Palatini variational principle for an extended Einstein-Hilbert action

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We consider a Palatini variation on a generalized Einstein-Hilbert action. We find that the Hilbert constraint—that the connection equals the Christoffel symbol—arises only as a special case of this general action, while, for particular values of the coefficients of this generalized action, the connection is completely unconstrained. We discuss the relationship between this situation and that usually encountered in the Palatini formulation. [S0556-2821(98)03108-7]

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I. INTRODUCTION

From the earliest days since the advent of general relativity, attempts have been made to generalize it. The original motivations for doing so were concerned with unifying gravitation and electromagnetism, which today have been superseded by the desire to construct a theory of quantum gravity. There are presently many attempts to this end, including the superstring-theoretic formulation [1], the connection dynamics proposal [2], non-commutative geometries [3], Chern-Simons formulations [4], gauge-theoretic formulations [5], quantization of topologies [6], topological geons [7], gravity as an induced phenonemon [8], and so on.

Throughout this history the Palatini variational principle has played a subtle but important role. As is well known, if one subjects the ordinary N-dimensional Einstein-Hilbert (EH) action

$$S_{EH}=\int d^N x[\sqrt{-g}(R(\Gamma)+16\pi\mathcal{L}_m)] \tag{1}$$

to a Palatini variation, i.e. assumes that there is no *a priori* relationship between the (torsion-free) affine connection $\Gamma^{\alpha}_{\mu\nu}$ and the metric, and thus subjects the action to a variation $\delta_\Gamma S=0$ as well as $\delta_g S=0$, one finds, in addition to the usual field equation resulting from the metric variation,

$$8\pi T_{\mu\nu}=G_{\mu\nu}(\Gamma), \tag{2}$$

from the connection variation the constraint

$$\partial_\lambda g_{\mu\nu}-\Gamma^{\eta}{}_{\lambda\mu}g_{\eta\nu}-\Gamma^{\eta}{}_{\lambda\nu}g_{\mu\eta}=0 \tag{3}$$

which is the familiar condition of metric compatibility, whose solution

$$\Gamma^{\eta}_{\mu\nu}=\begin{Bmatrix}\eta\\ \mu\quad\nu\end{Bmatrix} \tag{4}$$

is the Christoffel symbol. In other words the geometrical constraint (3) (henceforth called the "Hilbert constraint") is now a field equation that extremizes the action (1). The fact that this seemingly independent line of inquiry corroborated the metrically compatible choice of the Christoffel symbol has been viewed by many as a kind of "proof" of the validity of the Hilbert (or 2nd-order) variation of the EH action, in which Eq. (3) is given and Eq. (1) is therefore a functional only of the metric degrees of freedom. Certainly it alters the Lagrangian formulation of general relativity insofar as it removes the need to include a boundary term because there are no derivatives of field variations on the boundary [9].

However, as noted by Schrödinger long ago [10] and emphasized by Hehl *et al.* [5], in a generalized theory of gravitation one expects the geometrical relationship (3) to be modified in some manner that is typically not obvious. Hence the 2nd order variation is often not available, and one must resort to a Palatini-type of variational principle. Indeed, the Palatini approach has been employed in most of the generalized theories of (quantum) gravity mentioned above, either in terms of affine connection—metric variables or (as is common in supergravity theories [11]) spin connection—vielbien variables. Furthermore, although the physical relevance of the metrically compatible Christoffel symbol in general relativity is clear, from a geometrical perspective the singling out of the Christoffel connection is somewhat curious because the geometry is impervious to which particular connection is chosen (Christoffel or otherwise), as long as it is torsion-free.

Motivated by the above, we consider in this paper the relationship between the Palatini variational principle and the condition of metric compatibility. Since the key premise of the Palatini principle is that metric and connection are independent of one another at the outset, we consider a generalization of the EH action (1) which includes all possible terms that are at most quadratic in derivatives and/or connection variables. We then determine the circumstances under which a Palatini variational principle yields the compatibility condition (3), and what the consequent gravitational dynamics would be in situations that are more general. We work in N dimensions, and consider actions which are functionals only of the metric and the affine connection (although our approach could straightforwardly be extended to a vielbein formalism). For simplicity we consider only torsion-free connections.

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II. GENERALIZED ACTION AND CONNECTION CONSTRAINTS

If one assumes that metric and connection variables are independent of one another, then there is no longer any *a priori* reason to consider the EH action (1) as the action on which to base a theory of gravitation. One is guided only by principles of general covariance, minimal coupling, simplicity, and logical economy.

