Late time behavior of the maximal slicing of the Schwarzschild black hole

R. Beig*
Institut für Theoretische Physik, Universität Wien, Boltzmanngasse 5, A-1090 Vienna, Austria

N. Ó Murchadha†
Physics Department, University College Cork, Cork, Ireland
and Erwin Schrödinger International Institute for Mathematical Physics, Boltzmanngasse 9, A-1090 Vienna, Austria

(Received 16 June 1997; published 6 March 1998)

A time-symmetric Cauchy slice of the extended Schwarzschild spacetime can evolve into a foliation of the $r>3m/2$ region of spacetime by maximal surfaces with the requirement that time run equally fast at both spatial ends of the manifold. This paper studies the behavior of these slices in the limit as proper time at infinity becomes arbitrarily large. It is shown that the central lapse decays exponentially and an analytic expression is given both for the exponent and for the preexponential factor. [S0556-2821(98)02708-8]

PACS number(s): 04.20.Cv

I. INTRODUCTION

Maximal slices have been intensively studied, first to construct initial data for asymptotically flat solutions to the Einstein equations and second to investigate the evolving spacetime. In each case one obtains an elliptic equation with a unique solution (modulo boundary conditions) for a conformal factor and the lapse function, respectively.

A maximal slice is defined by the requirement that the trace of the extrinsic curvature vanish. This is equivalent to demanding that the Lie derivative along the normal to the slice of $\sqrt{g}$ vanish. The Schwarzschild solution has a surface-orthogonal timelike Killing vector in the exterior quadrants. Any spacelike slice perpendicular to this Killing vector has vanishing extrinsic curvatures and is obviously maximal. Thus each of the $t=$const slices in the standard coordinates is maximal.

However, this slicing is not a foliation. The lapse function is zero at the bifurcation ''point'' (actually, a two-sphere), which is a fixed point of the slicing, and this slicing looks antisymmetric in the extended Schwarzschild picture. As it runs forward in the right-hand quadrant, it runs backwards in the left-hand quadrant. It never enters the $r<2m$ region.

A very different spherically symmetric slicing exists. This is one where the lapse function along a central ''axis'' (actually, a central cylinder) does not vanish and the slices do enter the central quadrants. This slicing, or rather the special version which is symmetric across the central axis, has been investigated in the past by a mixture of numerical and analytic techniques [1,2].

It is apparent from the numerics that this slicing is a foliation; the slices do not seem to cross. It is a very unusual foliation, however, the first concrete example of the phenomenon that came to be called ''the collapse of the lapse.'' The central lapse goes to zero so that the slices never pass beyond $r=3m/2$. In this article we reanalyze this foliation focusing especially on the late time behavior of the central lapse. We show that it goes to zero exponentially quickly and explicitly display both the leading exponent and the coefficient multiplying it.

In this work we study the time function τ on the Schwarzschild black hole spacetime having the following properties.

(i) The level sets of τ result from evolution of a time-symmetric Cauchy slice of Schwarzschild spacetime by maximal surfaces under the additional requirement that the proper time for asymptotic observers at infinity, which are at rest relative to the slicing, runs equally fast at both spatial ends.

(ii) The time function τ is zero on the time-symmetric slice and coincides with the proper time of the infinite observers. (This means that α, the lapse of the time function, goes to 1 at both infinities along each slice.)

Note that (i) is really a property only of the slicing defined by τ rather than τ itself. This time function, which has first been considered in [1,2], has two key properties: The first property is that τ takes all real values or, in other words, the future singularity at $r=0$ does not prevent τ from assuming arbitrarily large positive values (and similarly for the past). It is believed that this property holds on vacuum spacetimes more general than Schwarzschild spacetime. Here it is important to realize that such spacetimes are not ''given'' to us. Rather, they have to be generated by a Cauchy problem: One first constructs regular asymptotically flat initial data, satisfying the vacuum constraints, say, maximal, and then tries to evolve these in time by analytical or numerical means. Doing this involves an *a priori* choice of gauge which in particular implies that the resultant globally hyperbolic spacetime comes already equipped with a specific time function. Suppose the initial data has a future-trapped surface. Then, by the Penrose singularity theorem [3], any Cauchy-evolved spacetime is singular in the sense of having future-incomplete null geodesics. (Similar conclusions, but in both the future and past directions, hold when the initial data has an outer-trapped surface [4] or when the topology is nontrivial, e.g., in the sense that there is more than one

*Email address: BEIG@PAP.UNIVIE.AC.AT; Fax: ++43-1-317-22-20.
†Email address: NIALL@BUREAU.UCC.IE; Fax: +353-21-276949.

asymptotic end [5].) Many maximal initial data sets having one of these properties exist (for trapped surfaces, see [6]). There is the conjecture, due to Moncrief and Eardley [7], that if one evolves the initial data in a gauge where the whole slicing is maximal and τ is the proper time at infinity, the evolution should be extendable to arbitrarily large values of τ, irrespective of whether singularities form or not. This global existence result, if true, would, in spirit at least, go a long way toward settling in the affirmative the Penrose cosmic censorship hypothesis [8] in the case of asymptotically flat vacuum data. The spacetime evolved in the way described, in the Schwarzschild case, has the second property that it is in fact extendable: There are no maximal spherically symmetric Cauchy slices of Schwarzschild spacetime reaching radii less than or equal to $r=3m/2$. Thus maximal slices of Schwarzschild spacetime ''avoid the singularity at $r=0$.'' It is this last property which numerical relativists expect to be true for evolutions of more general initial data and which is clearly desirable if numerical codes based on maximal slicings are used.

Take any observer at rest relative to the slicing defined by τ (''Eulerian observer''). Then $\int \alpha d\tau$ along the trajectory of that observer is her or his proper time. Since proper time is finite as the slicing approaches the limiting maximal slice at $r=3m/2$, we must have $\int \alpha d\tau<\infty$, and thus $\lim_{\tau\to\infty}\alpha(\tau)=0$ (''collapse of the lapse'' [9]). Our main result is that, along the Eulerian observers going through the bifurcation two-sphere,

$$\alpha(\tau)\sim\frac{4}{3\sqrt{2}}\exp\left(\frac{4A}{3\sqrt{6}}\right)\exp\left(-\frac{4\tau}{3\sqrt{6}m}\right)\quad\text{as}\quad\tau\to\infty, \tag{1.1}$$

where the constant A is given by Eq. (3.41), below. The exponent Eq. (1.1) has been estimated before [1,10] by a mixture of numerical and model calculations. The estimate in [10] of this exponent is 1.82, which agrees quite closely with our exact $3\sqrt{6}/4\sim1.83$. We hope that our result, Eq. (1.1), will be useful for the numerists as an accurate test for codes based on maximal slicings. An extension of the work here to the late time behavior of α along the trajectories of arbitrary Eulerian observers will appear elsewhere [17]. It remains to be seen whether our results, which are strongly tied to spherical symmetry, shed any light on the general situation.

