Quantum gravitationally induced stress tensor during inflation

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We derive non-perturbative relations between the expectation value of the invariant element in a homogeneous and isotropic state and the quantum gravitationally induced pressure and energy density. By exploiting previously obtained bounds for the maximum possible growth of perturbative corrections to a locally de Sitter background we show that the two loop result dominates all higher orders. We also show that the quantum gravitational slowing of inflation becomes non-perturbatively strong earlier than previously expected. [S0556-2821(98)05308-9]

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I. INTRODUCTION

Gauge-fixed perturbation theory is by far the simplest method for computing quantum corrections to a classical geometry. Even when the state of interest is not stationary this can be done using Schwinger's formalism for expectation values [1,2]. The procedure is first to compute the expectation value of the invariant element in the presence of the desired state:

$$\langle \psi | g_{\mu\nu}(t,\vec{x}) dx^\mu dx^\nu | \psi \rangle = \hat{g}_{\mu\nu}(t,\vec{x}) dx^\mu dx^\nu. \tag{1}$$

One then forms $\hat{g}_{\mu\nu}$ into gauge invariant and gauge independent observables to infer how quantum effects distort the geometry.

Geometrically significant *differences* between the classical and quantum backgrounds can be ascribed to a quantum-induced stress tensor. In pure gravity this is defined from the deficit by which $\hat{g}_{\mu\nu}$ fails to obey the classical Einstein equation:

$$8\pi G \hat{T}_{\mu\nu} \equiv \hat{R}_{\mu\nu} - \frac{1}{2}\hat{g}_{\mu\nu}\hat{R} + \hat{g}_{\mu\nu}\Lambda. \tag{2}$$

Here $\hat{R}_{\mu\nu}$ and $\hat{R}$ are the Ricci tensor and Ricci scalar constructed from $\hat{g}_{\mu\nu}$ and it should be noted that we have included a cosmological constant Λ in Einstein's equation. Note also that the relation between the induced stress tensor and the quantum background $\hat{g}_{\mu\nu}$ is, in principle, non-perturbative, even though the only practical way of computing $\hat{g}_{\mu\nu}$ is perturbatively.

The purpose of this paper is to derive the leading late time dependence, *to all orders*, for the induced stress tensor appropriate to a recent calculation of the quantum gravitational back-reaction on an initially empty and inflating universe [3]. That we can obtain an all-orders result arises from the conjunction of the non-perturbative relation (2) and explicit bounds on the maximum late time growth of perturbative corrections to a rather technical variant of the amputated 1-point function. Section II reviews the definition of this quantity and the procedure through which it is used to compute $\hat{T}_{\mu\nu}$. Section III shows how the perturbative bounds on the former imply an all-orders result for the latter. We discuss the consequences of this result in Sec. IV. In what remains of this Introduction we review the theoretical context of our previous work and its physical motivation.

Because the late time behavior is dominated by ultraviolet finite, non-local terms, we were able to use the Lagrangian of general relativity with a positive cosmological constant:

$$\mathcal{L} = \frac{1}{16\pi G}(R - 2\Lambda)\sqrt{-g} + \text{counter terms}, \tag{3}$$

absorbing ultraviolet divergences with local counterterms as required.[1] We worked on the manifold $T^3 \times R$ in the presence of a homogeneous and isotropic state for which the invariant element takes the following form in co-moving coordinates:

$$\hat{g}_{\mu\nu}(t,\vec{x}) dx^\mu x^\nu = -dt^2 + \exp[2b(t)] d\vec{x} \cdot d\vec{x}. \tag{4}$$

Our state is free de Sitter vacuum at $t=0$ in these coordinates, corresponding to the following classical background:

$$b_{\text{class}}(t) = Ht, \quad H^2 \equiv \frac{1}{3}\Lambda. \tag{5}$$

[1]Infrared phenomena can always be studied using the low energy effective theory. This is why Bloch and Nordsieck [4] were able to resolve the infrared problem in QED before the theory's renormalizability was suspected. It is also why Weinberg [5] was able to give a similar resolution for the infrared problem of quantum general relativity with zero cosmological constant. And it is why Feinberg and Sucher [6] were able to compute the long range force induced by neutrino exchange using Fermi theory. Extensive work along the same lines has recently been done on $\Lambda=0$ quantum gravity by Donoghue [7].

