Finite temperature perturbation theory and large gauge invariance

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(Received 16 December 1997; published 13 March 1998)

We examine finite temperature perturbation theory for Chern-Simons theories, in the context of an analogue $(0+1)$-dimensional model. In particular, we show how nonextensive terms arise in the perturbative finite temperature effective action, using both the real-time and imaginary-time formalisms. We illustrate how large gauge invariance is restored at all orders, despite being broken at any given order in perturbation theory. We discuss which aspects generalize to a perturbative analysis of finite temperature Chern-Simons terms in higher dimensions. [S0556-2821(98)06808-8]

PACS number(s): 11.10.Wx, 11.10.Kk

I. INTRODUCTION

Recently, some progress has been made in our understanding of the temperature dependence of the induced Chern-Simons terms [1–5]. On the face of it, a temperature dependent induced Chern-Simons term would seem to violate gauge invariance [6,7], because a temperature dependent Chern-Simons coefficient cannot be chosen to take discrete values, as invariance under large gauge transformations would require [8]. However, in [1] a mechanism was demonstrated, motivated by an exactly solvable $(0+1)$-dimensional Chern-Simons theory, whereby the full effective action *does* satisfy large gauge invariance, in spite of the fact that it contains temperature dependent terms which violate large gauge invariance at each order (of the field variable) in perturbation theory.

The essential new feature is that at finite temperature, other parity violating terms (other than the Chern-Simons term) can and do appear in the effective action, and if one takes into account all such terms to all orders (in the field variable) correctly, the full effective action can maintain gauge invariance even though it contains a Chern-Simons term with a temperature dependent coefficient. In fact, it is clear that if there are higher order terms present (which are not individually gauge invariant), one cannot ignore them in discussing the question of invariance of the effective action under a large gauge transformation. Remarkably, this mechanism requires the existence of nonextensive terms (i.e. terms that are not simply space-time integrals of a density) in the finite temperature effective action, although only extensive terms survive in the zero temperature limit. All these features have been demonstrated explicitly and exactly in the $0+1$ Chern-Simons model [1]. This idea has subsequently been analyzed in the framework of zeta function regularization [2], and has been extended to the Abelian $(2+1)$-dimensional QED (QED_{2+1}) fermion determinant in special gauge backgrounds that support gauge transformations with nontrivial winding numbers [2–5]. Once again, the full effective action is gauge invariant even though the induced Chern-Simons term has a temperature dependent coefficient.

However, in our opinion, the issue of large gauge invariance, at finite temperature, in a $(2+1)$-dimensional Chern-Simons system still poses many interesting and unsolved questions. The QED_{2+1} fermion determinants studied in [2–5] correspond to specially contrived backgrounds that essentially factorize the effective action into the $0+1$ model of [1] and a part corresponding to familiar 2 dimensional (Euclidean) fermion physics. For a general gauge background (and, more interestingly, for truly non-Abelian backgrounds) the effective action cannot be computed exactly. Furthermore, Chern-Simons terms may be induced not only in fermion systems, but also in purely gauge models [9,10] and in gauge-Higgs models with spontaneous symmetry breaking [11,12] where, again, the exact evaluation of the effective action may not be possible. Therefore, in such models, perturbation theory (at finite temperature) becomes a crucial and powerful tool. But as stressed clearly in [2], there is an inherent incompatibility between standard perturbation theory (at a given order) and large gauge invariance, because the coupling constant cannot be factored out of the large gauge transformation at finite temperature. The *really* interesting question, then, is to understand how to perform reliable and consistent finite temperature perturbative calculations when large gauge invariance is important. This is a very general question, and a deeper understanding of this phenomenon should have important implications for finite temperature QCD.

In this paper we make a modest first step in this direction by re-examining the $(0+1)$-dimensional model of [1] using the various standard forms of finite temperature perturbation theory [13]. At first sight, it may seem foolish to study perturbation theory for an exactly solvable model, but our goal is to explore the intricacies of finite temperature perturbation theory in the presence of large gauge invariance. We seek an understanding of how these somewhat unfamiliar nonextensive terms (in the effective action) arise in perturbation theory at finite temperature, and yet are all absent at zero temperature. We also seek to identify precisely which features of this model are special to $0+1$ dimensions and which may be generalized to a perturbative treatment of a $(2+1)$-dimensional model. In Sec. II we introduce the

model and review briefly the results of [1–5]. In Secs. III and IV we use the real time formalism of finite temperature perturbation theory to solve this system, in momentum space and then in coordinate space. In Sec. V we apply finite temperature perturbation theory in the imaginary time formalism. We conclude with some comments on the relative merits of these approaches and on which features may generalize to higher dimensions.

II. MODEL

Consider a $(0+1)$-dimensional field theory of N_f flavors of fermions ψ_j, $j=1,\ldots,N_f$, minimally coupled to a $U(1)$ gauge field A. It is not possible to write a Maxwell-like kinetic term for the gauge field in $0+1$ dimensions, but we can write a Chern-Simons term—it is linear in A [14]. [Such $(0+1)$-dimensional Chern-Simons models have also been studied recently in dimensional reductions of $(2+1)$-dimensional Chern-Simons theories [15].] In ''Minkowski space'' (i.e. real time) the Lagrangian is

$$L=\sum_{j=1}^{N_f}\ \overline{\psi}_j(i\partial_t-A-m)\psi_j-\kappa A. \tag{1}$$

This model supports gauge transformations with nontrivial winding number. Under the $U(1)$ gauge transformation $\psi\rightarrow e^{-i\lambda}\psi$, $A\rightarrow A+\partial_t\lambda$, the Lagrangian changes by a total derivative and the action changes by

$$\Delta S=-\kappa\int_{-\infty}^{+\infty}dt\,\partial_t\lambda=-2\pi\kappa N \tag{2}$$

where $N\equiv(1/2\pi)\int dt\,\partial_t\lambda$ is the integer-valued ''winding number'' of the topologically nontrivial gauge transformation. For example, the gauge transformation with

$$\lambda(t)=2N\ \arctan\ t=-iN\ \log\left(\frac{1+it}{1-it}\right) \tag{3}$$

has nonzero winding number N, and we see that N must be an integer so that the gauge transformation $\psi\rightarrow e^{-i\lambda}\psi$ preserves the single-valuedness of the field $\psi(t)$.

