Adiabatic invariants and mixmaster catastrophes

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We present a rigorous analysis of the role and uses of the adiabatic invariant in the mixmaster dynamical system. We propose a new invariant for the global dynamics which in some respects has an improved behavior over the commonly used one. We illustrate its behavior in a number of numerical results. We also present a new formulation of the dynamics via catastrophe theory. We find that the change from one era to the next corresponds to a fold catastrophe, during the Kasner shifts the potential is an implicit function form whereas, as the anisotropy dissipates, the mixmaster potential must become a Morse 0-saddle. We compare and contrast our results to many known works on the mixmaster problem and indicate how extensions could be achieved. Further exploitation of this formulation may lead to a clearer understanding of the global mixmaster dynamics. [S0556-2821(98)04408-7]

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I. INTRODUCTION

During the past eight years or so interest in the mixmaster universe [1] has increased dramatically. There are at least two reasons for this noticeable increase. First, there was already a considerable amount of background work concerning the basic dynamical issues of the model [2–5] and it is probably fair to say that this model (diagonal Bianchi type IX) was studied more than any other homogeneous cosmology (with a probable exception of inflationary issues). The picture drawn from that body of work was already quite rich (the actual system was also integrated numerically by Zardecki [6] but some of his results later criticized as incompatible with those of Belinskii, Khalatnikov, and Lifshitz (BKL) in [7,8]) and allowed for further generalizations to be considered.

Examples of this kind of generalized problems included Kaluza-Klein extensions [9] and the search for complicated mixmaster behavior in other theories of gravitation [10]. All these generalizations were motivated mainly by an idea and results, first obtained by Barrow [4] almost another decade earlier, that the well-known BKL-Misner oscillatory, mixmaster behavior should in fact be viewed as an example of the manifestation of chaotic (erratic, unpredictable) structures in general relativity. It was therefore natural to examine how common such a behavior could be. (It was also known [11] that departures from such an evolutionary scheme could be obtained with the addition of scalar fields.) In fact, many of those results were quite interesting and contributed much

to a sustained interest in the mixmaster dynamics during the eighties.

Interest in the mixmaster universe suddenly peaked with the appearance of the first works on numerical experiments in 1989–1990 [12]. Those results (and others which followed [13]) were conflicting in the sense that in most cases the standard picture [4] was challenged to the effect that many workers in the field felt that a reexamination of the original conclusions concerning the existence of chaos in the mixmaster approach to the initial singularity was necessary. At the same time other works appeared which either criticized [14] the use of some essentially coordinate dependent measures of chaoticity or indicated [15,16] that some sort of chaotic behavior should be present in the mixmaster dynamics.

It was felt necessary [17] that perhaps an analytical approach, known as *Painleve analysis*, which did not share the ''defects'' of numerical work could lead to more reliable results concerning the existence or nonexistence of chaotic behavior in the mixmaster dynamics. Initial results [18] pointed to the direction of integrability whereas later it was realized [19] that the situation was more complex. It is now understood that this analysis cannot be used to obtain reliable results concerning the question of chaoticity in this model.

Very recently Cornish and Levin [20] using fractal methods resolved the long-standing debate concerning the issue of chaoticity in the mixmaster universe, showing that the system is indeed chaotic. Their analysis not only confirms the earlier ergodic results of Barrow [4] but quantifies the chaotic behavior of the model by calculating a special set of numbers (topological entropy, multifractal dimension, etc.) relevant to the true dynamics. It therefore appears that all ambiguities concerning this issue have now disappeared.

It is perhaps encouraging that several issues about the mixmaster dynamics *not connected to the question of chaos* have been studied by several authors. It is indeed true that many problems in the dynamics of this model still remain. First, it is still unknown how local the BKL analysis is [21]. Another problem is borne out of earlier results of Moncrief [22] and is related to the role that the Geroch transformation plays for the true dynamics. Still another issue has to do with

