

Quantum inequalities and singular negative energy densities

L. H. Ford,* Michael J. Pfenning,† and Thomas A. Roman‡

Institute of Cosmology, Department of Physics and Astronomy, Tufts University, Medford, Massachusetts 02155

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There has been much recent work on quantum inequalities to constrain negative energy. These are uncertainty-principle-type restrictions on the magnitude and duration of negative energy densities or fluxes. We consider several examples of apparent failures of the quantum inequalities, which involve passage of an observer through regions where the negative energy density becomes singular. We argue that this type of situation requires one to formulate quantum inequalities using sampling functions with compact support. We discuss such inequalities, and argue that they remain valid even in the presence of singular energy densities. [S0556-2821(98)04108-3]

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I. INTRODUCTION

It has been known for some time that, unlike in classical physics, quantum field theory allows the local energy density to be negative [1,2] and even unboundedly negative at a single spacetime point. These situations imply violation of the weak energy condition [3] $T_{\mu\nu}u^\mu u^\nu \geq 0$, for all causal vectors u^μ . On the other hand, if field theory places no constraints on negative energy, then it might be possible to produce gross macroscopic effects. Such effects might include violation of the second law of thermodynamics [4,5], violation of the cosmic censorship hypothesis [6,7], traversable wormholes [8], warp drives [9,10], and time machines [11,12,13], to name a few. As a result, there has been much activity in recent years to determine what constraints, if any, quantum field theory places on negative energy.

One approach involves averaging the local energy conditions over timelike or null geodesics (see Refs. [14, 15] for a discussion and references). Another approach [4,16,17,18] entails multiplying the renormalized expectation value of the energy density (or flux) by a sampling function, i.e., a peaked function of time whose time integral is unity. One convenient choice is the Lorentzian function peaked around $\tau=0$,

$$q(\tau) = \frac{\tau_0}{[\pi(\tau^2 + \tau_0^2)]}, \quad (1)$$

where τ_0 is the characteristic width of the sampling function, i.e., the ‘‘sampling time.’’

Let $T_{\mu\nu}$ be the renormalized expectation value of the stress tensor taken in an arbitrary quantum state $|\psi\rangle$. Then $T_{\mu\nu}u^\mu u^\nu$ is the local energy density measured by an observer with four-velocity u^μ . For a quantized massless, minimally

coupled scalar field in four-dimensional Minkowski spacetime, the following inequality has been derived [17,18] for timelike geodesic observers:

$$\hat{\rho} = \frac{\tau_0}{\pi} \int_{-\infty}^{\infty} \frac{T_{\mu\nu}u^\mu u^\nu d\tau}{\tau^2 + \tau_0^2} \geq -\frac{3}{32\pi^2\tau_0^4}, \quad (2)$$

for all τ_0 , where τ is the observer’s proper time. (Our units are taken to be $\hbar = G = c = 1$.) Similar bounds have also been derived for the massive scalar and electromagnetic fields [18]. These constraints, which have come to be known as ‘‘quantum inequalities’’ (QIs), are uncertainty-principle-like bounds which restrict the magnitude and duration of negative energy effects. However, it should be noted that the energy-time uncertainty principle was not used as input to derive the QIs; they arise directly from quantum field theory. More recently, QI bounds have been proved in static curved spacetimes as well [19,20,21]. For the massless scalar field in two-dimensional Minkowski spacetime, generalized QI bounds have been derived for arbitrary sampling functions [22].

The original QI bounds were derived for Minkowski spacetime, and shown to hold for all choices of sampling time τ_0 . It was argued in Ref. [23] that the flat spacetime QIs should also hold in a curved spacetime and/or one with boundaries, if one restricts the choice of sampling time to be much less than either the smallest local proper radius of curvature or the smallest proper distance to any boundaries. In particular, it was shown that in this limit the Casimir effect satisfies the QI bound. This argument essentially says that one does not have to know the large-scale curvature of the universe in order to use flat space quantum field theory to predict the outcome of a laboratory-scale experiment. If the QI is applied in general spacetimes in the short sampling time limit, then it was shown that the bound severely constrains the geometry of traversable wormholes. More specifically, either the wormhole must be no larger than a few thousand Planck lengths in size, or, if the wormhole is macroscopic, there must be large discrepancies in the length scales which characterize the wormhole geometry, e.g., the negative energy must be confined to a band around the throat which can be no thicker than a few thousand Planck lengths. It was argued in Ref. [23] that, on dimensional grounds, one

