PHYSICAL REVIEW D VOLUME 57, NUMBER 8 15 APRIL 1998

de Broglie–Bohm interpretation for the wave function of quantum black holes

Masakatsu Kenmoku*
Department of Physics, Nara Women's University, Nara 630, Japan

Hiroto Kubotani†
Department of Mathematics, Nara University of Education, Nara 630, Japan

Eiichi Takasugi‡
Department of Physics, Osaka University, Osaka 560, Japan

Yuki Yamazaki§
Graduate School of Human Culture, Nara Women's University, Nara 630, Japan

(Received 10 November 1997; published 20 March 1998)

We study the quantum theory of spherically symmetric black holes. The theory yields the wave function inside the apparent horizon, where the role of time and space coordinates is interchanged. The de Broglie–Bohm interpretation is applied to the wave function and then the trajectory picture on the minisuperspace is introduced in the quantum as well as the semiclassical region. Around the horizon large quantum fluctuations on the trajectories of metrics U and V appear in our model, where the metrics are functions of the time variable T and are expressed as $ds^2=-(\alpha^2/U)dT^2+UdR^2+Vd\Omega^2$. On the trajectories, the classical relation $U=-V^{1/2}+2Gm$ holds, and the event horizon $U=0$ corresponds to the classical apparent horizon on $V=2Gm$. In order to investigate the quantum fluctuation near the horizon, we study a null ray on the dBB trajectory and compare it with the one in the classical black hole geometry. [S0556-2821(98)06108-6]

PACS number(s): 04.70.Dy, 03.65.Bz, 04.60.Ds, 97.60.Lf

I. INTRODUCTION

A quantum state with fluctuations of spacetime is described by a state vector Ψ, which is a solution to the Wheeler-DeWitt (WD) equation $H\Psi=0$, where H is the Hamiltonian of the gravitating system [1,2]. This equation implies that the wave function is a stationary state with zero energy, and there is no Schrödinger evolution of the physical state; that is, we cannot get the dynamical picture from this equation. This is called the problem of time in quantum gravity and has been researched extensively [3]. In addition to this, there are other problems on the interpretation for the wave function to be considered [4]. In particular, problems occur when we apply the ordinary Copenhagen interpretation to the wave function of the whole universe. The first one is that the observer is also an element of the quantum mechanical system. The second one is that in the Copenhagen interpretation one must consider many measurements performed on a pure ensemble, each element of which is characterized by the same state $|\Psi\rangle$, and when a measurement of observable A is made by an external observer described by the classical mechanics, the measurement causes the discontinuous change brought about by the collapse of the wave function into an eigenstate $|a\rangle$ with eigenvalue a. It is assumed that the quantity $|\langle a|\Psi\rangle|^2$ gives the probability for obtaining the measured value a. In quantum cosmology, however, since the wave function given by the WD equation describes a unique and whole system, one cannot accept the concept of probability.

Many people use the Wentzel-Kramers-Brillouin (WKB) approach to quantum cosmology to overcome the problem of time [5,6]. In the standard WKB approach, the system is assumed to separate into two parts: the gravitational part as the semiclassical system and the matter part as the quantum system. The time is introduced through the identification of the gradient of the classical action with the velocity variable. The Hamiltonian of quantum matter plays the time developing operator and the Schrödinger-like equation holds, though the total system does not develop the time. Some ambiguities and difficulties of this method were pointed out [7–9]. Though the WKB approach successfully introduces the time and the Schrödinger equation, the problem of observation remains unsolved because the standard WKB approach gives no alternative to the stand point of the Copenhagen interpretation.

The de Broglie–Bohm (dBB) interpretation [10,11] introduces the time in a very similar way to the WKB approach, whereas it takes very different interpretation to the wave function. The dBB interpretation defines a kind of trajectory by identifying the momentum with the gradient of the phase of the wave function. We shall call it the dBB or quantum trajectory. This interpretation seems to be able to avoid some of the problems mentioned above [12,13]. For the problem of the dynamical evolution, we can obtain it through the dBB trajectories, which describe the quantum evolution using the time coordinate appearing in the original Lagrangian. These trajectories are assumed to be the real entity. The quantum effects on the trajectories are represented by the quantum potential quantitatively which is defined by the second derivative of the amplitude of the wave function. By this quantum potential the dBB trajectories are modified from the

*Email address:kenmoku@cc.nara-wu.ac.jp
†Email address:kubotani@aquarius.nara-edu.ac.jp
‡Email address:takasugi@phys.wani.osaka-u.ac.jp
§Email address:yamazaki@phys.nara-wu.ac.jp

classical ones. If the quantum potential is small enough compared with the ordinary potential, the classical system is realized whether the observer exists or not. Therefore the dBB interpretation needs no observer and no collapse of the wave function. When we apply the dBB interpretation to quantum cosmology, it naturally represents how the classical universe emerges for a large cosmic scale factor [14,15,8,16].

In this paper, we apply the dBB interpretation to the quantum geometry of the Schwarzschild black hole, which is one of many other interesting issues in quantum gravity, because the quantum effect may become large near the singularity or the horizon. The quantum mechanical property of black hole geometry may affect the Hawking radiation [17–20] and the phenomenon of the smearing of a black hole singularity [21]. In order to treat the quantum theory of the black hole, Nambu and Sasaki [22] have proposed canonical quantum gravity inside the horizon of the black hole and introduced the mass scale using a dust collapse model. Nakamura, Konno, Oshiro and Tomimatsu (NKOT) [23] solve the WD equation with the limitation of the mass eigenstate to study the quantum fluctuation of the horizon and found an exact wave function for the minisuperspace model of the interior geometry of the black hole. They researched the quantum effect on horizon geometry on the basis of the WKB approach. The quantum fluctuation may become very large near the horizon, where the WKB method cannot be adopted. In order to estimate quantum fluctuation quantitatively beyond the semiclassical region, we study the dBB interpretation for the wave function of the Schwarzschild black hole.

In the classical theory, a spherically symmetric gravitational field in empty space must be static with its metric given by the Schwarzschild solution by the Birkhoff theorem [24]. In order to study the quantum theory, the canonical formalism must be formulated within the horizon region and the mass function must be expressed by canonical variables. The mass eigenvalue equation plays an important role in order to argue the quantum theory of empty space. Instead of imposing the momentum constraint, the mass eigenvalue equation guarantees the diffeomorphism invariance not only inside but also the outside of the horizon. The mass eigenvalue equation strongly restricts the solution space of the WD equation and plays the role of a kind of initial or boundary condition. We use the canonical definition of mass function considered by Fischler, Morgan, and Polchinski [25] and by Kuchař [26], who has examined the canonical quantization formalism by Dirac [27], and by Arnowitt, Deser and Misner [28] for the spherically symmetric spacetime. They have shown that mass function can be described by the canonical variables and is a constant of motion. We solved the WD equation and the mass eigenstate equation simultaneously and obtained the general wave function, which is essentially the same as that by NKOT but the general form of operator ordering is taken into account. We apply the dBB interpretation to this wave function and obtain the dBB trajectories on the minisuperspace, which represents the quantum feature of the geometry of the Schwarzschild black hole. We investigate the light ray on this quantum geometry.

This paper is organized as follows. In Sec. II, we review the canonical formalism in spherically symmetric spacetime. In Sec. III, we perform the canonical quantization of this system and obtain the wave function for the Schwarzschild black hole under a general consideration of the operator ordering. In Sec. IV, we consider the implication of the wave function with the help of the dBB interpretation. A summary and discussion are given in Sec. V.