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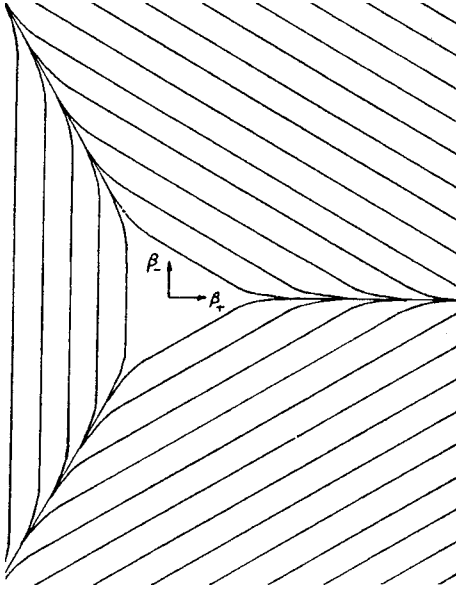
FIG. 1. Contour lines for the potential well $V(\beta_+,\beta_-)$ = *full potential*. The well exhibits C_3 symmetry and is almost an equilateral triangle except at the corners where the contour lines move off to infinity. The effective potential $\bar{V}=Ve^{-2\omega}$ and hence the "triangles" move outwards with time.

the description of the mixmaster universe from the point of view of the dynamical systems theory [23]. We therefore believe that the rich mixmaster behavior will continue to attract the interest of cosmologists for some time to come.

The purpose of this paper is to provide a rigorous analysis of the role and uses of adiabatic invariants in the mixmaster problem by carefully examining the adiabatic invariant commonly used and introducing a new and in a sense improved invariant for the global mixmaster dynamics. We then reformulate the main characteristic of the dynamics via a new, simpler, technique along the lines of catastrophe theory. We believe that further exploitation of this formulation may help to unravel certain global dynamical properties of the mixmaster universe.

The plan of this paper is as follows. In Sec. II we introduce our new invariant for the mixmaster system which behaves better than the standard adiabatic one and is explicitly time-independent in the appropriate coordinates. We also perform a numerical simulation of the corner-run part (cf. Fig. 1) of the evolution and we give an interpretation of the results using Misner's Hamiltonian picture. In Sec. III we apply catastrophe theory as a means to gain a better understanding of the complicated behavior of the model. The main result of this section is that the passage from one era to the next corresponds, in the language of catastrophe theory, to a fold catastrophe which in turn may provide a potentially new way to view the global evolution. In Sec. IV, we compare our results to previous work and point out how generalizations to higher dimensions could be obtained.

II. ADIABATIC ANALYSIS

Subject to a couple of overall approximations the motion of the "universe point" $\beta=(\beta_+,\beta_-)$ is governed by the mixmaster Lagrangian

$$L_{full}=1/2\Lambda^{1/2}(\beta_+'^2+\beta_0'^2)-2\Lambda^{-1/2}e^{-4\Omega}V(\beta), \quad (1)$$

where β_+ and β_- are related to the shape parameters of the model, the volume parameter, Ω plays the role of time, primes denote differentiation with respect to Ω, and Λ is a function of Ω which evolves according to

$$\Lambda'=-4e^{-4\Omega}V(\beta). \quad (2)$$

Here $V(\beta)$ is the standard, curvature anisotropy (mixmaster) potential given by

$$V(\beta)=\frac{1}{3}e^{-4\beta_+}-\frac{4}{3}e^{-\beta_+}\cosh(\sqrt{3}\beta_-)$$

$$+\frac{2}{3}e^{2\beta_+}[\cosh(2\sqrt{3}\beta_-)-1]+1. \quad (3)$$

There is also the energylike equation

$$4=\beta_+'^2+\beta_-'^2+4\Lambda^{-1}e^{-4\Omega}V(\beta). \quad (4)$$

We begin our analysis by reexamining the role and consequences of the adiabatic invariant used implicitly (or explicitly) in most mixmaster calculations. According to Misner [3] the most important asymptotic form of the mixmaster potential is when $|\beta_-|$ is small and slowly varying and $\beta_+ \to +\infty$, that is, $V(\beta)\sim 1+4\beta_-^2 e^{2\beta_+}$. In this case L_{full} can be considered as the Lagrangian for the β_- motion, that is, it can be supposed approximately that there exists a system with one degree of freedom, β_+, depending on the slowly varying parameter β_-. This reduced system is described by the Lagrangian,

$$L_{reduced}=1/2\Lambda^{1/2}\beta_-'^2-8\Lambda^{-1/2}e^{2\beta_+-4\Omega}\beta_-^2 \quad (5)$$