*Electronic mail address: ford@cosmos2.phy.tufts.edu

†Electronic mail address: mitchel@cosmos2.phy.tufts.edu

‡Permanent address: Department of Physics and Earth Sciences, Central Connecticut State University, New Britain, CT 060505; Electronic mail address: roman@ccsu.edu

would not expect nonlocal curvature terms to produce significant contributions to the renormalized energy density over macroscopic length scales unless one introduced large dimensionless coupling constants into the theory or enormous numbers (e.g., $\sim 10^{62}$) of fields. In this sense, the conclusions of Ref. [23] are not at odds with some recent claims [24,25]. Similar analyses apply the flat spacetime QI, in the short sampling time limit, to the ‘‘warp drive’’ spacetime of Alcubierre [9,26] and to the ‘‘superluminal subway’’ spacetime of Krasnikov [10,27], and arrive at even more stringent constraints on the physical realizability of these spacetimes. Strong evidence for the validity of the short sampling time approximation has been provided by recent analyses [20,21]. These show that for any static observer (geodesic or not) in any static spacetime, the QI reduces to the Minkowski spacetime form, Eq. (2), in the short sampling time limit.

Krasnikov [10] has recently pointed out that, in certain circumstances, the QIs might fail even in the short sampling time limit. He cites the specific example of a massless scalar field in the conformal vacuum state in two-dimensional Misner spacetime. For any geodesic observer and *any* sampling time, τ_0 , he observes that $\hat{\rho} = -\infty$ on the Cauchy horizon, and that $\hat{\rho}$ diverges to $-\infty$ as the observer approaches the Cauchy horizon. Krasnikov concludes that the QIs do not hold in this situation, and argues that similar failures should occur in the case where one ‘‘almost transforms’’ a traversable wormhole into a time machine [28].

In this paper we give other examples of apparent failures of the QIs, which arise when there are singular energy densities. We argue that the problem arises when one employs a sampling function, such as the Lorentzian function, with an infinite ‘‘tail.’’ If one formulates the quantum inequalities in terms of sampling functions with compact support, then the relevant integrals are finite, as long as one samples outside of the region where the energy density becomes singular. We further argue that the physical content of the quantum inequalities as restrictions on the magnitude and extent of negative energy is essentially the same as found in previous work.

II. SAMPLING FUNCTIONS AND DIVERGENT ENERGY DENSITIES

A. Representative example: The flat plate

Consider a minimally coupled scalar field in four-dimensional Minkowski spacetime with a single plane boundary, which is located at $z=0$. We assume that the field is in the vacuum state, and that an observer approaches the boundary at constant velocity along the z -axis. We take the observer’s equation of motion to be

$$z(\tau) = v \gamma (\tau - \tau_c), \quad (3)$$

where τ is the observer’s proper time, τ_c is the proper time at which the observer collides with the plate, and $\gamma = 1/\sqrt{1-v^2}$. The corresponding four-velocity is $u^\mu = \gamma(1, 0, 0, v)$. The renormalized expectation values of the stress-tensor components for the quantum field are given by [29]

$$T_{tt} = -T_{xx} = -T_{yy} = -\frac{1}{16\pi^2 z^4}, \quad (4)$$

and

$$T_{zz} = 0. \quad (5)$$

We see that the energy density diverges as z^{-4} . (Such a divergence does not occur for the massless conformally coupled scalar field or for the electromagnetic field, in the plane boundary case. However, divergences do occur in the case of curved boundaries in flat spacetime [30].)

The energy density in this observer’s frame is

$$T_{\mu\nu} u^\mu u^\nu = \gamma^2 T_{tt} = -\frac{1}{16\pi^2 v^4 \gamma^2 (\tau - \tau_c)^4}. \quad (6)$$

If we insert this expression into Eq. (2), we obtain

$$\hat{\rho} = -\frac{\tau_0}{16\pi^3 v^4 \gamma^2} \int_{-\infty}^{\infty} \frac{d\tau}{(\tau - \tau_c)^4 (\tau^2 + \tau_0^2)}. \quad (7)$$

This integral apparently diverges due to the singularity of the integrand as $\tau \rightarrow \tau_c$, corresponding to $z \rightarrow 0$. This is independent of the choices both of τ_0 and of τ_c .

