Quantum correction to the entropy of the (2+1)-dimensional black hole

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The thermodynamics of the (2+1)-dimensional nonrotating black hole of Bañados, Teitelboim, and Zanelli is discussed. The first quantum correction to the Bekenstein-Hawking entropy is evaluated within the on-shell Euclidean formalism, making use of the related Chern-Simons representation of three-dimensional gravity. Horizon and ultraviolet divergences in the quantum correction are dealt with by a renormalization of the Newton constant. It is argued that the quantum correction due to the gravitational field shrinks the effective radius of a hole and becomes more and more important as the evaporation process goes on, while the area law is not violated. [S0556-2821(98)00908-4]

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I. INTRODUCTION

It is well known that we do not have yet at our disposal a consistent and complete four-dimensional quantum gravity; nevertheless, a large number of interesting issues have been investigated, mainly within the semiclassical approximation. One of the most important issues is related to black hole physics and deals with the origin of entropy, its quantum corrections, the information loss paradox, and the validity of the area law (see, for example, Ref. [1]). However, it is well known that in 3+1 dimensions black hole quantum physics needs several approximations.

Recently, three-dimensional gravity has been studied in detail. Despite the simplicity of the three-dimensional case (no propagating gravitons), it is a common belief that it deserves attention as a useful laboratory. In fact, surprisingly, a black hole solution has been found by Bañados, Teitelboim, and Zanelli [2]. In particular, the simple geometrical structure of this black hole allows exact computations since its Euclidean counterpart is locally isomorphic to the constant-curvature three-dimensional hyperbolic space H^3.

In this paper, we shall compute a quantum correction to the semiclassical Bekenstein-Hawking entropy for the Bañados-Teitelboim-Zanelli (BTZ) black hole due to the one-loop gravitational fluctuations in an attempt to elucidate the statistical origin of black hole entropy [3–6] and to explore the possible relevance of quantum fluctuations during the late stages of the black hole evaporation process.

With regard to these issues, we recall that many papers have appeared in which the quantum entropy of matter fields, propagating in a black hole background, has been evaluated by means of several different techniques (see, for example, Refs. [7–17] and reference therein). We would like to stress here that we shall compute the one-loop contribution due to the quantization of the gravitational field itself. The tree-level approximation to the partition function has been discussed at length in [18], using Brown and York's approach to quasilocal thermodynamics for asymptotically anti–de Sitter black holes. It is found that in 2+1 dimensions there is a thermodynamically stable black hole solution and no negative heat capacity instantons. Thus one expects the quantum corrections to be well defined.

As far as the computation of these corrections is concerned, some work has been done in [19,20] and a motivation of our paper is to present a detailed and possibly complete discussion on this point. The quantum correction of the BTZ black hole will be evaluated by making use of the related Chern-Simons representation of three-dimensional gravity [21,22]. It should be stressed that within this approach a preliminary statistical mechanics explanation of the Bekenstein-Hawking entropy, counting boundary states at the horizon, has been given in Ref. [23].

The organization of the paper is as follows. In Sec. II we briefly review the geometry of the Euclidean BTZ black hole. In Sec. III we present a derivation of the Selberg trace formula, starting from an elementary derivation of the heat-kernel trace related to the Laplace operator, which is necessary for our regularization. In Sec. IV the computation of the quantum correction to the entropy is outlined. The paper ends with some concluding remarks in Sec. V. In the Appendix some explicit computations are included.

II. THE EUCLIDEAN BTZ BLACK HOLE

Following [19] we summarize here the geometrical aspects of the nonrotating BTZ black hole [2] that are relevant for our discussion. In the coordinates (t,r,ϕ), the static Lorentzian metric reads ($8G=1$ is assumed for the moment, thus the mass is dimensionless)

$$ds_L^2=-\left(\frac{r^2}{\sigma^2}-M\right)dt^2+\left(\frac{r^2}{\sigma^2}-M\right)^{-1}dr^2+r^2d\phi^2, \quad (1)$$

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where M is the standard Arnowitt-Deser-Misner mass and σ is a dimensional constant. A direct calculation shows that the above metric is a solution of the three-dimensional vacuum Einstein equation with a negative cosmological constant, i.e.,

$$R_{\mu\nu}=2\Lambda g_{\mu\nu}, \quad R=6\Lambda=-\frac{6}{\sigma^2}. \tag{2}$$

Thus the sectional curvature k is constant and negative, namely, $k=\Lambda=-1/\sigma^2$. The metric (1) has a horizon radius given by

$$r_+=\sqrt{M}\sigma \tag{3}$$

and describes a space-time locally isometric to the anti–de Sitter space.