in which a term $O(e^{-4\Omega})$ has been neglected and Λ and β_+ are again functions of Ω through the solutions of (2) and (4) with the potential as modified. The Lagrangian (5) is that of a time-dependent oscillator of slowly varying frequency given by

$$\omega_-=4\Lambda^{-1/2}e^{\beta_+-2\Omega}. \quad (6)$$

Then, for the reduced system, there exists the adiabatic invariant

$$\Sigma=\frac{E_-}{\omega_-}=\frac{1}{8}\Lambda e^{2\Omega-\beta_+}(\beta_-'^2+\omega_-^2\beta_-^2), \quad (7)$$

where E_- are the energy level sets and $2\pi\Sigma$ represents the area of appropriate domains bounded by curves passing through points in (the two dimensional phase) space (π_-,β_-). It can be shown, by adapting the methods of [24], that this is also an adiabatic invariant of the "full" mixmaster system (two degrees of freedom) with β_- slowly varying (but not necessarily small). We stress this point since it is important to remember that the motion of the universe point described by $L_{reduced}$ is only approximately true and the true dynamics in this case should be thought of as that given by L_{full} with β_- slowly varying.

The adiabatic invariant (7) (considered originally by Misner [3]) is exactly the one proposed by Lorentz at the first

Solvay Congress in 1911 [25,26]. A precise mathematical statement concerning its range of validity was given by Littlewood [27] in the sixties. Littlewood showed that (in our notation)

(i) $\Sigma = c + O(\varepsilon)$;

(ii) $\bar{\Sigma} = c + O(\varepsilon^2)$ ($\bar{\Sigma}$ is the average of Σ over the local period $2\pi/\omega_-$);

(iii) there is no improvement over (i) or (ii);

(iv) $\Sigma(\infty) - \Sigma(-\infty) = O(\varepsilon^n)$ for some specific n;

provided ω_- satisfied certain assumptions which give a measure, ε, to the expression ''slowly varying.'' These assumptions are

$$\omega_- > b_0, \quad \omega_-^{(n)} = \frac{d^n \omega_-}{dt^n} \to 0 \quad \text{as} \quad t \to \pm\infty \, (n \geq 1)$$

$$|\omega_-^{(n)}| < b_n \varepsilon^n \, (n \geq 1), \quad \int_{-\infty}^{\infty} |\omega_-^{(n)}| dt < b'_{n-1} \varepsilon^{n-1}, \quad (8)$$

where the b's are constants. (The notation b' is used to indicate that there need be no specific relationship between the two constants.) A consequence is that, since

$$|\omega_-(\tau_2) - \omega_-(\tau_1)| \leq \int_{\tau_1}^{\tau_2} |\dot{\omega}_-| dt \to 0,$$

where the overdot denotes differentiation with respect to the variable, t, as $\tau_1, \tau_2 \to -\infty, +\infty$ respectively, we have

$$\omega_-(-\infty) = \omega_-(+\infty).$$

Strictly we need not go as far as $\pm\infty$, but the in-channel time is supposed to be long.

Following Arnol'd [28] a function $I(q,p;\varepsilon t)$ is an adiabatic invariant of a Hamiltonian system

$$\dot{p} = -\frac{\partial H}{\partial q}, \quad \dot{q} = \frac{\partial H}{\partial p}, \quad H = H(q,p;\varepsilon t) \quad (9)$$

if $\forall \kappa > 0 \, \exists \varepsilon_0$ such that, if $\varepsilon < \varepsilon_0$ and $0 < t < 1/\varepsilon_0$,

$$|I(q(t),p(t);\varepsilon t) - I(q(0),p(0);0)| < \kappa. \quad (10)$$

The action variable of the corresponding autonomous problem is always an adiabatic invariant and it is for this reason that (7) can be selected as the adiabatic invariant.

However, the perpetuality (cf. [24]) must be calculated and verified. Previous works provide no information about the behavior of ω_- and certainly no measure of ε. Fortunately it is a fairly straightforward matter to test the validity of the use of the adiabatic invariant (7). We can simply integrate the equations of motion numerically and substitute the numbers into (7) to observe the variation of Σ with Ω.