From this example, one can see that the apparent failure of the QI occurs because the tail of the sampling function intersects the region of singular negative energy density. Furthermore, this problem may arise in more general cases for sampling functions which do not have compact support. For any sampling function which has a tail, one can always construct a quantum state designed so that the temporal asymptotic growth of the magnitude of the negative energy density overcomes any falloff of the chosen sampling function. However, the original choice of the Lorentzian sampling function, Eq. (1), was made simply for mathematical convenience. In this paper we will argue that the problem posed above can be remedied by using compactly supported sampling functions, that is, functions which are identically zero outside of a finite interval.

First, it is of interest to note that in some cases an alternative solution is available. Although the integral in Eq. (7) is apparently divergent, it can in fact be defined as a ‘‘generalized principal value’’ integral [31]. The basic idea is to perform successive integrations by parts. Consider the integral

$$I = \int_{-\infty}^{\infty} \frac{f(\tau)}{(\tau - \tau_c)^4} d\tau, \quad (8)$$

where $f(\tau)$ and its first three derivatives are finite everywhere, including as $|\tau| \rightarrow \infty$. If we perform three successive integrations by parts, the boundary terms all vanish and the result is

$$I = \frac{1}{6} \int_{-\infty}^{\infty} \frac{f'''(\tau)}{\tau - \tau_c} d\tau, \quad (9)$$

where the remaining integral may be defined as a conventional principal value. For the integral in Eq. (7), $f(\tau) = (\tau^2 + \tau_0^2)^{-1}$, and we find (with the aid of the symbolic algebra routine MACSYMA)

$$I = \pi \frac{(\tau_c^2 - 2\tau_0\tau_c - \tau_0^2)(\tau_c^2 + 2\tau_0\tau_c - \tau_0^2)}{\tau_0(\tau_c^2 + \tau_0^2)^4}. \quad (10)$$

Equation (7) now becomes

$$\hat{\rho} = -\frac{(b^2 - 2b - 1)(b^2 + 2b - 1)}{16\pi^2 v^4 \gamma^2 \tau_0^4 (b^2 + 1)^4}, \quad (11)$$

where $b = \tau_c / \tau_0$. We want to choose the sampling time to be small compared to the proper spatial distance to the plate, $\tau_0 \ll v\tau_c < \tau_c$. In this limit,

$$\hat{\rho} \approx -\frac{1}{16\pi^2 v^4 \gamma^2 \tau_c^4} \gg -\frac{1}{16\pi^2 \tau_0^4}. \quad (12)$$

Thus a quantum inequality of the form of Eq. (2) is in fact satisfied. This method can be used whenever the observer passes through an energy density which diverges symmetrically as an inverse integral power of proper time on either side of a boundary. It would not work, for example, if the observer were to stop abruptly at $z=0$.

B. Two-dimensional QI with a compactly supported sampling function

Flanagan has shown [22] that for a massless scalar field in two-dimensional Minkowski spacetime

$$\hat{\rho} \geq -(1/24\pi) \int_{-\infty}^{\infty} d\tau \frac{[g'(\tau)]^2}{g(\tau)}, \quad (13)$$

where $g(\tau)$ is an arbitrary sampling function. As a specific example, consider the sampling function given by

$$f(\tau) = \begin{cases} 0, & \tau < -\tau_0/2 \\ (1/\tau_0)[1 + \cos(2\pi\tau/\tau_0)], & -\tau_0/2 \leq \tau \leq \tau_0/2 \\ 0, & \tau > \tau_0/2 \end{cases} \quad (14)$$

If we substitute this sampling function into Eq. (13), we obtain the following QI:

$$\hat{\rho} \geq -\frac{\pi}{6\tau_0^2}. \quad (15)$$

Although a similar inequality using a compactly supported sampling function in four-dimensional Minkowski spacetime is not yet in hand, it is quite plausible to conjecture that such a QI exists. We now turn to showing that compactly supported sampling functions may be used to resolve the apparent difficulties posed by divergent energy densities.