The Euclidean section is obtained by the Wick rotation $t \to i\tau$ and reads

$$ds^2=\left(\frac{r^2}{\sigma^2}-M\right)d\tau^2+\left(\frac{r^2}{\sigma^2}-M\right)^{-1}dr^2+r^2d\phi^2. \tag{4}$$

Changing the coordinates $(\tau,r,\phi)\to(y,x_1,x_2)$ by means of

$$y=\frac{r_+}{r}e^{(r_+/\sigma)\,\phi},$$

$$x_1+ix_2=\frac{1}{r}\sqrt{r^2-r_+^2}\exp\left(i\frac{r_+}{\sigma^2}\tau+\frac{r_+}{\sigma}\phi\right), \tag{5}$$

the metric becomes the one of the upper-half space representation of H^3, i.e.,

$$ds^2=\frac{\sigma^2}{y^2}(d^2y+dx_1^2+dx_2^2). \tag{6}$$

As a consequence, the metric (4) describes a manifold locally isometric to the hyperbolic space H^3.

It is known that the group of isometries of H^3 is $SL(2,\mathbb{C})$. We shall consider a discrete subgroup $\Gamma\subset PSL(2,\mathbb{C}) \equiv SL(2,\mathbb{C})/\{\pm I\}$ (I is the identity element), which acts discontinuously at the point z belonging to the extended complex plane $\mathbb{C}\cup\{\infty\}$. We recall that a transformation $\gamma\neq I$, $\gamma\in\Gamma$, with

$$\gamma z=\frac{az+b}{cz+d}, \quad ad-bc=1, \quad a,b,c,d\in\mathbb{C}, \tag{7}$$

is called elliptic if $(\mathrm{Tr}\gamma)^2=(a+d)^2$ satisfies $0\leqslant(\mathrm{Tr}\gamma)^2<4$, hyperbolic if $(\mathrm{Tr}\gamma)^2>4$, parabolic if $(\mathrm{Tr}\gamma)^2=4$, and loxodromic if $(\mathrm{Tr}\gamma)^2\in\mathbb{C}\setminus[0,4]$. The element $\gamma\in SL(2,\mathbb{C})$ acts on $z=(y,w)\in H^3$, $w=x_1+ix_2$, by means of the linear-fractional transformation

$$\gamma z=\left(\frac{y}{|cw+d|^2+|c|^2y^2},\frac{(aw+b)(\overline{cw}+\bar{d})+a\bar{c}y^2}{|cw+d|^2+|c|^2y^2}\right). \tag{8}$$

The periodicity of the angular coordinate ϕ allows one to describe the BTZ black hole manifold as the quotient $\mathcal{H}^3\equiv H^3/\Gamma$, Γ being a discrete group of isometry possessing a primitive element $\gamma_h\in\Gamma$ defined by the identification

$$\gamma_h(y,w)=(e^{2\pi r_+/\sigma}y,e^{2\pi r_+/\sigma}w)\sim(y,w). \tag{9}$$

According to Eq. (8), this corresponds to the matrix

$$\gamma_h=\begin{pmatrix} e^{\pi r_+/\sigma} & 0 \\ 0 & e^{-\pi r_+/\sigma}\end{pmatrix}, \tag{10}$$

namely, to a hyperbolic element $(\mathrm{tr}\gamma_h>2)$ consisting in a pure dilatation. Furthermore, since in Euclidean coordinates τ becomes an angular-type variable with period β, one has the identification

$$\gamma_e(y,w)=(y,e^{i\beta r_+/\sigma^2}w)\sim(y,w). \tag{11}$$

This identification is generated by an elliptic element in the group Γ,

$$\gamma_e=\begin{pmatrix} e^{i\beta(r_+/2\sigma^2)} & 0 \\ 0 & e^{-i\beta(r_+/2\sigma^2)}\end{pmatrix}, \tag{12}$$

as soon as $(\mathrm{tr}\gamma_e)^2<4$, and a conical singularity will be present. However, if

$$\beta\frac{r_+}{2\sigma^2}=\pi, \tag{13}$$

then $\gamma_e\equiv I$ and the conical singularity is absent. As a result, the period is determined to be

$$\beta_H=2\pi\frac{\sigma^2}{r_+}, \tag{14}$$

which is interpreted as the inverse of the Hawking temperature [5]. Therefore, the on-shell BTZ black hole can be regarded as a strictly hyperbolic noncompact manifold $\mathcal{H}^3$. The mass, as a function of the black hole temperature $T=\beta_H^{-1}$, reads

$$M=4\pi^2\sigma^2T^2, \tag{15}$$

which shows that the stability condition $\partial M/\partial T>0$ is satisfied. The tree-level Bekenstein-Hawking entropy S_H may be simply obtained by making use of the relation

$$\beta_H=\frac{\partial S_H}{\partial M}. \tag{16}$$

Thus one has

$$S_H=4\pi r_+=2A. \tag{17}$$

Note that $A=2\pi r_+$ is the perimeter of the horizon. If we choose $G=1$ instead of $8G=1$, the entropy becomes $A/4$, which is the well-known ''area law'' for black hole entropy.