In the channel regime the equations governing the motion of the system point are

$$\Lambda' = -16 e^{2\beta_+ - 4\Omega} \beta_-^2, \quad (11)$$

$$\beta'_+ = \pm \left(4 - (\beta'_-)^2 - \frac{16}{\Lambda} e^{2\beta_+ - 4\Omega} \beta_-^2\right)^{1/2}, \quad (12)$$
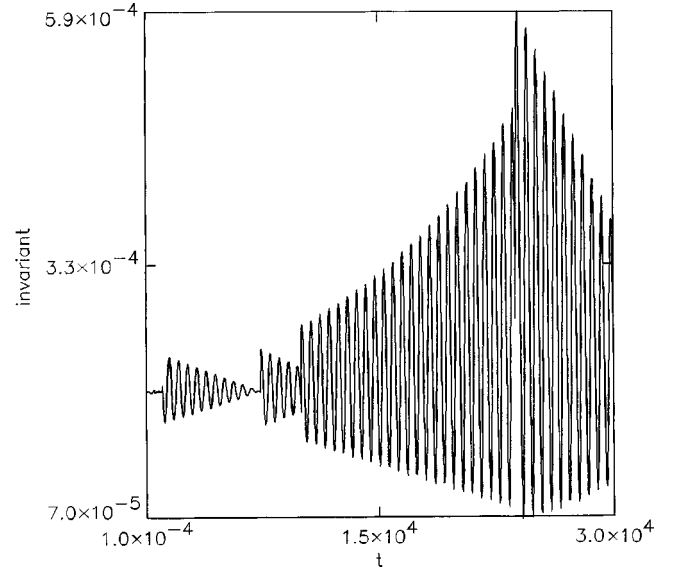

FIG. 2. A variation of the adiabatic invariant against ω in rescaled coordinates. Note that the adiabatic invariant varies from its initial value of $\pm 1.0 - \times 10^{-4}$ up to as much as 5.9×10^{-4} and as low as effectively zero.

$$\beta''_- - \frac{8}{\Lambda} e^{2\beta_+ - 4\Omega} \beta_-^2 \beta'_- + \frac{16}{\Lambda} e^{2\beta_+ - 4\Omega} \beta_- = 0. \quad (13)$$

The positive sign in (12) applies until the expression under the square root sign becomes zero. Equations (11)–(13) are those of Misner [3] except that β_0 has not been introduced. We take initial conditions consistent with the assumptions governing motion in the channel. We integrate the system (11)–(13) numerically using an implementation of the Runge-Kutta scheme [29]. The results are illustrated in Fig. 2. It is quite evident that these computations do not support the use of the adiabatic invariant over the whole time spent inside the channel.

Let us now consider some consequences of the above analysis. Upon the introduction of a relative coordinate,

$$\beta_0 = \beta_+ - 2\Omega, \quad (14)$$

and with the help of Eqs. (6) and (7), Eq. (4) becomes

$$0 = \beta_0'^2 + 4\beta_0' + \frac{\Sigma \omega_-^2}{2} e^{-\beta_0}. \quad (15)$$

On the assumption that β_0' is small, $\beta_0'^2$ is neglected and (15) gives immediately

$$\beta_0 = \log(\Omega_0 - \Omega) + \text{const} \quad (16)$$

which leads to $\beta_0 \to -\infty$ as Ω approaches the critical value Ω_0. This means [3] that the particle leaves the channel and returns bouncing in the triangular region.

Consider now the assumption that is usually made, namely that $\beta_0'^2$ may be neglected. We claim that this assumption leads to a solution which is only asymptotically correct. To see this notice that it follows easily that the solution of (15) is given implicitly by

$$\Omega_0 - \Omega = \frac{1}{4}\{a^{-1}e^{1/2\beta_0} + \sqrt{a^{-2}e^{\beta_0} - 1}\} - \frac{1}{2}\log\{a^{-1}e^{1/2\beta_0}$$
$$+ \sqrt{a^{-2}e^{\beta_0} - 1}\}, \tag{17}$$

where

$$a^2 = \frac{\omega_-^2 \Sigma}{8}, \tag{18}$$

is assumed constant. Misner's original solution (16) is, in a sense, asymptotic to (17) and our assertion follows.