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The physical motivation for our work is the possibility that the cosmological constant only appears to be unnaturally small today because it is screened by an infrared process in quantum gravity. This process is the buildup of gravitational interaction energy between virtual gravitons that are pulled apart by the inflationary expansion of the classical background [(4),(5)]. The effect acts to slow inflation because gravity is attractive. It requires a enormous time to become significant because gravity is a weak interaction, even for inflation on the grand unified theory (GUT) scale.[2] However, inflation must eventually be ended because the effect adds coherently for as long as exponential expansion persists. The effect is also unique to gravitons. Only massless particles can give a coherent effect, and the other phenomenologically viable quanta of zero mass are prevented from doing so by conformal invariance.

Our mechanism offers a natural explanation for how inflation can have lasted a long time, without fine tuning and without the need for fundamental scalars. Indeed, it results in such a long period of inflation that all energetically favorable phase transitions may have time to occur during this period, even if some are subsequently reversed by re-heating. If so, the cosmological constant which is finally screened would be that of the true vacuum, and the evolution after inflation would be almost that which is usually obtained by keeping gravity classical and fine tuning this parameter to zero.

Although perturbation theory must break down at the end of inflation, one can use the technique to partially verify our proposal. For example, the presence of infrared divergences in in-out matrix elements [8] and scattering amplitudes [9] invalidates the null hypothesis that inflation persists to asymptotically late times with only perturbatively small corrections. One can also use Schwinger's formalism to follow the evolution of the background until quantum corrections become non-perturbatively large [3]. It was previously believed that this occurred at the same time for all orders. The burden of this paper is to show that in fact the two loop effect becomes strong at a time when all higher orders are still insignificant. Of course one cannot extend past the breakdown of perturbation theory by using perturbation theory, but we now have precise information about how the breakdown occurs.

It is worth commenting on the relation between our mechanism and other schemes for screening. Proposals have been made for a one loop effect due to depletion by Hawking radiation [10–12]. Explicit computations with Schrödinger picture wave functions fail to show any such effect for free scalars in a classical geometry coupled to the expectation value of the stress tensor [13,14]. Nor can any temporally growing effect be discerned for free scalars or gravitons using canonical quantization [15,16]. In our mechanism the secular effect comes at second order, not from the continual injection of virtual gravitons into the expanding geometry but rather from the growth of the gravitational interaction between these gravitons.

Ford proposed a two-loop mechanism based on the assumption that the coincidence limit of the graviton propagator grows in time [15]. The result is a sort of diagrammatic mirror image of our mechanism: the biggest effect at any order comes from a single, many-point vertex with its legs joined by coincident propagators. This is the far ultraviolet where perturbative quantum gravity is certainly not reliable. Our mechanism is a genuine infrared effect: it comes from diagrams with many widely separated, low-point vertices.

It also turns out that the assumption of a growing coincidence limit is incorrect.[3] It was motivated by the discovery of an infrared divergence in the mode sum for the propagator on $R^3 \times R$ by Ford and Parker [17]. Subsequent authors suggested that an appropriate infrared regularization for the coincidence limit would be the co-moving momentum of the mode whose wavelength is redshifting beyond the horizon at the evaluation time. The time dependence of this *ad hoc* cutoff is the source of the temporal growth alleged for the coincidence limit. On our finite spatial manifold one can recognize the actual infrared cutoff (for all separations, not just coincidence) as the constant co-moving coordinate radius [9]. With the appropriate ultraviolet regularization our propagator shows no temporal growth at coincidence [3].

It has been suggested that one loop screening can occur due to the existence of an infrared fixed point in the effective gravity theory derived from integrating out conformal matter [18]. We do not understand the validity of various features of this analysis but there can be no doubt that its authors are alleging a different effect from the one which we have studied. Our mechanism occurs in pure gravity and is not changed by the addition of conformally invariant matter [3]. The only quanta that participate are those which are effectively massless and whose interactions are not conformally invariant on the classical level. Of the known massless particles, gravitons are unique in this respect.