Another important thermodynamical quantity is the off-shell Euclidean action of a black hole, namely, the action evaluated at $\beta\neq\beta_H$ (see Ref. [18] for the quasilocal formalism of black-hole thermodynamics):

$$I=-\frac{1}{2\pi}\int_{\mathcal{M}}(\mathcal{R}-2\Lambda)\sqrt{g}d^3x-\frac{1}{\pi}\int_{\partial\mathcal{M}}\mathcal{K}\sqrt{h}d^2x. \quad (18)$$

The boundary $\partial\mathcal{M}=S^1\otimes S^1$ (a torus) is identified with period β (the first circle) at some fixed radius $r=R$ (the second circle), which will go to the infinity at the end, and $\mathcal{K}$ is the trace of the extrinsic curvature of the boundary. The Euclidean action (18) is a divergent function of the boundary location and therefore it is necessary to subtract the action of a chosen background [24]. This will be the related ground-state solution, i.e., the $M=0$ line element

$$ds_0^2=\frac{r^2}{\sigma^2}d\tau^2+\frac{\sigma^2}{r^2}dr^2+r^2d\phi^2, \quad (19)$$

which also corresponds to the zero-temperature state; all quantities referring to this reference background will be denoted by the subscript 0. As mentioned above, at $\beta\neq\beta_H$ there exists a conical singularity whose contribution to the action is given by the Gauss-Bonnet theorem for a disk [25], which reads

$$\frac{1}{2\pi}\int_{\mathcal{M}}\mathcal{R}\sqrt{g}d^3x=\frac{2A}{\beta_H}(\beta_H-\beta)+(\text{volume contribution}). \quad (20)$$

The conical singularity contribution of the background vanishes since $A_0=0$. The volume contribution of the difference of two actions can be computed by matching the coordinate of the boundary location in the background $r=R_0$ to the coordinate of the boundary location in the black hole metric $r=R$ in such a way that the two metrics asymptotically agree. Finally, the surface contribution is seen to vanish. Therefore, the off-shell Euclidean action becomes

$$I=-\frac{2A}{\beta_H}(\beta_H-\beta)-\frac{r_+^2\beta}{\sigma^2}=M\beta-2A, \quad (21)$$

where (M,β) are now independent thermodynamics variables and $r_+=\sigma\sqrt{M}$. On shell we have $\beta=\beta_H$ and $I=-2\pi r_+$. If one identifies I with $-\ln Z$ [5], the partition function of the black hole, then the mean energy in the canonical ensemble will be

$$\langle E\rangle=-\partial_\beta\ln Z=M, \quad (22)$$

as expected, and the entropy will be again $S=2A=4\pi r_+=4\pi\sigma\sqrt{M}$. Because $S\sim\sqrt{M}$, the partition function as a ''sum over states'' in semiclassical quantum $2+1$ gravity will converge and the canonical ensemble for a black hole in equilibrium with thermal radiation will lead to a stable thermodynamics.

We conclude this section with a comment on the global geometry of the ground state. Looking at Eq. (19), it is clear that τ can be identified to any period β (in particular $\beta=\infty$) and that ϕ has period 2π. Changing the coordinates as $r=\sigma^2/y$, $\tau=x_1$, and $\phi=x_2/\sigma$, one gets the metric of hyperbolic space

$$ds_0^2=\sigma^2\frac{dx_1^2+dx_2^2+dy^2}{y^2} \quad (23)$$

and the identification $\gamma_p(w,y)=(w+\beta+i2\pi\sigma,y)\simeq(w,y)$. This identification is generated by elements of Γ of the form

$$\gamma_{p_1}=\begin{pmatrix}1 & \beta\\ 0 & 1\end{pmatrix},\quad \gamma_{p_2}=\begin{pmatrix}1 & 2\pi i\sigma\\ 0 & 1\end{pmatrix}, \quad (24)$$

which are parabolic. Thus our reference manifold can be regarded as the quotient $\mathcal{H}_0=H^3/\Gamma_0$, where a subgroup Γ_0 has primitive parabolic elements γ_{p_1} and γ_{p_2}.

We note that for negative mass, one gets solutions with naked conical singularity [26] unless one arrives at $M=-1$, namely, H^3, the Euclidean counterpart of the three-dimensional anti–de Sitter space-time. This solution is a permissible solution and can be regarded as a ''bound state'' [2].

III. THE TRACE FORMULA AND SPECTRAL ζ FUNCTION

In this section we investigate the spectral properties associated with the Laplace-type operator acting in a noncompact hyperbolic manifold $\mathcal{H}^3$. To go further, it is convenient to introduce spherical hyperbolic coordinates

$$y=\rho\cos\theta,\quad w=x_1+ix_2=\rho\sin\theta e^{i\varphi}. \quad (25)$$

It is easy to show that the fundamental domain of $\mathcal{H}^3$ is noncompact and reads [19]

$$F=\{1\leqslant\rho\leqslant N,\ 0\leqslant\theta<\pi/2,\ 0<\varphi<2\pi\}, \quad (26)$$

where $\ln N=2\pi r_+/\sigma$. Note that $z'=\gamma_h z=Nz$ and the corresponding transformation law for a scalar field Φ reads $\Phi(\gamma z)=\chi\Phi(z),\gamma\in\Gamma$, where χ is a finite-dimensional unitary representation (a character) of Γ.