II. CANONICAL FORMALISM IN SPHERICALLY SYMMETRIC SPACETIME

In this section, we review the canonical formalism in the spherically symmetric spacetime in four dimensions. The metric of the general spherically symmetric spacetime is

$$ds^2=-u(r)dt^2+\frac{\alpha(r)^2}{u(r)}dr^2+r^2d\Omega^2, \tag{2.1}$$

where the each metric depends only on the space variable r and $d\Omega^2=d\theta^2+\sin^2\theta d\phi^2$ denotes the line element on the unit sphere. The unique solution of the vacuum Einstein equation for this metric is known as the Schwarzschild solution for

$$\alpha=1, \quad \text{and} \quad u(r)=1-\frac{2Gm}{r}, \tag{2.2}$$

where G denotes the gravitational constant. The integration constant m represents the asymptotically observed mass of a spherically symmetric matter. The metric becomes singular when $r=0$ and $r=r_g\equiv 2Gm$. Here $r=0$ is a real physical singularity, whereas $r=r_g$ is a coordinate singularity and corresponds to the event horizon. Inside the black hole ($r<r_g$), the metric g_{tt} becomes positive and g_{rr} negative, and so the roles of time and space coordinate are interchanged. We denote this situation as

$$t\to R, \quad r\to T, \quad u(r)\to -U(T), \quad \alpha(r)\to\alpha(T). \tag{2.3}$$

Then the interior metric of a spherically symmetric black hole is represented as

$$ds^2=-\frac{\alpha(T)^2}{U(T)}dT^2+U(T)dR^2+V(T)d\Omega^2, \tag{2.4}$$

where the range of the variables are $T>0$, $-\infty<R<\infty$. The metric $V(T)$ is introduced in order to represents the quantum geometry of the spherically symmetric spacetime. In this metric, the classical solution is written as

$$\alpha=1, \quad U=-\left(1-\frac{r_g}{T}\right), \quad V^{1/2}=T. \tag{2.5}$$

The vacuum Einstein action

$$S=\frac{1}{16\pi G}\int d^4x\sqrt{-{}^{(4)}g}\,{}^{(4)}R \tag{2.6}$$

is written in the form of (3+1) decomposition in terms of the Arnowitt-Deser-Misner (ADM) analysis as

$$S=\int dTL, \tag{2.7}$$

where L is the Lagrangian

$$L=\frac{1}{16\pi G}\int d^3xN\sqrt{^{(3)}g}[K_{ab}K^{ab}-(K_a{}^a)^2+{}^{(3)}R], \tag{2.8}$$

where N and K_{ab} denote the lapse function and the extrinsic curvature, and $^{(3)}g$ and $^{(3)}R$ denote the metric and the curvature in three dimensional manifold respectively. By inserting the metric in Eq. (2.4), the Lagrangian Eq. (2.8) becomes

$$L=\frac{v_0}{4G}\left[-\frac{1}{\alpha}\left(\dot{U}\dot{V}+\frac{U\dot{V}^2}{2V}\right)+2\alpha\right], \tag{2.9}$$

where the dot denotes the derivative with respect to T. The volume of the system $v_0\equiv\int_0^\infty dR$ is treated to be finite.

Here we change the variables from U,V to z_+,z_- as follows:

$$z_+\equiv U\sqrt{V}, \quad z_-\equiv\sqrt{V}. \tag{2.10}$$

By using these new variables, the Lagrangian [Eq. (2.9)] becomes the simpler and symmetric form [29]

$$L=\frac{v_0}{2G}\left[-\frac{1}{\alpha}\dot{z}_+\dot{z}_-+\alpha\right]. \tag{2.11}$$

The canonical momentum conjugate to z_+, z_- and α are obtained from this Lagrangian:

$$\Pi_+\equiv\frac{\partial L}{\partial\dot{z}_+}=-\frac{v_0}{2G\alpha}\dot{z}_-, \tag{2.12}$$

$$\Pi_-\equiv\frac{\partial L}{\partial\dot{z}_-}=-\frac{v_0}{2G\alpha}\dot{z}_+, \tag{2.13}$$

$$\Pi_\alpha\equiv\frac{\partial L}{\partial\dot{\alpha}}=0. \tag{2.14}$$

The variable α plays the role of the lapse function and so its canonical conjugate momentum (2.14) becomes zero, which is the primary constraint. The Hamiltonian for this system becomes the form

$$H=-\alpha\left[\frac{2G}{v_0}\Pi_+\Pi_-+\frac{v_0}{2G}\right]. \tag{2.15}$$

The development of the primary constraint (2.14) by this Hamiltonian (2.15) yields the secondary constraint:

$$H|_{\alpha=1}\approx 0, \tag{2.16}$$

which is called the Hamiltonian constraint.

Next we discuss the mass of the system. In the usual classical theory, the unique solution of the vacuum Einstein equation for spherically symmetric spacetime is the Schwarzschild solution with one integration constant, which represents the mass of the black hole. In the canonical formalism for spherically symmetric spacetime, the mass can be a dynamical function, which has been introduced by Fischler, Morgan, and Polchinski [25] and extensively discussed by Kuchař [26]. The spherically symmetric hypersurface on which the canonical data is given is supposed to be embedded in a Schwarzschild black hole spacetime whose metrics are given by Eq. (2.5). This identification of the spacetime with the canonical data enables us to connect the Schwarzschild mass M with the canonical data on any small piece of a spacelike hypersurface. As result, the mass function is expressed by the canonical variables z_+,z_- as

$$M=\frac{2G}{v_0^2}z_+(\Pi_+)^2+\frac{z_-}{2G}. \tag{2.17}$$

The Poisson brackets of this mass function with H vanishes weakly:

$$\{H,M\}_{\mathbf{p}}=-i\hbar\frac{2G}{v_0^2}\Pi_+H\approx 0, \tag{2.18}$$

which shows that the mass is a constant of motion. Therefore we can consistently impose two equations: the Hamiltonian constraint and the mass constraint $M=m$ (constant) on the canonical data.

III. QUANTIZATION

In this section, we proceed with the canonical quantization treatment of spherically symmetric spacetime, that is, the Schwarzschild black hole system. We do not discuss the Hilbert space of quantum gravity. Instead of this, we consider a general form of the operator ordering for the canonical operators.

In the Schrödinger representation, the canonical momenta are quantized as

$$\hat{\Pi}_{+p}\equiv z_+^p\hat{\Pi}_+z_+^{-p}=\hat{\Pi}_++\frac{i\hbar p}{z_+}, \tag{3.1}$$

$$\begin{aligned}\hat{\Pi}_{-s}&\equiv(z_--r_g)^s\hat{\Pi}_-(z_--r_g)^{-s}\\&=\hat{\Pi}_-+\frac{i\hbar s}{z_--r_g},\end{aligned} \tag{3.2}$$

where $\hat{\Pi}_+$ and $\hat{\Pi}_-$ are usual differential operators

$$\hat{\Pi}_+=-i\hbar\frac{\partial}{\partial z_+}, \quad \hat{\Pi}_-=-i\hbar\frac{\partial}{\partial z_-}, \tag{3.3}$$

and p, s are integers in order to take account of the operator ordering. The Hamiltonian constraint (2.16) gives a condition imposed on the state vector Ψ

$$\hat{H}\Psi=-\left(\frac{2G}{v_0}\hat{\Pi}_{+r}\hat{\Pi}_{-s}+\frac{v_0}{2G}\right)\Psi=0, \tag{3.4}$$

where r and s are arbitrary integers. This is the Wheeler-DeWitt (WD) equation for the geometry of the Schwarzschild spacetime. In addition, we confine the state vector to the mass eigenvalue equation. Therefore,

$$(\hat{M}-m)\Psi=\left(\frac{2G}{v_0^2}\hat{\Pi}_{+p}z_+\hat{\Pi}_{+q}+\frac{z_-}{2G}-m\right)\Psi=0, \tag{3.5}$$

where p,q are arbitrary integers.