Our plan is as follows. In Sec. II we review some generalities on lapse functions and foliations. Then we give a precise definition of the time function under study. In Sec. III we perform the asymptotic analysis leading to Eq. (1.1). In Appendix A we essentially rederive the Schwarzschild metric in terms of spherically symmetric maximal Cauchy data. In Appendix B we prove a calculus lemma which is basic to our analysis.

II. GENERALITIES

Let (M,ds^2) be a globally hyperbolic spacetime and $\tau:M\to\mathbf{R}$ a time function, i.e., a function the level sets of which form a foliation $\mathcal{F}_\tau$ of M by Cauchy surfaces $\cong\Sigma$. Then the function $\alpha:M\to\mathbf{R}$ defined by

$$\alpha:=[-(\nabla\tau)^2]^{-1/2} \tag{2.1}$$

is called the *lapse* of $\mathcal{F}_\tau$. The reason for this name is that α measures the ''lapse of proper time'' along trajectories normal to the leaves of $\mathcal{F}_\tau$ as a function of τ. To make this explicit, define the vector field τ^μ by

$$\tau^\mu=-\alpha^2\nabla^\mu\tau\Rightarrow\tau^\mu\nabla_\mu\tau=1, \tag{2.2}$$

which is timelike and future (i.e., increasing τ) pointing. We assume for simplicity that the map τ is onto whence the vector field τ^μ is complete. Then the vector τ^μ yields an orthogonal decomposition of M as $M=\mathbf{R}\times\Sigma$, as follows. Construct a diffeomorphism $\varphi:\mathbf{R}\times\Sigma$, i.e., $\varphi:(\lambda,y^i)\in\mathbf{R}\times\Sigma\mapsto x^\mu=\varphi^\mu_\lambda(y^i)\in M$, by

$$\dot\varphi^\mu_\lambda(y):=\frac{d}{d\lambda}\varphi^\mu_\lambda(y)=\tau^\mu(\varphi_\lambda(y)),$$

$$\tau(\varphi_0(y))=0. \tag{2.3}$$

It follows from Eq. (2.2) that $\tau(\varphi_\lambda(y))=\lambda$, which further implies that

$$\tau_{,\mu}(\varphi_\lambda(y))\varphi^\mu_{\lambda,i}(y)=0\Rightarrow\dot\varphi^\mu_\lambda(y)\varphi^\nu_{\lambda,i}(y)g_{\mu\nu}(\varphi_\lambda(y))=0. \tag{2.4}$$

Thus λ, viewed as a function on M, coincides with τ and the lines of constant y^i are orthogonal trajectories to $\mathcal{F}_\tau$. Consequently, in (τ,y^i) coordinates, the metric takes the form

$$\begin{aligned}\varphi^*_\tau(ds^2)&=\dot\varphi^\mu_\tau\dot\varphi^\nu_\tau g_{\mu\nu}d\tau^2+\varphi^\mu_{\tau,i}\varphi^\nu_{\tau,j}g_{\mu\nu}dy^idy^j\\&=g_{\tau\tau}(\tau,y)d\tau^2+g_{ij}(\tau,y)dy^idy^j,\end{aligned} \tag{2.5}$$

where g_{ij} is the induced metric on the leaves and

$$g_{\tau\tau}(\tau,y)=-\alpha^2(\varphi_\tau(y)). \tag{2.6}$$

Thus, along $y^i=\text{const}$, the proper time s is given by

$$s=\int\alpha(\varphi_{\tau'}(y))d\tau'. \tag{2.7}$$

Note that, when τ' is another time function giving the same foliation, i.e., $\tau'=\tau'(\tau)$, the lapse α changes according to $\alpha'=(d\tau'/d\tau)^{-1}\alpha$. Suppose now we are given another vector field ξ^μ on M. This can be uniquely decomposed,

$$\xi^\mu=Nn^\mu+X^\mu,\quad X^\mu n_\mu=0, \tag{2.8}$$

where $n^\mu=-\alpha\nabla^\mu\tau$, is the future normal of $\mathcal{F}_\tau$. To distinguish N from α, we call N the *boost function* of ξ^μ relative to $\mathcal{F}_\tau$. If N is nonzero on some leaf Σ_{τ_0}, it can be viewed as the restriction to Σ_{τ_0} of the lapse of the time function t' obtained by $\xi^\mu\nabla_\mu t'=0$, $t'|_{\Sigma_{\tau_0}}=\text{const}$.

We have the relation

$$N=\alpha\xi^\mu\nabla_\mu\tau, \tag{2.9}$$

which is, of course, trivial in the present context, but will be extremely useful in our computation of the lapse α of a maximal foliation of the extended Schwarzschild spacetime, where ξ^μ can be chosen as the ''static'' Killing vector.

We now recall some features of Schwarzschild spacetime which are used in our construction. In the exterior region $r > 2m > 0$, we have

$$ds^2 = -\left(1-\frac{2m}{r}\right)dt^2 + \left(1-\frac{2m}{r}\right)^{-1}dr^2 + r^2 d\Omega^2,$$
$$-\infty < t < \infty. \quad (2.10)$$

Here ds^2 can be smoothly extended across $r=2m$ to the Kruskal spacetime M on which r is a globally defined function $r: M \rightarrow \mathbf{R}^+$, which has saddle points at $\mathcal{S}$, the bifurcation two-sphere of the horizon. The Killing vector field $\partial/\partial t$ extends to a global Killing vector field ξ^μ on M which is spacelike in the interior, i.e., black and white hole, regions, null on the horizon and zero on $\mathcal{S}$. Both the black hole region and the right exterior region can be written in the form (2.10) with the understanding that the functions (θ,φ) and r together with the retarded Eddington-Finkelstein coordinate

$$u = t - r - 2m \ln|r-2m| \quad (2.11)$$

covers both regions and the horizon at $r=2m$. The function t goes to ∞ at the right component (where ‘‘right’’ refers to the original unextended spacetime) and goes to $-\infty$ at the left horizon. The set where t vanishes is the union of S, the original $t=0$ spacelike hypersurface (extended in the obvious way to the left exterior region) and the timelike, totally geodesic cylinder Γ, which is ruled by timelike radial geodesics through $\mathcal{S}$ which are orthogonal to $\mathcal{S}$ and which hit the singularity as $r \rightarrow 0$. Since r is constant along the trajectories of ξ^μ and r is, by Eq. (2.10), an ‘‘areal radius,’’ it follows that every spherically symmetric spacelike slice has a spherical minimal surface (a ‘‘throat’’) exactly where it is tangential to ξ^μ [which, of course, can only happen in the interior and it necessarily has to happen there for slices leaving to the other (left) exterior region].

Consider the function $h(r,C)$ given by

$$h(r,C) = -\int_{r_C}^{r} \frac{C}{(1-2m/x)(x^4-2mx^3+C^2)^{1/2}}dx, \quad (2.12)$$

where the integral is to be understood in the Cauchy-principal-value sense for $r>2m$ and where $0<C<3(\sqrt{3}/4)m^2$, $r>r_C$, and r_C is the unique root of $P(x)=x^4-2mx^3+C^2$ for this range of C in the interval $3m/2<r_C<2m$. For $x>r_C$, we have $P(x)>0$. Thus $h(r,C)+r+2m \ln|r-2m|$ depends smoothly on (r,r_C). We easily infer that

$$t = h(r,C) \quad (2.13)$$

defines, for each fixed C, a spacelike slice Σ_C which smoothly extends to the black hole region, where it intersects Γ at $r=r_C$.