Let us consider an arbitrary integral operator, defined by a kernel $k(z,z')$. The operator is invariant [i.e., the operator commutes with all operators of the quasiregular representation of the group $PSL(2,\mathbb{C})$ in the space $C_0^\infty(H^3)$], if its kernel satisfies the condition $k(\gamma z,\gamma z')=k(z,z')$ for any $z,z'\in H^3$. Thus the kernel of the invariant operator, for example, the Laplace operator, is a function of the geodesic distance between z and z', namely,

$$d(z,z')=\cosh^{-1}\left[1+\frac{(y-y')^2+(x_1-x_1')^2+(x_2-x_2')^2}{2yy'}\right]. \quad (27)$$

In our case, the geodesic length between the point z and $z'=\gamma_h z$ is

$$l_0=\inf d(z,\gamma_h z)=\ln N=2\pi\frac{r_+}{\sigma}. \quad (28)$$

It is convenient to replace such a distance with the fundamental invariant

$$u(z,z')=\frac{1}{2}[\cosh d(z,z')-1], \quad u(z,z)=0, \tag{29}$$

and therefore $k(z,z')=k[u(z,z')]$. Finally, for the sake of simplicity, we set $\sigma=1$; thus $|k|=1/\sigma^2=1$ and all the quantities are dimensionless (the physical dimensions can be restored by dimensional analysis at the end of the calculations).

A. The heat-kernel trace formula

Let us start with the heat kernel of the Laplace operator acting in H^3. We shall use the method of images. The heat kernel reads (see, for example, Refs. [27] and [28])

$$K_t^{H^3}(z,z')=\frac{\exp(-t-d^2(z,z')/4t)}{(4\pi t)^{3/2}}\frac{d(z,z')}{\sinh d(z,z')}. \tag{30}$$

With regard to the heat kernel on $\mathcal{H}^3$, the method of images gives

$$K_t(z,z')=\sum_n \chi^n K_t^{H^3}(z,\gamma_h^n z')$$
$$=K_t^{H^3}(z,z')+\sum_{n\neq 0}\chi^n K_t^{H^3}(z,\gamma_h^n z')\chi^n, \tag{31}$$

where the separation between the identity and the nontrivial periodic geodesic contribution has been done. In our case, the volume $V(F_3)$ of the fundamental domain F_3 is divergent and we must introduce a regularization. The simplest one is to limit the integration in the variable θ between $0<\theta<\pi/2-\varepsilon$, with ε suitable. Thus we have

$$V_\varepsilon(F)=\int_1^N\frac{d\rho}{\rho}\int_0^{2\pi}d\varphi\int_0^{\pi/2-\varepsilon}\frac{\sin\theta}{(\cos\theta)^3}d\theta$$
$$=2\pi^2 r_+(\cot\varepsilon)^2=2\pi^2 r_+\left(\frac{1}{\varepsilon^2}-\frac{2}{3}+O(\varepsilon)\right). \tag{32}$$

We may determine ε choosing

$$\frac{1}{\varepsilon^2}=\frac{R^2}{r_+^2}-\frac{1}{3}. \tag{33}$$

Thus

$$V_R(F)=2\pi^2\frac{R^2}{r_+}-2\pi^2 r_+=\int_0^{\beta_H}d\tau\int_0^{2\pi}d\phi\int_{r_+}^R r\,dr, \tag{34}$$

where the cutoff parameter R has been introduced (see Sec. II for notation). The integration over the regularized (fundamental) domain of the diagonal part leads to

$$\mathrm{Tr}e^{-t\Delta_0}(R)\equiv \mathrm{Tr}K_t(R)$$
$$=V_R(F)\frac{e^{-t}}{(4\pi t)^{3/2}}+2\pi l_0 e^{-t}\sum_{n\neq 0}\frac{\chi^n}{(4\pi t)^{3/2}}$$
$$\times\int_0^{\pi/2}\frac{(\sin\theta)d(z,\gamma_h^n z)}{(\cos\theta)^3\sinh[d(z,\gamma_h^n z)]}e^{-d^2(z,\gamma_h^n z)/4t}d\theta, \tag{35}$$

where Δ_0 is a scalar Laplacian and $d(z,\gamma_h^n z)=\cosh^{-1}(1+b_n^2\cos^{-2}\theta)$. The integral over θ can be performed by changing the integration variable $\theta\to u$ given by $2\sqrt{ut}=\cosh^{-1}(1+b_n^2\cos^{-2}\theta)$. As a consequence, the resulting integral becomes elementary, i.e.,

$$\mathrm{Tr}K_t(R)=V_R(F)\frac{e^{-t}}{(4\pi t)^{3/2}}$$
$$+4\pi l_0\sum_{n=1}^\infty\frac{\chi^n e^{-t}}{b_n^2(4\pi t)^{3/2}}\int_{n^2l_0^2/4t}^\infty e^{-tu}du \tag{36}$$

since $\cosh^{-1}(1+b_n^2)=nl_0$. As a result, one obtains

$$\mathrm{Tr}K_t(R)=V_R(F)\frac{e^{-t}}{(4\pi t)^{3/2}}+\frac{l_0}{2}\sum_{n=1}^\infty\frac{\chi^n}{(\sinh nl_0/2)^2}\frac{e^{-t-l_0^2n^2/4t}}{(4\pi t)^{1/2}}. \tag{37}$$

The above heat-kernel trace has also been computed recently in Ref. [29].