Of more interest, however, is the actual equation for β_0', which follows immediately from (15), viz.,

$$\beta_0' = -2 \pm 2\sqrt{1 - a^2 e^{-\beta_0}}. \tag{19}$$

(The upper sign applies for the initial motion in the channel since $|\beta_+'| < 2$.) Equation (19) is valid only if

$$e^{\beta_0} \geq a^2. \tag{20}$$

Inserting this into Eq. (17) we find that

$$\Omega \leq \Omega_0 - \frac{1}{4} \quad \text{or} \quad \beta_0 \geq -0.60 + \text{const.} \tag{21}$$

In other words Ω never reaches the critical value Ω_0 and β_0 cannot go to $-\infty$ which is what is necessary for the particle to return to the triangle from the channel. [Of course, in practice, it is only sufficient to have β_0 large and negative for the particle to resume bouncing with the wall in the triangular box as soon as it leaves the channel, but even for this to happen the bound in Eq. (21) seems too stringent.]

We are now ready to introduce a new invariant which in many ways behaves better than Σ. It is a well-known fact that the time-dependent oscillator described by the Hamiltonian

$$H = \frac{1}{2}(p^2 + \omega^2(t)q^2) \tag{22}$$

possesses the first integral

$$I = \frac{1}{2}\left((\rho p - \dot{\rho}q)^2 + \left(\frac{q}{\rho}\right)^2\right), \tag{23}$$

which is known as the Ermakov-Lewis invariant [30], provided that the auxiliary variable, ρ, is a solution of

$$\ddot{\rho} + \omega^2(t)\rho = \frac{1}{\rho^3}. \tag{24}$$

Furthermore the solution of (24) has been given by Pinney [31] in terms of the linearly independent solutions of

$$\ddot{v} + \omega^2(t)v = 0. \tag{25}$$

The Lagrangian (5) gives directly the Hamiltonian

$$H_{reduced} = \frac{1}{2}\Lambda^{-1/2}\Pi_-^2 + 8\Lambda^{-1/2}e^{2\beta_+ - 4\Omega}\beta_-^2, \tag{26}$$

where

$$\Pi_- = \Lambda^{1/2}\beta_-'. \tag{27}$$

The reduced Hamiltonian (26) is not precisely of the form of (22) since the coefficient of Π_-^2 is not constant and we have the equivalent of an harmonic oscillator of variable mass. However, following the procedure detailed by Leach [32], for the treatment of a Hamiltonian of the form of (26) we introduce a change of time scale

$$T = \int \Lambda^{-1/2}(\Omega)d\Omega \tag{28}$$

so that the mixmaster system in this regime is described by

$$\widetilde{H}_{reduced} = \frac{1}{2}\Pi_-^2 + \frac{1}{2}\omega^2(T)\beta_-^2 \tag{29}$$

in which T is now the independent variable and

$$\omega^2(T(\Omega)) = 16e^{2\beta_+(\Omega) - 4\Omega}. \tag{30}$$

Under the generalized canonical transformation [33]

$$Q = \frac{\beta_-}{\rho}, \quad P = \rho p - \dot{\rho}q, \quad \tau = \int \rho^{-2}(T)dT, \tag{31}$$

Eq. (29) is transformed to

$$\bar{H} = \frac{1}{2}(P^2 + Q^2) \tag{32}$$

provided that $\rho(T)$ is a solution of (24) [with $\omega^2(T)$ from (30) instead of the $\omega^2(t)$]. $\bar{H}$ is the Ermakov-Lewis invariant in the canonical variables in which it becomes free of explicit dependence on time.

It is $\bar{H}$ which should be used instead of the adiabatic invariant used by Misner. However, there is a problem. In the new variables the evolution of the oscillator is easily described, but that of β_+ becomes unmanageable as the Ermakov-Lewis invariant does not lead to a significant simplification of (12). We must resort to numerical computation and this may as well be performed in the original variables.

Over the interval $0 < \omega < 30000$, β_+ increases essentially linearly with ω (see Fig. 3). There is no indication that it approaches $-\infty$ and, as the deviation from strict linearity is so small, it can only be expected to take a long time to approach zero. In the meantime the potential well proceeds outwards as is evident from Fig. 4 at which the contour is that corresponding to the energy of the motion. We see that the wall outmarches the β_+ value of the universe point. Over this period, as depicted in Fig. 5, the amplitude of the β_- motion increases in a strictly monotonic fashion with a rate of increase increasing with time.