In order to see that this surface extends smoothly and symmetrically through Γ, we use for $r<2m$ the parameter

$$l(r) = \int_{r_C}^{r} \frac{x^2 dx}{[P(x)]^{1/2}}, \quad (2.14)$$

which is the proper distance along the slice, as can either be seen from Eqs. (2.10), (2.12), or (2.13) or from Appendix A. Then, from Eqs. (2.12), (2.14), we have the system of ordinary differential equations (ODE's)

$$\frac{dh}{dl} = -\frac{C}{r^2-2mr},$$
$$\frac{d^2r}{dl^2} = \frac{m}{r^2} - \frac{2C^2}{r^5}, \quad (2.15)$$

with $h(0)=0$, $r(0)=r_C$, $(dr/dl)(0)=0$, which is regular at $l=0$. Thus the function r along the slice is symmetric with respect to $l=0$ and smooth. This implies that $dr/dl=(1-2m/r+C^2/r^4)^{1/2}$ is antisymmetric.

Next, we observe that the level sets of $\sigma = t - h(r,C)$, for fixed C in the allowed range, give rise to maximal surfaces on the Kruskal manifold; i.e., they satisfy

$$\nabla^\mu([-(\nabla\sigma)^2]^{-1/2}\nabla_\mu\sigma) = 0. \quad (2.16)$$

The function σ is not the time function of interest to us (in fact, σ being not differentiable at r_C, it does not define a global foliation). Rather this local foliation arises from moving a given maximal slice, say, $\sigma=0$, along the flow of $\xi^\mu = (\partial/\partial t)^\mu$. The function $N=[-(\nabla\sigma)^2]^{-1/2}$ is nothing but the boost function of $\partial/\partial t$ relative to $\sigma=0$. There exists an explicit solution of Eq. (2.14) due to Reinhart [2]. He, essentially by guessing, found N to be

$$N = \left(1-\frac{2m}{r}+\frac{C^2}{r^4}\right)^{1/2} \quad (2.17)$$

and from this inferred Eq. (2.12). For a more illustrative derivation from the initial-value point of view, see Appendix A. Note that N as a function of l is antisymmetric relative to $l=0$.

We now claim that the surface $t=h(r,C')$ lies everywhere in the future of $t=h(r,C)$ when $C'>C$ and that $t=h(r,C)$ lies to the future of $S=\Sigma_0$. It is interesting that we are unable to see this from the explicit integral (2.12). Instead, we first compute $(d/dC)r_C$ from

$$r_C^4 - 2mr_C^3 + C^2 = 0, \quad (2.18)$$

to yield

$$\frac{dr_C}{dC} = -\frac{2C}{4r_C^3(1-3m/2r_C)} < 0. \quad (2.19)$$

Thus the claimed behavior is true at least along the throat. Next, observe that our slices are asymptotically flat at both spatial ends and that $t_\infty(C) = \lim_{r\rightarrow\infty} h(r,C)$ exists. Suppose that $h(R,C)=h(R,C')$ for some $R>r_C$ to the right of Γ. Then, by the symmetry with respect to Γ, this would have to happen also to the left of Γ. Thus we would have a lens-shaped region spanned by two maximal slices. But this, by an elegant argument due to Brill and Flaherty [11], is impossible, except if the two slices are identical, which they are not in our case. This argument continues to be valid for $R=\infty$. Thus $h(r,C)$ monotonically increases with C for fixed r and so does $t_\infty(C)$.

It follows that the equation $t=h(r,C)$ can be solved for C to yield a smooth time function defined on the $r<3m/2$ subset of the part of Kruskal lying in the future of the Cauchy slice S. Here C labels the leaves of the foliation we are interested in, but it is not yet the time function we want: Rather, this is obtained by eliminating C in terms of τ using the relation

$$\tau=t_\infty(C)=-\int_{r_C}^{\infty}\frac{C}{(1-2m/x)(x^4-2mx^3+C^2)^{1/2}}\,dx. \tag{2.20}$$

Suppose we had started with the Cauchy slice $t=0$ which, being time symmetric, is in particular maximal and evolve it into a maximal slicing by a lapse function α going to 1 at both spatial ends. This is possible in a unique way (see [12]). Then the resultant time function is spherically symmetric and symmetric with respect to Γ, and so it has to coincide with the one obtained above. In particular, it follows that our τ can be smoothly extended to negative values of τ which would have been very nonobvious from the explicit formula (2.12).

We next compute the lapse function α of τ. Using Eq. (2.9), this involves computing

$$(\xi^\mu\nabla_\mu\tau)^{-1}=\frac{dC}{d\tau}\left.\frac{\partial h}{\partial C}\right|_r. \tag{2.21}$$

Note that the right-hand side (RHS) of Eq. (2.21) blows up at $r=r_C$, but in such a way that

$$\alpha=(\xi^\mu\nabla_\mu\tau)^{-1}N \tag{2.22}$$

has a smooth limit as $r\to r_C$, as it has to be. Using formula (B12) and Eqs. (2.17), (2.21), there results

$$\alpha=\left(\frac{d\tau}{dC}\right)^{-1}\frac{1}{2}\left[\frac{1}{r-3m/2}-(1-2m/r+C^2/r^4)^{1/2}\times\int_{r_C}^{r}\frac{x(x-3m)dx}{(x-3m/2)^2[x^4-2mx^3+C^2]^{1/2}}\right], \tag{2.23}$$

with

$$\frac{d\tau}{dC}=-\frac{1}{2}\int_{r_C}^{\infty}\frac{x(x-3m)dx}{(x-3m/2)^2[x^4-2mx^3+C^2]^{1/2}}. \tag{2.24}$$

Note that N and α are linearly independent radial solutions of

$$(D^iD_i-K_{ij}K^{ij})f=0, \tag{2.25}$$

where N goes to 1 at the right infinity and to -1 at the left one, whereas α goes to 1 at both ends.

We are interested in studying α along the trajectories of Eulerian observers. This requires choosing a coordinate $\rho=\rho(r,\tau)$ the level surfaces of which are timelike cylinders orthogonal to our slicing. (One such timelike cylinder is already known, namely, Γ given by $r=r_C$.) Such a coordinate can be found without any calculation. Recall that maximal slicings preserve spatial volumes along Eulerian observers. Thus a suitable coordinate will be the ''volume radius'' on each slice, defined by

$$\rho^3(r,\tau)=3\int_{r_{C(\tau)}}^{r}\frac{x^4dx}{[x^4-2mx^3+C^2]^{1/2}}, \tag{2.26}$$

using that the spatial metric on each slice has the form (see Appendix A)

$$g_{ij}dx^idx^j=\left(1-\frac{2m}{r}+\frac{C^2}{r^4}\right)^{-1}dr^2+r^2d\Omega^2. \tag{2.27}$$

In the coordinates (t,ρ,θ,φ), the Schwarzschild metric for $r>3m/2$ reads

$$ds^2=-\alpha^2d\tau^2+\left(\frac{\rho}{r}\right)^4d\rho^2+r^2d\Omega^2, \tag{2.28}$$

where $r=r(\rho,\tau)$ is given implicitly by Eq. (2.26) and α by Eq. (2.23). [To check Eq. (2.28) explicitly one should first observe that $C\partial h/\partial C|_r=\rho^2\partial\rho/\partial C|_r$.]