Hence we seek a Lagrangian which is a scalar under general coordinate transformations and which has the minimal number of derivatives and/or powers of the field variables in every term. Since the connection does not transform like a tensor, one must construct objects from it which have tensorial properties. The simplest of these are the Riemann curvature tensor

$$R^{\alpha}{}_{\beta\mu\nu}=\partial_{\nu}\Gamma^{\alpha}{}_{\beta\mu}-\partial_{\mu}\Gamma^{\alpha}{}_{\beta\nu}+\Gamma^{\alpha}{}_{\sigma\mu}\Gamma^{\sigma}{}_{\beta\nu}-\Gamma^{\alpha}{}_{\sigma\nu}\Gamma^{\sigma}{}_{\beta\mu} \quad (5)$$

and the covariant derivative of the metric

$$\nabla_{\lambda}g_{\mu\nu}=\partial_{\lambda}g_{\mu\nu}-\Gamma^{\eta}{}_{\lambda\mu}g_{\eta\nu}-\Gamma^{\eta}{}_{\lambda\nu}g_{\mu\eta} \quad (6)$$

where in Eqs. (5) and (6) the connection is assumed to be torsion-free.

The most general action in N dimensions that one can construct out of these objects subject to these constraints is

$$S=\int d^{N}x\sqrt{-g}[R+H(\nabla_{\nu}g^{\alpha\beta})(\nabla^{\nu}g_{\alpha\beta})+IV^{2}+J(\nabla_{\epsilon}g_{\mu\nu})\times(\nabla^{\mu}g^{\epsilon\nu})+KV\cdot Z+LZ\cdot Z], \quad (7)$$

where

$$V_{\rho}:=\frac{\nabla_{\rho}\sqrt{-g}}{\sqrt{-g}},\quad Z^{\lambda}:=\nabla_{\eta}g^{\eta\lambda} \quad (8)$$

and where the coefficients H, I, J, K and L are constants. Other scalar quantities exist, but they either can be rewritten as linear combinations of the terms in Eq. (7) up to total derivatives or they are at least cubic in derivatives and/or connection variables. Since we assume $\delta g_{\mu\nu}$ and $\delta\Gamma^{\alpha}_{\mu\nu}$ to vanish at the boundary, no additional boundary terms in Eq. (7) are required.

Variation of Eq. (1) with respect to the connection $\Gamma^{\lambda}_{\rho\sigma}$ leads to the following constraint:

$$\frac{1}{\sqrt{-g}}\left(\nabla_{\lambda}[\sqrt{-g}g^{\rho\sigma}]-\frac{1}{2}\nabla_{\epsilon}[\sqrt{-g}g^{\rho\epsilon}]\delta^{\sigma}_{\lambda}-\frac{1}{2}\nabla_{\epsilon}[\sqrt{-g}g^{\sigma\epsilon}]\delta^{\rho}_{\lambda}\right)+H[(\nabla^{\rho}g^{\sigma\gamma}+\nabla^{\sigma}g^{\rho\gamma})g_{\gamma\lambda}-\nabla^{\rho}g_{\lambda\gamma}g^{\sigma\gamma}-\nabla^{\sigma}g_{\lambda\gamma}g^{\rho\gamma}]+I[V^{\rho}\delta^{\sigma}_{\lambda}+V^{\sigma}\delta^{\rho}_{\lambda}]+J[g_{\nu\lambda}(\nabla^{\rho}g^{\sigma\nu})+\nabla_{\lambda}g^{\rho\sigma}-g^{\mu\rho}\{g^{\nu\sigma}(\nabla_{\lambda}g_{\mu\nu})+\nabla^{\sigma}g_{\mu\lambda}\}]+K\left[\frac{1}{2}(Z^{\sigma}\delta^{\rho}_{\lambda}+Z^{\rho}\delta^{\sigma}_{\lambda})-\frac{1}{2}(V^{\sigma}\delta^{\rho}_{\lambda}+V^{\rho}\delta^{\sigma}_{\lambda})-V_{\lambda}g^{\rho\sigma}\right]-L[Z^{\rho}\delta^{\sigma}_{\lambda}+Z^{\sigma}\delta^{\rho}_{\lambda}+2Z_{\lambda}g^{\sigma\rho}]=0 \quad (9)$$

whose solution determines the connection as a function of the metric in a manner which generalizes Eq. (4).