We note that the commutation relation between the Hamiltonian and the mass operator is calculated to be

$$[\hat{H},\hat{M}-m]=\frac{-2i\hbar G}{v_0^2}\hat{\Pi}_{+r}\hat{H}+\frac{4iG^2\hbar^2}{v_0^3}(r-p)(r-q)\times\frac{1}{z_+^2}\hat{\Pi}_{-s}. \tag{3.6}$$

For $r=p$ or $r=q$, the commutation relation vanishes weakly and the simultaneous requirement of the WD equation and the mass eigenvalue equation becomes compatible. In the following, we take the case of $r=q$. The case of $r=p$ is obtained by replacing $q\leftrightarrow p$ in the following calculation.

Instead of solving the mass eigenvalue equation directly, we consider the eigenvalue equation derived from the linear combination of the Hamiltonian and the mass operator:

$$[\hat{L}-\hbar(p-s)]\Psi=0, \tag{3.7}$$

where the operator $\hat{L}$ is defined as

$$\hat{L}\equiv-2iG\left[\frac{1}{v_0}\hat{\Pi}_{+p}z_+\hat{H}+\hat{\Pi}_{-s}(\hat{M}-m)\right]+\hbar(p-s)$$
$$=i[z_+\hat{\Pi}_+-(z_--r_g)\hat{\Pi}_-]. \tag{3.8}$$

As this equation is the first order differential equation, we can treat it easier. If we define the wave function ψ:

$$\Psi\equiv z_+^q(z_--r_g)^s\psi(z_+,z_-), \tag{3.9}$$

the equations we have to solve become the following two:

$$\left(\frac{2G}{v_0}\hat{\Pi}_+\hat{\Pi}_-+\frac{v_0}{2G}\right)\psi=0, \tag{3.10}$$

$$(\hat{L}-\hbar(p-q))\psi=0. \tag{3.11}$$

If we make change of variables

$$y\equiv\frac{v_0}{G\hbar}\sqrt{-z_+/(z_--r_g)},\quad z\equiv\frac{v_0}{G\hbar}\sqrt{-z_+(z_--r_g)}, \tag{3.12}$$

Eq. (3.11) becomes

$$\left[y\frac{\partial}{\partial y}-(p-q)\right]\psi(y,z)=0, \tag{3.13}$$

whose general solution is given by

$$\psi(y,z)=y^{p-q}u(z), \tag{3.14}$$

where $u(z)$ is an arbitrary function of z. By substituting the new variables (3.12) and the wave function Eq. (3.14) into Eq. (3.10), the WD equation is reduced to

$$\left[\frac{1}{z}\frac{\partial}{\partial z}z\frac{\partial}{\partial z}-\frac{y^2}{z^2}\left(\frac{1}{y}\frac{\partial}{\partial y}y\frac{\partial}{\partial y}\right)+1\right]y^{p-q}u(z)$$
$$=y^{p-q}\left[\frac{1}{z}\frac{\partial}{\partial z}z\frac{\partial}{\partial z}-\frac{(p-q)^2}{z^2}+1\right]u(z)$$
$$=0. \tag{3.15}$$

The equation for u is the Bessel's differential equation with order p-q. Then we obtain the eigenfunction for the WD and the mass eigenvalue equations

$$\Psi=y^{p-s}z^{q+s}[c_1H_{p-q}^{(1)}(z)+c_2H_{p-q}^{(2)}(z)], \tag{3.16}$$

where c_1,c_2 are integration constants. The Hankel functions $H^{(1)},H^{(2)}$ are linearly independent and complex conjugate each other.

NKOT have derived the quantum wave function for the Schwarzschild black hole. As the variables y,z in Eq. (3.12) take real values for the physical metrics U,V in Eq. (2.4), the complex phase factor comes only from the Hankel function. Note that our solution includes NKOT's solution [30].

IV. IMPLICATION OF WAVE FUNCTION

NKOT have discussed the quantum evolution of the metric variables from the wave function of the black hole in the WKB approach [23]. They further argued the possibility of a tunneling solution across the horizon, where the quantum fluctuation becomes large and the WKB approach is not applicable. In order to make the quantitative estimation of the quantum fluctuation, we apply the de Broglie–Bohm (dBB) interpretation to the wave function of the quantum black hole which is obtained in the previous section. This dBB interpretation has been applied to quantum cosmology by several authors in order to solve the problems of time and the observer [8,14–16].

In the following, we apply the dBB interpretation to the wave function of the spherically symmetric black hole. For this purpose, we rewrite the WD equation and the mass eigenequation explicitly in terms of the real phase and the real amplitude of the wave function:

$$\Psi(z_+,z_-)=R(z_+,z_-)\exp[iS(z_+,z_-)/\hbar]. \tag{4.1}$$

Inserting this expression into the WD equation (3.4), we obtain the real part equation:

$$\frac{2G}{v_0}\frac{\partial S}{\partial z_+}\frac{\partial S}{\partial z_-}+\frac{v_0}{2G}+Q=0, \tag{4.2}$$

where Q denotes the quantum potential for the quantum black hole as

$$Q=-\frac{2G\hbar^2}{v_0R}\left(\frac{\partial}{\partial z_+}-\frac{q}{z_+}\right)\left(\frac{\partial}{\partial z_-}-\frac{s}{z_--r_g}\right)R, \tag{4.3}$$

and the imaginary part equation:

$$\left(\frac{\partial}{\partial z_+}-\frac{q}{z_+}\right)\left(R^2\frac{\partial S}{\partial z_-}\right)+\left(\frac{\partial}{\partial z_-}-\frac{s}{z_--r_g}\right)\left(R^2\frac{\partial S}{\partial z_+}\right)=0. \tag{4.4}$$

If Q tends to zero, the real part of the WD equation (4.2) is reduced to the Hamilton-Jacobi equation for the black hole. This means that the quantum potential indicates the quantum effect quantitatively. In our quantum back hole model, we also demanded the mass eigenvalue equation (3.5). The real part of this equation can be rewritten as

$$\frac{2G}{v_0^2}z_+\left(\frac{\partial S}{\partial z_+}\right)^2+\frac{z_--r_g}{2G}+M_Q=0, \tag{4.5}$$

where M_Q is defined as

$$M_Q=-\frac{2G\hbar^2}{v_0^2R}\left(\frac{\partial}{\partial z_+}-\frac{p}{z_+}\right)\left[z_+\left(\frac{\partial}{\partial z_+}-\frac{q}{z_+}\right)R\right], \tag{4.6}$$

and the imaginary part is given as

$$\left(\frac{\partial}{\partial z_+}-\frac{q+p}{z_+}\right)\left(R^2z_+\frac{\partial S}{\partial z_+}\right)=0. \tag{4.7}$$

If M_Q tends to zero, Eq. (4.5) is reduced to the classical relation: M in Eq. (2.17) equals a constant m. Therefore M_Q represents the quantum effect to the mass function. The operator $\hat{L}$ in Eq. (3.8), which is the linear combination of the Hamiltonian and the mass operator, is easy to analyze, because it is the first order differential operator. Its real and imaginary parts are rewritten respectively as

$$\left[z_+\frac{\partial}{\partial z_+}-(z_--r_g)\frac{\partial}{\partial z_-}-p+s\right]R=0, \tag{4.8}$$

$$\left[z_+\frac{\partial}{\partial z_+}-(z_--r_g)\frac{\partial}{\partial z_-}\right]S=0. \tag{4.9}$$

Using Eq. (4.9), the quantum effect of the mass operator M_Q can be expressed as proportional to the quantum potential Q:

$$M_Q=\frac{z_--r_g}{v_0}Q, \tag{4.10}$$

so that all the quantum effect vanishes when Q tends to zero.