Nevertheless, even though the classical action changes [as in Eq. (2)] under a large gauge transformation, the quantum path integral, which involves e^{iS}, remains invariant provided the Chern-Simons coefficient κ is an integer. This is just the usual discreteness condition on the Chern-Simons coefficient, familiar from three dimensional non-Abelian Chern-Simons theories [8].

Induced Chern-Simons terms appear when we compute the fermion contribution to the effective action for this theory:

$$\Gamma[A]=-i\ \log\left[\frac{\det(i\partial_t-A-m)}{\det(i\partial_t-m)}\right]^{N_f}. \tag{4}$$

From [1], we know the *exact* finite temperature effective action for this theory to be [note that under the Euclidean rotation $\int d\tau A(\tau)\rightarrow-\int dtA(t)$]

$$\Gamma[A]=-iN_f\ \log\left[\cos\left(\frac{1}{2}\int dtA\right)+i\tanh\left(\frac{\beta m}{2}\right)\sin\left(\frac{1}{2}\int dtA\right)\right]. \tag{5}$$

Several comments are in order. First, notice that the effective action $\Gamma[A]$ is not an extensive quantity (i.e. it is not an integral of a density). Rather, it is a complicated function of the Chern-Simons action: $\int dtA$. Second, in the zero temperature limit, the effective action reduces to

$$\Gamma[A]_{T=0}=\frac{1}{2}\frac{m}{|m|}N_f\int dtA(t) \tag{6}$$

which is the usual zero temperature induced Chern-Simons term. At nonzero temperature the effective action is much more complicated. A formal perturbative expansion of the exact result (5) in powers of the gauge field yields

$$\Gamma[A]=\frac{N_f}{2}\left[\tanh\left(\frac{\beta m}{2}\right)a+\frac{i}{4}\ \mathrm{sech}^2\left(\frac{\beta m}{2}\right)a^2+\frac{1}{12}\tanh\left(\frac{\beta m}{2}\right)\mathrm{sech}^2\left(\frac{\beta m}{2}\right)a^3+\ldots\right] \tag{7}$$

where we have defined

$$a\equiv\int dtA(t). \tag{8}$$

The first term in this perturbative expansion (7) is the Chern-Simons action, but with a temperature dependent coefficient, just as was found in the perturbative computations in $(2+1)$-dimensional Chern-Simons theories [16,17]. If the computations stopped there, then we would arrive at the apparent contradiction mentioned in the Introduction—namely, the ''renormalized'' Chern-Simons coefficient

$$\kappa_R=\kappa-\frac{N_f}{2}\tanh\left(\frac{\beta m}{2}\right) \tag{9}$$

would be temperature dependent, and so could not take discrete values. Thus, it would seem that the effective action cannot be invariant under large gauge transformations. The flaw in this argument is clear. There are other terms in the effective action besides the Chern-Simons term which cannot be ignored, and these must all be taken into account when considering the question of the large gauge invariance of the effective action. Indeed, the effective action (5) shifts by $(N_fN)\pi$, independent of the temperature, under a large gauge transformation, for which $a\rightarrow a+2\pi N$. This is just the familiar global anomaly [18,19], which can be removed (for example) by taking an even number of flavors, and is not directly related to the issue of the temperature dependence of the Chern-Simons coefficient. The clearest way to understand this global anomaly is through zeta function regularization of the theory [2].

Finally, note that only the first term in the perturbative expansion (7) survives in the zero temperature limit. The higher order terms are all nonextensive—they are powers of the Chern-Simons action. The corresponding Feynman dia-

grams vanish identically at zero temperature, and this is usually understood by noting that they *must* vanish because there is no gauge invariant (even under infinitesimal gauge transformations) term involving more than one factor of $A(t)$ that can be written down. This, however, assumes that we only look for *extensive* terms; at nonzero temperature, this assumption breaks down and correspondingly we shall see that our notion of perturbation theory must be enlarged to incorporate nonextensive contributions to the effective action. For example, let us consider an action quadratic in the gauge fields which can have the general form

$$\Gamma_{(2)}=\frac{1}{2}\int dt_1 dt_2 A(t_1)F(t_1-t_2)A(t_2) \tag{10}$$

where we assume that $F(t_1-t_2)=F(t_2-t_1)$. Under an infinitesimal gauge transformation, this action will transform as

$$\delta\Gamma_{(2)}=-\int dt_1 dt_2 \lambda(t_1)\partial_{t_1}F(t_1-t_2)A(t_2). \tag{11}$$

Clearly, the action will be invariant under an infinitesimal gauge transformation if $F=0$. This corresponds to excluding the quadratic term (10) from the effective action. But as is clear from Eq. (11), the action can also be invariant under infinitesimal gauge transformations if $F=\text{const}$, which would make the quadratic action (10) nonextensive and, in fact, proportional to the square of the Chern-Simons action. The origin of such nonextensive terms will be discussed in detail in the following sections when we analyze this model using standard finite temperature perturbation theory techniques.

To conclude this section, we recall briefly the results of [2–5] concerning the fermionic determinant in QED_{2+1}. In Euclidean space, using the imaginary time formalism, it has been shown that for an Abelian gauge background of the form

$$A_3=\text{const}, \quad \vec{A}(\vec{x}) \quad (\text{independent of time}) \tag{12}$$

the parity odd part of the finite temperature effective action is

$$S_{\text{odd}}=i\Phi\ \arctan\left[\tanh\left(\frac{\beta m}{2}\right)\ \tan\left(\frac{eA_3}{2}\right)\right] \tag{13}$$

where $\Phi\equiv(e/2\pi)\int d^2x\,\epsilon_{ij}\partial_i A_j$ is the time-independent magnetic flux of the background field. Notice that this parity odd part of the effective action corresponds precisely to the real part of the effective action (5), with the natural identifications $A_3\to a$ and $\Phi\to N_f$.