We next seek to find the conditions under which Eq. (9) may be solved for Γ in terms of the metric. Tracing Eq. (9) on the (ρ,σ) indices yields

$$[(N-3)+2I-4J-(N+1)K]V_{\lambda}+[4H+2J+K-2L(N+1)-1]Z_{\lambda}=0, \quad (10)$$

while a $\rho-\lambda$ contraction of Eq. (9) gives

$$[(N-1)+8H-2(N+1)I+4J+(N+3)K]V_{\lambda}+[(N-1)-4H-6J-(N+1)K+2(N+3)L]Z_{\lambda}=0. \quad (11)$$

Equations (10) and (11) are two equations in the two unknown vector fields V_{λ} and Z_{λ}. Provided the determinant of coefficients is non-zero, the only possible simultaneous solutions of Eqs. (10) and (11) are

$$V_{\lambda}=Z_{\lambda}=0 \quad (12)$$

which implies

$$-\nabla_{\lambda}g^{\rho\sigma}[1+2J]+(2H+J)[g_{\lambda\gamma}(\nabla^{\rho}g^{\sigma\gamma}+\nabla^{\sigma}g^{\rho\gamma})]=0 \quad (13)$$

upon insertion of Eq. (12) into Eq. (9). It is straightforward to show that

$$\Gamma^{\alpha}_{\mu\nu}=\frac{1}{2}g^{\alpha\lambda}(\partial_{\mu}g_{\alpha\nu}+\partial_{\nu}g_{\alpha\mu}-\partial_{\alpha}g_{\mu\nu})\equiv\left\{\begin{matrix}\eta\\ \mu\quad\nu\end{matrix}\right\} \quad (14)$$

is the only solution to Eq. (13) provided that $3J+2H\neq-1$ or $H\neq\frac{1}{4}$. Consequently we see that metric compatibility arises within the Palatini formalism under quite general conditions unless $3J+2H=-1$, in which case, for $J\neq-\frac{1}{2}$, it can be shown that $\Gamma^{\lambda}{}_{\mu\nu}$ is of the form

$$\Gamma^{\lambda}{}_{\mu\nu}=\left\{\begin{matrix}\lambda\\ \mu\quad\nu\end{matrix}\right\}+g^{\lambda\gamma}[\Upsilon_{\mu\gamma\nu}+\Upsilon_{\nu\gamma\mu}-2\Upsilon_{\mu\nu\gamma}] \quad (15)$$

where $\Upsilon_{\mu\nu\gamma}$ is a tensor obeying $\Upsilon_{\mu\nu\gamma}=\Upsilon_{\nu\mu\gamma}$ and $g^{\mu\nu}\Upsilon_{\mu\nu\gamma}=g^{\mu\nu}\Upsilon_{\mu\gamma\nu}$ but is otherwise arbitary. Similarly, if $H=\frac{1}{4}$, we find, again for $J\neq-\frac{1}{2}$, that $\Gamma^{\lambda}{}_{\mu\nu}$ is of the form

$$\Gamma^{\lambda}{}_{\mu\nu}=\left\{\begin{matrix}\lambda\\ \mu\quad\nu\end{matrix}\right\}+g^{\lambda\gamma}[\Lambda_{\mu\nu\gamma}+\Lambda_{\nu\gamma\mu}+\Lambda_{\gamma\mu\nu}] \quad (16)$$

where $\Lambda_{\mu\nu\gamma}=\Lambda_{\nu\mu\gamma}$ is an arbitrary tensor that is traceless on all indices. We further note that the condition that trivializes Eq. (13), i.e. $J=-\frac{1}{2}$, $H=\frac{1}{4}$, is a simultaneous solution of both of the above special cases and thus leaves $\nabla_{\lambda}g^{\rho\sigma}$ completely undetermined modulo the conditions given in Eq. (12). In this case, the Palatini variation provides almost no information about the relationship between the metric and the connection, as Eq. (12) furnishes only 8 equations to determine the 24 unknowns Γ. Furthermore, Eq. (12) would not exist if the determinant of coefficients in Eqs. (10),(11) were set to zero, thereby yielding a redundancy.