Here we introduce the dBB interpretation, which is based on the following assumptions.

(1) The trajectory picture on the minisuperspace is introduced. The momenta of the quantum geometry of the black holes are assumed to be given by

$$\Pi_+=-\frac{v_0}{2G}\dot{Z}_-=\left.\frac{\partial S}{\partial z_+}\right|_{z_+=Z_+,z_-=Z_-}, \tag{4.11}$$

$$\Pi_-=-\frac{v_0}{2G}\dot{Z}_+=\left.\frac{\partial S}{\partial z_-}\right|_{z_+=Z_+,z_-=Z_-}, \tag{4.12}$$

where the velocities are determined by the classical relations (2.12) and (2.13) with $\alpha=1$. The trajectories Z_+,Z_- are obtained by integrating Eqs. (4.11) and (4.12). We emphasize that they are the differential equations with respect to T, which is the notion of time parameter appearing in Eq. (2.4), so that the notion of time is introduced even if the WD equation does not depend on the time explicitly. We shall call these trajectories the dBB trajectories hereafter.

(2) Consider a statistical ensemble of the trajectories. The probability distribution is assumed to be given by R^2. Note that the imaginary part equation guarantees the continuity for the probability density R^2.

(3) The quantum potential Q denotes the quantum effect quantitatively. This is not a usual local potential because it is defined through a wave function. If Q is negligible compared with the classical potential or kinetic term, the trajectory coincides with the classical one. This means that the quantum system behaves like the classical one spontaneously. It is indeed this situation that we call ''classical.'' Therefore, we need no classical observer which is an essential element in Copenhagen interpretation. On the other hand, if Q cannot be negligible, the dBB trajectory is modified by any quantum probe. In this case, we should explicitly include a specific quantum observer in the dynamical system to get information of the dBB trajectory. When the observer becomes classical, it plays the role of an observer in the Copenhagen interpretation which is classical by nature.

We note that in the dBB interpretation, an existence in nature is a dBB trajectory, and therefore, predictability of quantum mechanics is based on the probability distribution of their statistical ensemble. As a result, the reduction of the wave function or the loss of the quantum coherence by the measurement is not the matter with the dBB interpretation.

Next we insert the explicit form of the Hankel function (3.16) into the general form of the wave function (4.1):

$$\Psi=y^{p-s}z^{q+s}H_\nu^{(2)}(z). \tag{4.13}$$

Here we have chosen the Hankel function of the second kind $H^{(2)}$ with the index $\nu\equiv p-q$, which denotes the freedom of the operator ordering, since from the dBB point of view it corresponds to the classical relation $V^{1/2}=T$ in Eq. (2.5) in the semiclassical region, as will be seen in Fig. 2. Physically the selection of $H^{(2)}$ corresponds to solve the trajectory from the singularity at the origin to the outside. On the ansatz the dBB trajectory for the wave function of the linear combination of both $H^{(1)}$ and $H^{(2)}$ in Eq. (3.16) is shown not to approach a classical trajectory.

After inserting Eq. (4.13), the quantum potential is expressed by the Hankel function as

$$Q=-\frac{v_0}{2G}\left(1-\frac{4}{\pi^2z^2|H_\nu^{(2)}(z)|^4}\right). \tag{4.14}$$

Similarly, the continuity equations (4.4), (4.7), and (4.9) are reduced to one equation:

$$\frac{\partial}{\partial z}\left(z|H_\nu^{(2)}(z)|^2\frac{\partial}{\partial z}S\right)=0, \tag{4.15}$$

which corresponds to the identity of the Hankel function:

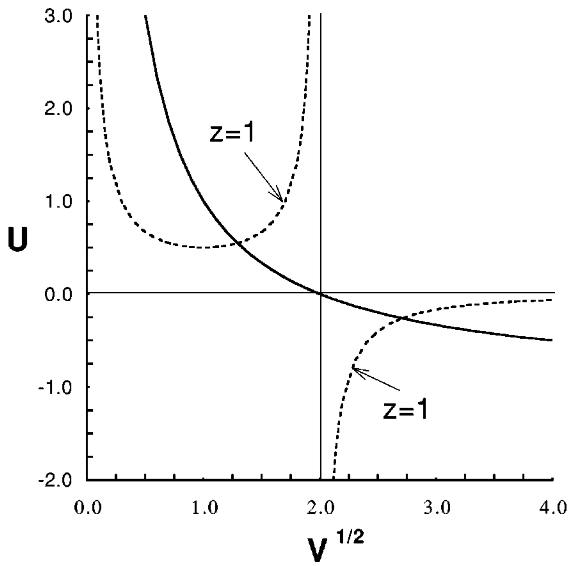

FIG. 1. The U-$V^{1/2}$ relation is shown. The semiclassical region is bounded by the dashed line. The event horizon ($U=0$) and the apparent horizon ($V^{1/2}=2Gm$) coincide on the dBB trajectory. The natural units $c=\hbar=G=1$ and $v_0=2$ are taken.

$$H^{(2)}\frac{\partial}{\partial z}H^{(1)}-H^{(1)}\frac{\partial}{\partial z}H^{(2)}=\frac{4i}{\pi z}. \tag{4.16}$$

Noticing that the phase S comes only from the Hankel function and that the derivative of the phase can be expressed by the identity equation (4.16), a couple of equations on the velocities (4.11) and (4.12) are obtained as

$$\dot{Z}_-=\frac{2\hbar G}{\pi v_0}\frac{1}{Z_+|H_\nu^{(2)}(Z)|^2}, \tag{4.17}$$

$$\dot{Z}_+=\frac{2\hbar G}{\pi v_0}\frac{1}{(Z_--r_g)|H_\nu^{(2)}(Z)|^2}. \tag{4.18}$$

We take the ratio of Eq. (4.17) to Eq. (4.18) to cancel the time dependence, and integrate it to get the Z_+-Z_- relation:

$$Z_+=c_0(Z_--r_g), \tag{4.19}$$

where c_0 is the integration constant. With the choice of $c_0=-1$ this relation is translated back to that of the original metric variables U,V in Eq. (2.4):

$$U=-\left(1-\frac{r_g}{V^{1/2}}\right), \tag{4.20}$$

which corresponds to the classical relation in Eq. (2.5). The U-$V^{1/2}$ relation is shown in Fig. 1, where the boundary of the classical region is also shown. We take the natural units $c=\hbar=G=1$ and $v_0=2$ in this and the following figures.