B. The Selberg-like trace formula

In the preceding subsection we have derived the heat-kernel trace formula. For our purpose it is important that Eq. (37) looks (formally) like the Selberg trace formula associated with the Laplace operator acting in a compact space $\mathcal{H}^3$ (a group Γ is cocompact). This statement is formal enough; nevertheless, let us verify it through a common style of presentation.

First of all, we may consider a given (regularized) compact Riemannian manifold as conformally equivalent to one of constant scalar curvature. This is known as the Yamabe problem [30]. This problem has been solved for the case of nonpositive scalar curvature in Ref. [31]. Furthermore, let $\{\lambda_j\}_{j=0}^\infty$ denote the nonzero isolated eigenvalues (appearing the same number of times as its multiplicity) of the positive self-adjoint Laplace operator. Let us introduce a suitable analytic function $h(r)$, where $r_j^2=\lambda_j-1$. It can be shown that $h(r)$ is related to the quantity $k[u(z,\gamma z)]$ by means of the Selberg transform (see, for example, Refs. [32] and [28] and references therein). Let $\hat{h}(p)$ be the Fourier transform of $h(r)$,

$$\hat{h}(p)=\frac{1}{2\pi}\int_{-\infty}^\infty e^{-irp}h(r)dr. \tag{38}$$

For the derivation of the Selberg trace formula, one has to consider the contributions coming from the identity element in Γ and all γ-type conjugacy classes (the method of images), namely,

$$\mathrm{Tr}h(\Delta_0)=\sum_j h(\lambda_j)=\mathcal{C}(I)+\mathcal{C}(H)$$
$$=V(F_3)k(0)+\sum_{\{\gamma\}}\chi(\gamma)\int_{F_3}k[u(z,\gamma z)]d\mu_3. \tag{39}$$

The first term on the right-hand side of Eq. (39), $\mathcal{C}(I)$, is the contribution of the identity element, while $V(F_3)$ is the (finite) volume of the fundamental domain with respect to the Riemannian measure $d\mu_3=dx_1dx_2dyy^{-3}$. Formally, for the noncompact manifold $\mathcal{H}^3$, whose fundamental domain is given by Eq. (26), one may set $V(F_3)\sim V_R(F)$, where $V_R(F)$ is given by Eq. (34).

Let us consider now the hyperbolic (a topologically nontrivial) contribution and show that it is finite. First it reduces to

$$\mathcal{C}(H)=\sum_{\{\gamma\}}\chi(\gamma)\int_{F_3}k[u(z,\gamma z)]d\mu_3$$
$$=\sum_{n\neq 0}\chi^n\int_{F_3}k[u(z,\gamma_h^n z)]d\mu_3. \tag{40}$$

Noting that $\chi^n=\chi^{-n}$ and

$$u(z,\gamma_h^n z)=\frac{1}{2}[\cosh d(z,\gamma_h^n z)-1]=b_n^2(1+\tan^2\theta), \tag{41}$$

with $b_n^2=\sinh^2(nl_0/2)$, one has

$$\mathcal{C}(H)=4\pi l_0\sum_{n=1}^{\infty}\chi^n\int_0^{\pi/2}\frac{\sin\theta}{(\cos\theta)^3}k[b_n^2(1+\tan^2\theta)]d\theta$$
$$=2\sum_{n=1}^{\infty}\frac{\chi^n}{b_n^2}\int_{b_n^2}^{\infty}k(x)dx. \tag{42}$$

Recalling the Selberg transform in the three-dimensional case [32,28] one gets

$$k(0)=\int_0^\infty\frac{r^2}{2\pi^2}h(r)dr,$$
$$\int_{b_n^2}^{\infty}k(x)dx=\frac{1}{4\pi}\hat{h}(n2r_+). \tag{43}$$

Thus the final trace formula reads

$$\mathrm{Tr}h(\Delta_0)(R)=V_R(F)\int_0^\infty\frac{r^2}{2\pi^2}h(r)dr$$
$$+l_0\sum_{n=1}^{\infty}\chi^n\frac{\hat{h}(n2r_+)}{(\sinh nl_0/2)^2}. \tag{44}$$

The trace formula (44) is valid for a large class of $h(r)$ functions. In particular, choosing $h(r)=e^{-t(r^2+1)}$ in Eq. (44), one obtains the result of Eq. (37).