We expect that this redundancy is made manifest by some symmetry on the connection coefficients. To this end, consider the following general transformation of the connection:

$$\Gamma^{\lambda}{}_{\mu\nu} \Rightarrow \hat{\Gamma}^{\lambda}{}_{\mu\nu} = \Gamma^{\lambda}{}_{\mu\nu} + Q^{\lambda}{}_{\mu\nu}, \tag{17}$$

where $Q^{\lambda}{}_{\mu\nu}$ is an arbitrary tensor field with the sole restriction that it, like $\Gamma^{\lambda}{}_{\mu\nu}$, is symmetric in its last two indices. This type of transformation is sometimes called a deformation transformation [5]. Under the above transformation we find that the action (7) is correspondingly transformed:

$$S \Rightarrow \hat{S} = S + \delta S, \tag{18}$$

where

$$\begin{aligned}\delta S = &-[1+2J](\nabla^{\lambda} g^{\mu\nu})Q_{\lambda\mu\nu} - [2H+J](\nabla^{\lambda} g^{\mu\nu})(Q_{\mu\lambda\nu} + Q_{\nu\lambda\mu}) - [1+2H+3J]Q^{\lambda\mu\nu}Q_{\nu\mu\lambda} - [2H+J]Q^{\lambda\mu\nu}Q_{\lambda\mu\nu}\\ &+[I-K+L]Q_{\lambda}{}^{\lambda\rho}Q^{\epsilon}{}_{\epsilon\rho} + [1-K+2L]Q^{\lambda}{}_{\lambda\rho}Q^{\rho\epsilon}{}_{\epsilon} + LQ_{\rho\epsilon}{}^{\epsilon}Q^{\rho\lambda}{}_{\lambda} + [1-2I+K]V_{\lambda}Q_{\epsilon}{}^{\epsilon\lambda} + [K-1]V_{\lambda}Q^{\lambda\eta}{}_{\eta} + 2LZ^{\lambda}Q_{\lambda\eta}{}^{\eta}\\ &+[1+2L-K]Z^{\lambda}Q^{\eta}{}_{\eta\lambda}.\end{aligned} \tag{19}$$

For $\Gamma^{\lambda}{}_{\mu\nu}$ to be completely unconstrained, we must have $\delta S = 0$ regardless of the choice of $Q^{\lambda}{}_{\mu\nu}(Q_{\lambda\mu\nu})$. This can only happen if

$$H = \frac{1}{4}, \quad J = -\frac{1}{2}, \quad I = K = 1, \quad L = 0, \tag{20}$$

which we note ensures that the determinant of coefficients in the system (10),(11) vanishes.

Conversely, consider subsitution of Eq. (17) for $\Gamma^{\lambda}{}_{\mu\nu}$, into the general action (7), and then varying the (transformed) action with respect to $Q^{\lambda}{}_{\mu\nu}$. This yields a set of complicated algebraic equations for $Q^{\lambda}{}_{\mu\nu}$. Insertion into Eq. (7) of their solution for $Q^{\lambda}{}_{\mu\nu}$ in terms of $\Gamma^{\lambda}{}_{\mu\nu}$ and $g_{\mu\nu}$ leads directly to a modified action of the form given in Eq. (7) whose specific values for H,I,J,K,L are given by Eqs. (20) above.[1]

In other words, Eqs. (20) clearly compose the unique set of values such that our action is invariant under the transformation (17) with $Q^{\lambda}{}_{\mu\nu}$ completely unconstrained other than being symmetric in its lower two indices. Accordingly, the values (20) will henceforth be called the ''maximally symmetric'' values.[2]

From this perspective one can say that the compatibility condition (3), obtained by applying the Palatini variational principle to the EH action, is an example of a constraint induced by a broken symmetry. That is, the EH action is a special case of our general action (7) above, with the particular requirement that $H = I = J = K = L = 0$. That these values of H,I,J,K,L break the general symmetry is obvious from the above analysis, and it is this breaking of this ''connection symmetry'' which singles out the Christoffel symbol.

III. EXTENDED ACTION DYNAMICS

Momentarily putting aside our consideration of the ''connection dynamics'' of our extended action and calculating the ordinary ''metric dynamics,'' we find