Using the U-$V^{1/2}$ relation (4.17), the remaining independent T-$V^{1/2}$ relation is obtained in the integral form:

$$T=\frac{\pi}{2}\int Z|H_\nu^{(2)}(Z)|^2dV^{1/2} \quad \text{with} \quad Z=\frac{v_0}{G\hbar}|V^{1/2}-r_g|. \tag{4.21}$$

Considering the asymptotic form of the Hankel function

$$H_\nu^{(2)}(Z)\to\exp(-iZ)\sqrt{\frac{2}{\pi Z}} \quad \text{for} \quad Z\gg 1, \tag{4.22}$$

we can show that the T-$V^{1/2}$ relation approaches the classical relation $T=V^{1/2}$ in the semiclassical region. Near the horizon $Z\approx 0$, the Hankel function is approximately expressed as

$$|H_\nu^{(2)}(Z)|^2\approx\begin{cases}\dfrac{4}{\pi^2}(\ln Z)^2 & \text{for} \quad \nu=0,\\[2ex] \dfrac{2^{2\nu}(\nu-1)!^2}{\pi^2Z^{2\nu}} & \text{for} \quad \nu=\text{positive integer}.\end{cases} \tag{4.23}$$

In the following, we discuss the case of $\nu=0$ and $\nu=1$. In the case for $\nu>1$ the behavior of the physical quantities is similar to the case for $\nu=1$. We estimate the T-$V^{1/2}$ relation (4.21) near the horizon using the approximate equation (4.23) as

$$T-T_0\approx\begin{cases}-\dfrac{v_0}{\pi G\hbar}(V^{1/2}-r_g)^2(\ln|V^{1/2}-r_g|)^2 & \text{for} \quad \nu=0,\\[2ex] -\dfrac{2G\hbar}{\pi v_0}\ln|V^{1/2}-r_g| & \text{for} \quad \nu=1,\end{cases} \tag{4.24}$$

where T_0 is the integration constant. From these expressions, we can see that T shows flat behavior for $\nu=0$ and steep behavior for $\nu=1$ near the horizon $V^{1/2}\approx r_g$. In Figs. 2(a) and 2(b), numerical estimations on the T-$V^{1/2}$ relation (4.21) are shown. Figure 2 implies that the net region of the horizon extends widely for $\nu=1$ whereas some part of the horizon region is cut off for $\nu=0$, in comparison with the classical relation $V^{1/2}=T$. We note that the inner metric for $\nu=1$ cannot connect the outside metric as T diverges near the horizon. In Fig. 2(b), the double wavy mark denotes this discontinuity between the inside and the outside. The U-T relation can be obtained from the $U-V^{1/2}$ relation in Fig. 1 by combining the T-$V^{1/2}$ relation in Fig. 2.

We next estimate the quantum potential Q, Eq. (4.14). Using the asymptotic behavior of the Hankel function in Eq. (4.22), the quantum potential becomes zero for $Z\gg 1$. The nonzero structure, which shows the quantum effects, exists only near the horizon $Z\approx 0$. The quantum potential Eq. (4.14) is estimated approximately near the horizon, using the relation (4.23), as

$$Q\approx\begin{cases}\dfrac{\pi^2 v_0}{8G}\dfrac{1}{Z^2(\ln Z)^4} & \text{for} \quad \nu=0,\\[2ex] -\dfrac{v_0}{2G} & \text{for} \quad \nu=1.\end{cases} \tag{4.25}$$

The quantum potential diverges positively for $\nu=0$ and is negative finite for $\nu=1$. In Figs. 3(a) and 3(b), the graphical representation of the quantum potential is shown. The quantum effect behaves very differently for each value of the index of the operator ordering $\nu=0$ and $\nu=1$.

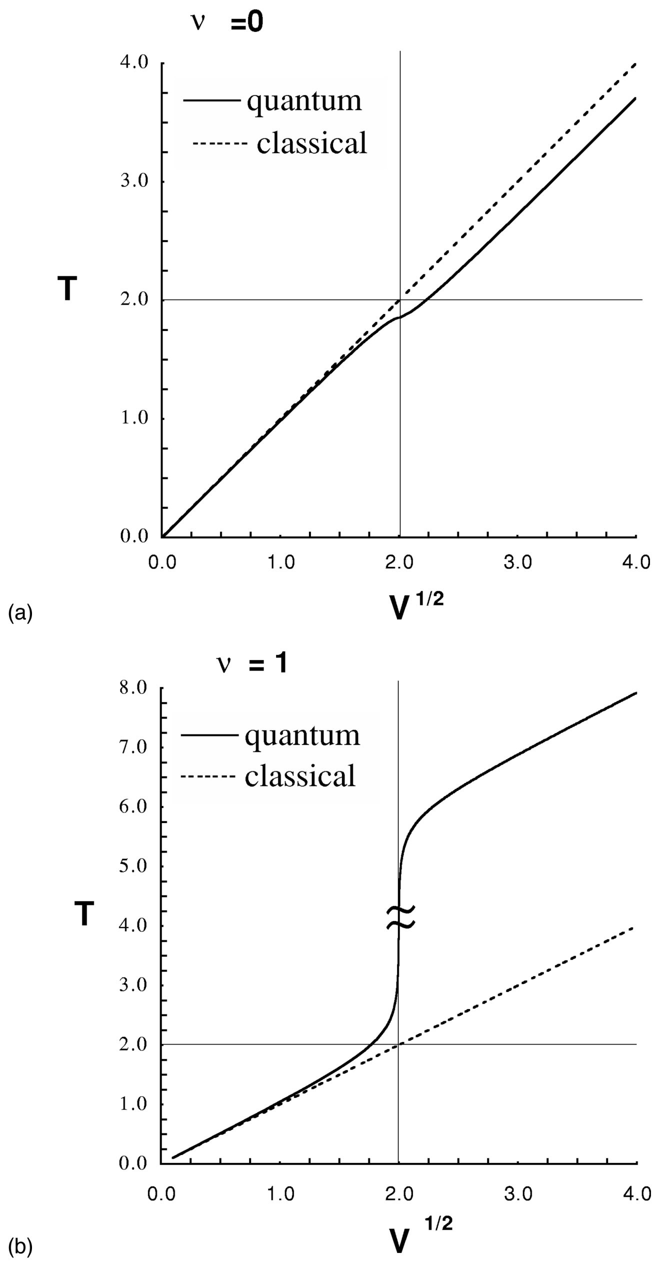

FIG. 2. The T-$V^{1/2}$ relation is shown. The ordering parameter is taken $\nu=0$ in (a) and $\nu=1$ in (b). The classical relation $T=V^{1/2}$ is denoted by the dashed line. The double wavy mark in (b) indicates the discontinuity between the inside and the outside of the horizon. The axis unit is the same as in Fig. 1.