It is clear from these results that, as the particle moves along the channel, the potential walls are receding. Initially this does not present a problem as the β_+ velocity of the particle is sufficiently large. However, there is a critical time, Ω_0, when the walls ''leave the particle behind.'' Hence the particle finds itself in the triangular region again and no
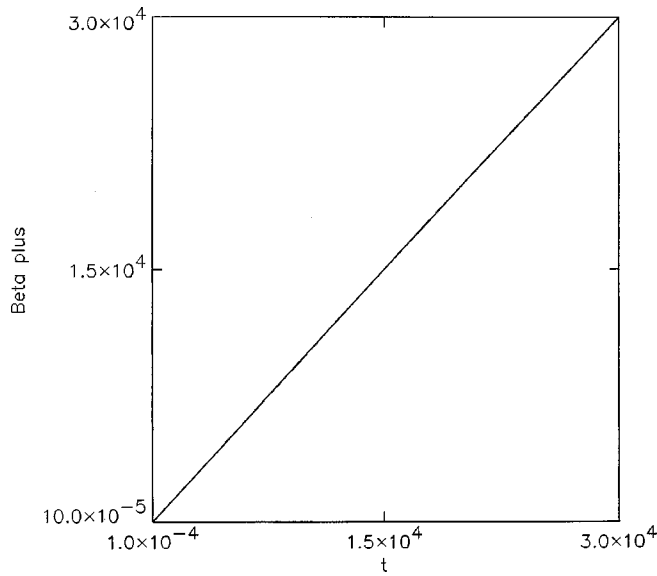
FIG. 3. Variation of β_+ with ω in rescaled coordinates over a time interval (time is numerically equal to ω) from zero to 30 000 (there is a variation of one in 30 000 in β_+ from linearity over the full interval).

longer in the channel. Note that, as β_+ is never less than zero, the particle does not ''turn around'' in the sense that is sometimes described.

III. A CATASTROPHE DESCRIPTION

In what follows we present a novel way to describe the qualitative differences of the main stages in the evolution of the mixmaster universe. As we shall see, the standard interpretation can be reached quite naturally and independently by this line of thought. This approach is established by the application of singularity theory [34] and in particular by that branch of singularity theory known as catastrophe theory [35]. Catastrophe theory studies changes in the equilibria of potentials as the control parameters of the system change. The local properties of the potential in a gradient or a dynamical system are determined by a sequence of theorems such as the implicit function theorem of advanced calculus, the Morse lemma [36] and the Thom theorem [37].
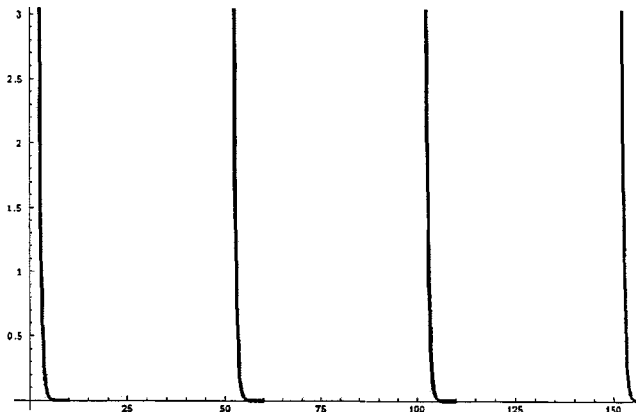

FIG. 4. Movement of the potential barrier as a function of β_+ at successive time intervals. (The large horizontal scale makes the potential walls appear vertical.)
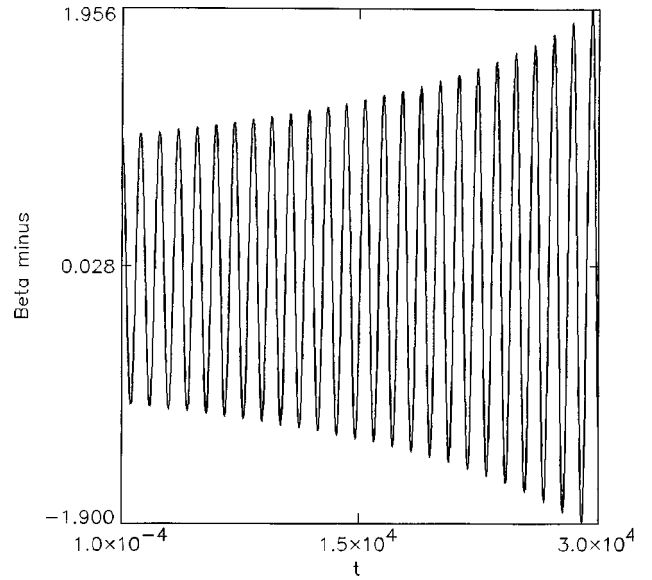