III. REAL-TIME FORMALISM: MOMENTUM SPACE CALCULATION

The perturbative computation of the fermionic contribution to the effective action requires computing all diagrams with one fermion loop and any number of external gauge fields. We begin by considering the first few such diagrams for this theory, in momentum space, using the real-time formalism [13], before giving a systematic method for evaluating them. The fermionic Feynman propagator is (we assume from now on that $m>0$, and since the propagator as well as the vertices are diagonal in the flavor index, we do not write it explicitly for simplicity)

$$S(p)=(p+m)\left(\frac{i}{p^2-m^2+i\epsilon}-2\pi n_F(|p|)\,\delta(p^2-m^2)\right) \tag{14}$$

where $n_F(|p|)$ is the Fermi statistical factor:

$$n_F(|p|)=\frac{1}{e^{\beta|p|}+1}. \tag{15}$$

This propagator simplifies dramatically in $0+1$ dimensions, due to the trivial one-dimensional nature of space-time. Using $\delta(p^2-m^2)=(1/2m)[\delta(p-m)+\delta(p+m)]$, we find that

$$S(p)=\frac{i}{p-m+i\epsilon}-2\pi n_F(m)\,\delta(p-m). \tag{16}$$

Each fermion-gauge-fermion vertex contributes a factor of $-i$. Thus, the contribution of the tadpole diagram to the linear term in the effective action is (with the negative sign for the fermion loop)

$$iI_{(1)}=-(-i)N_f\int\frac{dp}{2\pi}\left(\frac{i}{p-m+i\epsilon}-2\pi n_F(m)\,\delta(p-m)\right)$$
$$\equiv iI_{(1)}^{(T=0)}+iI_{(1)}^{(\beta)}. \tag{17}$$

The zero temperature piece is

$$iI_{(1)}^{(T=0)}=-N_f\int\frac{dp}{2\pi}\frac{1}{p-m+i\epsilon}=-N_f\int\frac{dp}{2\pi}\frac{p+m}{p^2-m^2+i\epsilon}$$
$$=\frac{i}{2}N_f. \tag{18}$$

Note that to evaluate the first form of this integral directly, we must include the contribution from the semicircle at infinity since the integrand does not fall off fast enough. The temperature dependent tadpole contribution can be evaluated trivially to give

$$iI_{(1)}^{(\beta)}=-2\pi i N_f n_F(m)\int\frac{dp}{2\pi}\,\delta(p-m)=-iN_f n_F(m). \tag{19}$$

Thus, the net tadpole diagram contribution is

$$iI_{(1)}=\frac{iN_f}{2}[1-2n_F(m)]=\frac{iN_f}{2}\tanh\left(\frac{\beta m}{2}\right) \tag{20}$$

which gives a linear contribution to the effective action,

$$\Gamma_{(1)}=-iA(k=0)iI_{(1)}=\frac{N_f}{2}\tanh\left(\frac{\beta m}{2}\right)\int dt A(t), \tag{21}$$

in agreement with the first term in the perturbative expansion (7) of the full effective action.

The two-point function also splits naturally into a zero temperature piece and a temperature dependent piece (this is, in fact, a general property of the real time formalism) and gives a contribution to the quadratic term in the effective action of the form

$$iI_{(2)}(k)=(-)\frac{(-i)^2}{2!}N_f\int\frac{dp}{2\pi}S(p)S(k+p)$$
$$\equiv iI_{(2)}^{(T=0)}(k)+iI_{(2)}^{(\beta)}(k). \tag{22}$$

The zero temperature piece is

$$iI_{(2)}^{(T=0)}(k)=-\frac{N_f}{2}\int\frac{dp}{2\pi}\left(\frac{1}{p-m+i\epsilon}\right)\left(\frac{1}{(k+p)-m+i\epsilon}\right)$$
$$=\frac{N_f}{4\pi}2\pi i\left[\frac{1}{k}+\frac{1}{-k}\right]$$
$$=0. \tag{23}$$

This is an explicit demonstration of the fact that the two-point function (and, therefore, its contribution to the effective action) vanishes identically at zero temperature, as required by (small) gauge invariance. However, the finite temperature contribution is

$$iI_{(2)}^{(\beta)}(k)=-\frac{N_f}{2}\int\frac{dp}{2\pi}\left[2\pi i n_F(m)\,\frac{\delta(p-m)}{(k+p)-m+i\epsilon}\right.$$
$$+2\pi i n_F(m)\frac{\delta(k+p-m)}{p-m+i\epsilon}$$
$$\left.-4\pi^2 n_F^2(m)\,\delta(p-m)\,\delta(k+p-m)\right]$$
$$=-\frac{i}{2}N_f n_F(m)\left[\frac{1}{k+i\epsilon}+\frac{1}{-k+i\epsilon}\right]$$
$$+\pi N_f n_F^2(m)\,\delta(k)$$
$$=-\pi N_f n_F(m)[1-n_F(m)]\delta(k)$$
$$=-2\pi\delta(k)\frac{N_f}{8}\mathrm{sech}^2\left(\frac{\beta m}{2}\right). \tag{24}$$

Here we have used the identity $\delta(k)=(1/\pi)\lim_{\epsilon\to 0}[\epsilon/(k^2+\epsilon^2)]$. Since $I_{(2)}^{(T=0)}(k)$ vanishes, this result (24) gives the entire two-point function. The resulting quadratic contribution to the effective action is

$$\Gamma_{(2)}=-i\int\frac{dk}{2\pi}A(k)A(-k)iI_{(2)}(k)$$
$$=i\frac{N_f}{8}\mathrm{sech}^2\left(\frac{\beta m}{2}\right)\left(\int dtA(t)\right)^2 \tag{25}$$

in agreement with the perturbative expansion (7).

There are two important things to observe from the structure of the two point function: (i) its dependence on the external momentum k is through a delta function $\delta(k)$, and (ii) it vanishes at zero temperature ($\beta\to\infty$) because of the ''sech'' factor. The first observation illustrates how nonextensive terms such as [cf. Eq. (10)]

$$a^2=\left(\int dtA(t)\right)^2=\int\frac{dk}{2\pi}A(k)A(-k)2\pi\delta(k) \tag{26}$$

arise in a perturbative approach to the finite temperature effective action, while the latter explains why these are not seen in zero temperature perturbation theory.