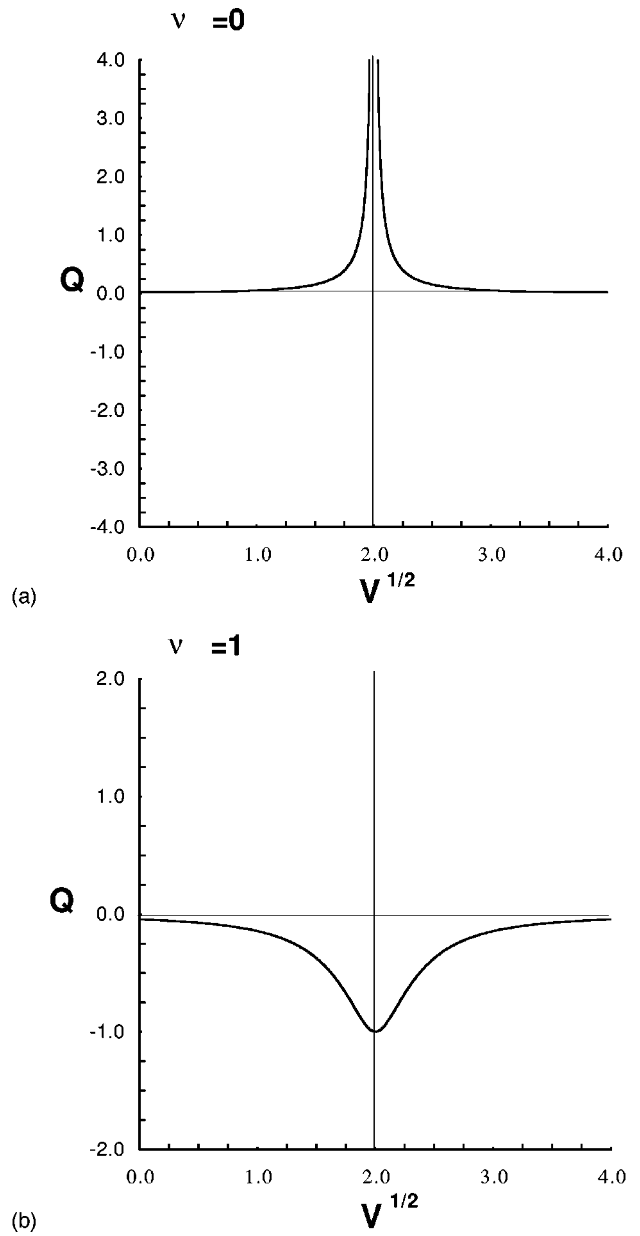

FIG. 3. The quantum potential Q is shown with respect to the variable $V^{1/2}$. The ordering parameter is taken; $\nu=0$ in (a) and $\nu=1$ in (b). The large positive effect is apparent for $\nu=0$ and the negative constant effect is apparent for $\nu=1$ near the horizon. The axis unit is the same as in Fig. 1.

Now we consider the horizons. The event horizon, which has the global meaning, is located at the null surface $U=0$. On the other hand, the apparent horizon, which has the local meaning, and is defined by the expansion for outgoing null rays becomes zero: $\theta_+=0$. The product of the expansion for outgoing and incoming null rays can be written as [23,31]

$$\theta_-\theta_+=UV^{-1}(\dot{\sqrt{V}})^2=\frac{1}{V}\left(1-\frac{2G\hat{M}}{V^{1/2}}\right). \qquad (4.26)$$

In the classical theory, the apparent horizon takes on $V^{1/2}=2Gm$ and agrees with the event horizon through the relation $U=-V^{1/2}+2Gm$. In quantum theory, $\theta_-\theta_+$ is an operator, and the quantum fluctuation becomes very large near $V^{1/2}=2Gm$. Then the estimation of the apparent horizon is not straightforward. Following by NKOT [23], we require the apparent horizon in the quantum theory as

$$\theta_-\theta_+\Psi=0, \qquad (4.27)$$

where Ψ denotes the simultaneous solution of the WD equation and the mass eigenstate equation. By this requirement the mass operator $\hat{M}$ in Eq. (4.26) is reduced to the eigen value m and the quantum apparent horizon takes on the classical value $V^{1/2}=2Gm$. Then the relation U-$V^{1/2}$, Eq. (4.20) means that the two horizons, the event horizon and the apparent horizon, coincide from the view point of the dBB interpretation.

In order to understand the property of the horizon, we consider the motion of the light ray on the geometry of the quantum black holes. In principle, the light ray as well as the black hole geometry should be treated in the quantum theory. It is because the massless particle curves the spacetime where it propagates, while it propagates in the curved spacetime. Instead of this, we treat the light ray as a test particle in order to probe the quantum property of the geometry. It means that we consider the low energy limit of the light ray and the quantum geometry does not take the influence from this test light ray.

The null condition for the metric in Eq. (2.4) becomes

$$\frac{dR}{dT} = \pm\frac{1}{U} \quad \text{with} \quad U = \frac{r_g}{V^{1/2}} - 1. \tag{4.28}$$

We take the variable $V^{1/2}$ instead of T as a independent variable, where the T-$V^{1/2}$ relation is seen in Eq. (4.21). Then we obtain the integral expression for the light ray as

$$R = \pm \int \frac{1}{U}\left(\frac{dV^{1/2}}{dT}\right)^{-1} dV^{1/2}$$

$$= \pm \frac{\pi v_0}{2G\hbar}\int V^{1/2} |H^{(2)}(Z)|^2 dV^{1/2}$$

$$\text{with} \quad Z = \frac{v_0}{G\hbar}(V^{1/2} - r_g). \tag{4.29}$$

The approximate behavior near the horizon $V^{1/2} \simeq r_g$ of this equation is estimated using Eq. (4.23) as

$$R - R_0 \simeq \begin{cases} \mp \dfrac{v_0}{\pi G\hbar}(V - r_g^2) \ (\ln|V_{1/2} - r_g|)^2 & \text{for} \quad \nu = 0, \\ \mp \dfrac{G^2\hbar m}{4\pi v_0} \dfrac{1}{V^{1/2} - r_g} & \text{for} \quad \nu = 1, \end{cases} \tag{4.30}$$

where R_0 denotes the integration constant. In Fig. 4, the numerical estimations of the R-$V^{1/2}$ relation for the light ray (4.29) are shown. For the comparison, the light ray in the classical case

$$R_{\text{cl}} = -T - 2\ln|V^{1/2} - r_g| + 2\ln r_g \quad \text{and} \quad V^{1/2} = T \tag{4.31}$$

is also shown in the figures. The integration constant is fixed so that the light ray on the dBB trajectory and that on the classical trajectory coincide at the origin ($T=0$) and the infinity ($T=\infty$). The classical ray is modified near the horizon region. In case of $\nu=0$ the light ray forms a cusp of a finite height for R at the horizon position $V^{1/2}=2Gm$. In the case of $\nu=1$ though the light ray behaves like that in the classical case, it diverges as the inverse power $(V^{1/2}-r_g)^{-1}$ near the horizon instead of the logarithm in the classical case. In this case the horizon region is just expanded.

