p-brane cosmology and phases of Brans-Dicke theory with matter

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We study the effect of the solitonic degrees of freedom in string cosmology following the line of Rama. The gas of a solitonic p-brane is treated as a perfect fluid in a Brans-Dicke-type theory. In this paper, we find exact cosmological solutions for any Brans-Dicke parameter ω and for a general parameter γ of the equation of state and classify the cosmology of the solutions on a parameter space of γ and ω. [S0556-2821(98)07408-6]

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I. INTRODUCTION

Recent developments of string theory suggest that in a regime of Planck length curvature, quantum fluctuation is very large so that string coupling becomes large and consequently the fundamental string degrees of freedom are not weakly coupled *good* ones [1]. Instead, solitonic degrees of freedom such as p-branes or D-p-branes [2] are more important. Therefore it is a very interesting question to ask what effect these new degrees of freedom might have on the space-time structure, and especially whether including these degrees of freedom resolves the initial singularity, which is a problem in standard general relativity.

For the investigation of p-brane cosmology, the usual low-energy effective action coming from the β function of the string world sheet would not be a good starting point. So there will be a difference from string cosmology [3]. We need to find the low-energy effective theory that contains gravity and at the same time reveals the effect of these solitonic objects. But those are not known. Therefore one can only guess the answer at this moment. The minimum requirement is that it should be a gravity theory; therefore it must be a generalization of general relativity. Brans-Dicke (BD) theory [4] is a generic deformation of general relativity allowing variable gravity coupling. In fact a low-energy theory of the fundamental string contains the Brans-Dicke theory with a fine tuned deformation parameter ($\omega=-1$). Moreover Duff, Khuri, and Lu [5] found that the natural metric that couples to the p-brane is the Einstein metric multiplied by certain power of the dilaton field. In terms of this new metric, the action that gives the p-brane solution becomes a Brans-Dicke action with a definite deformation parameter ω depending on p. Using this action, Rama [6] recently argued that the gas of the solitonic p-brane [5] treated as a perfect-fluid-type matter in a Brans-Dicke theory can resolve the initial singularity without any explicit solution. In a previous paper [7], we studied this model and found several analytic solutions for a few values of the parameter with which the coupled dilaton-graviton system could be decoupled by the simple ‘‘completing the square’’ method. In this paper, we give exact cosmological solutions for any Brans-Dicke parameter ω and for a general equation of state and classify the cosmology of the solutions according to the range of parameters involved.

The rest of this paper is organized as follows. In Sec. II, we set up the action for the p-brane cosmology. In Sec. III, we find an analytic solution for the equation of motion and constraint equation for the general case. In Secs. IV and V, we study the cosmology of the solution and classify them according to their behavior. In Sec. IV, t as a function of the dilaton time τ is studied and the behavior of the scale factor a with respect to the dilaton time τ in the asymptotic region is studied in Sec. V. In Sec. VI, using the results in Secs. IV and V, we classify the cosmology into several phases and investigate the behavior of the scale factor a as a function of the cosmic time t. In Sec. VII, we summarize and conclude with some discussions.

II. CONSTRUCTION OF THE ACTION WITH THE SOLITONIC MATTER

We consider the bosonic part of the effective string action and analyze the evolution of a D-dimensional homogeneous isotropic universe with the solitonic matter included. The action is given by

$$S=\int d^Dx\sqrt{-g}e^{-\phi}[\mathcal{R}-\omega\partial_\mu\phi\partial^\mu\phi]+S_m, \tag{1}$$

where ϕ is the dilaton field and S_m is the matter part of the action. In string theory the BD parameter ω is fixed as -1. In the high curvature regime, the string coupling is also big and the solitonic p-brane will be copiously produced since they become light and dominate the universe in that regime. Duff *et al.* [5] have shown that in terms of metric which couples minimally to the p-brane ($p=d-1$), the effective action can be written as Brans-Dicke theory with the BD parameter ω given by

$$\omega=-\frac{(D-1)(d-2)-d^2}{(D-2)(d-2)-d^2}. \tag{2}$$

In four dimensions, the BD parameter is given by $\omega=-\frac{4}{3}$ for the 0-brane ($p=0$) and $\omega=-\frac{3}{2}$ for the instanton ($p=-1$). Let us assume that the gas of solitonic p-brane can be considered as perfect fluid in the Brans-Dicke theory with the