Finally, the related ζ function can be calculated by means of the Mellin transform

$$\zeta(s|\Delta_0)(R)=\frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\mathrm{Tr}K_t(R)dt, \tag{45}$$

valid for Re$s>3/2$. A direct computation gives the analytic continuation of the ζ function in the neighborhood of the point $s=0$, i.e.,

$$\zeta(s|\Delta_0)(R)=V_R(F)\frac{\Gamma(s-3/2)}{(4\pi)^{3/2}\Gamma(s)}+\frac{l_0}{\Gamma(s)\Gamma(1-s)}$$
$$\times\int_0^\infty(2t+t^2)^{-s}\Psi(2+t)dt, \tag{46}$$

where the function

$$\Psi(s)=\sum_{n=1}^{\infty}\frac{\chi^n}{(\sinh nl_0/2)^2}e^{-(s-1)l_0 n} \tag{47}$$

has been introduced.

For transverse one-forms, there exists a similar trace formula (see, for example, Ref. [34]) and we quote here only the results: There is no gap in the spectrum of Laplace operator $\Delta_1^{\perp}$, the Plancherel measure is $(r^2+1)/2\pi^2$, and the heat-kernel trace formula reads

$$\mathrm{Tr}e^{-t\Delta_1^{\perp}}(R)$$
$$=Q(t)\left(V_R(F)+2l_0(4\pi t)\sum_{n=1}^{\infty}\chi^n\frac{e^{-n^2l_0^2/4t}}{(\sinh nl_0/2)^2}\right), \tag{48}$$

with

$$Q(t)=\frac{1}{4(\pi t)^{3/2}}+\frac{1}{2\pi^{3/2}t^{1/2}}. \tag{49}$$

IV. THE QUANTUM CORRECTION TO THE ENTROPY OF THE BTZ BLACK HOLE

The on-shell quantum correction to the Bekenstein-Hawking entropy may be computed within the Euclidean semiclassical approximation [5] and we shall follow this approach in this section. We have to mention that a more sophisticated approach has been proposed in Ref. [33], where the canonical and microcanonical partition functions of the black hole in a cavity with suitable boundary conditions have been investigated. This approach has the merit of a more direct physical understanding and has been applied to anti–de Sitter black holes in Ref. [18].

Within the Euclidean approach, making use of the Chern-Simons representation of the three-dimensional gravity [21,22], the one-loop approximation gives

$$\ln Z^{(1)}=\ln(T^{1/2})-I, \tag{50}$$

where, if one is dealing with a compact three-manifold M, the quantum prefactor T is the Ray-Singer torsion associated with M (see a more precise definition below). In our case, we assume the quantum prefactor to be the same, but, according the discussion of Sec. II, we write

$$\ln Z^{(1)}=\frac{1}{2}\ln T-(I_{BTZ}-I_0)-(B_{BTZ}-B_0)\equiv\frac{1}{2}\ln T-I_P\,, \tag{51}$$

in which B_{BTZ} is the usual boundary term that depends on the extrinsic curvature at a large spatial distance. We recall that the total classical action is divergent; the geometry is noncompact and we have introduced the ''reference'' background $\mathcal{H}^3_0$ at the tree level [24] and the related volume cutoffs R and R_0. With this proposal, in Eq. (51) the two boundary terms of the classical contribution cancel for large R and the difference of the on-shell Euclidean classical actions gives rise to (see Sec. II),

$$I_P=I_{BTZ}-I_0=-\frac{2}{\pi}[V(R)-V_0(R)]\to-2\pi r_+=-\ln Z^{(0)}. \tag{52}$$

Restoring the correct physical dimension in Eq. (52), it is easy to show that the on-shell tree-level partition function $Z^{(0)}$, Eq. (52), becomes

$$\ln Z^{(0)}=\frac{4\pi^2 r_+}{16\pi G}, \tag{53}$$

which leads to the semiclassical Bekenstein-Hawking entropy

$$S^{(0)}=S_H=\left(r_+\frac{\partial}{\partial r_+}+1\right)\ln Z^{(0)}=\frac{1}{4}\,\frac{2\pi r_+}{G}. \tag{54}$$

So far, we have neglected quantum fluctuations. The on-shell quantum correction of the gravitational quantum fluctuations is given by the square root of the Ray-Singer torsion of the manifold $\mathcal{H}^3$. For a compact hyperbolic manifold the Ray-Singer torsion is the ratio between functional determinants of Laplace operators Δ_k acting on k-forms on $\mathcal{H}^3$ (see, for example, Refs. [21], [22], and [34]); i.e.,

$$T=\frac{\det\Delta_0}{(\det\Delta_1^{\perp})^{1/2}}. \tag{55}$$

However, in our case a manifold is noncompact and a volume regularization previously introduced will be used. Thus we have

$$\ln Z^{(1)}=\frac{1}{2}\ln\det\Delta_0-\frac{1}{4}\ln\det\Delta_1^{\perp}-I_P\,. \tag{56}$$

For the tree-level term it is necessary to introduce the bare quantity G_B since the quantum correction is plagued by the ultraviolet divergences and a renormalization procedure must be used. The functional determinants are then calculated by means of a regularization. We shall use the proper-time regularization (the ζ-function regularization gives the same finite part), in order to deal explicitly with the ultraviolet divergences.