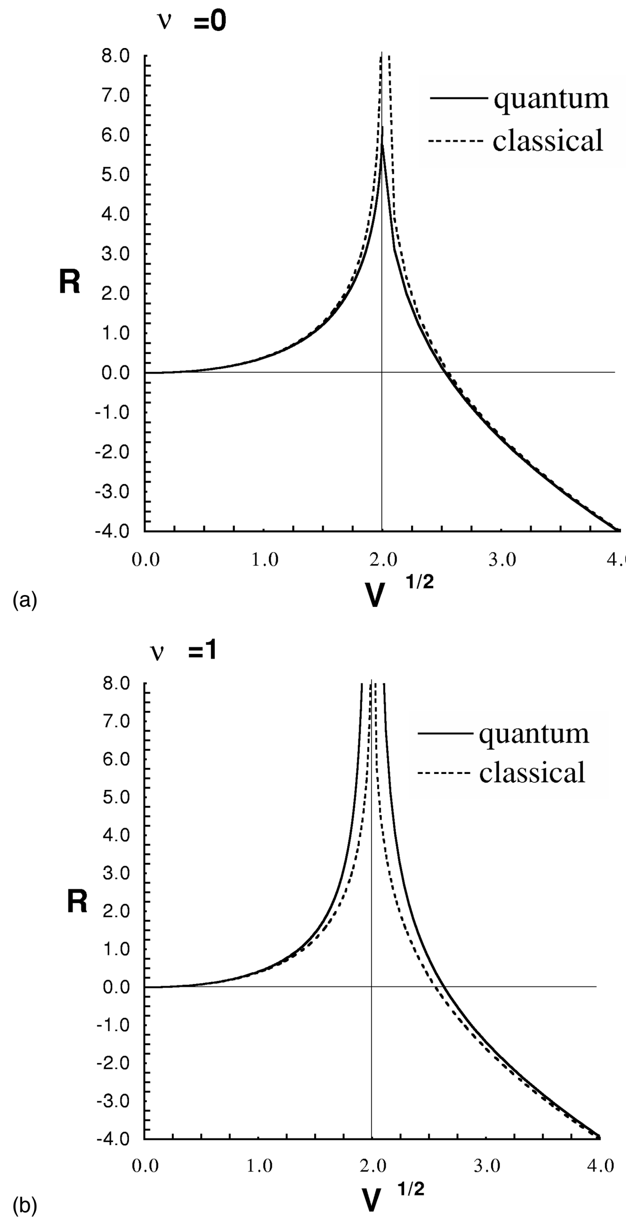

FIG. 4. The light ray on the dBB trajectory is shown. The light ray on the classical geometry is indicated by the dashed line. The ordering parameter is taken; $\nu=0$ in (a) and $\nu=1$ in (b). The light ray forms a cusp and reaches the horizon at finite R in the case of $\nu=0$. The light ray in the case of $\nu=1$ behaves like that in the classical case. The axis unit is the same as in Fig. 1.

V. SUMMARY AND DISCUSSION

We have studied the dBB interpretation for the quantum theory of the black hole geometry. We have estimated the dBB trajectories and the quantum potential. The dBB trajectories have been obtained from the phase and the quantum potential from the amplitude of the wave function. By considering both quantities, we can get the total picture of the quantum black holes.

Here we compare the dBB interpretation with the WKB approach. The WKB approach covers the semiclassical region. However, by means of the dBB interpretation we can define the dBB trajectories not only in the classical region but also in the quantum region and get the global picture through the dBB trajectories. The quantum effect can be estimated quantitatively and continuously from the quantum potential and from the difference between the dBB trajectories and the classical ones. In the case that the quantum potential is negligible, the trajectories in the dBB interpretation agree with those in the WKB approach, but its interpretation is different. For example, wave functions are peaked about strong correlation between coordinates and momenta along the trajectories in the WKB approach discussed by Halliwell [6] and Kadama [32]. On the other hand, the trajectories in the dBB interpretation have the causal meaning and therefore approach the classical ones without the measurement process.

In the region where the quantum effects cannot be negligible, any probe which plays the role of the observer in Copenhagen's interpretation may affect the dBB trajectories. We have considered the motion of the light ray on quantum geometry in order to study the quantum effect, which becomes very large near the horizon. In the evaluation in Figs. 4(a) and 4(b), we treat the light ray as a test particle, which means that it does not affect the geometry. In principle, the light ray as well as the black hole geometry should be treated in the quantum theory, because the light ray curves the spacetime which it propagates. In the Appendix, a model for the quantum theory of the massless particle and black hole geometry is considered and its connection to the calculation of the light ray in Sec. IV is studied.

We concluded that the classical U-$V^{1/2}$ relation (4.20) holds also in the quantum case on the dBB trajectory. Then the event horizon and the apparent horizon which is defined in Eq. (4.27) coincide. The reason why the classical and the quantum U-$V^{1/2}$ relation become the same is that two constraints, the Hamiltonian constraint and the mass constraint, are imposed on the wave function and then the phase factor S of the wave function becomes the function of the variable z in Eq. (3.12). The ratio of the two equations defining the momenta of the dBB trajectories, Eqs. (4.11) and (4.12), can be calculated unambiguously to derive the U-$V^{1/2}$ relation. The possibility of the inequality of the event horizon to the apparent horizon was discussed by NKOT [23]. Their main interest lay in the tunneling effect from the outside to the inside of the horizon. They argued that if the apparent horizon does not coincide with the event horizon, the U-$V^{1/2}$ curve outside the horizon can be connected to that inside the horizon through the tunneling of the forbidden region.

In our analysis, the quantum fluctuation largely depends upon the ordering parameter. In case of the ordering parameter $\nu \geqslant 1$, the net horizon region is enlarged to infinity as seen in Fig. 2(b) and the causal connection between the inside and the outside of the horizon is cut off. Its mathematical reason is the strong singular behavior of the Hankel function at the $z=0$. On the other hand, in case of $\nu=0$, the light ray on the quantum geometry of the black hole forms a cusp and reaches the horizon region within the finite R, which plays the role of time originally. So the $\nu=0$ case is especially interesting. We also note that if the Hermiticity is required for the quantum operators such as the Hamiltonian and the mass operator, even though the Hilbert space of quantum gravity is not well understood yet, the relation among the ordering indexes $\nu=p-q=0$ is preferred and then the quantum solution (3.16) is expressed by the Hankel function with the order $\nu=0$.

Concerning our work, an interesting phenomenon is on the Hawking radiation [17]. As we have obtained the quantum geometry of the black hole, it will give the effects on the scattering or the radiation of the electromagnetic and matter fields. Using the Vaidya metric black hole radiation in quantum geometry was recently discussed by Tomimatsu [19] and Hosoya and Oda [20]. The black hole radiation in our approach remains a future problem.

APPENDIX: MODEL FOR MASSLESS PARTICLE

We consider a quantum model of the massless particle and black hole geometry. The action consists of two parts: the gravity part Eq. (2.6) and the massless particle action S_M, which is

$$S_M=\int d^4x\sqrt{-{}^{(4)}g}\frac{1}{8\pi}{}^{(4)}g^{\mu\nu(4)}g_{\sigma\rho}(\partial_\mu X^\sigma)(\partial_\nu X^\rho), \tag{A1}$$

where X denotes the coordinate of the massless particle, which are assumed to take only the time and the space components as functions of T,

$$X^\sigma=(X^0(T)/\alpha,X^1(T),0,0). \tag{A2}$$

We substitute the metrics Eq. (2.4) into the matter action (A1) and change the variables to symmetric ones z_+,z_-, Eq. (2.10). Then the matter action becomes

$$S_M=\int dt dr\frac{V}{2\alpha}[(\dot X^0)^2-U^2(\dot X^1)^2]. \tag{A3}$$

The canonical momenta conjugate to X^0 and X^1 are calculated as

$$P_0=v_0\frac{V}{\alpha}\dot X^0, \quad P_1=-v_0\frac{VU^2}{\alpha}\dot X^1. \tag{A4}$$