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equation of state $p=\gamma\rho, \gamma<1$. Therefore our starting point is the equation of BD theory [8,9]:

$$\mathcal{R}_{\mu\nu}-\frac{g_{\mu\nu}}{2}\mathcal{R}=\frac{e^{\phi}}{2}T_{\mu\nu}+\omega\left\{\partial_\mu\phi\partial_\nu\phi-\frac{g_{\mu\nu}}{2}(\partial\phi)^2\right\}$$
$$+\{-\partial_\mu\partial_\nu\phi+\partial_\mu\phi\partial_\nu\phi+g_{\mu\nu}\mathcal{D}^2\phi-g_{\mu\nu}(\partial\phi)^2\},$$
$$\mathcal{R}-2\omega\mathcal{D}^2\phi+\omega(\partial\phi)^2=0, \tag{3}$$

where ϕ is the dilaton and $\mathcal{D}$ means a covariant derivative. $\mathcal{R}$ is the curvature scalar and cosmological metric is given as the following form:

$$ds_D{}^2=-\frac{1}{\mathcal{N}}dt^2+e^{2\alpha(t)}\delta_{ij}dx^i dx^j \quad (i,j=1,2,\ldots,D-1), \tag{4}$$

where $e^{\alpha(t)}[=a(t)]$ is the scale factor and $\mathcal{N}$ is the (constant) lapse function. Now, we assume that all variables are the functions of time only. The curvature scalar [10] in D dimension is given by

$$\mathcal{R}=g^{00}\mathcal{R}_{00}+g^{ij}\mathcal{R}_{ij},$$
$$g^{00}\mathcal{R}_{00}=\frac{D-1}{\mathcal{N}}[\ddot{\alpha}+\dot{\alpha}^2],$$
$$g^{ij}\mathcal{R}_{ij}=\frac{D-1}{\mathcal{N}}[\ddot{\alpha}+(D-1)\dot{\alpha}^2], \tag{5}$$

where $\dot{\alpha}$ means the time derivative of α.

The energy-momentum tensor of the solitonic matter is given by

$$T_{\mu\nu}=pg_{\mu\nu}+(p+\rho)U_\mu U_\nu, \tag{6}$$

where U_μ is the fluid velocity. The hydrostatic equilibrium condition of energy-momentum conservation is

$$\dot{\rho}+(D-1)(p+\rho)\dot{\alpha}=0. \tag{7}$$

Using $p=\gamma\rho$, we get the solution

$$\rho=\rho_0 e^{-(D-1)(1+\gamma)\alpha}. \tag{8}$$

The parameters γ and ω expressed in Eqs. (1) and (8) are free parameters. Our goal is to study how the metric variables change their behavior for various values of γ and ω.

If we consider only the time dependence, the action can be brought to the following form:

$$S=\int dt\, e^{(D-1)\alpha-\phi}\left[\frac{1}{\sqrt{\mathcal{N}}}\{-(D-2)(D-1)\dot{\alpha}^2+2(D-1)\dot{\alpha}\dot{\phi}+\omega\dot{\phi}^2\}-\sqrt{\mathcal{N}}\rho_0 e^{-(D-1)(1+\gamma)\alpha+\phi}\right], \tag{9}$$

where we eliminated p and ρ by Eqs. (7) and (8). The variation over the constant lapse function, which is only g_{00}, gives a constraint equation. When we set the lapse function $\mathcal{N}$ to be 1 after varying of the action over the lapse function $\mathcal{N}$, this constraint equation is the equation of motion of the g_{00} component in Eq. (5).