Note that, as C approaches $\sqrt{27/16}m^2$, r_C approaches the value $3m/2$, since

$$\begin{aligned}P(x)&=x^4-2mx^3+C^2\\&=\left(x-\frac{3m}{2}\right)^2\left(x^2+mx+\frac{3m^2}{4}\right)+O\left(\left(x-\frac{3m}{2}\right)^2\right).\end{aligned} \tag{2.29}$$

Equation (2.29) also shows that r_C approaches a double root of $P(x)$ as $C\to\sqrt{27/16}m^2$. Thus, as one lets τ tend to infinity for fixed ρ, the function r approaches $3m/2$. In that sense the slices approach the limiting maximal slice at $r=3m/2$. We are interested in estimating α in that limit. For simplicity, we will confine ourselves to $\rho=0$, i.e., the throat Γ.

III. LATE TIME ANALYSIS

It is convenient to replace the parameter C by δ defined by

$$\delta=r_C-\frac{3m}{2},\quad r_C^4-2mr_C^3+C^2=0. \tag{3.1}$$

As C ranges between 0 and $3(\sqrt{3}/4)m^2$, δ ranges monotonically from $m/2$ to 0. Using the rescaled quantities

$$\bar{C}=\frac{C}{m^2},\quad\bar{\tau}=\frac{\tau}{m},\quad\bar{\delta}=\frac{\delta}{m}, \tag{3.2}$$

we find that

$$\bar{\tau}(\bar{\delta})=-\bar{C}\int_{3/2+\bar{\delta}}^{\infty}\frac{ydy}{(y-2)(y^4-2y^3+\bar{C}^2)^{1/2}}, \tag{3.3}$$

where

$$\bar{C}=\left(\bar{\delta}+\frac{3}{2}\right)^{3/2}\left(\frac{1}{2}-\bar{\delta}\right)^{1/2}. \tag{3.4}$$

We have the following lemma:

Lemma.

$$\bar{\tau}(\bar{\delta}) = -\frac{3\sqrt{6}}{4}\ln\bar{\delta} + \frac{3\sqrt{6}}{4}\ln|18(3\sqrt{2}-4)| - 2\ln\left|\frac{3\sqrt{3}-5}{9\sqrt{6}-22}\right| + O(\bar{\delta})$$
$$= -\frac{3\sqrt{6}}{4}\ln\bar{\delta} + A + O(\bar{\delta}) \quad \text{as} \quad \bar{\delta}\to 0. \tag{3.5}$$

Proof. First note that

$$\frac{d}{d\bar{C}}\left[\frac{\bar{C}}{(y^4-2y^3+\bar{C}^2)^{1/2}}\right] = \frac{y^3(y-2)}{(y^4-2y^3+\bar{C}^2)^{3/2}}. \tag{3.6}$$

Thus, from the mean value theorem,

$$\left|\frac{y}{y-2}\left(\frac{\bar{C}}{(y^4-2y^3+\bar{C}^2)^{1/2}} - \frac{\sqrt{27/16}}{(y^4-2y^3+27/16)^{1/2}}\right)\right| \leqslant \frac{\sqrt{27}\bar{\delta}^2 y^4}{(y^4-2y^3+\bar{C}^2)^{3/2}}, \tag{3.7}$$

where we have used

$$\sqrt{27/16} - \bar{C} \leqslant \sqrt{27}\bar{\delta}^2. \tag{3.8}$$

Inequality (3.7) is valid for $y\neq 2$, but, by continuity, also for $y=2$. We will find it convenient to sometimes express $\bar{C}$ in terms of $\bar{\delta}$, using Eq. (3.4). Writing

$$Q(s) = s^2\left(s^2+4s+\frac{9}{2}\right) - \bar{\delta}^2\left(\bar{\delta}^2+4\bar{\delta}+\frac{9}{2}\right), \tag{3.9}$$

Eq. (3.3) can, after substituting $s=y-3/2$, be written as

$$\bar{\tau} = \left(\bar{\delta}+\frac{3}{2}\right)^{3/2}\left(\frac{1}{2}-\bar{\delta}\right)^{1/2}\int_{\bar{\delta}}^{\infty}\frac{(s+3/2)ds}{(1/2-s)[Q(s)]^{1/2}}. \tag{3.10}$$

It is elementary to see that, for $s\geqslant\bar{\delta}$,

$$0 \leqslant \frac{9}{2}(s^2-\bar{\delta}^2) \leqslant Q(s) \leqslant (s^2-\bar{\delta}^2)\left[\frac{9}{2}+2s(4+s)\right], \tag{3.11}$$

which, using $\sqrt{1+x}\leqslant 1+x/2$ for $x\geqslant 0$, implies

$$\left|\frac{1}{[Q(s)]^{1/2}} - \frac{1}{[\frac{9}{2}(s^2-\bar{\delta}^2)]^{1/2}}\right| \leqslant \frac{\frac{2s}{9}(4+s)}{[\frac{9}{2}(s^2-\bar{\delta}^2)]^{1/2}}. \tag{3.12}$$

The estimate (3.7) now takes the form

$$\left|\frac{s+3/2}{1/2-s}\left(\frac{(\bar{\delta}+3/2)^{3/2}(1/2-\bar{\delta})^{1/2}}{[Q(s)]^{1/2}} - \frac{\sqrt{27/16}}{[s^2(s^2+4s+9/2)]^{1/2}}\right)\right| \leqslant \frac{\sqrt{27}\bar{\delta}^2(s+3/2)^4}{[Q(s)]^{3/2}} \leqslant \frac{\sqrt{27}\bar{\delta}^2(s+3/2)^4}{[\frac{9}{2}(s^2-\bar{\delta}^2)]^{3/2}}. \tag{3.13}$$

The inequalities (3.12), (3.13) are the basic estimates we will be using. We now split the integration domain in Eq. (3.10),

$$\bar{\delta}\leqslant s\leqslant\sqrt{\bar{\delta}/2}, \quad \sqrt{\bar{\delta}/2}\leqslant s\leqslant\infty, \tag{3.14}$$

and write

$$\bar{\tau} = \bar{\tau}_1 + \bar{\tau}_2, \tag{3.15}$$

accordingly. We furthermore define $(0<\bar{\delta}<1/2)$

$$\bar{\tau}_1^0 = \sqrt{27/16}\int_{\bar{\delta}}^{\sqrt{\bar{\delta}/2}}\frac{s+3/2}{(1/2-s)[\frac{9}{2}(s^2-\bar{\delta}^2)]^{1/2}}\,ds, \tag{3.16}$$

$$\bar{\tau}_2^0 = \sqrt{27/16}\int_{\sqrt{\bar{\delta}/2}}^{\infty}\frac{s+3/2}{(1/2-s)[s^2(s^2+4s+9/2)]^{1/2}}\,ds. \tag{3.17}$$