The massless particle Hamiltonian H_M is obtained as

$$H_M=\frac{\alpha}{2v_0}\left(\frac{1}{z_-^2}\;P_0^2-\frac{1}{z_+^2}P_1^2\right). \tag{A5}$$

Again the total Hamiltonian constraint is obtained:

$$H+H_M\approx 0 \quad \text{with} \quad \alpha=1. \tag{A6}$$

After the canonical quantization $P_0=-i\hbar\partial_0$ and $P_1=-i\hbar\partial_1$, we obtain the WD equation for the wave function $\Psi(z_+,z_-,X^0,X^1)$ as

$$(\hat H+\hat H_M)\Psi(z_+,z_-,X^0,X^1)=0. \tag{A7}$$

By solving this WD equation we get the total quantum picture of the massless particle and the black hole geometry. Now we investigate the relation of this equation to the analysis which has been done in Sec. IV. First we make the approximation that the wave function is assumed to be of the form in the separation of the variables

$$\Psi(z_+,z_-,X^0,X^1)\simeq\Psi(z_+,z_-)\Phi(X^0,X^1). \quad \text{(A8)}$$

In this approximation, the wave functions $\Psi(z_+,z_-)$ and $\Phi(X^0,X^1)$ satisfy the WD equation separately, which are Eq. (3.4) and

$$-\frac{\hbar^2}{2v_0 z_-}\left(\partial_0^2-\frac{z_-^2}{z_+^2}\partial_1^2\right)\Phi(X^0,X^1)=0, \quad \text{(A9)}$$

where z_+ and z_- are assumed to take values of the dBB trajectories: Eqs. (4.11) and (4.12). We further make the WKB approximation for the phase of the wave function $\Phi(X^0,X^1)\simeq\exp(iS_M/\hbar)$ and identify its derivative with the momenta (A4), and get the equation for the massless particle as

$$\frac{dX^1}{dX^0}=\frac{z_-}{z_+}=\frac{1}{U}. \quad \text{(A10)}$$

Identifying $X^0\equiv\lambda T$ and $X^1\equiv\lambda R$, where λ is introduced as a scaling parameter, we obtain the light ray equation (4.28) in Sec. IV. In this approximation, the massless particle and the geometry do not influence each other. If we improve the approximation to include the correlation, the mass operator cannot take constant value. The change of the mass operator is expressed through the commutation relation between the mass operator and the Hamiltonian as

$$\dot{\hat{M}}=\frac{1}{i\hbar}[\hat{H}+\Delta\hat{H},\hat{M}]$$

$$=\frac{-2G}{v_0^2}\hat{\Pi}_+\hat{H}+\frac{2G}{v_0^2}\left(\Pi_+H_M+\frac{1}{v_0}(z_+^{-2}\Pi_+\Pi_+z_+^{-2})P_1^2\right),$$

$$\approx\frac{2G}{v_0^2}\left(\Pi_+H_M+\frac{1}{v_0}(z_+^{-2}\Pi_+\Pi_+z_+^{-2})P_1^2\right), \quad \text{(A11)}$$

where we take the Weyl operator ordering. This equation shows that the mass function changes according to the interaction of the masslass particle and the geometry. The analysis of this model remains our future problem.

---

[1] J. A. Wheeler, in *Batelle Rencontres: 1967 Lectures in Mathematics and Physics*, edited by C. DeWitt and J. A. Wheeler (Benjamin, New York, 1968).

[2] B. S. Dewitt, Phys. Rev. **160**, 1113 (1967).

[3] C. J. Isham, ''Canonical quantum gravity and the problem of time,'' Report Imperial/TP/91-92/25, gr-qc/9210011, 1992.

[4] K. V. Kuchař, ''Canonical quantum gravity,'' e-print gr-qc/9304012, 1993.

[5] T. Banks, Nucl. Phys. **B249**, 332 (1985).

[6] J. J. Halliwell, Phys. Rev. D **36**, 3626 (1987).

[7] T. P. Singh and T. Padmamabhan, Ann. Phys. (N.Y.) **196**, 296 (1989).

[8] S. P. Alwis and D. A. MacIntire, Phys. Rev. D **50**, 5164 (1994).

[9] M. Kenmoku and E. Takasugi (in preparation).

[10] L. de Broglie, *Tentative d'interpretation causale et nonlineaire de la mechanique ondulaire* (Gauthier-Villars, Paris, 1956).

[11] D. Bohm, Phys. Rev. **85**, 166 (1952); **85**, 180 (1952).

[12] J. S. Bell, *Speakable and Unspeakable in Quantum Mechanics* (Cambridge University Press, Cambridge, England, 1987).

[13] P. R. Holland, *The Quantum Theory Of Motion* (Cambridge University Press, Cambridge, England, 1993).

[14] J. C. Vink, Nucl. Phys. **B369**, 707 (1992).

[15] T. Horiguchi, Mod. Phys. Lett. A **9**, 1429 (1994).

[16] M. Kenmoku, K. Otsuki, K. Shigemoto, and K. Uehara, Class. Quantum Grav. **13**, 1751 (1996).

[17] S. W. Hawking, Commun. Math. Phys. **43**, 199 (1975).

[18] P. Thomi, B. Isaak, and P. Hajicek, Phys. Rev. D **30**, 1168 (1984); P. Hajicek, *ibid.* **30**, 1178 (1984).

[19] A. Tomimatsu, Phys. Lett. B **289**, 283 (1992).

[20] A. Hosoya, and I. Oda, Prog. Theor. Phys. **97**, 233 (1997).

[21] A. Hosoya, Class. Quantum Grav. **12**, 2967 (1995).

[22] Y. Nambu and M. Sasaki, Prog. Theor. Phys. **79**, 96 (1988).

[23] K. Nakamura, S. Konno, Y. Oshiro, and A. Tomimatsu, Prog. Theor. Phys. **90**, 861 (1993).

[24] S. Weinberg, *Gravitation and Cosmology* (Wiley, New York, 1972), Sec. II, pp. 335.

[25] W. Fischler, D. Morgan, and J. Polchinski, Phys. Rev. D **42**, 4042 (1990).

[26] K. V. Kuchař, Phys. Rev. D **50**, 3961 (1994).

[27] P. A. M. Dirac, *Lectures on Quantum Mechanics* (Yeshiva University, New York, 1964).

[28] R. Arnowitt, S. Deser, and C. W. Misner, in *Gravitation: An Introduction to Current Research*, edited by L. Witten (Wiley, New York, 1962).

[29] H. Suzuki, E. Takasugi, and Y. Takayama, Mod. Phys. Lett. A **11**, 1281 (1996).

[30] Precise correspondence between our solution and NKOT's solution is the following. The definition of the momentum in Eq. (3.2) can be generalized to

$$\hat{\Pi}_{-f}\equiv f(z_-)\hat{\Pi}_- f(z_-)^{-1}=\hat{\Pi}_-+\frac{i\hbar f'}{f},$$

where f is an arbitrary function of z_-. We get the wave function as

$$\Psi=f(z_-)\ y^p z^q[c_1' H_{p-q}^{(1)}(z)+c_2' H_{p-q}^{(2)}(z)].$$

If we set $f=z_-^{q-p}$, $q=0$, and $c_2'=0$, our solution coincides with NKOT's one.

[31] James W. York, Jr., Phys. Rev. D **28**, 2929 (1983).

[32] H. Kodama, report KUCP-0014 (1988).