Very recently a true one loop analogue of our mechanism was discovered in the context of chaotic inflation by Mukhanov, Abramo and Brandenberger [19,20]. The earlier negative results for scalar systems are avoided by two features of their class of models:

(1) The scalar is far from its vacuum state (in fact it is the inflaton whose classical evolution regulates inflation); and

(2) Classical gravity is coupled to the scalar stress tensor *operator*, rather than to its expectation value.

The first point means that the scalar stress tensor contains terms *linear* in the fluctuation fields. Even though the expectation value of these linear terms is zero, the second point means that they engender a linearized response in the Newtonian potential. At the next order, the interaction between this potential and the linearized fluctuations slows inflation because gravity is attractive. This is the same mechanism as

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[2]One traditionally defines the ''scale of inflation'' M so that M^4 equals the energy density of the cosmological constant, $\Lambda/(8\pi G)$. Since the Planck mass is $M_P = G^{-1/2}$, the dimensionless coupling constant that characterizes quantum gravitational effects on inflation can be expressed as

$$G\Lambda = 8\pi\left(\frac{M}{M_P}\right)^4. \tag{6}$$

For GUT scale inflation this works out to about $G\Lambda \sim 10^{-11}$. The comparable figure for inflation on the electroweak scale would be about $G\Lambda \sim 10^{-67}$.

[3]We emphasize that Ford bore no responsibility for this assumption.

ours but it is permitted to occur an order lower in perturbation theory because the scalar is far from vacuum. In our model the first response of the Newtonian potential is *quadratic*—i.e., one loop—in the dynamical graviton fields, which postpones the deflationary interaction term to two loops. We believe that the work of Mukhanov, Abramo and Brandenberger has important implications for scalar-driven inflation, however, it does not seem to be capable of nulling a pre-existing cosmological constant.

II. PERTURBATION THEORY REVISITED

The purpose of this section is to explain the connection between the induced stress tensor of co-moving coordinates and the quantities we actually computed. We begin with the coordinate system of the classical background. For a variety of reasons, it is simplest to formulate perturbation theory in conformal coordinates:

$$-dt^2+\exp[2Ht]d\vec{x}\cdot d\vec{x}=\Omega^2(-du^2+d\vec{x}\cdot d\vec{x}), \tag{7}$$

$$\Omega\equiv\frac{1}{Hu}=\exp(Ht). \tag{8}$$

Note the temporal inversion and the fact that the onset of inflation at $t=0$ corresponds to $u=H^{-1}$. The infinite future is $u\rightarrow 0^+$.

Perturbation theory is organized most conveniently in terms of a ''pseudo-graviton'' field, $\psi_{\mu\nu}$, obtained by conformally re-scaling the metric:

$$g_{\mu\nu}\equiv\Omega^2\tilde{g}_{\mu\nu}\equiv\Omega^2(\eta_{\mu\nu}+\kappa\psi_{\mu\nu}). \tag{9}$$

Our notation is that pseudo-graviton indices are raised and lowered with the Lorentz metric, and that the loop counting parameter is $\kappa^2\equiv 16\pi G$. After some judicious partial integrations the invariant part of the bare Lagrangian takes the following form [21]:

$$\mathcal{L}_{\rm inv}=\sqrt{-\tilde{g}}\tilde{g}^{\alpha\beta}\tilde{g}^{\rho\sigma}\tilde{g}^{\mu\nu}\left(\frac{1}{2}\psi_{\alpha\rho,\mu}\psi_{\nu\sigma,\beta}-\frac{1}{2}\psi_{\alpha\beta,\rho}\psi_{\sigma\mu,\nu}+\frac{1}{4}\psi_{\alpha\beta,\rho}\psi_{\mu\nu,\sigma}-\frac{1}{4}\psi_{\alpha\rho,\mu}\psi_{\beta\sigma,\nu}\right)\Omega^2 -\frac{1}{2}\sqrt{-\tilde{g}}\tilde{g}^{\rho\sigma}\tilde{g}^{\mu\nu}\psi_{\rho\sigma,\mu}\psi_\nu^{\ \alpha}(\Omega^2)_{,\alpha}. \tag{10}$$

