Continuum version of ϕ^4_{1+1} theory in light-front quantization

Pierre Grangé
Laboratoire de Physique Mathématique et Theórique, Université Montpellier II, F-34095, Montpellier, Cedex 05, France

Peter Ullrich and Ernst Werner
Institut für Theoretische Physik, D-93040, Universität Regensburg, Germany

(Received 22 October 1997; published 5 March 1998)

A genuine continuum treatment of the massive ϕ^4_{1+1} theory in light-cone quantization is proposed. Fields are treated as operator-valued distributions, thereby leading to a mathematically well-defined handling of ultraviolet- and light-cone-induced infrared divergences and of their renormalization. Although nonperturbative, the continuum light-cone approach is no more complex than usual perturbation theory in lowest order. Relative to discretized light-cone quantization, the critical coupling increases by 30% to a value $r=1.5$. Conventional perturbation theory at the corresponding order yields $r_1=1$, whereas the RG-improved fourth-order result is $r_4=1.8\pm0.05$. [S0556-2821(98)00308-7]

PACS number(s): 11.10.Kk, 11.10.Ef, 11.10.St, 11.30.Rd

I. INTRODUCTION

Discretized light-cone quantization (DLCQ) [1] has played an important role in clarifying infrared aspects of the theory which are decisive for the appearance of the vacuum sector field, the LC counterpart of the nontrivial ground state of equal-time (ET) quantization [2–7]. The popularity of DLCQ resides in the easy and conceptually simple treatment of the necessary infrared regularization. However, it has never been demonstrated that the limit where the periodicity length L goes to infinity is identical to the genuine continuum theory where momentum space discretization is avoided from the start. The reason lies in the infrared behavior of the continuum theory, which has not yet been understood. Our aim is to clarify this issue on the basis of a mathematically well-defined procedure.

As an example, we treat explicitly ϕ^4_{1+1} theory in the continuum and compare its results for the phase transition to the DLCQ case. It turns out that with the same type of physical approximations the characteristics of the phase transition are the same in both cases, whereas the critical coupling strength and the dependence of the order parameter on the coupling strength are substantially different.

In connection with phase transitions, there is a vital interest to dispose of a continuum version of the theory, if one is interested in the study of critical phenomena in the framework of effective theories. This is the point of view of statistical theories of fields in which cutoffs are introduced in order to define a momentum or mass scale below which the effective theory is valid. Critical points of phase transitions are determined from zeros of the β function. To do this requires complete knowledge of the cutoff dependence of the critical mass, which can be given only by the continuum theory.

In Sec. II we make use of the concept of field-operator-valued distributions to have a mathematically well-defined Fock expansion. This can be done in a chart-independent manner, expressing the field as a surface integral over a manifold. Comparing the expressions for the Minkowski and light-front cases, one sees that the regularization properties of the test functions are automatically transferred from the first to the second case; thus, it is ensured that, if the field is regular in the Minkowski case, it is also regular in the LC case—it is the same field expressed by different surface integral. Actually, what is called IR divergence in the unregularized LC-field expansion is an UV divergence in the LC energy; it is only the special choice of coordinates which makes it look like an IR divergence. Therefore, there is no extra IR singularity in the LC case which would have to be treated separately: The UV behavior of the field on the Minkowski manifold dictates the UV *and* IR behavior on the LC manifold. There is absolutely no freedom in the LC case beyond the choice of test functions relevant in the Minkowski case. Moreover, because of general properties known from functional analysis, the independence of physical results from the special form of the test functions is ensured. In Sec. III we use the Haag expansion of field operators to define the decomposition into the particle sector field φ and the vacuum sector field Ω. In Sec. IV we discuss the equation of motion (EM) for φ and the constraint for Ω which are coupled equations. Finally, in Sec. V we discuss the so-called mean field solution of these equations and the results for the phase transition. In Sec. VI we rewrite the condition for the phase transition in the language of an effective theory and compare the results with the literature. In the Appendix we collect a number of results from the theory of distributions to substantiate the discussions in Sec. II.

II. FUNDAMENTALS: DEFINITIONS AND CONVENTIONS

The physical system under consideration is defined by the Lagrangian

$$\mathcal{L}=\frac{1}{2}(\partial_\mu\phi)(\partial^\mu\phi)-\frac{m^2}{2}\phi^2(x)-\frac{\lambda}{4!}\phi^4(x),\qquad m^2>0,\quad \lambda>0. \tag{2.1}$$

From Eq. (2.1) follows the EM

$$\partial_\mu \partial^\mu \phi + m^2 \phi + \frac{\lambda}{3!}\phi^3 = 0. \tag{2.2}$$

The free field EM defines the free field $\varphi_0(x)$, which can be expanded in terms of free field creation and annihilation operators.

We start with the Minkowski case (ET quantization). The field $\varphi_0(x)$ is defined in the sense of distributions by the functional

$$\varphi_0\{u\} = (\varphi_0(x), u(x)) = \int d^4x\ \varphi_0(x) u(x), \tag{2.3}$$

where $u(x)$ is an element of the test function space in coordinate spaces. Plugging in the Fock expansion of $\varphi_0(x)$ yields

$$(\varphi_0(x), u(x)) = \frac{1}{(2\pi)^3} \int \frac{d^4x d^3p}{2\omega(\vec{p})} \cdot [a(\vec{p}) e^{-i\langle x,p\rangle_M} + a^\dagger(\vec{p}) e^{+i\langle x,p\rangle_M}] \cdot f(\omega(\vec{p}), \vec{p}), \tag{2.4}$$

where $\langle x,p\rangle_M$ is the scalar product of x and p in the Minkowski metric, $\omega(\vec{p}) = \sqrt{m^2 + \vec{p}^2}$ is the on-shell energy, and $f(p_0, \vec{p})$ is a test function in momentum space which is required to fall off sufficiently fast as a function of the arguments p_0, p_1, p_2, p_3 as any one of the p_i's goes to ∞.