$$\begin{aligned}8\pi T_{\mu\nu} = &G_{(\mu\nu)}(\Gamma) + (I-K)V_{\mu}V_{\nu} - \frac{1}{2}K[\nabla_{\mu}V_{\nu} + \nabla_{\nu}V_{\mu}] - 2(\nabla_{\lambda} + V_{\lambda})g^{\lambda\epsilon}\left[H\nabla_{\epsilon}g_{\mu\nu} + \frac{1}{2}J(\nabla_{\mu}g_{\nu\epsilon} + \nabla_{\nu}g_{\mu\epsilon})\right] - L[V_{\mu}Z_{\nu}\\ &+V_{\nu}Z_{\mu} + \nabla_{\mu}V_{\nu} + \nabla_{\nu}V_{\mu} + Z_{\mu}Z_{\nu}] + H[(\nabla_{\mu}g^{\alpha\beta})(\nabla_{\nu}g_{\alpha\beta}) + 2g^{\alpha\beta}(\nabla^{\lambda}g_{\alpha\mu})(\nabla_{\lambda}g_{\beta\nu})] + \frac{1}{2}J[(\nabla^{\eta}g_{\alpha\mu})(\nabla^{\alpha}g_{\nu\eta})\\ &+(\nabla^{\eta}g_{\alpha\nu})(\nabla^{\alpha}g_{\mu\eta})] + g_{\mu\nu}\left\{-\frac{1}{2}H(\nabla_{\rho}g^{\alpha\beta})(\nabla^{\rho}g_{\alpha\beta}) + \frac{1}{2}IV^{2} - \frac{1}{2}J(\nabla_{\epsilon}g_{\alpha\beta})(\nabla^{\alpha}g^{\epsilon\beta}) - \frac{1}{2}LZ^{2} + \nabla_{\epsilon}\left(IV^{\epsilon} + \frac{1}{2}KZ^{\epsilon}\right)\right\}\end{aligned} \tag{21}$$

[1]See the Appendix for more explicit details.

[2]''Maximally'' symmetric to distinguish them from other partial symmetries which may occur when one assumes some particular tensorial structure in $Q^{\lambda}{}_{\mu\nu}$.

upon variation of Eq. (7) with respect to the metric. Provided the constants H,I,J,K,L are chosen so that Eq. (14) is satisfied (i.e. the coefficients are chosen so that $3J+2H\neq-1$ and $H\neq\frac{1}{4}$), then all terms on the right-hand side of Eq. (21) vanish except for the first one, which becomes the usual expression for the Einstein tensor in terms of the metric.

Consider next the condition of maximal symmetry. Insertion of our maximally symmetric values, Eqs. (20), into the above dynamical equation yields

$$\begin{aligned}8\pi T_{\mu\nu}=G_{(\mu\nu)}(\Gamma)&+\frac{1}{4}[\nabla_{\mu}P_{\nu\eta}{}^{\eta}+\nabla_{\nu}P_{\mu\eta}{}^{\eta}]\\&+\left[\nabla_{\lambda}-\frac{1}{2}(P_{\lambda\eta}{}^{\eta})\right](E^{\lambda}{}_{\mu\nu})+\frac{1}{4}[2(P^{\lambda\beta}{}_{\mu})(P_{\lambda\beta\nu})\\&-(P_{\mu}{}^{\lambda\eta})(P_{\nu\lambda\eta})-2(P^{\lambda}{}_{\eta\mu})(P^{\eta}{}_{\lambda\nu})]\\&+\frac{1}{8}g_{\mu\nu}[2(P_{\lambda\eta}{}^{\eta})(P^{\lambda\rho}{}_{\rho})-2(P_{\epsilon\lambda\eta})(P^{\lambda\epsilon\eta})\\&+(P_{\lambda\eta\epsilon})(P^{\lambda\eta\epsilon})+4\nabla_{\epsilon}(P_{\lambda}{}^{\lambda\epsilon}-P^{\epsilon\eta}{}_{\eta})]\end{aligned}\tag{22}$$

where

$$P_{\eta}{}^{\mu\nu}:=\nabla_{\eta}g^{\mu\nu}\tag{23}$$

and

$$E^{\lambda}{}_{\mu\nu}:=\frac{1}{2}[P^{\lambda}{}_{\mu\nu}-P_{\mu\nu}{}^{\lambda}-P_{\nu\mu}{}^{\lambda}]=\left[\left\{\begin{matrix}\lambda\\ \mu\ \ \nu\end{matrix}\right\}-\Gamma^{\lambda}{}_{\mu\nu}\right],\tag{24}$$

thus enabling us to put some terms directly in terms of the Christoffel symbol.

Hence the field equations in the case of maximal symmetry consist of Eq. (22) alone—there is no equation which determines the connection in terms of the metric. In this sense the maximally symmetric action is a theory of gravity determined in terms of metric dynamics alone, with the connection freely specifiable.