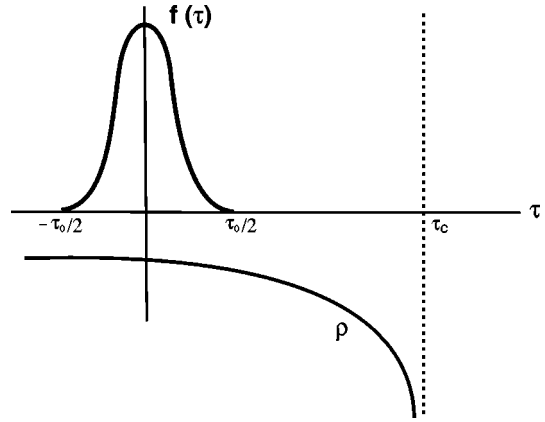


FIG. 1. The compactly supported sampling function $f(\tau)$, and the energy density, ρ , seen by an ingoing geodesic observer, plotted as a function of the proper time, τ . The width of the sampling function, i.e., the sampling time, is τ_0 . The energy density seen by the observer diverges negatively as $\tau \rightarrow \tau_c$.

III. APPLICATIONS OF QIs WITH COMPACTLY SUPPORTED SAMPLING FUNCTIONS

A. Flat spacetime with boundaries

1. Two-dimensional plate

In a two-dimensional flat spacetime, unlike in the case of four dimensions, the energy density for a minimally coupled scalar field does not diverge at a boundary, under the imposition of Dirichlet boundary conditions. However, a non-minimally coupled field does have such a divergence. The stress tensor is given by

$$T_{tt} = -\frac{A}{z^2}, \quad (16)$$

$$T_{zz} = 0, \quad (17)$$

where $A = -\xi/2\pi$, and ξ is the conformal coupling parameter. Here we assume that $A > 0$ ($\xi < 0$) and of order 1. Consider a geodesic observer whose equation of motion is given by Eq. (3). The energy density in this observer's frame is given by

$$\rho(\tau) = T_{\mu\nu} u^\mu u^\nu = -\frac{A}{v^2(\tau - \tau_c)^2}. \quad (18)$$

If we fold this into the compactly supported sampling function given by Eq. (14), we obtain

$$\hat{\rho} = -\frac{A}{v^2\tau_0} \int_{-\tau_0/2}^{\tau_0/2} \frac{d\tau}{(\tau - \tau_c)^2} [1 + \cos(2\pi\tau/\tau_0)]. \quad (19)$$

However, since the energy density is decreasing monotonically, as shown in Fig. 1, then $\hat{\rho}$ must be greater than the energy density at the end of the sampling interval, i.e.,

$$\hat{\rho} > \rho(\tau_0/2). \quad (20)$$

We can now ask under what circumstances will it be true that

$$\rho(\tau_0/2) \geq -\frac{\pi}{6\tau_0^2}. \tag{21}$$

Using the fact that the proper distance to $z=0$, as measured from the end of the sampling interval, is given by

$$\ell = v[\tau_c - \tau_0/2], \tag{22}$$

and that in the present case

$$\rho(\tau_0/2) = -\frac{A}{v^2[\tau_c - \tau_0/2]^2} = -\frac{A}{\ell^2}, \tag{23}$$

we see that Eq. (21) will be satisfied when

$$\ell \geq \left(\frac{6A}{\pi}\right)^{1/2} \tau_0. \tag{24}$$

Hence we see that our two-dimensional flat spacetime QI, Eq. (15), will be satisfied when

$$\tau_0 \ll \ell, \tag{25}$$

that is, when the observer’s sampling time is much smaller than the proper distance to the boundary. Although we have used the particular form given in Eq. (14), this conclusion should not depend upon the specific choice of sampling function, as long as it is compactly supported.

2. Four-dimensional plate

Although a QI which uses a compactly supported sampling function has not been derived in four-dimensional flat spacetime as yet, it is reasonable to assume that an analogue of Eq. (15) exists. On dimensional grounds, a QI for a massless scalar field in four dimensions should have the form

$$\hat{\rho} \geq -\frac{\alpha}{\tau_0^4}, \tag{26}$$

where α is a positive constant. For the purposes of the following argument we shall assume the existence of such a QI, valid for all quantum states, with α of the order of or less than unity.

Let us return to the case of the observer approaching a flat plate in four-dimensional flat spacetime, discussed in Sec. II A, and fold Eq. (6) into the compactly supported sampling function given by Eq. (14). The energy density in the observer’s frame now decreases as $-1/(\tau - \tau_c)^4$. Repeating the argument for the two-dimensional case, we find that our conjectured QI will hold if

$$\hat{\rho} \geq \rho(\tau_0/2) \geq -\frac{\alpha}{\tau_0^4}. \tag{27}$$

From Eq. (6), we find that this will be true if

$$\tau_0 \leq (16\pi^2 \alpha \gamma^2)^{1/4} \ell, \tag{28}$$

where ℓ is again the proper distance to the plate. (Unlike in the two-dimensional case, the factors of γ do not cancel out.) When $v \rightarrow 0$, $\gamma \rightarrow 1$, and therefore the QI is satisfied when $\tau_0 \ll \ell$. As γ gets larger, the condition, Eq. (28), becomes easier to satisfy.