The minimal conditions for the attenuation factor $f(p)$ are

$$\int \frac{d^3p}{2\omega(p)} |f(\omega(\vec{p}), \tilde{p})| |\langle s|a^\dagger(\vec{p})|s'\rangle| < \infty$$

and

$$\int \frac{d^3p}{2\omega(p)} |f(\omega(\tilde{p}), \vec{p})| |\langle r|a(\vec{p})|r'\rangle| < \infty.$$

Here $(|s\rangle, |s'\rangle)$ and $(|r\rangle, |r'\rangle)$ are arbitrary pairs of states yielding nonvanishing matrix elements for $a^\dagger$ and a, respectively [see the Appendix, Eq. (A15)]. This condition is necessary, if one wants to guarantee that the Fourier integral Eq. (2.4) is finite. $\varphi_0(x)$ is then defined as

$$\varphi_0(x) = \frac{1}{(2\pi)^3} \int \frac{d^3p}{2\omega(\vec{p})} [a(\vec{p}) e^{-i\langle x,p\rangle_M} + a^\dagger(\vec{p}) e^{i\langle x,p\rangle_M}] f(p_0, \vec{p}). \tag{2.5}$$

which can also be written as a surface integral over the manifold defined by $p_0^2 - \vec{p}^{\,2} - m^2 = 0$ (see the Appendix). The positive and negative frequency parts in Eqs. (2.4) and (2.5) are a consequence of the necessity to introduce two charts—corresponding to $p_0 = \pm\sqrt{p^2 + m^2}$—if one wants to cover the manifold.

In the light-cone case the corresponding expression becomes

$$\varphi_0(x) = \frac{1}{(2\pi)^3} \int d^3\tilde{p}\, \frac{\theta(p^+)}{2p^+} [\tilde{a}(\vec{\tilde{p}}) e^{-i\langle \bar{x}, \tilde{p}\rangle_{LC}} + \tilde{a}^+(\vec{\tilde{p}}) e^{i\langle \tilde{x}, \tilde{p}\rangle_{LC}}] \tilde{f}(p_0(\tilde{p}), \vec{p}(\tilde{p})). \tag{2.6}$$

Here $\tilde{x}$ and $\tilde{p}$ designate the light-cone variables

$$\tilde{x}^{\,0} =:= x^+ + \frac{1}{\sqrt{2}}(x^0 + x^3),$$

$$\tilde{x}^{\,3} := x^- = \frac{1}{\sqrt{2}}(x^0 - x^3),$$

$$\tilde{x}^{\,2} := x^i, \quad i = 1,2, \tag{2.7}$$

$$\tilde{p}^{\,3} := p^+ = \frac{1}{\sqrt{2}}(p^0 + p^3),$$

$$\tilde{p}^{\,0} := p^- = \frac{1}{\sqrt{2}}(p^0 - p^3),$$

$$\tilde{p}^{\,i} = p^i, \quad i = 1,2. \tag{2.8}$$

$\langle \tilde{x}, \tilde{p}\rangle_{LC}$ is the scalar product within the LC metric. The creation and annihilation operators in Eqs. (2.5) and (2.6) are related through

$$\tilde{a}(\vec{\tilde{p}}) = a(\vec{v}(\vec{\tilde{p}})), \quad \tilde{a}^\dagger(\vec{\tilde{p}}) = a^\dagger(\vec{v}(\vec{\tilde{p}})), \quad \forall \vec{\tilde{p}} | p^+ > 0, \tag{2.9}$$

where $\vec{v}(\vec{\tilde{p}})$ is the three-vector defined by

$$\vec{v}(\vec{\tilde{p}}) = \left(\vec{p}_\perp, \frac{1}{\sqrt{2}}\left(p^+ - \frac{(p_\perp^2 + m^2)}{2p^+}\right)\right). \tag{2.10}$$

Finally, $\tilde{f}(p_0(\vec{p}), \vec{p}(\tilde{p}))$ is the transformed test function.

Explicitly, we have

$$\tilde{f} = f\left[\frac{1}{\sqrt{2}}\left(p^+ + \frac{p_\perp^2 + m^2}{2p^+}\right), \left(p_\perp^1, p_\perp^2, \frac{1}{\sqrt{2}}\left(p^+ - \frac{p_\perp^2 + m^2}{2p^+}\right)\right)\right], \tag{2.11}$$

from where it is clear that there is no infrared singularity in Eq. (2.6), if there is none in Eq. (2.5), since the singular behavior of $1/p^+$ is completely damped out by the behavior of the test function for $p^+ \to 0$. This is also clear from the fact that the two integrals in Eqs. (2.5) and (2.6) are equal (see the Appendix). Therefore, if $\varphi_0(x)$ is a bounded operator in the ET case, it is also guaranteed to be bounded in the light-cone case. Whereas the integral in the Minkowski case is composed of contributions from two charts, the final expression (2.6) in the LC case goes only over one chart (the one with $p^+ > 0$). Originally, there were also two charts, corresponding to $p^+ < 0$ and $p^+ > 0$, but the integral over the former one turns out to be equal to the integral over $p^+ > 0$, hence merging in a single expression; this is due to the different topologies in the Minkowski and LC cases: In the LC case the sign of p^- is the same as the sign of p^+, whereas in the Minkowski case the sign of the energy is not correlated with signs of momentum components; instead, there is a sign ambiguity.

III. DECOMPOSITION OF FIELDS INTO PARTICLE SECTOR AND VACUUM SECTOR FIELDS

In the DLCQ case the total field $\phi(x)$ can be naturally decomposed into the particle sector $\varphi(x)$—to be constructed from polynomials in $\widetilde{a}^{\dagger}(k^+)$ and $\widetilde{a}(k^+)$ with total momentum $K^+>0$—and the vacuum sector Ω—to be constructed from polynomials with total momentum $K^+=0$ [4]. In the continuum case this decomposition can be achieved with the help of the Haag expansion [8].

As in the DLCQ case, we decompose $\phi(x)=\varphi(x)+\Omega$, where for fixed LC time we have

$$\varphi(x)=\varphi_0(x)+\int dy_1^- dy_2^- g_2(x^- - y_1^-, x^- - y_2^-):\varphi_0(y_1)\varphi_0(y_2):$$
$$+\int dy_1^- dy_2^- dy_3^- g_3(x^- - y_1^-, x^- - y_2^-, x^- - y_3):\varphi_0(y_1)\varphi_0(y_2)(y_3):+\cdots . \tag{3.1}$$

All fields are taken at a fixed time, e.g., $x^+=0$; the argument y_i means therefore $y_i=(y_i^-, x^+)$.