Since the connection may be freely specified, one choice is to make it compatible with the metric, i.e. to demand that Eq. (14) hold. In this case all $P_{\eta}{}^{\mu\nu}=0$, and Eq. (22) reduces to

$$8\pi T_{\mu\nu}=G_{(\mu\nu)}(\{\ \ \})\tag{25}$$

which are the field equations for general relativity. Alternatively, suppose we choose $\Gamma^{\eta}{}_{\mu\nu}=0$. In this case Eq. (22) becomes

$$\begin{aligned}8\pi T_{\mu\nu}=&+\frac{1}{4}[\nabla_{\mu}\hat{P}_{\nu\eta}{}^{\eta}+\nabla_{\nu}\hat{P}_{\mu\eta}{}^{\eta}]\\&+\left[\nabla_{\lambda}-\frac{1}{2}\hat{P}_{\lambda\eta}{}^{\eta}\right]\left\{\begin{matrix}\lambda\\ \mu\ \ \nu\end{matrix}\right\}+\frac{1}{4}[2(\hat{P}^{\lambda\beta}{}_{\mu})(\hat{P}_{\lambda\beta\nu})\\&-(\hat{P}_{\mu}{}^{\lambda\eta})(\hat{P}_{\nu\lambda\eta})-2(\hat{P}^{\lambda}{}_{\eta\mu})(\hat{P}^{\eta}{}_{\lambda\nu})]\\&+\frac{1}{8}g_{\mu\nu}[2(\hat{P}_{\lambda\eta}{}^{\eta})(\hat{P}^{\lambda\rho}{}_{\rho})-2(\hat{P}_{\epsilon\lambda\eta})(\hat{P}^{\lambda\epsilon\eta})\\&+(\hat{P}_{\lambda\eta\epsilon})(\hat{P}^{\lambda\eta\epsilon})+4\nabla_{\epsilon}(\hat{P}_{\lambda}{}^{\lambda\epsilon}-\hat{P}^{\epsilon\eta}{}_{\eta})]\end{aligned}\tag{26}$$

where $\hat{P}_{\eta}{}^{\mu\nu}:=\partial_{\eta}g^{\mu\nu}$. Further simplification of the right-hand side of Eq. (26) yields

$$8\pi T_{\gamma\sigma}=G_{(\gamma\sigma)}(g)\tag{27}$$

where $G_{(\gamma\sigma)}(g)$ is the Einstein tensor expressed as a functional of the metric, i.e. $G_{(\gamma\sigma)}(g)=G_{(\gamma\sigma)}(\{\ \ \})$. Hence Eq. (27) also yields the equations of general relativity.

The above case of examining $\Gamma=0$ raises an interesting curiosity. Clearly, as the maximally symmetric case only restricts the connection to be torsion-free, $\Gamma=0$ is an available option. But the fact that we are able to choose such a connection *globally* enables us to say something additional about the geometry of our manifold—namely that it is flat or, rather, that it can be made flat with no physical sacrifice.

The preceding situation is also a generalization of a result obtained by Gegenberg *et al.* for $(1+1)$ gravity [12]. Consider the action (7) for $N=2$ with each of H,I,J,K,L set to zero. In this case the determinant of coefficients in Eqs. (10) and (11) vanishes, and the general solution to Eq. (9) is given by [12]

$$\Gamma^{\alpha}_{\mu\nu}=\overline{G}^{\alpha}_{\mu\nu}=\left\{\begin{matrix}\eta\\ \mu\ \ \nu\end{matrix}\right\}+(\delta^{\alpha}_{\mu}B_{\nu}+\delta^{\alpha}_{\nu}B_{\mu}-g_{\mu\nu}B^{\alpha})\tag{28}$$

where B_{μ} is an arbitrary vector field. The Einstein tensor is given by

$$\begin{aligned}G_{(\sigma\gamma)}(\overline{G})&=G_{(\sigma\gamma)}(\{\ \ \})\\&=0\end{aligned}\tag{29}$$

and so renders the $(1+1)$-dimensional field equations trivial, as in the usual Hilbert case. We see from the preceding analysis of Eq. (22) that an analogous situation holds in higher dimensions for the maximally symmetric action: although the field equations do not determine the connection in terms of the metric, one can choose the connection to be compatible with the metric by appropriately choosing $Q^{\alpha}{}_{\mu\nu}$ in Eq. (17) and recover the metric field equations of general relativity.

More generally, the choice of connection is completely irrelevant to the theory in the maximally symmetric case. One has only Eq. (22), which determines the evolution of the metric in terms of the basic matter fields.

IV. CONCLUSIONS

From the connection-dynamics perspective we have adopted in this paper, the most general action which is 2nd order in connection and derivatives is given by Eq. (7). In the usual formulation of the Palatini principle the H,I,J,K,L coefficients are all set to zero. We have shown that there exists a unique choice of these coefficients, given by Eq. (20), such that the action is invariant under Eq. (17). This case of maximal symmetry yields a theory of gravity which is independent of the connection.