Since $\Omega\sim u^{-1}$, it might seem as if the final term is stronger at late times than the others. In reality it is only comparable because its undifferentiated pseudo-graviton field must always contain a ''0'' index—$\psi_\nu^{\ \alpha}(\Omega^2)_{,\alpha}=2u^{-1}\psi_{\nu 0}\Omega^2$—and ''0'' components of the pseudo-graviton propagator are suppressed by a factor of u [9].

Gauge fixing is accomplished through the addition of $-\frac{1}{2}\eta^{\mu\nu}F_\mu F_\nu$ where

$$F_\mu\equiv\left(\psi^\rho_{\ \mu,\rho}-\frac{1}{2}\psi^\rho_{\ \rho,\mu}+2\psi^\rho_{\ \mu}(\ln\Omega)_{,\rho}\right)\Omega. \tag{11}$$

The resulting gauge fixed kinetic operator has the form

$$D_{\mu\nu}^{\ \ \rho\sigma}\equiv\left(\frac{1}{2}\bar{\delta}_\mu^{(\rho}\bar{\delta}_\nu^{\sigma)}-\frac{1}{4}\eta_{\mu\nu}\eta^{\rho\sigma}-\frac{1}{2}\delta_\mu^0\delta_\nu^0\delta_0^\rho\delta_0^\sigma\right)\mathrm{D}_A +\delta_{(\mu}^{\ 0}\bar{\delta}_{\nu)}^{\ (\rho}\delta_0^{\sigma)}\mathrm{D}_B+\delta_\mu^0\delta_\nu^0\delta_0^\rho\delta_0^\sigma\mathrm{D}_C. \tag{12}$$

A variety of conventions in this relation deserve comment. First, indices enclosed in a parenthesis are symmetrized. Second, the presence of a bar over a Kronecker delta or a Lorentz metric indicates that the temporal components of these tensors are deleted:

$$\bar{\delta}^\mu_{\ \nu}\equiv\delta^\mu_{\ \nu}-\delta^\mu_0\delta^0_{\ \nu}, \quad \bar{\eta}_{\mu\nu}\equiv\eta_{\mu\nu}+\delta^0_{\ \mu}\delta^0_{\ \nu}. \tag{13}$$

The symbol D_A stands for the kinetic operator of a massless, minimally coupled scalar:

$$\mathrm{D}_A\equiv\Omega\left(\partial^2+\frac{2}{u^2}\right)\Omega, \tag{14}$$

while $D_B=D_C$ denote the kinetic operator of a conformally coupled scalar:

$$\mathrm{D}_B=\mathrm{D}_C\equiv\Omega\,\partial^2\Omega. \tag{15}$$

What we actually computed was the amputated expectation value of $\kappa\psi_{\mu\nu}(u,\vec{x})$ which, on general grounds, must have the following form:

$$D_{\mu\nu}^{\ \ \rho\sigma}\langle 0|\kappa\psi_{\rho\sigma}(x)|0\rangle=a(u)\,\bar{\eta}_{\mu\nu}+c(u)\,\delta^0_{\ \mu}\delta^0_{\ \nu}. \tag{16}$$

Attaching the external leg gives the invariant element, but in a perturbatively corrected version of conformal coordinates:

$$\hat{g}_{\mu\nu}(t,\vec{x})dx^\mu dx^\nu=-\Omega^2[1-C(u)]du^2+\Omega^2[1+A(u)]d\vec{x}\cdot d\vec{x}. \tag{17}$$