Equation (3.17) is in the principal-value sense at $s=1/2$. These integrals and the one following later in Eq. (3.32) can be explicitly computed using the formulas (see, e.g., [14])

$$\int\frac{dx}{x^2\sqrt{x^2-\bar{\delta}^2}} = \frac{\sqrt{x^2-\bar{\delta}^2}}{x\bar{\delta}^2}, \quad x>\bar{\delta}>0, \tag{3.18}$$

$$\int\frac{dx}{\sqrt{x^2-\bar{\delta}^2}} = \ln|x+\sqrt{x^2-\bar{\delta}^2}|, \quad x>\bar{\delta}>0, \tag{3.19}$$

$$\int\frac{dx}{x\sqrt{ax^2+bx+c}} = \frac{1}{\sqrt{c}}\ln\frac{|-2\sqrt{c(ax^2+bx+c)}+2c+bx|}{2|x|}, \quad c>0. \tag{3.20}$$

Using $(s+3/2)/s(1/2-s) = 3/s - 4/(s-1/2)$, there results, after straightforward manipulations,

$$\bar{\tau}_1^0 = -\frac{3\sqrt{6}}{4}\ln\bar{\delta} + \frac{3\sqrt{6}}{4}\ln\sqrt{\bar{\delta}/2} + o(1) \quad \text{as} \quad \bar{\delta}\to 0, \tag{3.21}$$

$$\bar{\tau}_2^0 = -\frac{3\sqrt{6}}{4}\ln\sqrt{\bar{\delta}/2} + \frac{3\sqrt{6}}{4}\ln 2 + \frac{3\sqrt{6}}{4}\ln\left|\frac{18}{4+3\sqrt{2}}\right| - 2\ln\left|\frac{3\sqrt{3}-5}{9\sqrt{6}-22}\right| + o(1) \quad \text{as} \quad \bar{\delta}\to 0. \tag{3.22}$$

Next, we have to estimate the remainders. We have

$$\Delta\bar{\tau}_1=\int_{\bar{\delta}}^{\sqrt{\bar{\delta}/2}}\frac{s+3/2}{1/2-s}\left[\frac{\bar{C}(\bar{\delta})}{[Q(s)]^{1/2}}-\frac{\sqrt{27/16}}{[\frac{9}{2}(s^2-\bar{\delta}^2)]^{1/2}}\right]ds. \quad (3.23)$$

Using $\bar{C}(\bar{\delta})=\sqrt{27/16}+O(\bar{\delta}^2)$ and Eqs. (3.11), (3.12), this has

$$|\Delta\bar{\tau}_1|\leqslant\text{const}\times\int_{\bar{\delta}}^{\sqrt{\bar{\delta}/2}}\frac{s\,ds}{\sqrt{s^2-\bar{\delta}^2}}=O(\bar{\delta}^{1/2}). \quad (3.24)$$

Next,

$$\Delta\bar{\tau}_2=\int_{\sqrt{\bar{\delta}/2}}^{\infty}\frac{s+3/2}{1/2-s}\left[\frac{\bar{C}(\bar{\delta})}{[Q(s)]^{1/2}}-\frac{\sqrt{27/16}}{[s^2(s^2+4s+9/2)]^{1/2}}\right]ds. \quad (3.25)$$

By inequality (3.13), this has a bound of the form

$$|\Delta\bar{\tau}_2|\leqslant\text{const}\times\bar{\delta}^2\int_{\sqrt{\bar{\delta}/2}}^{\infty}\frac{(s+3/2)^4}{[s^2(s^2+4s+9/2)]^{3/2}}ds=\text{const}\times\bar{\delta}^2 I. \quad (3.26)$$

The integral I in Eq. (3.26) can be further split as $I=I_2+I_2'$, where

$$I_2=\int_{\sqrt{\bar{\delta}/2}}^{1}\frac{(s+3/2)^4}{[s^2(s^2+4s+9/2)]^{3/2}}ds\leqslant\text{const}\times\int_{\sqrt{\bar{\delta}/2}}^{1}\frac{ds}{(s^2-\bar{\delta}^2)^{3/2}}$$
$$=\text{const}\times\frac{1}{\bar{\delta}^2}\int_{\sqrt{2/\bar{\delta}}}^{1/\bar{\delta}}\frac{df}{(f^2-1)^{3/2}}=O\left(\frac{1}{\bar{\delta}}\right). \quad (3.27)$$

Now

$$I_2'=\int_1^{\infty}\frac{(s+3/2)^4}{[s^2(s^2+4s+9/2)]^{3/2}}ds\leqslant\text{const}\times\int_1^{\infty}\frac{ds}{s^2}<\infty. \quad (3.28)$$

Thus $\Delta\bar{\tau}_2=O(\bar{\delta})$. Putting all this together implies

$$\bar{\tau}(\bar{\delta})=-\frac{3\sqrt{6}}{4}\ln\bar{\delta}+A+o(1)\quad\text{as}\quad\bar{\delta}\to 0, \quad (3.29)$$

which is not quite good enough. From Eq. (B12) in the limit that r goes to infinity and

$$2\bar{C}\frac{d\bar{C}}{d\bar{\delta}}=-4\bar{\delta}\left(\bar{\delta}+\frac{3}{2}\right)^2, \quad (3.30)$$

we see that

$$\frac{d\bar{\tau}}{d\bar{\delta}}=\frac{\bar{\delta}(\bar{\delta}+3/2)^{1/2}}{(1/2-\bar{\delta})^{1/2}}\int_{\bar{\delta}}^{\infty}\frac{(s+3/2)(s-3/2)}{s^2[Q(s)]^{1/2}}ds$$
$$=\sqrt{3}\,\bar{\delta}\int_{\bar{\delta}}^{1}\frac{s^2-9/4}{s^2}\frac{ds}{[Q(s)]^{1/2}}+O(\bar{\delta})=\sqrt{3}\,\bar{\delta}J+O(\bar{\delta}). \quad (3.31)$$

Here J can in turn be split as $J=J^0+\Delta J$, where

$$J^0=\int_{\bar{\delta}}^{1}\frac{s^2-9/4}{s^2}\frac{ds}{[\frac{9}{2}(s^2-\bar{\delta}^2)]^{1/2}}$$
$$=\sqrt{2/9}\int_1^{1/\bar{\delta}}\frac{df}{\sqrt{f^2-1}}-9/4\sqrt{2/9}\frac{1}{\bar{\delta}^2}\int_1^{\infty}\frac{df}{f^2\sqrt{f^2-1}}+O\left(\frac{1}{\bar{\delta}}\right)$$
$$=O(\ln\bar{\delta})-\sqrt{9/8}\frac{1}{\bar{\delta}^2}\times 1. \quad (3.32)$$

Finally,

$$\Delta J=\int_{\bar{\delta}}^{1}\frac{s^2-9/4}{s^2}\left[\frac{1}{[Q(s)]^{1/2}}-\frac{1}{[\frac{9}{2}(s^2-\bar{\delta}^2)]^{1/2}}\right]ds. \quad (3.33)$$