Note that in this and the previous subsection, we have defined the proper distance to the plate to be measured from the end of the sampling interval. However, if $\tau_0 \ll \ell$, this is approximately the distance from the middle of the sampling interval. In subsequent subsections, we shall use the latter definition.

B. Boulware vacuum at $r=2M$

1. Two dimensions

The two-velocity of an ingoing geodesic observer in two-dimensional Schwarzschild spacetime is given by

$$u^\mu = (u^t, u^r) = \left(\frac{dt}{d\tau}, \frac{dr}{d\tau}\right) = \left(\frac{k}{C}, -\sqrt{k^2 - C}\right), \tag{29}$$

where $C = 1 - 2M/r$ and k is the energy per unit rest mass of the observer. From Eq. (33) of Ref. [15], we have that

$$T_{\mu\nu} u^\mu u^\nu = \frac{1}{24\pi} C^{-2} \left\{ k^2 \left[\frac{6M^2}{r^4} - \frac{4M}{r^3} \right] + \frac{CM^2}{r^4} \right\}. \tag{30}$$

Note that this quantity is negative everywhere for $r \geq 2M$ [32], and diverges at $r = 2M$.

From $r = \infty$ to $r = 2M$, C varies from 1 to 0. Consider the ultrarelativistic limit, $k \gg 1$, corresponding to an observer shot inward at high velocity. Then from Eq. (29), we have, to first order in k^{-1} ,

$$\tau \sim \tau_c - \frac{rC}{k}, \tag{31}$$

where τ_c is the proper time at which the observer reaches $r = 2M$. As the observer approaches the horizon, the local energy density varies as

$$T_{\mu\nu} u^\mu u^\nu \sim -\frac{1}{48\pi(\tau_c - \tau)^2}. \tag{32}$$

We can think of the horizon in the Boulware vacuum as a singular boundary analogous to the flat space examples discussed in Sect. III A. Note that in the infalling observer’s rest frame near the horizon, the $r = 2M$ boundary is approaching at nearly the speed of light. As the horizon is approached, the proper distance in this observer’s frame to the boundary from the point $\tau = 0$ (which is always the midpoint of our sampling interval) is approximately $\ell = \tau_c$. If we select a sampling time $\tau_0 \ll \ell$, the sampling function is zero at the horizon and the flat space form of the quantum inequality, Eq. (15), is satisfied, just as in the example in Sec. III A 1.

2. Four dimensions

Visser has recently given an approximate analytic expression for the renormalized stress-tensor components of a conformally coupled scalar field in the Boulware vacuum state in four-dimensional Schwarzschild spacetime [33]. Our

original QI was proved for the minimally coupled, rather than the conformally coupled scalar field. In light of the recent proofs of similar QIs for the massive scalar field and the electromagnetic field [18,19], it seems highly likely that such a bound should also hold for the conformally coupled scalar field as well. For the sake of the following argument, we will assume this to be true.

If Visser's equations (8) and (9) are transformed from the static orthonormal frame back into the usual Schwarzschild t, r coordinates, one obtains

$$T_{tt} = -3p_\infty x^6 \frac{[40 - 72x + 33x^2]}{(1-x)}, \quad (33)$$

$$T_{rr} = p_\infty x^6 \frac{[8 - 24x + 15x^2]}{(1-x)^3}, \quad (34)$$

where $x \equiv 2M/r$, and

$$p_\infty = \frac{1}{90(16\pi)^2(2M)^4}. \quad (35)$$

For an infalling geodesic observer with $u^\mu = (k/(1-x), -\sqrt{k^2 - (1-x)}, 0, 0)$, we have that

$$T_{\mu\nu}u^\mu u^\nu = \frac{-p_\infty x^6(15x^3 - 84k^2x^2 - 39x^2 + 192k^2x + 32x - 112k^2 - 8)}{(x-1)^3}. \quad (36)$$

If take $k \gg 1$, and use Eq. (31), we can express the energy density near the horizon as

$$T_{\mu\nu}u^\mu u^\nu \approx -\frac{32M^3 p_\infty}{k(\tau_c - \tau)^3}. \quad (37)$$

As discussed in the previous subsection, near the horizon, the observer's proper distance to $r=2M$ is $\ell \approx \tau_c$. If $\tau_0 \ll \ell$, then the quantum inequality (26) will be satisfied if

$$\tau_0 < [45(16\pi)^2 k \alpha M \ell^3]^{1/4}. \quad (38)$$

However, this will in fact be the case because $k \gg 1$ and $\tau_0 \ll \ell < M$.