Because of the properties of $\varphi_0(x)$, the support of φ in Fourier space is determined by the support of the test functions in $\varphi_0(x)$. The coefficient functions $g_2, g_3, \ldots$—or rather their Fourier transforms—have to be determined from the equation of motion and the constraints.

In order to obtain the vacuum sector field Ω—which by definition is x^- independent—we perform an additional integration over x^- and add a constant c-number part ϕ_0:

$$\Omega := \phi_0 + \frac{1}{V}\int dx^- dy_1^- dy_2^- g_2(x^- - y_1^-, x^- - y_2^-):\varphi_0(y_1)\varphi_0(y_2):$$
$$+\frac{1}{V}\int dx^- dy_1^- dy_2^- g_3(x^- - y_1^-, x^- - y_2^-, x^- - y_3^-):\varphi_0(y_1)\varphi_0(y_2)\varphi_0(y_3):+\cdots , \tag{3.2}$$

V being the integration volume.

Apparently, the operator valued part of Ω is nonlocal. Because of the fact that φ_0 is defined as an operator-valued distribution, the integrations in Eqs. (3.1) and (3.2) are well defined.

Substituting the expansion (2.6) into the definition (3.2,) one obtains, after a lengthy but completely standard calculation the Fourier expansion of Ω,

$$\Omega=\phi_0+\int_0^\infty \frac{dk^+}{4\pi k^+} f^2(k^+,\hat{k}^-(k^+))C(k^+)$$
$$\times \widetilde{a}^{\dagger}(k^+)\widetilde{a}(k^+)+\cdots , \tag{3.3}$$

where the coefficient $C(k^+)$ is given by

$$C(k^+)=\frac{2}{V}\int\int g_2(x^- - y_1^-, x^- - y_2^-)$$
$$\times \cos\left[\frac{k^+}{2}(y_2-y_1)\right]dy_1 dy_2 .$$

The higher terms of the expansion are not reproduced here because they will not be considered in this paper. Apparently, one has, as in DLCQ,

$$\langle 0|\Omega|0\rangle=\langle 0|\phi|0\rangle=\phi_0$$

and

$$\Omega|q_1^+, q_2^+, \cdots q_N^+\rangle=\lambda(q_1^+\cdots q_N^+)|q_1^+, q_2^+, \ldots q_N^+\rangle, \tag{3.4}$$

the eigenvalues $\lambda(q_1^+\cdots q_N^+)$ being given by

$$\lambda(q_1^+\cdots q_N^+)=\phi_0+\sum_{i=1}^N \frac{f^2(q_i^+, \hat{q}^-(q_i^+))}{4\pi q_i^+} C(q_i^+). \tag{3.5}$$

This shows that within the bilinear approximation Ω acts like a momentum-dependent mass term.

IV. DETERMINATION OF ϕ_0 AND $C(k^+)$

The field $\phi(x)=\varphi(x)+\Omega$ satisfies the LC form of the equation of motion

$$2\partial_+\partial_-(\varphi(x)+\Omega)+m^2[\varphi(x)+\Omega]+\frac{\lambda}{3!}[\varphi(x)+\Omega]^3=0. \tag{4.1}$$

We first define an operator P which projects an operator $\mathbb{F}(x)$ onto the vacuum sector according to

$$P*\mathbb{F}(x):=\frac{1}{V}\int_{-\infty}^{+\infty}\mathbb{F}(x)dx. \tag{4.2}$$

Acting with P on Eq. (4.1) yields the constraint (the derivative term vanishes)

$$\theta_3 := m^2\Omega+\frac{\lambda}{3!}\Omega^3+\frac{\lambda}{3!}\frac{1}{V}\int_{-\infty}^{+\infty}[\varphi^3(x)\Omega+2\Omega\varphi^2(x)$$
$$+\varphi(x)\Omega\varphi(x)]dx=0. \tag{4.3}$$

Projection with the complementary operator $Q := 1 - P$ yields the equation of motion for $\varphi(x)$,

$$2\partial_+\partial_-\varphi(x) + m^2\varphi(x) + \frac{\lambda}{3!}\, Q*[\varphi(x)+\Omega]^3 = 0, \quad (4.4)$$

Equations (4.3) and (4.4) are coupled operator-valued equations which are solved by taking matrix elements between Hilbert space states. Technically, this is very similar to the DLCQ case [4]:

One replaces $\varphi \to \varphi_0$ in Eqs. (4.3) and (3.2) and calculates the matrix elements $\langle 0|\theta_3|0\rangle$ and $\langle k^+|\theta_3|k^+\rangle$.

The results are

$$\langle 0|\theta_3|0\rangle = \mu^2\phi_0 + \frac{\lambda}{3!}\,\phi_0^3 + \frac{\lambda}{24\pi}\int_0^\infty dk^+\,\frac{C(k^+)\hat{f}^4(k^+)}{k^+} = 0 \quad (4.5)$$

and

$$\langle k^+|\theta_3|k^+\rangle = \frac{\lambda}{6}\,C^3(k^+)\hat{f}^6(k^+) + \frac{\lambda}{2}\,\phi_0 C^2(k^+)\hat{f}^4(k^+) + \left[\mu^2 f^2(k^+) + \frac{\lambda}{2}\,\phi_0^2\hat{f}^2(k^+) + \frac{\lambda}{4\pi k^+}\,\hat{f}^4(k^+)\right]C(k^+) + \frac{\lambda\phi_0}{4\pi k^+}\,\hat{f}^2(k^+) = 0, \quad \forall k^+. \quad (4.6)$$

Here we use the notation $f(k^+,\hat{k}^-(k^+)) = \hat{f}(k^+)$. μ^2 is defined by

$$\mu^2 = m^2 + \frac{\lambda}{8\pi}\int_0^\infty \frac{dk^+}{k^+}\,\hat{f}^2(k^+), \quad (4.7)$$

which is nothing but the tadpole renormalization of the mass. In order to keep things as close as possible to the DLCQ case, we use dimensionless momenta, which we measure in units of μ.

Apparently, Eq. (4.5) is an equation for ϕ_0 as a functional of $C(k^+)$; in turn, Eq. (4.6) determines $C(k^+)$ as a function of ϕ_0 in the form of a cubic equation. Whereas the "exact" solution of Eqs. (4.5) and (4.6) has to found numerically, important qualitative features of the solution can be discussed analytically.