It is a straightforward matter to go ahead and evaluate diagrams with more than two external gauge fields. However, motivated by the above result that the two-point diagram is proportional to a delta function in the external momentum, we appeal to the Ward identities for (small) gauge invariance, which state that the N-leg diagram (with $N\geqslant 2$), which is a function of $N-1$ external momenta, satisfies the relations

$$k_j I_{(N)}(k_1,\ldots,k_j,\ldots,k_{N-1})=0\quad\text{for } j=1,\ldots N-1. \tag{27}$$

This is a generalization of Eq. (11) to N-point functions in momentum space and here there is no contraction of space-time indices since we are in a one-dimensional space-time. This implies that $I_{(N)}(k_1,\ldots k_{N-1})$, for $N\geqslant 2$, must be proportional to a product of delta functions in the $N-1$ external momenta:

$$iI_{(N)}(k_1,\ldots,k_{N-1})=C_N(\beta m)\,\delta(k_1)\,\delta(k_2)\cdots\delta(k_{N-1}) \tag{28}$$

where the coefficient C_N is a function of βm by dimensional reasoning. But this immediately implies that the N^{th} order contribution to the effective action in the perturbative expansion is proportional to the N^{th} power of the first order term, which is just the Chern-Simons action. This is exactly the nonextensive structure that we observe in the perturbative expansion (7). Here we see that it is a direct consequence of the Ward identities for small gauge invariance.

The result (28) suggests that the coefficients $C_N(\beta m)$ could be calculated from the N-leg diagram with zero external energies:

$$(-)\frac{(-i)^N}{N}N_f\int\frac{dp}{2\pi}[S(p)]^N. \tag{29}$$

These are, however, extremely singular integrals because of the product of delta functions with coincident arguments and have to be evaluated carefully. We have done this, but it is rather complicated, and we shall see in the following sections that it is much easier to evaluate these coefficients using the coordinate space representation or using the imaginary-time formalism.

IV. REAL-TIME FORMALISM: COORDINATE SPACE CALCULATION

In this section we describe the coordinate space analysis of the perturbative calculation in this model, using the real-time formalism. The real-time propagator in the coordinate space can be obtained simply from the Fourier transform of the momentum space propagator (16):

$$S(t)\equiv\int\frac{dp}{2\pi}e^{-ipt}S(p)=[\theta(t)-n_F(m)]e^{-imt}. \quad (30)$$

Here $\theta(t)$ is the standard Heaviside step function.

Let us now consider the N-leg diagrams contributing to the effective action. The contribution of the tadpole diagram is

$$iI_{(1)}=-(-i)N_fS(0)=iN_f\left(\frac{1}{2}-n_F(m)\right)=\frac{iN_f}{2}\tanh\left(\frac{\beta m}{2}\right). \quad (31)$$

The contribution of the two-point function to the quadratic action has the form

$$\begin{aligned}iI_{(2)}(t_1,t_2)&=-(-i)^2\frac{N_f}{2}S(t_1-t_2)S(t_2-t_1)\\&=\frac{N_f}{2!}[\theta(t_1-t_2)-n_F(m)][\theta(t_2-t_1)\\&\quad-n_F(m)]\\&=-\frac{N_f}{2}n_F(m)[1-n_F(m)]\\&=-\frac{N_f}{8}\mathrm{sech}^2\left(\frac{\beta m}{2}\right). \quad (32)\end{aligned}$$

The three-point function gives a contribution of the form

$$\begin{aligned}iI_{(3)}(t_1,t_2,t_3)&=-(-i)^3\frac{N_f}{3!}[S(t_1-t_2)S(t_2-t_3)S(t_3-t_1)\\&\quad+S(t_1-t_3)S(t_3-t_2)S(t_2-t_1)]\\&=-\frac{iN_f}{6}[-n_F(m)+3n_F^2(m)-2n_F^3(m)]\\&=\frac{iN_f}{6}n_F(m)[1-n_F(m)][1-2n_F(m)]\\&=\frac{iN_f}{24}\tanh\left(\frac{\beta m}{2}\right)\mathrm{sech}^2\left(\frac{\beta m}{2}\right). \quad (33)\end{aligned}$$

We notice that the N-point functions are independent of the external coordinates. This is the coordinate space analogue of the statement (28) that the momentum space N-point functions are proportional to products of delta functions in the external momenta [or the generalization of Eq. (11) to N-leg diagrams]. In coordinate space, it is easy to see explicitly how this works. Clearly, the tadpole is independent of the external coordinate. For the 2-point function (and its contribution to the quadratic part of the action),

$$\begin{aligned}\frac{\partial}{\partial t_1}iI_{(2)}(t_1,t_2)&=\frac{N_f}{2}\frac{\partial}{\partial t_1}[(\theta(t_1-t_2)-n_F)(\theta(t_2-t_1)-n_F)]\\&=\frac{N_f}{2}\delta(t_1-t_2)[\theta(0)-n_F]\\&\quad-\frac{N_f}{2}\delta(t_2-t_1)[\theta(0)-n_F]\\&=0. \quad (34)\end{aligned}$$

Similarly, for the 3-point function,

$$\begin{aligned}\frac{\partial}{\partial t_1}iI_{(3)}(t_1,t_2,t_3)&=-\frac{iN_f}{6}\delta(t_1-t_2)[S(t_2-t_3)S(t_3-t_1)\\&\quad-S(t_1-t_3)S(t_3-t_2)]+\frac{iN_f}{6}\delta(t_1-t_3)\\&\quad\times[S(t_1-t_2)S(t_2-t_3)\\&\quad-S(t_3-t_2)S(t_2-t_1)]\\&=0. \quad (35)\end{aligned}$$

In general, the derivative of the higher N-point functions with respect to any one, say t_1, of the external coordinates vanishes because for any diagram contributing to the N-point function with a given ordering of the external coordinates, there is another diagram with t_1 interchanged with another coordinate, say t_2. But $(\partial/\partial t_1)S(t_1-t_2)=(-\partial/\partial t_1)S(t_2-t_1)$, and so these diagrams cancel pairwise. We recognize that this is just a manifestation of the coordinate space Ward identities, which corresponds to a generalization of Eq. (11).