The external leg of the 1-point function is a retarded Green's function in Schwinger's formalism. From the gauge fixed kinetic operator (12) we see that the coefficient functions $A(u)$ and $C(u)$ have the following expressions in terms of the scalar retarded propagators acting on $a(u)$ and $c(u)$ [22]:

$$A(u)=-4G_A^{\rm ret}[a](u)+G_C^{\rm ret}[3a+c](u), \tag{18}$$

$$C(u)=G_C^{\rm ret}[3a+c](u). \tag{19}$$

It is simple to work out what the retarded propagators of D_A and D_C give when acting on any power of the conformal time:

$$G_A^{\rm ret}[u^{-4}(Hu)^\varepsilon]=\frac{H^2}{\varepsilon(3-\varepsilon)}\left\{(Hu)^\varepsilon-1+\frac{1}{3}\varepsilon-\frac{1}{3}\varepsilon(Hu)^3\right\}, \tag{20}$$

$$G_C^{\rm ret}[u^{-4}(Hu)^\varepsilon]=\frac{H^2}{(1-\varepsilon)(2-\epsilon)}\{-(Hu)^\varepsilon+(2-\varepsilon)Hu-(1-\varepsilon)(Hu)^2\}. \tag{21}$$

One-particle-irreducible diagrams containing l loops can be shown to contribute to $a(u)$ and $c(u)$ at late times no more strongly than some number times [3]:

$$\kappa^{2l}H^{2l-2}u^{-4}\ln^l(Hu). \tag{22}$$

The action of the scalar retarded propagators on such a term is obtained by differentiating (20) and (21) l times with respect to ε and then taking the limit $\varepsilon\to 0$. Because of the factor of ε^{-1} on the right hand side of (20), $G_A^{\rm ret}$ acquires an extra logarithm whereas $G_C^{\rm ret}$ does not. For example, the leading contributions at two loops give

$$G_A^{\rm ret}[\kappa^4H^2u^{-4}\ln^2(Hu)]=(\kappa H)^4\left\{\frac{1}{9}\ln^3(Hu)+\frac{1}{9}\ln^2(Hu)+\frac{2}{27}\ln(Hu)+\frac{2}{81}-\frac{2}{81}(Hu)^3\right\}, \tag{23}$$

$$G_C^{\rm ret}[\kappa^4H^2u^{-4}\ln^2(Hu)]=(\kappa H)^4\left\{-\frac{1}{2}\ln^2(Hu)-\frac{3}{2}\ln(Hu)-\frac{7}{4}+2Hu-\frac{1}{4}(Hu)^2\right\}. \tag{24}$$

This phenomenon has great significance. Its physical origin is the fact that A-type Green's functions receive contributions from throughout the timelike region inside the past light cone while the C-type Green's functions have support only on the lightlike surface of the past lightcone [21].

Comparison between (4) and (17) results in the following formulas for the conversion to co-moving time:

$$d(Ht)=-\sqrt{1-C(u)}\,d[\ln(Hu)], \tag{25}$$

$$b(t)=-\ln(Hu)+\frac{1}{2}\ln[1+A(u)]. \tag{26}$$

It is then straightforward to work out the relation between physically interesting quantities defined in co-moving coordinates and the things one actually computes in perturbation theory. For example, the effective Hubble constant is[4]

$$H_{\rm eff}(t)\equiv\frac{db(t)}{dt}=\frac{H}{\sqrt{1-C(u)}}\left\{1-\frac{1}{2}u\frac{d}{du}\ln[1+A(u)]\right\}. \tag{27}$$

Two particularly interesting quantities come from the induced stress tensor: the energy density $T_{00}=\rho(t)$ and the pressure $T_{ij}=p(t)g_{ij}$. The task of this section is completed by first using (2) to express these in terms of $b(t)$ and then converting to the coefficient functions $A(u)$ and $C(u)$:

$$\rho(t)=\frac{1}{8\pi G}(3\dot{b}^2(t)-3H^2),$$
$$=\frac{1}{8\pi G}\frac{3H^2}{1-C}\left\{C-\frac{uA'}{1+A}+\frac{1}{4}\left(\frac{uA'}{1+A}\right)^2\right\}, \tag{28}$$