From this perspective the condition of metric compatibility (3) in the usual Palatini formulation arises as a field equation because this formulation breaks the maximal symmetry condition (20), hence uniquely determining the connection. The equations of general relativity are recovered as a consequence of this broken symmetry.

In the maximally symmetric case we also recover the field equations of general relativity but for a different reason. In this case the connection may be freely chosen by an appropriate choice of $Q^{\alpha}{}_{\mu\nu}$ in Eq. (17), and so choosing it to be metrically compatible obviously yields the metric field equations of general relativity. However, these equations are recovered even if one does not choose the connection to be compatible, as shown by the choice $\Gamma^{\alpha}{}_{\mu\nu}=0$ in the preceding section.

Classically, then, it would appear that maximally symmetric theories in the Palatini formulation are classically equivalent to their broken counterparts, at least insofar as metric dynamics is concerned. The role of maximally symmetric theories in quantum gravity is, however, not clear, and would be interesting to study further.

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APPENDIX

The following is a proof of the claim made at the end of Sec. II:

If one begins with our usual generalized action, with H,I,J,K,L arbitrary, that is,

$$S=\int d^Nx\sqrt{-g}[R+H(\nabla_{\nu}g^{\alpha\beta})(\nabla^{\nu}g_{\alpha\beta})+IV^2+J(\nabla_{\epsilon}g_{\mu\nu})\times(\nabla^{\mu}g^{\epsilon\nu})+KV\cdot Z+LZ\cdot Z], \tag{A1}$$

and applies to it the variation

$$\Gamma^{\lambda}{}_{\mu\nu}\Rightarrow\hat{\Gamma}^{\lambda}{}_{\mu\nu}=\Gamma^{\lambda}{}_{\mu\nu}+Q^{\lambda}{}_{\mu\nu}, \tag{A2}$$

we find that S consequently transforms to

$$S\Rightarrow\hat{S}=S+\delta S, \tag{A3}$$

where

$$\delta S=-[1+2J](\nabla^{\lambda}g^{\mu\nu})Q_{\lambda\mu\nu}-[2H+J](\nabla^{\lambda}g^{\mu\nu})(Q_{\mu\lambda\nu}+Q_{\nu\lambda\mu})-[1+2H+3J]Q^{\lambda\mu\nu}Q_{\nu\mu\lambda}-[2H+J]Q^{\lambda\mu\nu}Q_{\lambda\mu\nu}$$
$$+[I-K+L]Q_{\lambda}{}^{\lambda\rho}Q^{\epsilon}{}_{\epsilon\rho}+[1-K+2L]Q^{\lambda}{}_{\lambda\rho}Q^{\rho\epsilon}{}_{\epsilon}+LQ_{\rho\epsilon}{}^{\epsilon}Q^{\rho\lambda}{}_{\lambda}+[1-2I+K]V_{\lambda}Q_{\epsilon}{}^{\epsilon\lambda}+[K-1]V_{\lambda}Q^{\lambda\eta}{}_{\eta}+2LZ^{\lambda}Q_{\lambda\eta}{}^{\eta}$$
$$+[1+2L-K]Z^{\lambda}Q^{\eta}{}_{\eta\lambda}. \tag{A4}$$

Now if we subject this new action, $\hat{S}$, to a variation with respect to $Q^{\lambda}{}_{\alpha\beta}$, we clearly have

$$\delta_{Q^{\lambda}{}_{\alpha\beta}}\hat{S}=\delta_{Q^{\lambda}{}_{\alpha\beta}}(\delta S),$$

since $\delta_{Q^{\lambda}{}_{\alpha\beta}}(S)=0$. Now, from the above we see that $\delta_{Q^{\lambda}{}_{\alpha\beta}}\hat{S}=0=\delta_{Q^{\lambda}{}_{\alpha\beta}}(\delta S)$ can be expressed as