Furthermore, we notice from the results (31),(32),(33) that $I_{(N)}$ is essentially the derivative of $I_{(N-1)}$ with respect to βm, for $N\leqslant 3$. To prove this in general, for any N, note that in any diagram contributing to the effective action, the product of the phase factors e^{-imt} in a loop simply cancels out. Therefore, we can effectively consider the ''reduced'' propagator without this phase factor in our computations:

$$\tilde{S}(t_1-t_2)\equiv\theta(t_1-t_2)-n_F(m). \quad (36)$$

Then it is clear that

$$\frac{\partial}{\partial m}\tilde{S}(t_1-t_2)=-n_F'(m)=-\beta n_F(1-n_F) \quad (37)$$

where in the last step we have used the identity explicitly satisfied by the Fermi factor in Eq. (15). Since the Feynman amplitudes are independent of the external time coordinates, we can choose any time ordering for these quantities and we choose $t_1>t_2$. This gives

$$\frac{\partial}{\partial m}\tilde{S}(t_1-t_2)=-\beta\tilde{S}(t_1-t_3)\tilde{S}(t_3-t_2),\quad t_1>t_2>t_3, \quad (38)$$

where t_3 is arbitrary, but strictly less than t_2. Similarly,

$$\frac{\partial}{\partial m}\tilde{S}(t_2-t_1)=-\beta\tilde{S}(t_2-t_3)\tilde{S}(t_3-t_1),\quad t_1>t_2>t_3. \quad (39)$$

Thus, the effect of differentiating an N-point function with respect to m is to introduce another coordinate (and, therefore, an external gauge field), lower than all the others, in all possible lines on the original diagrams. This is a generaliza-

tion of the zero temperature Ward identity and relates the $(N+1)$-point function to the N-point function through the following recursion relation:

$$\frac{\partial}{\partial m} I_{(N)} = -i\beta(N+1) I_{(N+1)} \tag{40}$$

where we have suppressed the external coordinates since the N-point functions are in fact independent of them. But this means that *all* the coordinate space N-point diagrams are given as derivatives of the tadpole diagram with respect to the mass m. Thus, the full effective action is

$$\begin{aligned}\Gamma &= -i \sum_{N=1}^{\infty} \left(\int dt A(t) \right)^N (i I_{(N)}) \\ &= -\frac{i\beta N_f}{2} \sum_{N=1}^{\infty} \frac{[(i/\beta)\int dt A(t)]^N}{N!} \\ &\quad \times \left(\frac{\partial}{\partial m} \right)^{N-1} \tanh\left(\frac{\beta m}{2} \right).\end{aligned} \tag{41}$$

Remarkably, this expansion may be resummed, to yield the full exact effective action in Eq. (5).

V. IMAGINARY TIME: MOMENTUM SPACE CALCULATION

In this section we present a perturbative analysis of the model using the imaginary time formalism for finite temperature perturbation theory. The coordinate space approach, in the imaginary time formalism, was already given in [1], and so here we discuss the momentum space analysis.

Defining an imaginary time coordinate $\tau = it$, the Lagrangian (1) becomes

$$L_E = \sum_{j=1}^{N_f} \psi_j^{\dagger} (\partial_\tau - iA + m) \psi_j - i\kappa A. \tag{42}$$

The imaginary time coordinate τ is restricted to the range $\tau \in [0,\beta]$, where β is the inverse temperature. Fermi fields are antiperiodic in τ, $\psi(0) = -\psi(\beta)$, while gauge fields are periodic, $A(0) = A(\beta)$. The propagators are the same as the zero temperature propagators, but the antiperiodicity and periodicity conditions on the fields imply that the corresponding energies take discrete values, being odd (even) multiples of $2\pi T$ for fermions (bosons) [13].

The photon vertex, in this theory, contributes a factor of i so that the contribution of the ''Euclidean'' tadpole diagram to the effective action is

$$I_{(1)} = (-)(i)\mathrm{tr}\left(\frac{1}{ip+m} \right) = -i\,\mathrm{tr}\left(\frac{-ip+m}{p^2+m^2} \right). \tag{43}$$

There is no Dirac index except for the flavor index whose trace is trivial and at zero temperature the energy trace is an integral: so

$$I_{(1)} = -iN_f \int_{-\infty}^{\infty} \frac{dp}{2\pi} \left(\frac{-ip+m}{p^2+m^2} \right) = -\frac{i}{2} \frac{m}{|m|} N_f. \tag{44}$$

Notice that the real part of the tadpole vanishes identically, while the imaginary part is proportional to the sign of the mass m. At nonzero temperature, the trace is a sum over the discrete fermionic energies:

$$\begin{aligned} I_{(1)} &= (-iN_f) T \sum_{n=-\infty}^{\infty} \frac{-(2n+1)i\pi T + m}{(2n+1)^2\pi^2 T^2 + m^2} \\ &= -\frac{iN_f m}{\pi^2 T} \sum_{n=-\infty}^{\infty} \frac{1}{(2n+1)^2 + m^2\beta^2/\pi^2} \\ &= -\frac{iN_f}{2} \tanh\left(\frac{\beta m}{2} \right). \end{aligned} \tag{45}$$

The infinite sum, here, is just a standard representation [20] of the tanh function and this agrees exactly with the first term of Eq. (14) in [1]. In the zero temperature limit, this reduces smoothly to the zero temperature result (44) for the tadpole.