[4]Note that the effective Hubble constant is an invariant by virtue of its relation to the Einstein tensor, $G_{00}=3\dot{b}^2$, and by the fact that co-moving coordinates are unique up to constant rescalings of space. It can also be shown to be gauge independent [3].

$$p(t)=-\frac{2\ddot{b}(t)}{8\pi G}-\rho(t),$$
$$=\frac{1}{8\pi G}\frac{H^2}{1-C}\left\{\frac{uC'}{1-C}\left[1-\frac{1}{2}\frac{uA'}{1+A}\right]-\frac{u(uA')'}{1+A}+\left(\frac{uA'}{1+A}\right)^2\right\}-\rho(t). \tag{29}$$

A dot in these formulas indicates differentiation with respect to t, while a prime denotes differentiation with respect to u.

III. TWO LOOP DOMINANCE

Our perturbative work [3] produced explicit results for the late time $(u\to 0^+)$ behavior of the coefficient functions $a(u)$ and $c(u)$ at two loops:

$$a(u)=H^{-2}\left(\frac{\kappa H}{4\pi u}\right)^4\{-43\ln^2(Hu)+(\text{subleading})\}+O(\kappa^6), \tag{30}$$

$$c(u)=H^{-2}\left(\frac{\kappa H}{4\pi u}\right)^4\{15\ln^2(Hu)+(\text{subleading})\}+O(\kappa^6). \tag{31}$$

We also obtained the following limit on the maximum possible late time correction to $a(u)$ and $c(u)$ from one-particle-irreducible (1PI) graphs containing l loops:

$$\kappa^{2l}H^{2l-2}u^{-4}\ln^l(Hu). \tag{32}$$

This bound seemed to suggest that all orders become strong at the same time:

$$-\ln(Hu)\sim\frac{1}{\kappa^2H^2}=\frac{3}{8\pi}\frac{1}{G\Lambda}\gg 1. \tag{33}$$

That conclusion is valid for the non-invariant quantities $a(u)$ and $c(u)$, but not for invariants such as the effective Hubble constant, the induced energy density and the induced pressure. The purpose of this section is to show that, for these quantities, two loop effects become strong at a time when higher loop corrections are still insignificant. We will also use the two loop results to derive explicit formulas for the physical invariants which are valid until perturbation theory breaks down.

The key is the extra logarithm which the spatial trace coefficient $A(u)$ acquires from the A-type Green's function. The 1PI amputated coefficient functions have the following form:

$$a_{1PI}(u)=\sum_{l=2}^{\infty}a_l\kappa^{2l}H^{2l-2}u^{-4}\ln^l(Hu)+\text{subdominant}, \tag{34}$$

$$c_{1PI}(u)=\sum_{l=2}^{\infty}c_l\kappa^{2l}H^{2l-2}u^{-4}\ln^l(Hu)+\text{subdominant}, \tag{35}$$

where a_l and c_l are pure numbers. The non-amputated coefficient functions are defined by acting retarded Green's functions according to relations (18) and (19). From the general action of the retarded propagators (20) and (21), we obtain expansions for the leading terms induced by 1PI graphs:[5]

$$A_{1PI}(u) = -\frac{4}{3}\ln(Hu)\sum_{l=2}^{\infty}\frac{a_l}{l+1}(\kappa^2H^2\ln(Hu))^l + \text{subdominant}, \tag{36}$$

$$C_{1PI}(u) = -\frac{1}{2}\sum_{l=2}^{\infty}(3a_l+c_l)(\kappa^2H^2\ln(Hu))^l + \text{subdominant}. \tag{37}$$

From expression (26) for $b(t)$ we see that inflation stops when $A(u)$ approaches -1. The $l=2$ term in $A(u)$ passes through -1 when

$$-\ln(Hu) = \left(\frac{-9}{4a_2}\right)^{1/3}\left(\frac{1}{\kappa H}\right)^{4/3}. \tag{38}$$

At this time the higher l effects in $A(u)$ are of strength

$$\ln^{l+1}(Hu)(\kappa H)^{2l} \sim (\kappa H)^{(2/3)l-4/3}. \tag{39}$$

This is insignificant when one recalls that $\kappa H \sim 10^{-5}$, even for GUT scale inflation. And *all* the terms in $C(u)$ are insignificant because they have one fewer power of the large logarithm.