In the case of zero-forms, one can compute a functional determinant by means of Eq. (37). Thus we have

$$\begin{aligned}\ln\det\Delta_0&=-\int_\varepsilon^\infty t^{-1}\mathrm{Tr}e^{-t\Delta_0}dt\\&=-\frac{V_R}{(4\pi)^{3/2}}\Gamma\left(-\frac{3}{2},\varepsilon\right)+\sum_{n=1}^\infty\frac{\chi^n}{n(\sinh n l_0/2)^2}e^{-l_0 n}\\&=-\frac{V_R}{(4\pi)^{3/2}}\Gamma\left(-\frac{3}{2},\varepsilon\right)+\ln\mathcal{Z}_0(2),\end{aligned} \tag{57}$$

where $\Gamma(-\frac{3}{2},\varepsilon)$ is the incomplete Gamma function, which has two divergent terms as $\varepsilon\to0$, namely,

$$\Gamma\left(-\frac{3}{2},\varepsilon\right)=\Gamma\left(-\frac{3}{2}\right)-\frac{1}{4(\pi\varepsilon)^{3/2}}+\frac{1}{(4\pi)^{3/2}\varepsilon^{1/2}}+O(\varepsilon^{1/2}) \tag{58}$$

and

$$\ln\mathcal{Z}_0(2)=\sum_{k=1}^\infty k\ln(1-\chi e^{-2(k+1)(r_+/\sigma)}). \tag{59}$$

In a similar way, using Eq. (48), one has

$$\begin{aligned}\ln\det\Delta_1^{\perp}&=-\int_\varepsilon^\infty t^{-1}\mathrm{Tr}e^{-t\Delta_1^{\perp}}dt\\&=-\frac{V_R}{8(4\pi\varepsilon)^{3/2}}+\frac{V_R}{2(4\pi)^{3/2}\varepsilon^{1/2}}-\ln\mathcal{Z}_1(1),\end{aligned} \tag{60}$$

with

$$\ln\mathcal{Z}_1(1)=\sum_{n=1}^\infty\frac{\chi^n}{[\sinh(nr_+/\sigma)]^2}\left[\frac{1}{n}+8n\left(\frac{r_+}{\sigma}\right)^2\right]. \tag{61}$$

As a result,

$$\ln Z^{(1)}=\frac{4\pi^2 r_+}{16\pi G}+g(r_+)-F_\varepsilon\,, \tag{62}$$

where

$$g(r_+)=\frac{1}{2}\ln\mathcal{Z}_0(2)+\frac{1}{4}\ln\mathcal{Z}_1(1) \tag{63}$$

and

$$F_\varepsilon=V_R\left[\frac{1}{2(4\pi)^{3/2}}\Gamma\left(-\frac{3}{2},\varepsilon\right)+\frac{1}{32(4\pi\varepsilon)^{3/2}}-\frac{1}{8(4\pi)^{3/2}\varepsilon^{1/2}}\right]. \tag{64}$$

If we define the renormalized quantity

$$\frac{1}{16\pi G_r}=\frac{1}{16\pi G}+\frac{F_\varepsilon}{4\pi^2 r_+}, \tag{65}$$

we arrive at

$$\ln Z^{(1)}=\frac{\pi r_+}{4G_r}+g(r_+). \tag{66}$$

This renormalized one-loop effective action may be thought to describe an effective classical geometry belonging to the same class of nonrotating BTZ black hole solution. This stems from the results contained in Ref. [35], where it has been shown that the constraints for pure gravity have a unique solution. As a consequence, one may introduce a new effective radius by means of

$$\ln Z^{(1)}=\frac{\pi R_+}{4\pi G_r}, \tag{67}$$

where

$$R_+=r_++\frac{4G_r}{\pi}g(r_+), \tag{68}$$

mimicking in this way the back reaction of the quantum gravitational fluctuations. As a consequence, the new entropy is given by an effective Bekenstein-Hawking term, namely,

$$S^{(1)}=\frac{1}{4}\,\frac{2\pi R_+}{G_r}. \tag{69}$$

One can evaluate the asymptotics of the quantity $g(r_+)$ for $r_+\to\infty$ and $r_+\to 0$ and then compute the effective radius. Note that $\ln\mathcal{Z}_1(1)$ and $\ln\mathcal{Z}_0(2)$ are exponentially small for large r_+. Thus

$$R_+\simeq r_+ \tag{70}$$

and nothing of interest is present in this limit.

Making use of the results of the Appendix, for small r_+, one has

$$R_+\simeq r_++\frac{4G_r}{\pi}\left\{\frac{\sigma^2}{16r_+^2}\left[-\ln\left(\frac{2r_+}{\sigma}\right)+2\gamma+\Psi(2)-\zeta(3)\right]+\frac{\sigma\pi^2}{24r_+}+\frac{1}{4}\ln\left(\frac{r_+}{\sigma\pi}\right)+O(r_+)\right\}. \tag{71}$$

One can see that for r_+ sufficiently small the effective radius becomes larger and positive. This means that R_+ (as a function of r_+) reaches a minimum for suitable r_+^*, the solution of the equation

$$\frac{4G_r}{\pi}g'(r_+^*)=-1. \tag{72}$$

This result is in qualitative agreement with a very recent computation of the off-shell quantum correction to the entropy due to a scalar field in the BTZ background [29] and all the qualitative considerations contained there are also valid for the gravitational case we are dealing with. In particular, it appears that the quantum gravitational corrections could become more and more important as the evaporation process continues and thus they cannot be neglected.