Thus, using inequality (3.12),

$$|\Delta J|\leqslant\text{const}\times\int_{\bar{\delta}}^{1}\frac{s}{s^2}\frac{ds}{\sqrt{s^2-\bar{\delta}^2}}=\frac{\text{const}}{\bar{\delta}}. \quad (3.34)$$

Putting Eqs. (3.31), (3.32), (3.33) together, there results

$$\frac{d\bar{\tau}}{d\bar{\delta}}=-\frac{3\sqrt{6}}{4}\frac{1}{\bar{\delta}}+O(1)\quad\text{as}\quad\bar{\delta}\to 0. \quad (3.35)$$

Integrating Eq. (3.35), we obtain

$$\bar{\tau}(\bar{\delta})=-\frac{3\sqrt{6}}{4}\ln\bar{\delta}+A'+O(\bar{\delta}), \quad (3.36)$$

for some constant A'. Comparing with Eq. (3.29), we infer $A=A'$ and the proof of the estimate (3.5) is complete. ■

From $A=A'$ and Eq. (3.36) it is elementary to infer that

$$\bar{\delta}=\exp\left(-\frac{4}{3\sqrt{6}}(\bar{\tau}-A)\right)+O\left[\exp\left(-\frac{8}{3\sqrt{6}}\bar{\tau}\right)\right]\quad\text{as}\quad\bar{\tau}\to\infty. \quad (3.37)$$

Using Eq. (3.30), Eq. (3.35) can be written as

$$\frac{d\bar{\tau}}{d\bar{C}}=\frac{3}{4\sqrt{2}}\frac{1}{\bar{\delta}^2}+O\left(\frac{1}{\bar{\delta}}\right). \quad (3.38)$$

We want to evaluate the lapse α of the time function $\tau=m\bar{\tau}$ along the central throat $r=r_C$. This, using Eqs. (2.23) and (2.24), is given by

$$\alpha(\tau)=\frac{1}{2m\bar{\delta}}\left(\frac{d\bar{\tau}}{d\bar{C}}\right)^{-1}. \quad (3.39)$$

Using Eq. (3.36), this finally leads to

$$\alpha(\tau)=\frac{4}{3\sqrt{2}}\,\overline{\delta}+O(\overline{\delta}^2)$$

$$=\frac{4}{3\sqrt{2}}\exp\left(\frac{4A}{3\sqrt{6}}\right)\exp\left(-\frac{4\tau}{3\sqrt{6}m}\right)$$

$$+O\left[\exp\left(-\frac{8\tau}{3\sqrt{6}m}\right)\right]\quad\text{for}\quad \tau\to\infty. \qquad (3.40)$$

We sum up our results in the following theorem.

Theorem. For the chosen maximal foliation, with the time function τ coinciding with proper time at infinity and being zero on the time-symmetric leaf S, the lapse along the central geodesics orthogonal to the leaves behaves, as a function of τ, according to Eq. (3.40) with A given by

$$A=\frac{3\sqrt{6}}{4}\ln|18(3\sqrt{2}-4)|-2\ln\left|\frac{3\sqrt{3}-5}{9\sqrt{6}-22}\right|=-0.2181. \qquad (3.41)$$

It would be interesting to estimate the lapse for large τ along arbitrary Eulerian observers rather than just the ones along Γ. In terms of the coordinate ρ introduced in Sec. II, we conjecture that

$$\alpha(\rho,\tau)=B(\rho)\exp\left(-\frac{4}{3\sqrt{6}}\frac{\tau}{m}\right)+O\left[B^2(\rho)\exp\left(-\frac{8}{3\sqrt{6}}\frac{\tau}{m}\right)\right], \qquad (3.42)$$

where $B(\rho)$ behaves for large ρ as

$$B(\rho)\sim\text{const}\times\cosh\frac{4}{3\sqrt{6}}\left(\frac{\rho}{m}\right)^3. \qquad (3.43)$$

The form of $B(\rho)$ in Eq. (3.43) is motivated by the solution to the lapse equation (2.25) on the limiting slice at $r=3m/2$, which is symmetrical with respect to the throat.

ACKNOWLEDGMENTS

The authors thank Piotr Chruściel for helpful discussions. One of us (N.Ó.) gratefully acknowledges the support and hospitality of ESI. N.Ó. was partially supported by Forbairt Grant SC/96/750.

APPENDIX A

The following discussion is similar in spirit to [15]. Let Σ be the manifold $\mathbf{R}\times S^2$ with a Riemannian, spherically symmetric metric, which we write in the ''radial'' gauge, i.e.,

$$g=dl^2+r^2(l)d\Omega^2,\quad r\in(0,\infty). \qquad (A1)$$

The unit vector $l^i=(\partial/\partial l)^i$ is geodesic and satisfies ($r'=dr/dl$)

$$D_i l_j=\frac{r'}{r}\,q_{ij}, \qquad (A2)$$

where $q_{ij}=g_{ij}-l_i l_j$ and a prime means derivative with respect to l. After a calculation, which most easily follows the lines of Besse [13], we find, for the Riemann tensor,

$$R_{ijkl}l^j l^k=\frac{r''}{r}\,q_{il} \qquad (A3)$$

and

$$q_i^{i'}q_j^{j'}q_k^{k'}q_l^{l'}R_{i'j'k'l'}=\frac{2}{r^2}(1-r'^2)q_{k[i}q_{j]l}. \qquad (A4)$$

Identities (A3), (A4) imply that

$$R_{ij}=-\frac{r''}{r}(2l_i l_j+q_{ij})+\frac{1-r'^2}{r^2}\,q_{ij}, \qquad (A5)$$

$$R=-4\frac{r''}{r}+2\frac{1-r'^2}{r^2}. \qquad (A6)$$

The extrinsic curvature on Σ, in order to be spherically symmetric, has to be of the form

$$K_{ij}=v\,l_i l_j+wq_{ij}. \qquad (A7)$$

The condition $K_{ij}g^{ij}=0$ implies that $v+2w=0$. Using Eq. (A2), we have

$$D^iK_{ij}=\left(v'+3\frac{r'}{r}v\right)l_j. \qquad (A8)$$

Thus the maximal momentum constraint implies $v=2C/r^3$ for some constant C. Consequently,

$$K_{ij}=\frac{2C}{r^3}\,l_i l_j-\frac{C}{r^3}\,q_{ij}, \qquad (A9)$$

$$K_{ij}K^{ij}=6\frac{C^2}{r^6}. \qquad (A10)$$

Inserting Eqs. (A10) and (A6) into the Hamiltonian constraint, there results

$$-4\frac{r''}{r}+2\frac{1-r'^2}{r^2}=6\frac{C^2}{r^6}. \qquad (A11)$$

Next, we define $m(r)$ by

$$m(r):=\frac{r}{2}(1-r'^2)+\frac{C^2}{2r^3}. \qquad (A12)$$

Now Eq. (A11) implies that dm/dr is zero. Thus

$$r'=\left(1-\frac{2m}{r}+\frac{C^2}{r^4}\right)^{1/2}. \qquad (A13)$$

Assuming $m>0$ and $0\leqslant|C|<3(\sqrt{3}/4)m^2$, there are two initial-data sets consistent with Eqs. (A9) and (A13). One starts at $r=0$, expands to an $r_{\max}<3m/2$, and collapses back to $r=0$. The other is an asymptotically flat complete metric on $\mathbf{R}\times S^2$ with mass m at both ends which is symmetric with respect to the throat at $r=r_C>3m/2$ with