The ''Euclidean'' two-point diagram gives a contribution to the quadratic action of the form

$$I_{(2)}(k) = (-)\frac{(i)^2}{2!}\,\mathrm{tr}\left(\frac{m^2 - p(p+k) + im(2p+k)}{[m^2+p^2][m^2+(p+k)^2]} \right). \tag{46}$$

First consider this diagram at zero temperature. The imaginary part vanishes identically, as can be seen by replacing p by $-(k+p)$. The real part also vanishes, as a result of a cancellation between the two terms:

$$\begin{aligned} &\int_{-\infty}^{\infty} \frac{dp}{2\pi} \left(\frac{m^2 - p(p+k)}{[m^2+p^2][m^2+(p+k)^2]} \right) \\ &\quad = \frac{|m|}{k^2+4m^2} - \frac{|m|}{k^2+4m^2} = 0. \end{aligned} \tag{47}$$

At nonzero temperature, the trace in Eq. (46) involves a sum over the discrete fermionic energies. Once again, the imaginary part vanishes after a shift of the energy variable, but we note that this relies on the fact that the (bosonic) external energy k is an even multiple of $2\pi T$, while the (fermionic) loop energy p is an odd multiple of $2\pi T$. The real part requires considerably more care, as we must distinguish between the case when the external energy vanishes and when it is nonzero. When $k=0$, it is trivial to evaluate

$$\begin{aligned} I_{(2)}(k=0) &= \frac{N_f}{2} T \sum_{n=-\infty}^{\infty} \frac{m^2 - (2n+1)^2\pi^2 T^2}{[m^2 + (2n+1)^2\pi^2 T^2]^2} \\ &= -\beta \frac{N_f}{8} \mathrm{sech}^2\left(\frac{\beta m}{2} \right). \end{aligned} \tag{48}$$

When $k \neq 0$, it is surprisingly tricky to evaluate this two-point diagram (46) at finite temperature. A natural approach is to use Schwinger's parametric representation of the integrand, in which case the diagram becomes a parametric integral involving a Jacobi theta function. The advantage of this approach is that the only difference between the zero temperature and nonzero temperature result is the absence or presence of the theta function factor. Represent the integrand as

$$\frac{m^2-p(p+k)}{[m^2+p^2][m^2+(p+k)^2]}=[m^2-p(p+k)]\int_0^\infty d\alpha_1\int_0^\infty d\alpha_2 e^{-\alpha_1(m^2+p^2)}e^{-\alpha_2[m^2+(p+k)^2]}$$

$$=\int_0^\infty d\alpha_1\int_0^\infty d\alpha_2\left[2m^2+\frac{1}{2}k^2+\frac{1}{2}\frac{\partial}{\partial\alpha_1}+\frac{1}{2}\frac{\partial}{\partial\alpha_2}\right]e^{-\alpha_1(m^2+p^2)}e^{-\alpha_2[m^2+(p+k)^2]}. \tag{49}$$

At zero temperature, the integration over the loop energy p is simply a Gaussian integral:

$$\int_{-\infty}^{\infty}\frac{dp}{2\pi}e^{-\alpha_1 p^2}e^{-\alpha_2(p+k)^2}=\frac{1}{\sqrt{4\pi(\alpha_1+\alpha_2)}}\exp\left[-k^2\left(\frac{\alpha_1\alpha_2}{\alpha_1+\alpha_2}\right)\right]. \tag{50}$$

After a change of variables

$$u=\alpha_1+\alpha_2,\quad v=\frac{\alpha_2}{\alpha_1+\alpha_2} \tag{51}$$

for which the Jacobian equals u, we arrive at a simple parametric integral representation for the two-point function:

$$I_{(2)}^{(T=0)}(k)=\frac{N_f}{4\sqrt{\pi}}\int_0^1 dv\int_0^\infty udu\left[2m^2+\frac{1}{2}k^2+\frac{\partial}{\partial u}+\frac{1-2v}{2u}\frac{\partial}{\partial v}\right]\left(\frac{1}{\sqrt{u}}e^{-u[m^2+v(1-v)k^2]}\right)$$

$$=\frac{N_f}{4\sqrt{\pi}}\int_0^1 dv\int_0^\infty udu\left[m^2+v(1-v)k^2-\frac{1}{2u}\right]\left(\frac{1}{\sqrt{u}}e^{-u[m^2+v(1-v)k^2]}\right)$$

$$=0. \tag{52}$$

Thus, the two-point function vanishes at zero temperature, as in Eq. (47).

At nonzero temperature, the energy trace is a sum, not an integral, and so we cannot perform a Gaussian integration as in Eq. (50). Rather, the summation over the loop energy produces a Jacobi theta function. Write the loop energy as $p=(2n+1)\pi T$ and the external gauge momentum as $k=2l\pi T$. Then

$$\sum_{n=-\infty}^{\infty}e^{-\alpha_1\pi^2T^2(2n+1)^2}e^{-\alpha_2\pi^2T^2(2n+1+2l)^2}$$

$$=e^{-4\pi^2l^2T^2\alpha_2}\Theta_2(4\pi ilT^2\alpha_2|4\pi iT^2(\alpha_1+\alpha_2)) \tag{53}$$

where Θ_2 is the second Jacobi theta function [20]:

$$\Theta_2(v|\tau)\equiv\sum_{n=-\infty}^{\infty}e^{i\pi\tau(n+1/2)^2}e^{i\pi v(2n+1)}. \tag{54}$$

We now exploit the Poisson summation formula

$$\Theta_4\left(\frac{v}{\tau}\middle|\frac{-1}{\tau}\right)=\sqrt{-i\tau}e^{i\pi v^2/\tau}\Theta_2(v|\tau) \tag{55}$$

where Θ_4 is the fourth Jacobi theta function:

$$\Theta_4(v|\tau)\equiv 1+2\sum_{n=1}^{\infty}(-1)^n e^{i\pi\tau n^2}\cos(2nv). \tag{56}$$

Then, the finite temperature two-point function becomes

$$I_{(2)}(k)=\frac{N_f}{4\sqrt{\pi}}\int_0^1 dv\int_0^\infty udu\left[2m^2+\frac{1}{2}k^2+\frac{\partial}{\partial u}+\frac{1-2v}{2u}\frac{\partial}{\partial v}\right]$$

$$\times\left[\frac{1}{\sqrt{u}}e^{-u[m^2+v(1-v)k^2]}\Theta_4\left(\frac{kv}{2\pi T}\middle|\frac{i}{4\pi T^2u}\right)\right]. \tag{57}$$