III. ANALYTIC SOLUTION

Now, we introduce a new time variable τ as

$$dt=e^{(D-1)\alpha-\phi}d\tau. \tag{10}$$

Then the action becomes

$$S=\int d\tau\left[\frac{1}{\sqrt{\mathcal{N}}}\{(D-1)\kappa\dot{Y}^2+\mu\dot{X}^2\}-\sqrt{\mathcal{N}}\rho_0 e^{-2X}\right], \tag{11}$$

where new variables presented in the action are given by

$$\kappa=(D-1)(1-\gamma)^2(\omega-\omega_\kappa),$$
$$\nu=2(1-\gamma)(\omega-\omega_\nu),$$
$$\mu=-\frac{4(D-2)}{\kappa}(\omega-\omega_{-1}),$$
$$-2X=(D-1)(1-\gamma)\alpha-\phi,$$
$$Y=\alpha+\frac{\nu}{\kappa}X,$$
$$\omega_\kappa=-\frac{D-2D\gamma+2\gamma}{(D-1)(1-\gamma)^2},$$
$$\omega_\nu=-\frac{1}{1-\gamma},$$
$$\omega_{-1}=-\frac{D-1}{D-2}. \tag{12}$$

The constraint equation is written as

$$0=(D-1)\kappa\dot{Y}^2+\mu\dot{X}^2+\rho_0 e^{-2X}, \tag{13}$$

where ρ_0 is a positive real constant. The equations of motion are written as

$$0=\ddot{Y},$$
$$0=\ddot{X}-\frac{\rho_0}{\mu}e^{-2X}. \tag{14}$$

Note that ω_{-1} in Eq. (12) happens to be the value of the instanton. If ω is less than ω_{-1}, the kinetic term of the dilaton has a negative energy in Einstein frame. So we will consider the case where ω is larger than ω_{-1}. According to the sign of κ, the types of solutions are very different.

When κ is negative, an exact solution becomes

$$X=\ln\left[\frac{q}{c}\cosh(c\tau)\right],$$
$$Y=A\tau+B, \tag{15}$$

where c, A, B, and $q=\sqrt{\rho_0/|\mu|}$ are arbitrary real constants. Using the constraint equation, we determine A in terms of other variables

$$A=\frac{c}{\delta}, \quad \text{with} \quad \delta=\sqrt{-\frac{(D-1)\kappa}{\mu}}$$
$$=\frac{|\kappa|}{2\sqrt{1+\omega(D-2)/(D-1)}}. \tag{16}$$

If κ is zero, then we can obtain a solution of the equations of motion, but it does not satisfy the constraint equation. If κ is positive, the solution is

$$X=\ln\left[\frac{q}{c}|\sinh(c\tau)|\right],$$
$$Y=\frac{c}{\delta}\tau+B. \tag{17}$$

IV. COSMOLOGY OF THE SOLUTION

Now, we investigate the relation between the cosmic time t and the dilaton time τ. Since the solutions of the equations of motion have different forms, we study the behavior of t as a function of τ case by case.

A. $\kappa<0$ case

In this region, $\omega<\omega_\kappa$. γ is always less than 1. We find the relation between t and τ using Eq. (10)

$$t-t_0=\int_{\tau_0}^{\tau}d\tau'\exp\left[\frac{(D-1)\gamma c}{\delta}\tau' - \left(2+\frac{(D-1)\gamma\nu}{\kappa}\right)\ln\left\{\frac{q}{c}\cosh(c\tau')\right\}+(D-1)\gamma B\right], \tag{18}$$

where $(D-1)\gamma B$ is a constant. This constant can be ignored in the limit $\tau\to\pm\infty$. Because $dt/d\tau$ is always positive definite, t is a monotonic function of τ. The behavior of $a(t)$ as a function of t depends crucially on the relation between t and τ. When τ goes to $\pm\infty$, t is reduced to

$$t-t_0\approx\frac{1}{T_\pm}(e^{T_\pm\tau}-e^{T_\pm\tau_0}), \tag{19}$$

where

$$T_\pm=\frac{2c}{|\kappa|}\left[(D-1)\gamma\sqrt{1+\omega\frac{D-2}{D-1}} \pm\{\kappa+(D-1)\gamma[1+\omega(1-\gamma)]\}\right]. \tag{20}$$