C. Singularity at $r=0$ in black hole spacetimes

Perhaps the most serious example of a singular energy density arises at $r=0$, the curvature singularity of a black hole. Unlike the other examples discussed in this paper, this singular energy density cannot be explained away as being due to an unphysical choice of quantum state, as is the case of the Boulware vacuum at the horizon, or an unphysical boundary condition, as in the case of the perfectly reflecting plate in Sect. III A 2. This singular energy density is essentially independent of the quantum state. In this case, the generalized principal value method discussed in Sect. II A cannot be utilized, as the observer cannot pass beyond $r=0$. However, the use of compactly supported sampling functions is still successful. We will restrict our attention to the case of a two-dimensional black hole, as the form of the stress tensor near $r=0$ is not known in the four-dimensional case. From Eq. (30), or from the corresponding expression for the Unruh vacuum state, one finds that, near the origin of a two-dimensional black hole,

$$T_{\mu\nu}u^\mu u^\nu \sim -\frac{M}{48\pi r^3}. \quad (39)$$

The geodesic equation (29) implies that for small r

$$r(\tau) \approx \left[\frac{3\sqrt{2M}}{2}(\tau_c - \tau) \right]^{2/3}, \quad (40)$$

where τ_c is again the proper time at which the singularity is reached. The energy density can be expressed in terms of the proper time as

$$T_{\mu\nu}u^\mu u^\nu \sim -\frac{1}{216\pi(\tau_c - \tau)^2}. \quad (41)$$

Note that this result has the same form as Eqs. (18) and (32). A two-dimensional spacetime is characterized by a single component of the Riemann tensor, which we may take to be the scalar curvature. In the case of 2D Schwarzschild spacetime, this is

$$R = \frac{4M}{r^3}. \quad (42)$$

Define the proper local radius of curvature by

$$r_c = \frac{1}{\sqrt{R}} \sim \frac{3\sqrt{2}}{4}(\tau_c - \tau) \quad \text{as } \tau \rightarrow \tau_c. \quad (43)$$

We see that near $r=0$, this local radius of curvature and the proper time for an observer to reach the singularity, $\tau_c - \tau$, are proportional to one another. Thus if we require that the sampling time satisfy $\tau_0 \ll r_c$, we again find that the flat space form of the quantum inequality (15) is satisfied.

D. Misner space

A further example of a singular energy density arises in Misner space. The two-dimensional version of this example was cited by Krasnikov as the possible counterexample to quantum inequalities based upon noncompactly supported sampling functions. The stress tensor in this two-dimensional version has recently been discussed in detail by Cramer and Kay [34]. Hiscock and Konkowski [35] have calculated the

quantum stress tensor for a massless conformally coupled scalar field in the four-dimensional version of Misner space: so we may use their results to demonstrate that quantum inequalities using compactly supported sampling functions are meaningful in this space. Misner space is a locally flat spacetime with periodic identifications. It may be represented by the metric

$$ds^2 = -dt^2 + t^2 dx^2 + dy^2 + dz^2, \quad (44)$$

with the points (t, x, y, z) and $(t, x + na, y, z)$ identified with one another, where n is any integer and a is a positive constant. Misner space is a portion of Minkowski space, and the metric may be transformed to the Minkowski form

$$ds^2 = -dy_0^2 + dy_1^2 + dy_2^2 + dy_3^2 \quad (45)$$

by means of the transformation

$$y_0 = t \cosh x, \quad y_1 = t \sinh x, \quad y_2 = y, \quad y_3 = z. \quad (46)$$