In the region where k^+ is very small but $\hat{f}(k^+) \approx 1$, the solution of Eq. (4.6) is simply

$$C(k^+) = -\frac{\lambda\phi_0}{4\pi k^+}\,\frac{f^2(k^+)}{(\lambda/4\pi k^+) f^4(k^+)} \approx -\frac{\phi_0}{f^2(k^+)}; \quad (4.8)$$

i.e., the order parameter ϕ_0 determines the infrared behavior of the vacuum vector part of the field.

Using Eq. (4.8) in the small-k^+ region in Eq. (4.5) yields an integrand which behaves in the IR as $(-\lambda\phi_0/24\pi)\hat{f}^2(k^+)/k^+$ which has, up to numerical factors, the same behavior as the tadpole contribution in Eq. (4.7). Therefore, any divergence in Eq. (4.5) arising from sending cutoffs present in $\hat{f}(k^+)$ to infinity can be cured by an appropriate mass counterterm. It is important to note that this result is independent of the particular form chosen for the test function. In the UV region the integral in Eq. (4.5) causes no problem whatsoever, since for $k^+ \to \infty$ $C(k^+) \sim 1/k^+$, which yields an integrand $\sim 1/k^{+2}$.

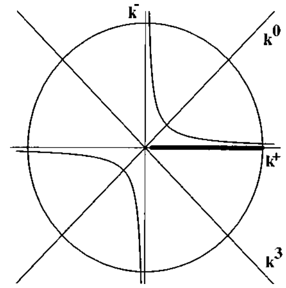

FIG. 1. The area inside the circle is the support of the test function $f(k^+,k^-)$. f equals unity inside a circle of radius $\Lambda - \epsilon$. The falloff to zero takes place in the interval $[\Lambda-\epsilon,\Lambda]$. The hyperbola represents the on-shell condition $\hat{k}^- = m^2/k^+$. Its intersections with the circle determine the IR and UV cutoffs for the variable k^+. The resulting support for the function $\hat{f}(k^+)$ is indicated by the thick line. Only the right half is physically realized due to the kinematical condition $k^+ > 0$.

Nevertheless, in order to be able to evaluate the integrals, one has to make a choice for $\hat{f}(k^+) = f(k^+,\hat{k}^-(k^+))$; without the on-shell condition $\hat{k} = \hat{k}^-(k^+)$, the test functions depend on the two variables k^+ and k^-. f can be chosen to have compact support and to be unlimitedly differentiable. In our case the support is the interior of a circle of radius Λ—which later on will be identified with a cutoff. A possible form is [9]

$$f(k^+,k^-) = \exp\left(\frac{1}{\Lambda^2}\right)\exp\left[-\frac{1}{\Lambda^2-(k^{+2}+k^{-2})}\right] \quad (k^{+2}+k^{-2} \leqslant \Lambda^2) = 0 \quad (k^{+2}+k^{-2} > \Lambda^2).$$

From this function one can construct another one which has the property that it equals 1 inside the two-sphere of radius $\Lambda - \epsilon$ and falls to zero within the interval $[\Lambda-\epsilon,\Lambda]$, ϵ can be chosen as small as one likes without affecting the C^∞ character of the function [9].

In order to make the comparison with the DLCQ case, we use from now on the convention $\hat{k}^- = m^2/k^+$ instead of $\hat{k}^- = m^2/2k^+$ (see Sec. II). This means the use of the factor $\frac{1}{2}$ in Eqs. (2.7) and (2.8) instead of $1/\sqrt{2}$.

Taking into account the on-shell condition, we arrive at the situation depicted in Fig. 1. The final result for $\hat{f}(k^+)$ is shown in Fig. 2.

In the limit where ϵ is arbitrarely small, $\hat{f}$ acts like a cutoff at $1/\Lambda$ and Λ. Using this cutoff form does not influence the physical results, since any other test function having the same support would yield the same results. Of course, instead of the divergent integral in Eq. (4.9), we would get something else, but the counterterm of Eq. (4.8) would change correspondingly. We have tested and verified this statement by doing numerical integrations with small, but finite values of ϵ.

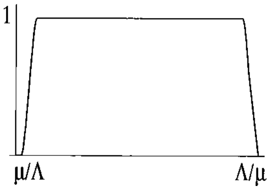

FIG. 2. Generic form of the test function $\hat{f}(k^+)$.

Using from now on the cutoff form for $\hat{f}$ and the dimensionless coupling $g=\lambda/4\pi\mu^2$, the integral to be renormalized becomes

$$I(\Lambda)=-\frac{g}{6}\int_{\mu/\Lambda}^{\Lambda/\mu}dk^+\,\frac{C(k^+)}{k^+}. \tag{4.9}$$

which is UV finite, but IR divergent. Using the IR limit of $C(k^+)$ given in Eq. (4.8), $I(\Lambda)$ becomes

$$I(\Lambda)=-\frac{g\phi_0}{6}\log\left(\frac{\Lambda}{\mu}\right)-\frac{\phi_0}{6(\Lambda/\mu)}+0\left(\frac{\mu^2}{\Lambda^2}\right)\cdots. \tag{4.10}$$

Given that

$$\langle 0|\varphi_0^2|0\rangle=\frac{1}{8\pi}\int_{\mu/\Lambda}^{\Lambda/\mu}\frac{dk^+}{k^+}=\frac{1}{4\pi}\log(\Lambda/\mu), \tag{4.11}$$

the divergence in Eq. (4.9) can be compensated for by the substraction of a mass-type counterterm $2\pi g\phi_0\varphi_0^2/3$.

V. CRITICAL COUPLING AND NATURE OF THE PHASE TRANSITION

In the vicinity of the phase transition where $\phi_0 \ll 1$, one can linearize Eqs. (4.5) and (4.6), yielding

$$C(k^+)=-\frac{\lambda\phi_0}{4\pi k^+}\,\frac{1}{\mu^2+\lambda/4\pi k^+}=-g\phi_0\,\frac{1}{(g+k^+)}. \tag{5.1}$$

The phase transition being determined by the vanishing of the mass term, the critical coupling g_c is determined by the condition

$$1=\frac{g_c^2}{6}\int_{\mu/\Lambda}^{\Lambda/\mu}\frac{dk^+}{k^+(k^++g_c)}-\frac{2\pi}{3}\,g_c\langle 0|\varphi_0^2|0\rangle, \tag{5.2}$$

which follows from Eq. (4.5) after division by μ^2 and substraction of the mass counterterm.