FIG. 5. Variation of β_- over time. The amplitude increases strictly monotonically. The plot was made at every 500th value of t and hence understates the frequency of oscillation. A plot of all the points fills the region between the upper and lower envelope.

We consider the mixmaster universe as a gradient system described by the potential (3). We choose as a control parameter the volume (time) parameter Ω. We set $\nabla = (\partial/\partial\beta_+, \partial/\partial\beta_-)$ and denote the Hessian matrix by

$$V_{ij} = \begin{pmatrix} \dfrac{\partial^2 V}{\partial\beta_+^2} & \dfrac{\partial^2 V}{\partial\beta_+\partial\beta_-} \\ \dfrac{\partial^2 V}{\partial\beta_+\partial\beta_-} & \dfrac{\partial^2 V}{\partial\beta_-^2} \end{pmatrix}. \tag{33}$$

Then, for the Kasner-to-Kasner evolution described by the potential

$$\beta_+ \to -\infty, \quad V(\beta) \sim \frac{1}{3} e^{-8\beta_+}, \tag{34}$$

we find

$$\nabla V = \left(-\frac{8}{3} e^{-8\beta_+}, 0 \right) \neq 0. \tag{35}$$

This means that during an era the implicit function theorem applies and there are no critical points. [This also implies that there is a smooth change of coordinates which makes the potential (3) depend on only one of the variables, say β_+. We see that the form of the potential (34) can be deduced from the general form (3) by using this argument without resorting to any sort of approximations.]

Secondly, we examine the structure of the potential in the neighborhood of the isotropy point (0,0) given by

$$(\beta_+, \beta_-) \sim (0,0), \quad V(\beta) \sim 16(\beta_+^2 + \beta_-^2). \tag{36}$$

After some straightforward manipulations we find that

$$\nabla V = 0. \tag{37}$$

We see that the conditions for the validity of the implicit function theorem are no longer satisfied. Equation (37) implies that near the isotropic point (0,0) the mixmaster universe is in a stable equilibrium state. To see this we determine the stability properties of this state by finding the eigenvalues of the Hessian matrix, V_{ij}. First after a tedious calculation we find

$$\det_{(0,0)} V_{ij} = \frac{256}{3} \neq 0. \tag{38}$$

Also the eigenvalues of V_{ij} are $\lambda_1 = \lambda_2 = 16$. This means that due to the Morse theorem [36] there is a smooth change of variables so that the potential in this case takes the form

$$V = M_0^2 = \lambda_1^2 + \lambda_2^2 = 16(\beta_+^2 + \beta_-^2). \tag{39}$$

Not surprisingly this is exactly the form of the potential that Misner found in this case. M_0^2 stands for the Morse 0-saddle which is the only i-saddle that is stable for two-dimensional gradient systems (cf. [35]). Thus the point (0,0) is a Morse critical point (isolated, nondegenerate). In particular, the potential in this case is structurally stable.

Lastly we examine the corner-run evolution which turns out to be the most interesting from the point of view of catastrophe theory. In this case we find

$$\nabla V = 0 \tag{40}$$

and

$$\det V_{ij} = -8172 \beta_-^2 e^{8\beta_+}. \tag{41}$$

It is clear that for all points on the β_+-axis ($\beta_- = 0$) we have

$$\det_{(\beta_+, 0)} V_{ij} = 0. \tag{42}$$

This implies that in the channel region all points which lie on the β_+ axis are non-Morse critical points. In this case we can cast the potential in a canonical form by adopting a procedure known as the Thom splitting lemma [38]. We split the potential into a Morse part and a non-Morse part according to the number of the vanishing eigenvalues of the Hessian matrix for this case. These are found to be