$$1-\frac{2m}{r_C}+\frac{C^2}{r_C^4}=0. \tag{A14}$$

Here we restrict ourselves to asymptotically flat data. These constitute a two-parameter family of solutions to the spherically symmetric, maximal vacuum constraints. Of course, we know from the Birkhoff theorem that members of this family with different C but the same m have all to lie in the same spacetime, namely, the extended Schwarzschild spacetime. ''Discovering'' this fact in the present context amounts to finding the ''height function'' written down in Sec. II. The trick is to try to find the remaining Killing vector and to seek the Σ_C's as graphs over the surfaces orthogonal to this Killing vector. If (g_{ij},K_{ij}) evolve to a spacetime having another Killing vector, there must be a function N, not identically zero, and a vector field X^i so that

$$2NK_{ij}+2D_{(i}X_{j)}=0. \tag{A15}$$

Assuming X^i to be again spherical, i.e.,

$$X_i=\mu l_i, \quad \mu=\mu(r), \tag{A16}$$

and again using Eqs. (A2) and (A9), we infer that

$$-2\frac{NC}{r^3}+2r'\frac{\mu}{r}=0, \tag{A17}$$

$$4\frac{NC}{r^3}+2r'\frac{d\mu}{dr}=0. \tag{A18}$$

After combining Eqs. (A17) and (A18), there results

$$\mu(r)=\frac{D}{r^2}, \quad D=\text{const}, \tag{A19}$$

$$N=\frac{D}{C}r', \tag{A20}$$

where we have assumed $C\neq 0$. We assume without loss that $D=C$. The existence of (N,X^i) solving Eq. (A15) does not necessarily imply that the vacuum evolution of the initial-data set has a static Killing vector. There also has to be satisfied

$$\mathcal{L}_X K_{ij}+D_iD_jN=N(R_{ij}-2K_{il}K_j{}^l). \tag{A21}$$

It is straightforward to check that Eqs. (A19), (A20) do satisfy Eq. (A21). [In the case where C is zero, $X^i=0$, and Eq. (A21) implies that $N\sim r'$.]

We remark in passing that the function N, by virtue of Eqs. (A15) and (A21), satisfies

$$D^iD_iN=NK_{ij}K^{ij}. \tag{A22}$$

[Of the two linearly independent spherical solutions of Eq. (A22), N is that combination which vanishes on the throat.]

It now follows that for $r>r_C$ the metrics

$$ds^2=-(N^2-g_{ij}X^iX^j)d\sigma^2+2g_{ij}X^jdx^id\sigma+g_{ij}dx^idx^j, \tag{A23}$$

with N_i, X^i, g_{ij} extended in a σ-independent way to $\mathbf{R}\times\Sigma$, are vacuum solutions evolving from the above initial-data sets. They have $\xi^\mu=(\partial/\partial\sigma)^\mu$ as a Killing vector. More explicitly, since

$$N^2-X_iX^i=1-\frac{2m}{r}, \tag{A24}$$

we have

$$ds^2=-\left(1-\frac{2m}{r}\right)d\sigma^2+2\frac{C}{r^2}\,dl\,d\sigma+dl^2+r^2d\Omega^2, \tag{A25}$$

where $r(l)$ is given implicitly by

$$l(r)=\int_{r_C}^{r}\frac{dx}{\sqrt{1-2m/x+C^2/x^4}}. \tag{A26}$$

[For $C=0$, $l(r)$ can be written as $l(r)=r\sqrt{1-2m/r}+m\,\ln|(1+\sqrt{1-2m/r})/(1-\sqrt{1-2m/r})|$.]

Note that for $C\neq 0$ the above metrics extend smoothly across $r=2m$. We now seek a function t with level surfaces orthogonal to $\partial/\partial\sigma$. Writing this function as

$$t=F(r)+\sigma, \tag{A27}$$

we obtain from

$$g_{\mu\nu}\xi^\mu dx^\nu=-(N^2-X_iX^i)d\sigma+X_idx^i=\omega(dF+d\sigma), \tag{A28}$$

for some function ω, the equation

$$-D_iF=\frac{X_i}{N^2-X_jX^j}, \tag{A29}$$

which makes sense only off the horizon. Using Eqs. (A16), (A19), this leads to

$$\frac{dF}{dr}=-\frac{C}{r^2-2mr}\frac{1}{\sqrt{1-2m/r+C^2/r^4}}. \tag{A30}$$

Now consider the coordinate transformation

$$\sigma=t-F. \tag{A31}$$

Then

$$ds^2=-(N^2-X_jX^j)dt^2+\overline{g}_{ij}dx^idx^j, \tag{A32}$$

with

$$\overline{g}_{ij}=g_{ij}+2X_{(i}F_{,j)}-(N^2-X_lX^l)F_{,i}F_{,j} \tag{A33}$$

$$=g_{ij}+(N^2-X_lX^l)^{-1}X_iX_j, \tag{A34}$$

where $X_i = g_{ij}X^j$. Using Eqs. (A29) and (A30),

$$\overline{g_{ij}}dx^i dx^j = \left[1+\left(1-\frac{2m}{r}\right)^{-1}\frac{C^2}{r^4}\right]dl^2 + r^2 d\Omega^2$$
$$= \left(1-\frac{2m}{r}\right)^{-1} dr^2 + r^2 d\Omega^2. \tag{A35}$$

We have thus recovered the Schwarzschild metric. In particular, this calculation shows that the parameter C in our initial-data sets is ''pure gauge'': Initial data with the same m lie in the same spacetime, namely, as level sets of the function σ. They can also be written as

$$t = F_C(r) \tag{A36}$$

and its translates under $\xi^\mu = (\partial/\partial t)^\mu$, where

$$F_C(r) = -C\int_{r_0}^{r}\frac{dx}{(1-2m/x)(1-2m/x+C^2/x^4)^{1/2}}, \tag{A37}$$

for some r_0. Taking $r_0 = r_C$, we have, with $h(r,C) = F_C(r)$, recovered Eq. (2.12).

It is shown in Sec. II that

$$t = h(r,C) \tag{A38}$$

implicitly defines a smooth time function on the $r > 3m/2$ subset of the future half of Kruskal. The boost function N obtained in this Appendix satisfies the same equation, on each leaf Σ_C, as the lapse function of C, namely Eq. (A22). The reason for this is that, for fixed Σ_C, ξ^μ defines another local foliation, which is again maximal since ξ^μ is a Killing vector.