We define a new concept for our purpose: t is the supermonotonic function of τ if it is monotonic and t runs the entire real line when τ does. When t is a supermonotonic function of τ, the universe evolves from infinite past to infinite future. Otherwise the scale factor $a(t)$ has a starting (ending) point at a finite cosmic time t_i (t_f) which corresponds to initial (final) singularity. As a mapping, t maps the real line of τ to

$$(-\infty,\infty) \quad \text{if} \quad T_-<0<T_+,$$
$$(-\infty,t_f) \quad \text{if} \quad T_-<0 \quad \text{and} \quad T_+<0,$$
$$(t_i,\infty) \quad \text{if} \quad T_->0 \quad \text{and} \quad T_+>0,$$
$$(t_i,t_f) \quad \text{if} \quad T_+<0<T_-.$$

In the limit $\tau\to\pm\infty$, the condition $T_\pm<0$ is expressed as

$$(D-1)\gamma\sqrt{1+\omega\frac{D-2}{D-1}}<\mp[\kappa+(D-1)\gamma\{1+\omega(1-\gamma)\}]. \tag{21}$$

This inequality is divided into two cases according to the sign of γ. In each case we obtain the different region of ω satisfying the condition $T_\pm<0$.

1. $\gamma>0$ case

Because we have considered only the case $\omega>\omega_{-1}$, the left-hand side in Eq. (21) is positive definite. To satisfy the inequality $T_-<0$, the conditions

$$\kappa+(D-1)\gamma\{1+\omega(1-\gamma)\}>0$$

and

$$\left((D-1)\gamma\sqrt{1+\omega\frac{D-2}{D-1}}\right)^2 <[\kappa+(D-1)\gamma\{1+\omega(1-\gamma)\}]^2, \tag{22}$$

must be satisfied. If they do not, we know that T_- is positive. The first inequality in Eq. (22) is reduced to the following inequality:

$$\omega>\omega_{-\infty}:=-\frac{D-(D-1)\gamma}{(D-1)(1-\gamma)}. \tag{23}$$

It is remarkable that the second inequality in Eq. (22) is written as

$$(\omega-\omega_0)(\omega-\omega_\kappa)>0, \tag{24}$$

where ω_κ appeared in the definition of κ and

$$\omega_0=-\frac{D}{D-1} \tag{25}$$

is the value of ω for the 0-brane, see Eq. (2).

As shown in Fig. 1, $\omega_\kappa<\omega_0$ ($\omega_\kappa>\omega_0$) in the region $0<\gamma<2/D$ ($\gamma>2/D$). Therefore the solution of Eq. (24) becomes

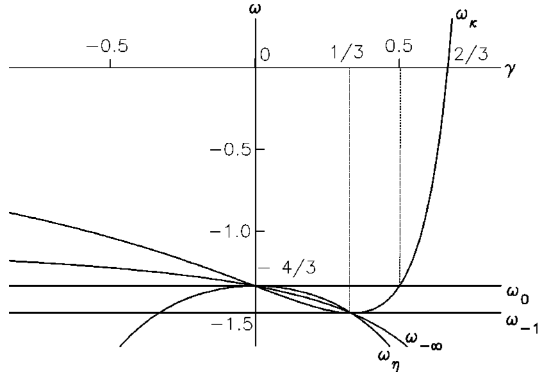

FIG. 1. In four dimensions $(D=4)$, all functions $(\omega_\kappa, \omega_\eta, \ldots)$, defined by the relation between t and τ, are presented on a parameter space of γ and ω.

$$\omega_{-1}<\omega<\omega_\kappa \quad \text{for} \quad 0<\gamma<\frac{2}{D},$$

$$\omega_{-1}<\omega<\omega_0 \quad \text{for} \quad \gamma>\frac{2}{D}. \tag{26}$$

Combining Eqs. (23) and (26), we find the region of ω satisfying the condition $T_-<0$ as the following:

$$\omega_{-1}<\omega<\omega_\kappa \quad \text{for} \quad \frac{1}{D-1}<\gamma<\frac{2}{D},$$

$$\omega_{-1}<\omega<\omega_0 \quad \text{for} \quad \gamma>\frac{2}{D}. \tag{27}$$