We have still to account for tadpoles coming from the shift of the background. One does this by shifting the fields of the interaction Lagrangian (10) and studying the effect of the induced interactions. For example, most 3-point vertices have the generic form: $\psi\partial\psi\partial\psi$. Suppose that the two differentiated fields are taken by the lowest order $A(u)$ terms. This gives a 1-point interaction whose coefficient is

$$\kappa\Omega^2\frac{d}{du}(\kappa^3H^4\ln^3(Hu))\frac{d}{du}(\kappa^3H^4\ln^3(Hu)) \sim \kappa^7H^6u^{-4}\ln^4(Hu). \tag{40}$$

When one accounts for the extra factor of κ in our definition (16) the result is no stronger than the 1PI terms already allowed for at $l=4$ loops. In fact one can do considerably better at higher order, but never good enough to catch up with the two loop effect. The fastest possible growth for either $A(u)$ or $C(u)$ is

$$(\kappa H)^{4N+8}\ln^{3N+5}(Hu), \tag{41}$$

starting at $N=0$. [The order $(\kappa H)^4$ and $(\kappa H)^6$ terms are purely 1PI.] When the two loop term becomes of order one these contributions are still suppressed by a factor of the small number $(\kappa H)^{4/3} \lesssim 10^{-7}$.

It remains to obtain the promised all-orders results for the dominant late time behavior of $H_{\text{eff}}(t)$, $\rho(t)$ and $p(t)$. To simplify the formulas we define the following small parameter:

$$\epsilon \equiv \left(\frac{\kappa H}{4\pi}\right)^2 = \frac{G\Lambda}{3\pi} = \frac{8}{3}\left(\frac{M}{M_P}\right)^4. \tag{42}$$

Our explicit two loop results (30) and (31) imply

$$A(u) = \epsilon^2\left\{\frac{172}{9}\ln^3(Hu) + (\text{subleading})\right\} + O(\epsilon^3), \tag{43}$$

$$C(u) = \epsilon^2\{57\ln^2(Hu) + (\text{subleading})\} + O(\epsilon^3). \tag{44}$$

From $C(u)$ and relation (25) we infer the transformation to co-moving time:

$$Ht = -\left\{1 - \frac{19}{2}\epsilon^2\ln^2(Hu) + \cdots\right\}\ln(Hu). \tag{45}$$

This can be inverted to give

$$\ln(Hu) = -\left\{1 + \frac{19}{2}(\epsilon Ht)^2 + \cdots\right\}Ht. \tag{46}$$

It follows that we may set ln(*Hu*) to $-Ht$, to a very good approximation, for as long as perturbation theory remains valid.

We can now write $A(u)$ as a function of the co-moving time:

$$A(u) = -\frac{172}{9}\epsilon^2(Ht)^3 + \cdots. \tag{47}$$

The higher corrections are again insignificant when the first term becomes of order unity. We can also obtain $b(t)$ as an explicit function of time:

$$b(t) \approx Ht + \frac{1}{2}\ln(1+A). \tag{48}$$

Substituting into (27) gives the effective Hubble constant:

$$H_{\text{eff}}(t) \approx H + \frac{1}{2}\frac{d}{dt}\ln(1+A), \tag{49}$$

$$\approx H\left\{1 - \frac{\frac{86}{3}\epsilon^2(Ht)^2}{1-\frac{172}{9}\epsilon^2(Ht)^3}\right\}. \tag{50}$$

Note that the numerator of the correction term is still quite small when the denominator blows up. This is why we are justified in neglecting other terms—from $C(u)$—which are also of order $(\epsilon Ht)^2$.