Remarkably, this expression only differs from the zero temperature parametric expression (52) by the presence of the theta function factor. When $T\to 0$, this theta factor reduces to 1 and we get the zero temperature expression. Furthermore, when the external momentum k vanishes, it is a straightforward exercise to show that we regain the (nonzero) answer in Eq. (48). However, when $k\neq 0$, we can integrate by parts in v, and use the fact that the Jacobi theta functions satisfy a heat equation

$$\frac{\partial}{\partial\tau}\Theta(v|\tau)=-i\frac{\pi}{4}\frac{\partial^2}{\partial v^2}\Theta(v|\tau) \tag{58}$$

to convert the parametric integral (57) into

$$I_{(2)}(k\neq 0)=\frac{N_f}{4\sqrt{\pi}}\int_0^1 dv\int_0^\infty \sqrt{u}\,du\,e^{-u[m^2+v(1-v)k^2]}$$
$$\times\left[m^2+v(1-v)k^2-\frac{1}{2u}-\frac{\partial}{\partial u}\right]$$
$$\times\Theta_4\left(\frac{kv}{2\pi T}\middle|\frac{i}{4\pi T^2 u}\right)$$
$$=0. \tag{59}$$

Therefore, we have shown that the two-point function vanishes when the external momentum k is nonzero, but is nonzero when the external momentum k vanishes. That is, the two-point function is proportional to a Kronecker delta in the external momentum k:

$$I_{(2)}(k)=-\beta\delta_{k,0}\frac{N_f}{8}\mathrm{sech}^2\left(\frac{\beta m}{2}\right). \tag{60}$$

This is the imaginary time analogue of the real time result (24), and it agrees exactly with the second term in Eq. (14) of [1]. Similarly, using Schwinger's parametric representation, we can show that the N-point function vanishes if any one of the external momenta is nonzero. This is an explicit illustration of the Ward identity (28) in the imaginary-time formalism.

When all the external momenta vanish it is very easy to evaluate the N-point function using the imaginary time formalism:

$$I_{(N)}(k_1=0,\ldots,k_{N-1}=0)$$
$$=(-)\frac{(i)^N}{N}N_f T\sum_{n=-\infty}^{\infty}\left(\frac{1}{i(2n+1)\pi T+m}\right)^N$$
$$=\frac{N_f}{2}\frac{(-i)^N}{N!}\left(\frac{\partial}{\partial m}\right)^{N-1}\tanh\left(\frac{\beta m}{2}\right). \tag{61}$$

As before, this permits the effective action to be resummed:

$$\Gamma_E=\sum_{N=1}^{\infty}T^{N-1}\sum_{k_1}\cdots\sum_{k_{N-1}}A(k_1)\cdots A(k_{N-1})$$
$$\times A(-k_1-\cdots-k_{N-1})I_{(N)}(k_1,\ldots,k_{N-1})$$
$$=\sum_{N=1}^{\infty}T^{N-1}[A(k=0)]^N I_{(N)}(k_1=0,\ldots,k_{N-1}=0)$$
$$=\beta\frac{N_f}{2}\sum_{N=1}^{\infty}\frac{\left[-(i/\beta)\int_0^\beta A(\tau)\,d\tau\right]^N}{N!}$$
$$\times\left(\frac{\partial}{\partial m}\right)^{N-1}\tanh\left(\frac{\beta m}{2}\right). \tag{62}$$

This agrees with the exact effective action derived in [1], and is the imaginary time version of the real time result (41).

VI. CONCLUSIONS

In conclusion, we have analyzed the fermion contribution to the effective action in a $(0+1)$-dimensional model with a Chern-Simons term, using various standard techniques of finite temperature perturbation theory. Each formalism has its own advantages and disadvantages. In the real-time formalism, the momentum space calculation gives an immediate derivation, via the Ward identities (28), of the existence of nonextensive terms in the effective action. But the actual computation of the coefficients of these nonextensive terms is rather messy in the momentum space approach. In particular, the diagrams with zero external momenta are singular and must be treated with great care. Rather, in the coordinate space approach (in the real time formalism) it is very easy to evaluate these coefficients using the recursion relation (40) that relates all higher order diagrams back to the tadpole diagram. It is also rather straightforward to demonstrate the independence of the N-point diagram on the external time coordinates, which is the coordinate space statement of the Ward identities. Finally, the momentum space calculation in the imaginary time formalism gives a trivial evaluation of the N-point diagram when the external momenta are all zero. However, it is surprisingly difficult to verify explicitly the Ward identities by showing that the diagrams vanish if any of the external momenta are nonzero.

Nevertheless, each of these different approaches works. The final result is (as in [1]) that although each term in finite temperature perturbation theory gives a temperature dependent contribution that violates large gauge invariance, we can resum all orders of perturbation theory to obtain the full effective action (5), which respects large gauge invariance. We stress that gauge invariance alone is not enough to determine the exact form of the effective action. Small gauge invariance implies, Eq. (28), that the effective action Γ is a *function* of the Chern-Simons action a [i.e. that $\Gamma=\Gamma(a)$ is nonextensive]. But to satisfy large gauge invariance all we need is that the fermion determinant $e^{i\Gamma(a)}$ change by at most a sign under a large gauge transformation: $a\to a+2\pi N$. This is satisfied by a general expression

$$\exp[i\Gamma(a)/N_f]=\sum_{j=0}^{\infty}\left[d_j\cos\left(\frac{(2j+1)a}{2}\right)\right.$$
$$\left.+f_j\sin\left(\frac{(2j+1)a}{2}\right)\right]. \tag{63}$$

The actual answer (5) gives as the only nonzero coefficients $d_0=1$ and $f_0=i\tanh(\beta m/2)$. This fact can only be deduced by computation, not solely from gauge invariance requirements.