The condition $T_+<0$ is

$$(D-1)\gamma\sqrt{1+\omega\frac{D-2}{D-1}}<-[\kappa+(D-1)\gamma\{1+\omega(1-\gamma)\}]. \tag{28}$$

For this, two conditions,

$$\kappa+(D-1)\gamma\{1+\omega(1-\gamma)\}<0$$

and

$$\left((D-1)\gamma\sqrt{1+\omega\frac{D-2}{D-1}}\right)^2 <(-[\kappa+(D-1)\gamma\{1+\omega(1-\gamma)\}])^2, \tag{29}$$

must be satisfied at the same time. In Eq. (29), the first inequality gives

$$\omega<\omega_{-\infty} \tag{30}$$

and the second inequality gives Eq. (24) again. Therefore using Eqs. (26) and (30), we find the region of ω satisfying $T_+<0$:

$$\omega_{-1}<\omega<\omega_\kappa \quad \text{for} \quad 0<\gamma<\frac{1}{D-1}. \tag{31}$$

2. $\gamma<0$ case

In this case, the condition $T_-<0$ is written as

$$(D-1)|\gamma|\sqrt{1+\omega\frac{D-2}{D-1}} >-[\kappa+(D-1)\gamma\{1+\omega(1-\gamma)\}]. \tag{32}$$

For this, we need

$$\kappa+(D-1)\gamma\{1+\omega(1-\gamma)\}>0$$

or

$$\left((D-1)\gamma\sqrt{1+\omega\frac{D-2}{D-1}}\right)^2 >(-[\kappa+(D-1)\gamma\{1+\omega(1-\gamma)\}])^2. \tag{33}$$

Equation (33) can be simplified as

$$\omega_{-\infty}<\omega \quad \text{or} \quad \omega_0<\omega<\omega_\kappa. \tag{34}$$

Thus the solution, which is the sum of two regions in Eq. (34), is reduced to

$$\omega_0<\omega<\omega_\kappa \quad \text{for} \quad \gamma<0. \tag{35}$$

This solution includes the region of the first inequality of Eq. (34).

Similarly, the condition $T_+<0$ is written as

$$(D-1)|\gamma|\sqrt{1+\omega\frac{D-2}{D-1}}>\kappa+(D-1)\gamma\{1+\omega(1-\gamma)\}, \tag{36}$$

which gives

$$\kappa+(D-1)\gamma\{1+\omega(1-\gamma)\}<0$$

or

$$\left((D-1)\gamma\sqrt{1+\omega\frac{D-2}{D-1}}\right)^2 >[\kappa+(D-1)\gamma\{1+\omega(1-\gamma)\}]^2. \tag{37}$$

The solution of these can be written as $\omega<\omega_{-\infty}$ or $\omega_0<\omega<\omega_\kappa$. From these, the region satisfying $T_+<0$ is

$$\omega_{-1}<\omega<\omega_\kappa \quad \text{for} \quad \gamma<0. \tag{38}$$

B. $\kappa>0$ case

Now we consider positive κ, which means

$$\omega>\omega_\kappa. \tag{39}$$

Since the solution $X(\tau)$ has a singularity at $\tau=0$, we have to treat carefully the behavior of t near $\tau=0$. The relation between t and τ is given by

$$t-t_0=\int_{\tau_0}^{\tau}d\tau'\exp\left[\frac{(D-1)\gamma c}{\delta}\tau' -\left(2+\frac{(D-1)\gamma\nu}{\kappa}\right)\ln\left\{\frac{q}{c}|\sinh(c\tau')|\right\} +(D-1)\gamma B\right]. \tag{40}$$

In the limit $\tau\to 0$, the above equation is reduced to

$$t-t_0\approx \mathrm{sgn}(\tau)\frac{q^{-\eta}e^{(D-1)\gamma B}}{1-\eta}[|\tau|^{1-\eta}-|\tau_0|^{1-\eta}], \tag{41}$$

where $\eta=2+(D-1)\gamma\nu/\kappa$ and τ_0 and t_0 are real constants. In case of $\eta>1$, t has a singularity at $\tau\to 0$. In the other case, t has no singularity. So we consider two cases $\eta<1$ and $\eta>1$.

