mathcal{Y}') |_{\mathcal{Y}=\mathcal{Y}_0, \mathcal{Y}'=\mathcal{Y}'_0}. \quad (\text{C1})$$

The Poissonian distribution is given by $X = \exp[\mathcal{Y} - \mathcal{Y}_0]$, the situation described in Eq. (4.1) corresponds to

$$X = \exp[(\mathcal{Y} - \mathcal{Y}_0) + \frac{1}{2} \lambda (\mathcal{Y} - \mathcal{Y}_0)^2], \quad (\text{C2})$$

Then by applying Eq. (C1) to Eq. (C2), one gets as intermediate step

$$\sigma_H = \exp[-T\partial' + \frac{1}{2} \lambda (T\partial')^2] \exp[(\mathcal{Y}' - \mathcal{Y}'_0) + \frac{1}{2} \lambda (\mathcal{Y}' - \mathcal{Y}'_0)^2] |_{\mathcal{Y}'=\mathcal{Y}'_0}. \quad (\text{C3})$$

By the use of the shift formula $\exp[a\partial_x]F(x) = F(x+a)$ and of the relation.

$$\exp[-p\partial_x^2] \exp[qx^2] = \frac{1}{\sqrt{1-4pq}} \exp\left[\frac{qx^2}{1+4pq}\right],$$

it is seen that the expression appearing in Eq. (C3) reduces to the form already given in Eq. (4.3).

One could also choose the form

$$X = [1 - (\mathcal{Y} - \mathcal{Y}_0)]^{-\alpha},$$

which corresponds to a negative binomial distribution for the initial partons and obtain in this way the result

$$\sigma_H(\beta) = 1 - T^{-\alpha} U[\alpha; 1; 1/T],$$

by representing the incoming distributions as Laplace transforms as shown in Ref. [10]. Note that a slight simplification has been introduced in the distribution, with respect to that reference, and this reflects into a slight simplification of the result, which keep, however, its main properties; in particular U is the irregular confluent hypergeometric function.

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- [1] CDF Collaboration, F. Abe *et al.*, Phys. Rev. D **56**, 3811 (1997).
- [2] C. Goebel, F. Halzen, and D. M. Scott, Phys. Rev. D **22**, 2789 (1980); N. Paver and D. Treleani, Nuovo Cimento A **70**, 215 (1982); B. Humpert, Phys. Lett. **131B**, 461 (1983); B. Humpert and R. Odorico, Phys. Lett. **154B**, 211 (1985); T. Sjostrand and M. Van Zijl, Phys. Rev. D **36**, 2019 (1987).
- [3] AFS Collaboration, T. Akesson *et al.*, Z. Phys. C **34**, 163 (1987); UA2 Collaboration, J. Alitti *et al.*, Phys. Lett. B **268**, 145 (1991); CDF Collaboration, F. Abe *et al.*, Phys. Rev. D **47**, 4857 (1993).
- [4] L. Ametller and D. Treleani, Int. J. Mod. Phys. A **3**, 521 (1988).
- [5] A. Capella and J. Tran Thanh Van, Phys. Rev. Lett. **58**, 2015 (1987).
- [6] V. Abramovskii, V. N. Gribov, and O. V. Kancheli, Yad. Fiz. **18**, 595 (1973) [Sov. J. Nucl. Phys. **18**, 308 (1974)]; I. G. Halliday and C. T. Sacharajda, Phys. Rev. D **8**, 3598 (1973); J. Koplik and A. H. Mueller, Phys. Lett. **58B**, 166 (1975); L. D. McLerran and J. H. Weiss, Nucl. Phys. **B100**, 329 (1975); L. Bertocchi and D. Treleani, J. Phys. G **3**, 147 (1977); V. M. Braun and Yu. M. Shabelski, Int. J. Mod. Phys. A **3**, 2417 (1988); G. Calucci and D. Treleani, Phys. Rev. D **49**, 138 (1994); **50**, 4703 (1994); J. Bartels and M. Wüsthoff, Z. Phys. C **66**, 157 (1995).
- [7] A. Capella and A. Krzywicki, Phys. Lett. **67B**, 84 (1977); Phys. Rev. D **18**, 3357 (1978).
- [8] R. Blankenbecler, A. Capella, C. Pajares, J. Tran Thanh Van, and A. V. Ramallo, Phys. Lett. **107B**, 106 (1981).
- [9] D. Treleani, Int. J. Mod. Phys. A **11**, 613 (1996).
- [10] G. Calucci and D. Treleani, in *Proceedings of the Topical Workshop on the Small-x Behavior of Deep Inelastic Scattering Structure Functions in QCD*, Hamburg, Germany, 1990, edited by A. Ali and J. Bartels [Nucl. Phys. B, Proc. Suppl. **18C**, 187 (1990)]; Int. J. Mod. Phys. A **6**, 4375 (1991).
- [11] M. Abramowitz and I. A. Stegun, *Handbook of Mathematical Functions* (Dover, New York, 1965).
- [12] L. Lewin, *Dilogarithms and Associated Functions* (McDonald, London, 1985).