The integral in Eq. (5.2) can be evaluated analytically and yields, with Eq. (4.11),

$$\frac{g_c}{6}\log(g_c)=1,$$

which has the solution $g_c=4.19\ldots$.

The corresponding value for DLCQ [4] is $g_{c_{\mathrm{DLCQ}}}=3.18$; i.e., there is a 30% deviation between the two cases. On the other hand, there is no change in the nature of the phase transition (which is of second order) and of the critical exponents.

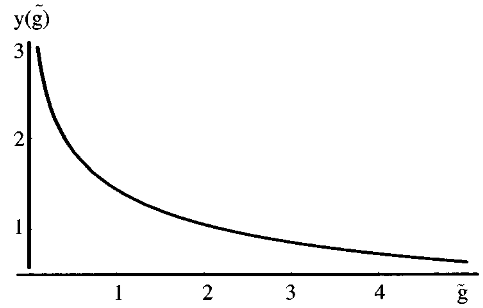

FIG. 3. The function $y(g)$ solution of Eq. (6.5).

VI. COMPARISON TO THEORIES OF CRITICAL BEHAVIOR

In order to compare our value for the critical coupling g_c to results obtained earlier in ET quantization, we have to rewrite the constraint (4.5). The most complete study of the critical behavior of ϕ^4_{1+1} theory has been performed by Parisi [10] in the scenario of a theory of critical phenomena. In this context the field theory is interpreted as an effective theory with a cutoff Λ, which defines the scale of validity of the theory. In the spirit of such a theory, one has to keep in the constraints (4.5) the dependence on Λ and consider this equation as a prescription for the calculation of the critical mass $M(\tilde{g},\Lambda)$. From this quantity one obtains the β function

$$\beta(\tilde{g})=M\left.\frac{\partial M}{\partial\tilde{g}}\right|_{\lambda,\Lambda}. \tag{6.1}$$

Here the definition of the coupling $\tilde{g}$ differs from our g—all momenta and masses are measured in units of Λ, and distances are replaced by the dimensionless quantity $x\Lambda$. g and $\tilde{g}$ are related by

$$g=\frac{\lambda}{4\pi\mu^2}=\frac{\lambda}{4\pi\Lambda^2}\,\frac{\Lambda^2}{\mu^2}:=\tilde{g}\,\frac{\Lambda^2}{\mu^2}. \tag{6.2}$$

We consider the constraint θ_3 as an equation for M^2:

$$M^2=\mu^2+\frac{\lambda}{24\pi}\int_{\mu/\Lambda}^{\Lambda/\mu}\frac{C(k^+)}{k^+}\,dk^+. \tag{6.3}$$

Satisfying the constraint $\theta_3=0$ amounts to $M^2(\Lambda,\tilde{g})=0$, which in turn means $\beta(\tilde{g},\Lambda)=0$, i.e., the condition which defines the phase transition via the fixed point of the β function.

We define the critical mass μ_c by

$$\begin{aligned}0&=\mu_c^2+\frac{\lambda}{24\pi}\int_{\mu_c/\Lambda}^{\Lambda/\mu_c}\frac{C(k^+)}{k^+}\,dk^+\\&=\mu_c^2+\frac{\lambda}{24\pi}\left[\log\frac{g_c+k^+}{k^+}\right]\Bigg|_{\mu_c/\Lambda}^{\Lambda/\mu_c}.\end{aligned} \tag{6.4}$$

Using $g_c=\tilde{g}_c\Lambda^2/\mu_c^2$ and the new variable $y(\tilde{g}_c)=\log(\Lambda/\mu_c)^2$, we obtain, in the limit of large Λ,

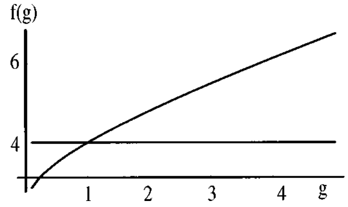

FIG. 4. The solution of Eq. (6.6).

$$ye^{y}=\frac{6}{\tilde{g}_c}. \tag{6.5}$$

The numerical solution of Eq. (6.5) is shown in Fig. 3 as a function of $\tilde{g}_c$.

Knowing $y(\tilde{g}_c)$, we can now relate $\tilde{g}_c$ and g_c by

$$\tilde{g}_c=g_c\left(\frac{\mu_c}{\Lambda}\right)^2=g_c e^{-y(\tilde{g}_c)}$$

or

$$\tilde{g}_c e^{y(\tilde{g}_c)}=g_c. \tag{6.6}$$

In Fig. 4 the left hand side of Eq. (6.6) is shown together with the straight line corresponding to $g_c=4.19$. The numerical value for $\tilde{g}_c$ is 1.

Parisi [10] uses in his calculation still another coupling, called r, defined by $r=3\lambda/8\pi\Lambda^2=\frac{3}{2}\tilde{g}$. It is normalized in such way that the critical coupling at the order of one loop is $r_1=1.0$. He pushed his calculations up to four loops (with a Borel improvement of the convergence of the asymptotic series) and obtained the value for r_4 reproduced in Table I.

As far as the solution of the equation of motion is concerned, our result corresponds to the one-loop result of Parisi (tadpole correction of the mass). On the other hand, it is nonperturbative as far as the solution of the constraint is concerned. This is reflected by the considerable improvement relative to $r_1=1$, which brings us with $r_{lc}=1.5$ already rather close to the four-loop result $r_4=1.85$ and to the lattice result $n_{\rm lat}=1.80\pm0.05$.