$$\lambda_1 = 0, \quad \lambda_2 = 32 e^{4\beta_+}, \tag{43}$$

which due to a theorem of Thom [37] guarantees that there is a smooth change of variables that puts the potential (in the channel) in the decomposed form

$$V(\beta) = \beta_+^3 + \alpha \beta_+ + 32 \beta_-^2, \tag{44}$$

with $\alpha \neq 0$. The first two terms in this potential (the non-Morse part) form what is known as the fold catastrophe (A_2) and it is the simplest of the seven elementary catastrophes first discussed by Thom in [37]. The Morse part of the above decomposition is unaffected by perturbations so it is only necessary to study how the qualitative properties of the catastrophe function $A_2 = \beta_+^3 + \alpha \beta_+$ are changed as the control parameter changes. When $\alpha > 0$ there are no critical points whereas $\alpha < 0$ gives two critical points namely, $\beta_\pm = \pm \sqrt{-\alpha}$. The case $\alpha = 0$ is the separatrix in the control pa-

rameter space between functions of two qualitatively different types (no critical points and two critical points).

Our interpretation of the above results uses the *delay convention* of catastrophe theory (see for instance [35]). Since, as we have shown, during the bounces of the point with the walls in the triangular box there are no critical points, we imagine that when the point enters the channel has $\alpha < 0$. Then, as it moves inside the channel, the degenerate point $\alpha = 0$ is reached (the stable minimum disappears into the degenerate critical point). At this instant β_- is no longer small, there are no critical points and the system jumps to the lowest of the two minima (the stable attractor) of $\alpha > 0$. This produces a (point) shock wave which is the simplest elementary catastrophe (fold). This, in turn, means that the system (point) has found itself bouncing again inside the (now larger) triangular box.

IV. CONCLUSIONS

Our adiabatic analysis relates to the well-known issue of the so-called ''anomalous'' behavior discussed previously analytically by Berger in Ref. [13] and numerically in Refs. [39] and [40]. Physically, this in-channel behavior appears only when the initial value of the so-called BKL parameter u is sufficiently large (a BKL ''long era''). However, the required value of u becomes larger (and therefore less probable) as the singularity is approached. This in turn means that the in-channel behavior becomes less probable as the mixmaster singularity is approached.

The recent demonstration of chaoticity by Cornish and Levin [20] via the existence of a mixmaster fractal strange repellor may be seen in the light of the nonadiabatic mixmaster evolution discussed here. It is interesting to point out that the problem of the existence of an adiabatic invariant for higher dimensional generalizations of the mixmaster universe [9] or in higher derivate extensions [10] (wherein chaotic behavior may be absent) is a nontrivial one and one expects that the usual difficulties [24] present in dynamical systems with more than two degrees of freedom exist in this problem too.

Our numerical results parallel those given in [39,40] in the following respects: In those references, figures equivalent to our Fig. 5 are given, but the variables $\beta_\pm / \Omega$ are plotted there (mixing bounces) rather than our variables $\beta_\pm$. We stress that no confusion must arise in this respect since, in the former variables the trajectory associated with a single era appears to move outwards along a corner and then inwards again while here the motion is strictly outward. Further, as is clearly emphasized by Berger in [40], the angle of the minisuperspace trajectory becomes ever closer to the perpendicular to the ray down the corner as the era progresses towards the singularity. A change of era occurs when the trajectory points inwards rather than outwards with respect to this perpendicular direction.

We hope that our reformulation of the problem in terms of catastrophe theory in Sec. III may be further used to examine questions of current interest such as, for instance, issues connected with the occurrence of chaotic behavior. In some sense, our catastrophe results correspond to just mixmaster *statics*. Further dynamical issues could be addressed if one considers the mixmaster system as a gradient system as is

usual in catastrophe discussions of dynamical systems.

Another issue that is raised by our formulation is the influence of changing the time parameter on the character of the catastrophe. It is well known that there exist different time parametrizations for the description of the mixmaster dynamics (see, for instance, [40]). It is therefore appropriate to ask how the catastrophe profile of the mixmaster dynamics is affected by different choices of time. Although the answer to this question is uncertain at present, we believe that a physically relevant formulation should be unaffected by different time choices.

ACKNOWLEDGMENTS

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