Going through the same exercise for the induced energy density gives

[5]These terms are 1PI except for the external propagator.

$$\rho(t) \approx \frac{\Lambda}{8\pi G}\left\{-\frac{1}{H}\frac{\dot{A}}{1+A}+\frac{1}{4H^2}\left(\frac{\dot{A}}{1+A}\right)^2\right\}, \quad (51)$$

$$\approx \frac{\Lambda}{8\pi G}\left\{-\frac{\frac{172}{3}\epsilon^2(Ht)^2}{1-\frac{172}{9}\epsilon^2(Ht)^3}+\left(\frac{\frac{86}{3}\epsilon^2(Ht)^2}{1-\frac{172}{9}\epsilon^2(Ht)^3}\right)^2\right\}. \quad (52)$$

Note that we cannot neglect the single denominator term compared to the double one; in fact the former dominates the latter. The most useful form in which to give the pressure is added to the energy density:

$$\rho(t)+p(t) \approx -\frac{1}{8\pi G}\frac{d^2}{dt^2}\ln(1+A), \quad (53)$$

$$\approx \frac{1}{8\pi G}\left(\frac{\dot{A}}{1+A}\right)^2, \quad (54)$$

$$\approx \frac{H^2}{8\pi G}\left(\frac{\frac{172}{3}\epsilon^2(Ht)^2}{1-\frac{172}{9}\epsilon^2(Ht)^3}\right)^2. \quad (55)$$

Note that in passing to the middle expression we have neglected the term, $\ddot{A}/(1+A)$, which is still insignificant when $\dot{A}/(1+A)$ is of order one. Note also that when $\dot{A}/(1+A)$ is small, its square is even smaller. Therefore $\rho+p$ is quite near zero until screening becomes significant.

IV. DISCUSSION

Our previous work [3,9,21] has established that quantum gravitational corrections slow the expansion of an initially inflating universe by an amount that becomes nonperturbatively large at late times. In this paper we have exploited exact relations between the objects which are actually computed in perturbative quantum gravity and the invariant quantities of physical interest. We conclude that the mechanism by which perturbation theory breaks down is the approach to -1 of the spatial trace coefficient $A(u)$. Furthermore, this approach is effected by two loop corrections at a time well before the higher loop corrections have become significant.

This insight has a number of consequences, starting with a revised estimate for the number of inflationary e-foldings:

$$N \sim \left(\frac{9}{172}\right)^{1/3}\left(\frac{3\pi}{G\Lambda}\right)^{2/3} = \left(\frac{81}{11008}\right)^{1/3}\left(\frac{M_P}{M}\right)^{8/3}, \quad (56)$$

where M is the mass scale of inflation and M_P is the Planck mass. For inflation on the GUT scale this gives $N \sim 10^7$ e-foldings. Electroweak inflation should last about $N \sim 10^{45}$ e-foldings. These numbers are smaller than our previous estimates, but still much longer than in typical models. We stress that this long period of inflation is a natural consequence of the fact that gravity is a weak interaction.

One can also estimate the rapidity with which inflation ends once the effect becomes noticeable. Suppose we expand around the critical time:

$$Ht = N - H\Delta t. \quad (57)$$

When $H\Delta t \ll N$ our expression (50) for the effective Hubble constant becomes

$$H_{\text{eff}}(t) \approx H\left\{1-\frac{1}{2H\Delta t}\right\}. \quad (58)$$

It follows that inflation must end rapidly. To be precise, let us define the end of inflation as the period from when the effective Hubble constant falls from $\frac{9}{10}$ to $\frac{1}{10}$ of its initial value. From the previous formula, H_{eff} reaches $\frac{9}{10}H$ at $H\Delta t = 5$, and it falls to $\frac{1}{10}H$ at $H\Delta t = \frac{5}{9}$, making for a transition time of $4\frac{4}{9}$ e-foldings.

Of course one cannot trust perturbation theory during this period but it is reasonable to conclude that the end of inflation is likely to be sufficiently violent to give a substantial amount of re-heating. The end of inflation is also likely to be sudden enough to justify assuming that the observationally relevant density perturbations crossed the causal horizon during the period when our perturbative expressions are still valid. This means that we do not need to solve the nonperturbative problem in order to make predictions.

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