VII. CONCLUSIONS

We have shown for scalar fields that the continuum quantum field theory quantized on the light cone does not suffer from divergence problems beyond those present in conventional quantization, if the field operators are treated properly as operator-valued distributions. The treatment is quite generic and should be rather easily generalizable to other types of fields. Apart from giving substantially different results for the critical coupling of the ϕ^4_{1+1} theory as compared to DLCQ, the continuum version has the advantage that it can be rewritten as an effective theory for critical phenomena. This is important for a detailed comparison with the literature because the most elaborate studies in conventional quantization have been performed in this domain. Our results compare very favorably with the best values of renormalization-group-improved fourth-order perturbation theory and of lattice calculations which are reached up to 20%. Given the calculational simplicity of our approach—which on the technical level corresponds to first-order perturbation theory—this is an encouraging success. It is attributed to the existence of an operator-valued vacuum sector field which has to be added to the usual particle sector field and which is the LC signature of nonperturbative physics.

TABLE I. Critical couplings by different methods. r_{lc} is the value from the present continuous light-cone calculation.

r_1 [a,b]	r_4 [a,b]	$r_{\rm lat}$ [c]	r_{lc}
1	1.85	1.80 ± 0.05	1.5

[a]Ref. [10].
[b]Ref. [11].
[c]Ref. [12].

ACKNOWLEDGMENTS

This work has been supported by the NATO Grant No. CRG920472.

APPENDIX

In this appendix we review some concepts from the theory of distributions which are at the basis of the results of Sec. II.

1. Pullback of distributions

Here we can give only a very short version. For more details, the reader can consult, e.g., Ref. [13].

We consider two open subsets U and V of R^n, $U,\ V\subset\mathcal{R}^n$, and a C^∞ diffeomorphism κ between them:

$$\kappa{:}U\rightarrow V.$$

Given a distribution

$$(\varphi,f)=\int_V \varphi(x)f_V(x)dx, \tag{A1}$$

with $f_V(x)$ a test function of compact support on V, the distribution φ can be ''pulled back'' to U with the help of the pullback mapping $\kappa^*=\kappa\circ\varphi=\varphi(\kappa(x))$ (see Fig. 5).

$$(\kappa^*[\varphi],f)=\int_U \varphi(\kappa(x))f_U(x)dx. \tag{A2}$$

Introducing the inverse mapping

$$\tau:=\kappa^{-1}{:}V\rightarrow U$$

and making the coordinate transformation $\xi:=\kappa(x)$, Eq. (A2) goes over into

$$(\kappa^*[\varphi],f)=\int_V \varphi(\xi)f(\tau(\xi))\left|\det D(\tau(\xi))\right|d\xi, \tag{A3}$$

where $D(\tau(\xi))$ is the Jacobian of the coordinate transformation. Equation (A2) assigns to each distribution φ on V ''pulled back'' distribution $\kappa^*[\varphi]$ on U and Eq. (A3) is the prescription for its evaluation in terms of φ. The application which one has in mind in connection with Eqs. (A2) and

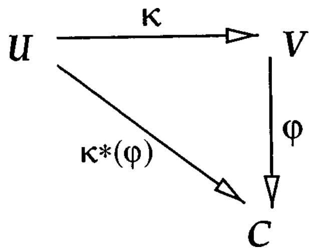

FIG. 5. Illustration of the pullback mapping: $C(V)\to C(U)$; $\kappa^*(\varphi):=\varphi\circ\kappa$.

(A3) is the frequent case where one works with distributions which depend on functions of the integration variables, e.g., $\delta(p^2-m^2)$.

In order to treat such a case in the pullback framework, we take $U\subset\mathcal{R}^n$, $n>1$; $V\subset\mathcal{R}^1$ is obtained from U through the mapping Q: $U\to V$ with a C^∞ function which we call now Q instead of κ. Moreover, we introduce the $(n-1)$-dimensional submanifold Σ defined by $Q(x)=0$,

$$\sum :=\{x\in U|Q(x)=0\},$$

and a Dirac δ^1 distribution δ_0 in $\mathcal{R}^1$.

The following theorem can be proved [13]: If $\nabla Q(x)\neq 0 \quad \forall x\in U$, then the pullback of δ^1 exists and is defined by

$$\delta\circ Q=\frac{1}{|\nabla Q|}\,\delta_\Sigma\,, \tag{A4}$$

where δ_Σ is a distribution defined by

$$(\delta_\Sigma\,,f\,)=\int_\Sigma f dS \quad \forall f\in U: \tag{A5}$$

here, dS is the Euclidian surface measure on Σ; this yields the distribution $\delta\circ Q$ as a surface integral:

$$(\delta\circ Q,f\,)=\int_\Sigma \frac{f}{|\nabla Q|}\,dS. \tag{A6}$$

Specializing Q to $Q(p)=p^2-m^2$, p being the energy-momentum four-vector, and using on Σ the charts

$$\Omega_\pm(\pm\omega(\vec{p}),\vec{p}),$$

the distribution (A6) can be rewritten as (in Minkowsky space)

$$(\delta\circ Q,f\,)=\int \frac{d^3p}{2\omega(\vec{p})}\,[f(\omega(\vec{p}),\vec{p})+f(-\omega(\vec{p}),\vec{p})], \tag{A7}$$

where $\omega(\vec{p})=+\sqrt{p^2+m^2}$.

We introduce the two tempered distributions

$$\delta_\pm(p^2-m^2)=1(\vec{p})\,\theta(\pm p^0)\,\delta(p^2-m^2), \tag{A8}$$

where $1(\vec{p})$ is the unit distribution defined by

$$(1,f\,)=\int f(\vec{p})d^3p$$

and $\delta_\pm$ obey

$$(\delta_\pm\,,f\,)=\int \frac{d^3p}{2\omega(\vec{p})}\,f(\pm\omega(\vec{p}),\vec{p}). \tag{A9}$$

Comparison with Eq. (A7) shows that

$$(\delta\circ Q)(p)=\delta_+(p^2-m^2)+\delta_-(p^2-m^2).$$

Going back to Eq. (A6) in the form

$$(\delta_+(p^2-m^2)+\delta_-(p^2-m^2),f(p))=(\delta\circ Q,f(p))$$
$$=\int_{\Sigma^+}\frac{f}{|\nabla Q|}\,ds+\int_{\Sigma^-}\frac{f}{|\nabla Q|}\,ds,$$

we see that δ_+ and δ_- can be written as surface integrals:

$$(\delta_\pm(p^2-m^2),f(p))=\int_{\Sigma_\pm}\frac{f}{|\nabla Q|}\,ds. \tag{A10}$$

Here the two integrals $\int_{\Sigma^\pm} ds$ are over the two surfaces defined by the two signs of p_0 in $p_0=\pm\sqrt{\vec{p}^2+m^2}$ with charts $\Omega^\pm$, Eq. (A5). It is important to note that the integrals in Eq. (A10) are independent of the special choice of the charts $\Omega^\pm$ which one makes on the surfaces $\Sigma^\pm$.