The quantum stress tensor is divergent everywhere on the Cauchy horizon at $t=0$ (see Fig. 2), including the ‘‘quasi-regular singularity’’ at $y_0 = y_1$. Hiscock and Konkowski show that the expectation value of the stress tensor in the conformal vacuum state is given in the coordinates of Eq. (44) by

$$T_{\mu\nu} = \frac{K}{12\pi^2 t^4} \text{diag}(-1, -3t^2, 1, 1), \quad (47)$$

where

$$K = \sum_{n=1}^{\infty} \frac{2 + \cosh(na)}{[\cosh(na) - 1]^2}. \quad (48)$$

They further show that in the frame of a geodesic observer, the energy density diverges on the Cauchy horizon as τ^{-3} , where τ is proper time measured from the horizon. Note that here we are discussing a situation where the divergent energy density is in the observer’s past, as illustrated in Fig. 2. However, one could equally well discuss the time-reversed situation where the singular energy is encountered in the future. In the special case in which a geodesic observer meets the quasiregular singularity, the energy density diverges as τ^{-4} . As this case seems to pose the strongest challenge for quantum inequalities, we will focus our attention here.

The observer in question moves along the path

$$y_1 = v_x y_0, \quad y_2 = v_y y_0, \quad y_3 = v_z y_0. \quad (49)$$

In the coordinates of Eq. (44), $v_x = \tanh x$, and hence the observer’s path is a line of constant x . The components of the observer’s four-velocity in these coordinates are

$$u^t = \frac{\gamma}{\cosh x}, \quad u^x = 0, \quad u^y = \gamma v_y, \quad u^z = \gamma v_z, \quad (50)$$

where, as usual, $\gamma = 1/\sqrt{1 - (v_x^2 + v_y^2 + v_z^2)}$. (Note that Hiscock and Konkowski use an unconventional definition of γ .) The local energy density in this observer’s frame near the singularity is

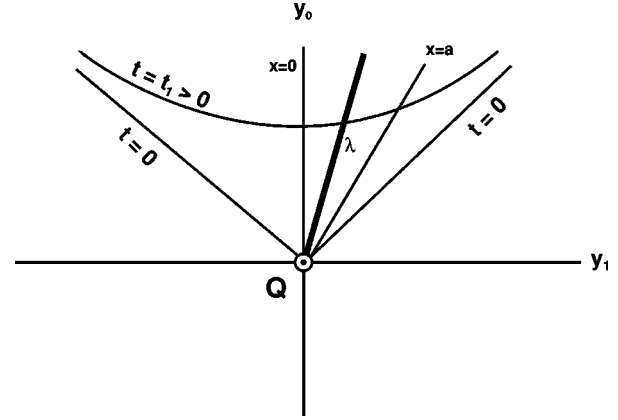


FIG. 2. A two-dimensional section of (four-dimensional) Misner spacetime, shown as the upper quadrant of Minkowski spacetime. The coordinates (y_0, y_1) are Minkowski coordinates, whereas (t, x) are Misner space coordinates. The straight lines with slope less than 45 are lines of constant x . The lines $x=0$ and $x=a$ are identified with one another. The line labeled λ is the worldline of a geodesic observer who passes through the quasi-regular singularity, Q . The Cauchy horizon is represented by the $t=0$ lines. (Note that in this representation the singularity is in the past.) The curve labeled $t=t_1 > 0$ is an arbitrary $t = \text{const}$ curve.

$$\rho = T_{\mu\nu} u^\mu u^\nu = -\frac{K}{12\pi^2 \gamma^4 (1 - v_x^2)^2 \tau^4}. \quad (51)$$

Note that

$$\frac{1}{\gamma^4 (1 - v_x^2)^2} = \left(\frac{1 - v_x^2 - v_y^2 - v_z^2}{1 - v_x^2} \right)^2 \leq 1, \quad (52)$$

so that unless $K \gg 1$, the quantum inequality (26) is satisfied whenever the sampling time is chosen so that $\tau_0 \ll \tau$.