V. CONCLUDING REMARKS

In this paper the quantum correction of the BTZ black hole has been evaluated making use of the appropriate Chern-Simons representation of three-dimensional gravity. The quantum prefactor, i.e., the Ray-Singer torsion, has been evaluated by means of the proper-time regularization.

In our computation the one-loop ultraviolet and horizon divergences, generally present in the quantum correction, have also been found and they have been accounted for by means of the introduction of the standard one-loop renormalization procedure of the Newton constant [8]. Also, the semiclassical Bekenstein-Hawking entropy has been rederived by the improved Euclidean method suggested in Ref. [24].

With regard to this, we have to stress again that we have been working within the so-called on-shell Euclidean approach [5], namely, our one-loop approximation has been evaluated on the regular Euclidean BTZ instanton. For a critical comparison between on-shell and off-shell methods in the black hole entropy issue, see the review [36]. Here we would like to mention only that our result for quantum corrections to the black hole entropy differs from the one reported in Ref. [19] and is consistent with the detailed off-shell computation of the entropy of scalar fields in the BTZ classical background given in Ref. [29]. Horizon divergences of the entropy for scalar fields in the same background have also been investigated in Ref. [37].

Finally, our result, even though obtained in the one-loop approximation, may be interpreted with a nonviolation of the area law, but with an effective radius that is the classical one for large black hole masses, but that shrinks as the black hole evaporation goes on. This seems to suggest that the quantum corrections of the gravitational field become more and more important near the end of the evaporation process. As far as this issue is concerned, we observe that the final effective geometrical configuration is the reference space $\mathcal{H}_0^3$, which admits a naked singularity at the origin. As a consequence, the quantum correction seems to have a tendency to avoid the appearance of the naked singularity, in agreement with the ''cosmic censorship'' hypothesis.

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APPENDIX

In this appendix we shall investigate the small $t=2r_+/\sigma$ asymptotics for the quantity $g(t)$, making use of the standard Mellin transform technique [38]. For the sake of simplicity we set $\chi=1$. To begin with, we observe that $\ln\mathcal{Z}_0(2)$ may be rewritten as

$$\ln\mathcal{Z}_0(2)=\sum_{n=1}^{\infty} n\ln(1-e^{-tn})-\mathbb{H}(t), \tag{A1}$$

where $\mathbb{H}(t)$ is the Hardy-Ramanujan modular function, given by

$$\mathbb{H}(t)=\sum_{n=1}^{\infty}\ln(1-e^{-tn}). \tag{A2}$$

It satisfies the functional equation

$$\mathbb{H}(t)=-\frac{\pi^2}{6t}-\frac{1}{2}\ln\left(\frac{t}{2\pi}\right)+\frac{t}{24}+\mathbb{H}\left(\frac{4\pi^2}{t}\right). \tag{A3}$$

For the first term, the Mellin transform representation gives

$$\sum_{n=1}^{\infty} n\ln(1-e^{-tn})=-\frac{1}{2\pi i}\int_{\mathrm{Re}z>2} t^{-z}\Gamma(z)\zeta(z+1)\times\zeta(z-1)dz. \tag{A4}$$

Shifting the vertical contour to the left, one has a simple pole at $z=2$, a double pole at $z=0$, and simple poles at $z=-2m$, $m=1,2,\ldots$. The residue theorem gives, for small t,

$$\sum_{n=1}^{\infty} n\ln(1-e^{-tn})=-\frac{\zeta(3)}{t^2}+\zeta(-1)\ln t-\zeta'(-1)+O(t^2). \tag{A5}$$

With regard to the quantity $\ln\mathcal{Z}_1(1)$, the same technique gives

$$\begin{aligned}\ln\mathcal{Z}_1(1)&=\sum_{n=1}^{\infty}\frac{1}{(\sinh n\, t/2)^2}\left(\frac{1}{n}+2nt^2\right)\\ &=\frac{1}{2\pi i}\int_{\mathrm{Re}z>2} dz\; t^{-z}\Gamma(z)\zeta(z-1)[\zeta(1+z)\\ &\quad+2t^2\zeta(z-1)]\\ &=\frac{\zeta(3)}{t^2}-\zeta(-1)\ln t+\zeta'(-1)\\ &\quad+\frac{1}{t^2}[2\gamma+\Psi(2)-\ln t]+O(t^2).\end{aligned} \tag{A6}$$

As a result, for small t the asymptotics for the quantity $g(t)$ [see Eq. (63)] reads

$$g(t)=\frac{1}{4t^2}[-\ln t+2\gamma+\Psi(2)-\zeta(3)]+\frac{\pi^2}{12t}+\frac{1}{4}\ln\left(\frac{t}{2\pi}\right)+\frac{t}{48}+O(t^2). \tag{A7}$$

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