2. Solutions of the KG equation in Minkowski space

The tempered distribution $\chi(p)\,\delta(p^2-m^2)$ defined by

$$\chi(p)\,\delta(p^2-m^2)=\chi_+(p)\,\delta_+(p^2-m^2)+\chi_-(p)\,\delta_-(p^2-m^2) \tag{A11}$$

satisfies

$$(\chi(p)\,\delta(p^2-m^2),f(p))=\int_\Sigma \frac{\chi f}{|\nabla Q|}\,ds \tag{A12}$$

and solves the Klein-Gordon (KG) equation in momentum space:

$$(p^2-m^2)v(p)=0,$$

with

$$v(p)=v_1(\vec{p})\,\theta(p^0)\,\delta(p^2-m^2)+v_2(\vec{p})\,\theta(-p^0)\,\delta(p^2-m^2).$$

From the distribution $\chi(p)\,\delta(p^2-m^2)$, one obtains the solution of the coordinate space KG equation as a distribution $\phi(x)$ defined by (Minkowsky metric)

$$\phi(x):=2\pi\mathcal{F}_{M_x}[\chi(p)\,\delta(p^2-m^2)], \tag{A13}$$

where $\mathcal{F}_\mathcal{M}$ symbolizes the Fourier transform with Minkowski metric or, more explicitly,

$$(\phi(x),f(x))=2\pi(\chi(p)\delta(p^2-m^2),\mathcal{F}_M(f)(p))$$
$$=2\pi\int_\Sigma \frac{\chi(\mathcal{F}_M f)}{|\nabla Q|}\,ds$$
$$=2\pi\int \frac{d^3p}{2\omega(\vec p)}[\chi(\omega(\vec p),\vec p)\cdot(\mathcal{F}_M f)(\omega(\vec p),\vec p)$$
$$+\chi(-\omega(\vec p),\vec p)(\mathcal{F}_M f)(-\omega(\vec p),\vec p)]. \quad \text{(A14)}$$

In order to guarantee the existence of the two last integrals, one has to impose the condition on χ:

$$\int \frac{|\chi|}{|\nabla Q|}\,ds<\infty \quad \text{and} \quad \int \frac{d^3p}{2\omega(\vec p)}|\chi(\pm\omega(\vec p),\vec p)|<\infty. \quad \text{(A15)}$$

As it stands, this is valid for classical fields. After quantization—where the χ's are replaced by creation and annihilation operators—Eq. (A15) becomes a condition for matrix elements of these operators.

With Eq. (A15) the functions

$$H_\pm(x,p)=\frac{1}{2\omega(\vec p)}\chi(\pm\omega(\vec p),\vec p)e^{-i\langle x,\hat p_\pm\rangle_M}f(x),$$
$$\hat p_\pm=(\pm\omega(\vec p),\vec p)$$

are integrable; consequently, one can inject the Fourier representation of $\mathcal{F}_{\mathcal{M}}f$ into Eq. (A14) to obtain the regular distribution

$$(\phi(x),f(x))=\frac{1}{(2\pi)^3}\int d^4x f(x)$$
$$\times\int \frac{dp^3}{2\omega(\vec p)}[\chi(\hat p_+)e^{-i\langle x,\hat p_+\rangle_M}$$
$$+\chi(\hat p_-)e^{-i\langle x,\hat p_-\rangle_M}]. \quad \text{(A16)}$$

From Eq. (A16) one identifies the field $\phi(x)$ as

$$\phi=\frac{1}{(2\pi)^3}\int \frac{dp^3}{2\omega(\vec p)}[\chi(\hat p_+)e^{-i\langle x,\hat p_+\rangle_M}$$
$$+\chi(\hat p_-)e^{-i\langle x,\hat p_-\rangle_M}]. \quad \text{(A17)}$$

The chart-independent form of Eq. (A17) is

$$\phi=\frac{1}{(2\pi)^3}\int_\Sigma \frac{\chi(p)}{|\nabla Q(p)|}e^{-i\langle x,\hat p\rangle_M}ds. \quad \text{(A18)}$$

3. Solutions of the KG equation on the *LC*

We go from the coordinates x, p to the corresponding LC coordinates $\tilde x$, $\tilde p$ via the mapping

$$\tilde x=\kappa\circ x, \quad \tilde p=\kappa\circ p,$$

which leads to the LC-KG equation

$$(\Box_{LC}+m^2)\tilde\phi(\tilde x)=0,$$

with

$$\Box_{LC}=2\partial_+\partial_--\partial_\perp^2.$$

Introducing the quadratic form

$$\tilde Q(\tilde p):=\tilde p^{\,2}-m^2, \quad \text{(A19)}$$

one can define the distribution $\delta(\tilde p^{\,2}-m^2)$ as the pullback of the δ distribution under the mapping (A19):

$$\delta(\tilde p^{\,2}-m^2):=\delta\circ\tilde Q,$$

$$(\delta(\tilde p^{\,2}-m^2),f(\tilde p))=\int_{\tilde\Sigma}\frac{f(\tilde p)}{|\nabla\tilde Q|}\,d\tilde s, \quad \text{(A20)}$$

where $\nabla\tilde Q(\tilde p)=2(p^-,-\vec p_\perp,p^+)\neq 0$ on the mass shell. The manifold $\tilde\Sigma$ is a sum of two disconnected parts $\tilde\Sigma^+,p^+>0$ and $\tilde\Sigma^-,p^+<0$.