$$0=\int d^Nx\sqrt{-g}(\delta Q^{\lambda}_{\alpha\beta})\Bigg[-(1+2J)\nabla_{\lambda}g^{\alpha\beta}-(2H+J)g_{\lambda\mu}[\nabla^{\alpha}g^{\mu\beta}+\nabla^{\beta}g^{\mu\alpha}]-[1+2H+3J](Q^{\alpha\beta}{}_{\lambda}+Q^{\beta\alpha}{}_{\lambda})$$
$$-2(2H+J)Q_{\lambda}{}^{\alpha\beta}+[I-K+L][(Q_{\epsilon}{}^{\epsilon\beta})\delta^{\alpha}_{\lambda}+(Q_{\epsilon}{}^{\epsilon\alpha})\delta^{\beta}_{\lambda}]+[1-K+2L]\left(\frac{1}{2}[(Q^{\beta\epsilon}{}_{\epsilon})\delta^{\alpha}_{\lambda}+(Q^{\alpha\epsilon}{}_{\epsilon})\delta^{\beta}_{\lambda}]+Q^{\epsilon}{}_{\epsilon\lambda}g^{\alpha\beta}\right)$$
$$+g^{\alpha\beta}[2L(Q_{\lambda\epsilon}{}^{\epsilon}+Z_{\lambda})+(K-1)V_{\lambda}]+\frac{1}{2}[(1-2I+K)V^{\beta}+(1+2L-K)Z^{\beta}]\delta^{\alpha}_{\lambda}+\frac{1}{2}[(1-2I+K)V^{\alpha}+(1+2L-K)Z^{\alpha}]\delta^{\beta}_{\lambda}\Bigg]. \tag{A5}$$

Clearly, for arbitrary $\delta Q^{\lambda}{}_{\alpha\beta}$, we have the constraint that the coefficient in square brackets vanishes. Taking the $g_{\alpha\beta}$ trace of this quantity yields

$$AQ^{\epsilon}{}_{\epsilon\lambda}+B[Q_{\lambda\epsilon}{}^{\epsilon}+Z_{\lambda}]+CV_{\lambda}=0 \tag{A6}$$

while contracting over, say, λ and α yields

$$DQ^{\epsilon}{}_{\epsilon\lambda}+E[Q_{\lambda\epsilon}{}^{\epsilon}+Z_{\lambda}]+FV_{\lambda}=0 \tag{A7}$$

where

$$A=[(N-2)-4H+2I-6J-K(N+2)+2L(N+1)] \tag{A8}$$

$$B=[1-4H-2J-K+(1+N)L] \tag{A9}$$

$$C=[(3-N)-2I+4J+(1+N)K] \tag{A10}$$

$$D=[-6H+(N+1)I-5J-(N+2)K+(N+3)L] \tag{A11}$$

$$E=\left[\frac{1}{2}(N-1)-2H-3J-\frac{1}{2}(N+1)K+(N+3)L\right] \tag{A12}$$

$$F=\left[\frac{1}{2}(N-1)+4H-(N+1)I+2J+\frac{1}{2}(N+3)K\right]. \tag{A13}$$

We note the following relationships:

$$BD-AE=CE-BF \tag{A14}$$

and

$$F+D=E. \tag{A15}$$

Meanwhile, together Eqs. (A6) and (A7) imply the following:

$$[BD-AE]Q^{\epsilon}{}_{\epsilon\lambda}+[BF-CE]V_{\lambda}=0. \tag{A16}$$

Therefore, Eqs. (A14), (A15) and (A16) in turn imply

$$Q^{\epsilon}{}_{\epsilon\lambda}=V_{\lambda} \tag{A17}$$

and

$$Q_{\lambda\epsilon}{}^{\epsilon}=-(V_{\lambda}+Z_{\lambda}). \tag{A18}$$

Inserting Eqs. (A17) and (A18) into the coefficient of δQ in Eq. (A5) yields, after a bit of symmetrization and manipulation,

$$Q_{\lambda\alpha\beta}=\frac{1}{2}[P_{\lambda\alpha\beta}-P_{\alpha\beta\lambda}-P_{\beta\lambda\alpha}] \tag{A19}$$

where $P_{\mu}{}^{\nu\rho}=\nabla_{\mu}g^{\nu\rho}$ and $P_{\mu\nu\lambda}=-\nabla_{\mu}g_{\nu\lambda}$. Inserting Eqs. (A17), (A18) and (A19) into Eq. (19) gives

$$\delta S=-S+\int d^{N}x\sqrt{-g}\left[R+\frac{1}{4}(\nabla_{\alpha}g^{\mu\nu})(\nabla^{\alpha}g_{\mu\nu})-\frac{1}{2}(\nabla_{\epsilon}g_{\mu\nu})\times(\nabla^{\mu}g^{\epsilon\nu})+V^{2}+V\cdot Z\right], \tag{A20}$$

in other words, our maximally symmetric values.

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