APPENDIX B

Consider

$$F(x,E) = \int_{x_E = V^{-1}(E)}^{x}\frac{W(y)}{[E-V(y)]^{1/2}}\,dy, \tag{B1}$$

where V is a smooth function $V:[x_0,\infty)\to\mathbf{R}$ with

$$0 < V(x_0), \quad V'(x) < 0 \quad \text{for } x > x_0, \quad V(\overline{x}) = 0, \tag{B2}$$

and

$$0 < E < V(x_0), \quad V(x) < E. \tag{B3}$$

The function W is smooth except perhaps at $x = \overline{x}$, where it may have a simple pole. (Thus the pole of $\sqrt{E}W(y)/[E - V(y)]^{1/2}$ is independent of E.) In the latter case, Eq. (B1) is to be understood in the principal-value sense and the following operations valid for $x \neq \overline{x}$. Next, define (we follow [16] in spirit)

$$J(x,E) = \int_{x_E}^{x}[E-V(y)]^{1/2}V(y)W(y)\,dy. \tag{B4}$$

Note that $V\cdot W$ is smooth. Equation (B4) can be rewritten as follows:

$$J(x,E) = -\frac{2}{3}\int_{x_E}^{x}\frac{d}{dy}[E-V(y)]^{3/2}\frac{V(y)W(y)}{V'(y)}\,dy, \tag{B5}$$

$$J(x,E) = -\frac{2}{3}[E-V(x)]^{3/2}\frac{V(x)W(x)}{V'(x)} + \frac{2}{3}\int_{x_E}^{x}[E-V(y)]^{3/2}\frac{d}{dy}\left[\frac{V(y)W(y)}{V'(y)}\right]dy. \tag{B6}$$

Differentiating Eq. (B6) with respect to E twice, we obtain

$$\frac{\partial^2}{\partial E^2}J(x,E) = -\frac{1}{2}\frac{1}{[E-V(x)]^{1/2}}\frac{V(x)W(x)}{V'(x)} + \frac{1}{2}\int_{x_E}^{x}\frac{1}{[E-V(y)]^{1/2}}\frac{d}{dy}\left[\frac{V(y)W(y)}{V'(y)}\right]dy. \tag{B7}$$

On the other hand, differentiating Eq. (B4) once with respect to E, it follows that

$$\frac{\partial}{\partial E}J(x,E) = \frac{1}{2}\int_{x_E}^{x}\frac{V(y)}{[E-V(y)]^{1/2}}W(y)dy$$
$$= \frac{1}{2}\int_{x_E}^{x}\frac{V(y)-E+E}{[E-V(y)]^{1/2}}W(y)dy$$
$$= -\frac{1}{2}\int_{x_E}^{x}[E-V(y)]^{1/2}W(y)dy + \frac{1}{2}EF(x,E). \tag{B8}$$

Differentiating Eq. (B8) once more with respect to E and comparing with Eq. (B7), we finally find

$$\frac{1}{4}F(x,E) + \frac{1}{2}E\frac{\partial}{\partial E}F(x,E) = -\frac{1}{2}\frac{1}{[E-V(x)]^{1/2}}\frac{V(x)W(x)}{V'(x)} + \frac{1}{2}\int_{x_E}^{x}\frac{1}{[E-V(y)]^{1/2}}\frac{d}{dy}\left[\frac{V(y)W(y)}{V'(y)}\right]dy. \tag{B9}$$

In our case we will have that $V'(x_0) = 0$ and we study the blowup of $F_\infty(E) = \lim_{x\to\infty}F(x,E)$ as E tends to $E_0 = V(x_0)$. As for a mechanical analogue, we could think of a particle on a half-line in a repulsive potential $V(x)$ and imagine $F(x,E)$ to be the time it takes a particle of energy E to travel from x_0 to x. [If it were not for the presence of $W(y)$ in Eq. (B1), this interpretation would be literally true.] The force on the particle grows so fast for large x that the particle reaches infinity in finite time $F_\infty(E)$. There is an unstable equilibrium point at $x = x_0$. We ask for the way in which $F_\infty(E)$ blows up as E approaches $V(x_0)$. If the energy E is further increased, the orbits reach $x=0$: This corresponds to maximal slices hitting the singularity.

To make contact with our function $h(r,C)$, set

$$V(x)=-x^4+2mx^3,\quad E=C^2,\quad W(x)=-\frac{1}{1-2m/x},$$

$$h(r,C)=CF(r,C^2), \tag{B10}$$

$$x_0=\frac{3m}{2},\quad \bar{x}=2m.$$

Thus

$$\frac{\partial}{\partial C}h(r,C)=2E\frac{\partial}{\partial E}F(r,E)\Big|_{E=C^2}+F(r,C^2), \tag{B11}$$

which, combined with Eq. (B9), gives

$$\frac{\partial}{\partial C}h(r,C)=\frac{1}{2(r-3m/2)\sqrt{1-2m/r+C^2/r^4}}-\frac{1}{2}\int_{r_C}^{r}\frac{x(x-3m)dx}{(x-3m/2)^2(x^4-2mx^3+C^2)^{1/2}}. \tag{B12}$$

[1] F. Estabrook, A. Wahlquist, S. Christensen, B. de Witt, L. Smarr, and E. Tsiang, Phys. Rev. D **7**, 2814 (1973); D. Eardley and L. Smarr, *ibid.* **19**, 2239 (1979).

[2] B. L. Reinhart, J. Math. Phys. **14**, 719 (1972).

[3] R. Penrose, Phys. Rev. Lett. **14**, 57 (1965).

[4] G. Totschnig, Ph.D. thesis, University of Vienna, 1994.

[5] D. Gannon, J. Math. Phys. **16**, 2364 (1975); C. W. Lee, Commun. Math. Phys. **51**, 157 (1976).

[6] R. Beig and N. Ó Murchadha, Phys. Rev. Lett. **66**, 2421 (1991); Class. Quantum Grav. **11**, 419 (1994); **13**, 739 (1996).

[7] V. Moncrief and D. Eardley, Gen. Relativ. Gravit. **13**, 887 (1981).

[8] R. Penrose, Riv. Nuovo Cimento **1**, 252 (1969).

[9] J. York, in *Sources of Gravitational Radiation*, edited by L. Smarr (Cambridge University Press, Cambridge, England, 1979).

[10] L. Smarr and J. York, Phys. Rev. D **17**, 2529 (1978).

[11] D. Brill and F. J. Flaherty, Commun. Math. Phys. **50**, 157 (1976).

[12] Y. Choquet-Bruhat, Ann. Scuola Norm. Sup. Pisa **3**, 361 (1976); R. Bartnik, Commun. Math. Phys. **94**, 155 (1984); R. Bartnik, P. T. Chruściel, and N. Ó Murchadha, *ibid.* **130**, 95 (1990).

[13] A. Besse, *Einstein Manifolds* (Springer, New York, 1987), Chap. 3.

[14] *Teubner-Taschenbuch der Mathematik*, edited by E. Zeidler (Teubner, Stuttgart, 1996), Vol. 1.

[15] J. Guven and N. Ó Murchadha, Phys. Rev. D **52**, 758 (1995).

[16] S.-N. Chow and D. Wang, Časopis pro pěstování matematiky (Praha) **111**, 14 (1986).

[17] R. Beig, P. Campbell, and N. Ó Murchadha (to be published).