The latter condition is certainly necessary in order that the spacetime be Minkowskian over the time of the sampling, but it is by no means sufficient. Space is compact in the x -direction, with a proper periodicity length which goes to zero near $t=0$. We see this from the following considerations. The element of proper length in the x -direction is $d\ell = t dx$, and from the first relation in Eq. (50), we have that $t = \tau \gamma / \cosh x$. Thus, the proper periodicity length is

$$\ell = \gamma \tau \int_x^{x+a} \frac{dx}{\cosh x} = 2\gamma \tau [\tan^{-1}(e^{x+a}) - \tan^{-1}(e^x)]. \quad (53)$$

In the limit of small a , this may be expressed as

$$\ell \approx \gamma \tau \frac{a}{\cosh x}. \quad (54)$$

Spacetime is Minkowskian only on scales small compared to this length; so we must also require that $\tau_0 \ll \ell$. The argument in the previous paragraph works except when $K \gg 1$. However, a large K arises only when a is small, which is precisely when $\ell \ll \tau$. From Eq. (48) we see that, when $a \ll 1$,

$$K \approx \sum_{n=1}^{\infty} \frac{12}{a^4 n^4} = \frac{12\zeta(4)}{a^4}. \quad (55)$$

The local energy density in the observer's frame, Eq. (51), can be expressed as

$$\rho = -\frac{K \cosh^4 x}{12\pi^2 \gamma^4 \tau^4}. \quad (56)$$

Thus the quantum inequality Eq. (27) is satisfied if

$$\tau_0 < \frac{\gamma(12\pi^2 \alpha K^{-1})^{1/4}}{\cosh x} \tau. \quad (57)$$

Equations (55) and (54), and the relation $\zeta(4) = \pi^4/90$, imply that for $a \ll 1$, Eq. (57) becomes

$$\tau_0 < \left(\frac{90\alpha}{\pi^2}\right)^{1/4} \ell, \quad (58)$$

which is indeed satisfied if $\tau_0 \ll \ell$. This confirms that the quantum inequality holds in Misner space.

IV. DISCUSSION

In the preceding sections, we have seen that various examples of singular energy densities obey quantum inequalities, provided that these inequalities are formulated using sampling functions with compact support. In order to sample the region around a singular energy density, it is desirable that the sampling function be identically zero at the singularity. Sampling functions with infinite tails, such as the Lorentzian function, Eq. (1), lead to divergent integrals because the tail encounters the singularity. Clearly, this behavior is not realistic. In the case of a particle falling into a black hole, for example, one will get a divergent integral regardless of where on the worldline one samples. A more reasonable outcome would be that the result of sampling while the particle is still very far away from the black hole is independent of the future fate of the particle. Quantum inequalities based upon compactly supported sampling functions achieve this outcome. The generalized principal value method discussed in Sec. II A is capable of rendering the integrals associated with noncompactly supported sampling functions finite in some cases, but not in the case of the singularity at $r=0$ in a black hole.

In any case, one does not expect truly divergent stress-energies to occur in reality. Various effects would be ex-

pected to smear out the divergences in physically realizable cases. For example, in the flat plate case, one expects the perfectly reflecting boundary condition imposed on the quantized electromagnetic field to break down for wavelengths smaller than about $\lambda_p = 1/f_p$, where f_p is the plasma frequency. Similarly, one would not expect the Boulware vacuum at $r=2M$ to be physically realizable, since the divergent stress-energy would produce a large backreaction which would presumably drastically alter the spacetime. A related argument could be made for the Misner and "almost-time-machine" wormhole spacetimes. These cases are all pathological in the sense that the quantum states do not have the Hadamard form on the horizon [36,37].

One can understand the origin of singular energy densities, such as Eq. (4), on a surface on which the quantum scalar field, φ , satisfies vanishing boundary conditions as follows: φ and its time derivative, $\dot{\varphi}$, are conjugate variables. Hence they satisfy an uncertainty relation such that if φ is precisely determined, then $\dot{\varphi}$ must be completely uncertain. This means that $\langle \dot{\varphi}^2 \rangle$ and hence T_{tt} must diverge. This situation is analogous to that of a position eigenstate in single particle quantum mechanics; such a state would have to have a completely uncertain momentum, and hence an infinite mean energy. This suggests that the singular energy density may disappear if the boundary's position is uncertain. This has recently been proved to be the case for the flat plate example of Sec. II A. In Ref. [38] it is shown that if the plate is in a quantum state where the position has a Gaussian probability distribution of finite width, then the mean energy density is finite everywhere and approaches Eq. (4) in the limit that this width vanishes.

As long as the energy density is bounded below, one expects that even quantum inequalities based upon noncompact sampling functions, such as Eq. (2), to hold provided that the sampling time is sufficiently short. Here sufficiently short presumably means $\tau_0 \ll \ell$, where $\ell \sim (\rho_{\max})^{-1/4}$ and ρ_{\max} is the maximum magnitude of the negative energy density.

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