1. $\eta<1$ case

In this case, t has no singularity at $\tau=0$, so we investigate the behavior of t at $\tau\to\pm\infty$ only.

(i) $\gamma>0$ case. In the case $\kappa>0$, Eq. (40) is reduced to

$$t-t_0\approx\frac{1}{T_\pm}(e^{T_\pm\tau}-e^{T_\pm\tau_0}), \tag{42}$$

where

$$T_\pm=\frac{2c}{|\kappa|}\left[(D-1)\gamma\sqrt{1+\omega\frac{D-2}{D-1}} \mp\{\kappa+(D-1)\gamma[1+\omega(1-\gamma)]\}\right]. \tag{43}$$

The condition $T_-<0$ is written as Eq. (28) and gives the solution

$$\omega<\omega_{-\infty}\quad\text{and}\quad\omega>\omega_0\quad\text{for}\quad 0<\gamma<\frac{2}{D},$$

$$\omega<\omega_{-\infty}\quad\text{and}\quad\omega>\omega_\kappa\quad\text{for}\quad\gamma>\frac{2}{D}, \tag{44}$$

where we use $\omega>\omega_\kappa$. As shown Fig. 1, $\omega_0>\omega_{-\infty}$ for $0<\gamma<2/D$ and $\omega_\kappa>\omega_{-\infty}$ for $\gamma>2/D$. Therefore there is no solution satisfying the condition $T_-<0$. Hence T_- is positive.

Now we investigate the behavior of t at $\tau\to+\infty$. The condition $T_+<0$ is written as in Eq. (22). Applying a similar method used in the above analysis, the region of ω satisfying $T_+<0$ is summarized as the following:

$$\omega_0<\omega\quad\text{for}\quad 0<\gamma<\frac{2}{D},$$

$$\omega_\kappa<\omega\quad\text{for}\quad\gamma>\frac{2}{D}. \tag{45}$$

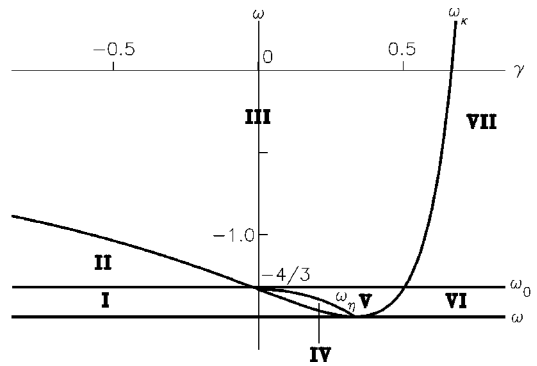

FIG. 2. In four dimensions ($D=4$), the parameter space is classified by the relation between t and τ.

(ii) $\gamma<0$ case. Through the same calculation, we can show that T_- is positive and T_+ is negative for all negative γ.

2. $\eta>1$ case

In this case, the behavior of t is singular at $\tau=0$. τ_0 and t_0 can be ignored due to the divergence of $|\tau|^{1-\eta}$. From Eqs. (40) or (41), we know that $dt/d\tau$ is always positive definite except a singular point $\tau=0$. The condition $\eta>1$ is reduced to

$$\omega>-\frac{D}{(D-1)(1-\gamma^2)}:=\omega_\eta. \tag{46}$$

Under this condition, the region of τ is divided into $-\infty<\tau<0$ and $0<\tau<\infty$. Near $\tau=0$, we obtain the behavior of t characterized by the sign of τ. When τ goes to zero from below, t in Eq. (41) is written as

$$t\approx\frac{q^{-\eta}e^{(D-1)\gamma B}}{(\eta-1)}\frac{1}{(-\tau)^{\eta-1}}. \tag{47}$$

When τ goes to zero from above, t is reduced as the following:

$$t\approx-\frac{q^{-\eta}e^{(D-1)\gamma B}}{(\eta-1)}\frac{1}{\tau^{\eta-1}}. \tag{48}$$

Thus $t\to+\infty$ as $\tau\to-0$ but $t\to-\infty$ as $\tau\to+0$. We also have to examine the behaviors of t at $\tau\to\pm\infty$. However, these were already described when we discussed the case $\eta<1$.