Choosing on $\tilde\Sigma^\pm$ the charts

$$[\tilde\Omega^+:(\tilde\omega(\vec{\tilde p}),\vec{\tilde p});p^+>0)]$$

and

$$[\tilde\Omega^-:(\tilde\omega(\vec{\tilde p}),\vec{\tilde p});p^+<0),]$$
$$\tilde\omega(\vec{\tilde p})=\frac{p_\perp^2+m^2}{2p^+},$$

one obtains

$$(\delta(\tilde p^{\,2}-m^2),f(\tilde p))=\int_{\Sigma^+}\frac{f(\tilde p)}{|\nabla\tilde Q|}\,d\tilde s+\int_{\Sigma^-}\frac{f(\tilde p)}{|\nabla\tilde Q|}\,d\tilde s$$
$$=\int d^3\tilde p\,\frac{\theta(p^+)}{2|p^+|}f(\tilde\Omega+(\vec{\tilde p}))$$
$$+\int d^3\tilde p\,\frac{\theta(-p^+)}{2|p^+|}f(\tilde\Omega^-(\vec{\tilde p})). \quad \text{(A21)}$$

The two integrals in Eq. (A13) (with $\chi=1$) and Eq. (A20) can be shown to be equal. More generally, it can be shown that for $\chi\in\mathcal{L}^1(\Sigma)$ and $\tilde\chi=\chi\circ u^{-1}\in\mathcal{L}^1(\tilde\Sigma)$, one has the identity

$$\int_{\Sigma^\pm}\frac{\chi(p)}{|\nabla Q|}\,ds=\int_{\tilde\Sigma^\pm}\frac{\tilde\chi(\tilde p)}{|\nabla\tilde Q|}\,d\tilde s. \quad \text{(A22)}$$

Using this identity, one arrives immediately at the result [see Eq. (A17)]

$$\phi(x)=\frac{1}{(2\pi)^3}\int_\Sigma\frac{\chi(p)}{|\nabla Q(p)|}e^{-i\langle x,\hat p\rangle_M}ds$$
$$=\frac{1}{(2\pi)^3}\int_{\tilde\Sigma}\frac{\tilde\chi(\tilde p)}{|\nabla\tilde Q(\tilde p)|}e^{-i\langle\hat x,\hat p\rangle_L}d\bar s. \quad \text{(A23)}$$

Introducing the same charts as in Eq. (A21), one obtains

$$\tilde{\phi}(\tilde{x}) = \frac{1}{(2\pi)^3}\int d^3p\,\frac{1}{2|p^+|}\,\tilde{\chi}(\tilde{\Omega}(\tilde{p}))e^{-i\langle\tilde{\Omega}(\tilde{p}),\tilde{x}\rangle_L}, \tag{A24}$$

where

$$\tilde{\chi}(\tilde{\Omega}(\vec{\tilde{p}})) = \chi\left[\frac{1}{\sqrt{2}}\left(p^+ + \frac{p_\perp^2+m^2}{2p^+}\right),\vec{p}_\perp,\frac{1}{\sqrt{2}}\left(p^+ - \frac{p_\perp^2+m^2}{2p^+}\right)\right].$$

The transition to the quantized field is made in Eq. (A17) by the substitution

$$\chi(\hat{p}_+)\rightarrow a(\vec{p}),$$

$$\chi(-\hat{p}_-)\rightarrow a^\dagger(\vec{p}),$$

yielding

$$\hat{\phi}(x) = \frac{1}{(2\pi)^3}\int\frac{d^3p}{2\omega(\vec{p})}\,[a(p)e^{-i\langle x,\hat{p}_-\rangle_M} + a^\dagger(\vec{p})e^{i\langle x,\hat{p}_+\rangle_M}].$$

In the LC case the two contributions from $\tilde{p}^{\,+}>0$ and $\tilde{p}^+<0$ in Eq. (A23) can be recast into a single one by first changing the integration variable in the second term $\vec{\tilde{p}}\rightarrow -\vec{\tilde{p}}$ and then making the substitution

$$\tilde{\chi}(\tilde{\Omega}(\vec{\tilde{p}}))\rightarrow\tilde{a}(\vec{\tilde{p}}),\quad \tilde{p}^{\,+}>0,$$

$$\tilde{\chi}(\tilde{\Omega}(-\vec{\tilde{p}}))\rightarrow\tilde{a}^{\,+}(\vec{\tilde{p}}),\quad \tilde{p}^{\,+}>0,$$

yielding

$$\hat{\tilde{\phi}}(\tilde{\chi}) = \frac{1}{(2\pi)^3}\int d^3\tilde{p}\,\frac{\theta(\tilde{p}^{\,+})}{2\tilde{p}^{\,+}}\times[\tilde{a}(\vec{\tilde{p}})e^{-i\langle\tilde{\Omega}(\tilde{p}),\tilde{x}\rangle_L} + \tilde{a}^\dagger(\vec{\tilde{p}})e^{+i\langle\tilde{\Omega}(\tilde{p}),\tilde{x}\rangle_L}]. \tag{A25}$$

For completeness we add a remark: In the literature [14] operator-valued distributions have been introduced on the level of the Fock-space operators $a(p)$, $a^\dagger(p)$ by defining the distributions

$$(a,f) = \frac{1}{(2\pi)^3}\int\frac{d^3p}{2\omega(\vec{p})}\,a(\vec{p})f(\omega(\vec{p}),\vec{p}),$$

$$(a^\dagger,f) = \frac{1}{(2\pi)^3}\int\frac{d^3p}{2\omega(\vec{p})}\,a^\dagger(\vec{p})f(\omega(\vec{p}),\vec{p}). \tag{A26}$$

For the LC case this reads

$$(\tilde{a},\tilde{f}) = \frac{1}{(2\pi)^3}\int\frac{d^3\tilde{p}}{\tilde{p}^{\,+}}\,\tilde{a}(\vec{\tilde{p}})\tilde{f}(\omega(\vec{p}),\vec{p}),$$

$$(\tilde{a}^\dagger,\tilde{f}) = \frac{1}{(2\pi)^3}\int\frac{d^3\tilde{p}}{\tilde{p}^{\,+}}\,\tilde{a}^{\,\dagger}(\vec{\tilde{p}})\tilde{f}(\omega(\vec{p}),\vec{p}). \tag{A27}$$

Decomposing the field $\varphi_0(x)$ into positive and negative frequency parts $\varphi_0(x) = \varphi_0^+(x) + \varphi_0^-(x)$, we see that

$$\varphi_0^+(0) = (a,f) = (\tilde{a},\tilde{f}),$$

$$\varphi_0^-(0) = (a^\dagger,f) = (\tilde{a}^{\,\dagger},\hat{f}).$$

This shows again that the field $\varphi_0(x)$ is well defined in the sense of operator-valued distributions.

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