We now ask, which features of this $(0+1)$-dimensional model will extend to the general $(2+1)$-dimensional case? In the $(0+1)$-dimensional case, the Ward identities (27) have the simple consequence that nonextensive terms will appear in the effective action (although the actual coefficients, zero or nonzero, must be determined by a calculation). This argument does not immediately generalize to 2 $+1$ dimensions, although it does apply to certain special backgrounds such as the static ones used in [2–5]; for these backgrounds the finite temperature fermionic determinant

can be calculated explicitly. However, the nonextensive terms that appear are nonextensive in time only, and this is probably a consequence of the fact that the backgrounds themselves are static. Another dramatic simplification of the $(0+1)$-dimensional model is that the propagator has a very simple structure, both in momentum space (16) and in coordinate space (30), primarily because there is contribution only from the positive energy terms. This is due to the fact that the on-shell condition is very simple for one-dimensional space-time and also the fact that the Dirac spinor structure is trivial. This simplicity is the key to deriving the recursive relations (40) between the N-leg diagrams. These recursion relations are essential for the resummation (41) of the perturbative expansion to yield the full effective action (5), which is necessary to demonstrate that large gauge invariance is satisfied. Another simplifying feature of the $(0+1)$-dimensional model is that there is no ambiguity in taking the zero momentum limit of the N-leg diagrams (because there is just energy, no spatial momentum). However, in higher dimensions the dependence of N-leg diagrams on external momenta is nonanalytic at finite temperature [21], and great care must be used in extracting Chern-Simons-like terms via a zero external momentum limit [22]. This problem simply does not arise in the $(0+1)$-dimensional model. Finally, an important feature of the $2+1$ induced Chern-Simons term at *zero* temperature is the Coleman-Hill theorem [23], which essentially states that only one-loop graphs contribute to the induced Chern-Simons term. But finite temperature violates the assumptions used for the Coleman-Hill result, and it is not clear what role this will play in a finite temperature perturbative analysis of $(2+1)$-dimensional systems. However, this issue simply does not even arise in the $(0+1)$-dimensional model, as the ''photon'' does not propagate; thus, there are no higher loop diagrams to consider.

A direct perturbative analysis of $(2+1)$-dimensional models will reveal whether these various simplifications of the $(0+1)$-dimensional model are matters of convenience or if they are crucial to restoring the large gauge invariance that finite temperature perturbation theory breaks order by order [24].

ACKNOWLEDGMENTS

This work was supported in part by U.S. Department of Energy Grant No. DE-FG-02-91ER40685 (A.D.) and Grant No. DE-FG-02-92ER40716 (G.D.).

[1] G. Dunne, K. Lee and C. Lu, Phys. Rev. Lett. **78**, 3434 (1997).
[2] S. Deser, L. Griguolo and D. Seminara, Phys. Rev. Lett. **79**, 1976 (1997); S. Deser, L. Griguolo and D. Seminara, ''Effective QED Actions: Representations, Gauge Invariance, Anomalies and Mass Expansions,'' hep-th/9712066.
[3] C. Fosco, G. Rossini and F. Schaposnik, Phys. Rev. Lett. **79**, 1980 (1997); C. Fosco, G. Rossini and F. Schaposnik, Phys. Rev. D **56**, 6547 (1997).
[4] I. Aitchison and C. Fosco, Phys. Rev. D **57**, 1171 (1998).
[5] R. González-Felipe, ''On the Chern-Simons Topological Term at Finite Temperature,'' hep-th/9709079.
[6] R. Pisarski, Phys. Rev. D **35**, 664 (1987).
[7] N. Bralić, C. Fosco and F. Schaposnik, Phys. Lett. B **383**, 199 (1996); D. Cabra, E. Fradkin, G. Rossini and F. Schaposnik, *ibid.* **383**, 434 (1996).
[8] S. Deser, R. Jackiw and S. Templeton, Phys. Rev. Lett. **48**, 975 (1982); S. Deser, R. Jackiw and S. Templeton, Ann. Phys. (N.Y.) **140**, 372 (1982).
[9] R. Pisarski and S. Rao, Phys. Rev. D **32**, 2081 (1985).
[10] E. Witten, Commun. Math. Phys. **121**, 351 (1989).
[11] S. Yu. Khlebnikov, JETP Lett. **51**, 81 (1990); S. Yu. Khlebnikov and M. Schaposhnikov, Phys. Lett. B **254**, 148 (1991).
[12] L. Chen, G. Dunne, K. Haller and E. Lim-Lombridas, Phys. Lett. B **348**, 468 (1995); A. Khare, R. MacKenzie, P. Panigrahi and M. Paranjape, *ibid.* **355**, 236 (1995).
[13] A. Das, *Finite Temperature Field Theory* (World Scientific, Singapore, 1997).
[14] G. Dunne, R. Jackiw and C. Trugenberger, Phys. Rev. D **41**, 661 (1990).
[15] R. Jackiw and S.-Y. Pi, ''Reducing the Chern-Simons term by a symmetry,'' hep-th/9712087.
[16] K. Babu, A. Das and P. Panigrahi, Phys. Rev. D **36**, 3725 (1987).
[17] I. Aitchison, C. Fosco and J. Zuk, Phys. Rev. D **48**, 5895 (1993); I. Aitchison and J. Zuk, Ann. Phys. (N.Y.) **242**, 77 (1995).
[18] E. Witten, Phys. Lett. **117B**, 324 (1982).
[19] R. Jackiw, in *Current Algebra and Anomalies*, edited by S. Treiman, R. Jackiw, B. Zumino and E. Witten (Princeton University Press, Princeton, 1985); S. Elitzur, Y. Frishman, E. Rabinovici and A. Schwimmer, Nucl. Phys. **B273**, 93 (1986).
[20] I. Gradshteyn and I. Ryzhik, *Table of Integrals, Series, and Products* (Academic, New York, 1980).
[21] H. A. Weldon, Phys. Rev. D **47**, 594 (1993); P. Bedaque and A. Das, *ibid.* **47**, 601 (1993); A. Das and M. Hott, *ibid.* **50**, 6655 (1994).
[22] Y-C. Kao and M-F. Yang, Phys. Rev. D **47**, 730 (1993).
[23] S. Coleman and B. Hill, Phys. Lett. **159B**, 184 (1985).
[24] A. Das and G. Dunne (in progress).