To describe the behavior of t as a function of τ at $\tau\to\pm\infty$ and $\tau\to 0$, we classify the parameter space of γ and ω using all results obtained in this section. These are shown in Fig. 2.

The behavior of the t as a function of τ in Fig. 2 is summarized as the following.

(1) In region I, $T_->0$ and $T_+<0$ ($\omega<\omega_\kappa$). t evolves from finite initial time t_i to finite final time t_f as τ runs $(-\infty,+\infty)$.

(2) In region II, $T_-<0$ and $T_+<0$ ($\omega<\omega_\kappa$). t evolves from negative infinity to finite final time t_f as τ runs $(-\infty,+\infty)$.

(3) In region III, $T_- > 0$ and $T_+ < 0$ ($\omega > \omega_\kappa$ and $\omega > \omega_\eta$). In this region, because t has a singular behavior at $\tau = 0$, the region of τ divided into $-\infty < \tau < 0$ and $0 < \tau < \infty$. Therefore t has two branches for any given values of γ and ω. For $-\infty < \tau < 0$, t evolves from finite initial time t_i to positive infinity. For $0 < \tau < \infty$, t evolves from negative infinity to finite final time t_f.

(4) In region IV, $T_- > 0$ and $T_+ > 0$ ($\omega > \omega_\kappa$ and $\omega < \omega_\eta$). In this region, because t has no singularity, t evolves from finite initial time t_i to positive infinity as τ runs $(-\infty, +\infty)$.

(5) In region V, $T_- > 0$ and $T_+ > 0$ ($\omega > \omega_\kappa$ and $\omega > \omega_\eta$). For the same reason as explained in region III, t evolves from finite initial time t_i to positive infinity for $-\infty < \tau < 0$ and t evolves from negative infinity to positive infinity for $0 < \tau < \infty$.

(6) In region VI, $T_- < 0$ and $T_+ > 0$ ($\omega < \omega_\kappa$). t evolves from negative infinity to positive infinity as τ runs $(-\infty, +\infty)$.

(7) In region VII, $T_- > 0$ and $T_+ > 0$ ($\omega < \omega_\kappa$). t evolves from finite initial time t_i to positive infinity as τ runs $(-\infty, +\infty)$.

V. THE BEHAVIOR OF THE SCALE FACTOR

Now we study the behavior of the scale factor a as a function of τ.

A. κ<0 case

We consider the exponent of the scale factor $\alpha(\tau)$. Using Eq. (10), $\alpha(\tau)$ is given by

$$\alpha(\tau) = \frac{2}{|\kappa|}\left[\sqrt{1+\omega\frac{D-2}{D-1}}c\tau + \{1+\omega(1-\gamma)\}\ln\left\{\frac{q}{c}\cosh(c\tau)\right\}\right] + B. \quad (49)$$

In the limit $\tau \to \pm\infty$, the scale factor $a(\tau) = e^{\alpha(\tau)}$ is rewritten as

$$a(\tau) \approx e^{H_\pm \tau}, \quad (50)$$

where $H_\pm$ is defined as

$$H_\pm = \frac{2c}{|\kappa|}\left[\sqrt{1+\omega\frac{D-2}{D-1}} \pm \{1+\omega(1-\gamma)\}\right]. \quad (51)$$

Just as $t(\tau)$, the behavior of $a(\tau)$ at $\tau \to \pm\infty$ is determined by the sign of $H_\pm$. Using this and the sign of $T_\pm$, we can read the behavior of the scale factor $a(t)$ as a function of t in the asymptotic regions.

For negative κ ($\omega < \omega_\kappa$), $H_- > 0$ can be written as

$$\sqrt{1+\omega\frac{D-2}{D-1}} > 1+\omega(1-\gamma). \quad (52)$$

This means

$$1+\omega(1-\gamma) < 0$$

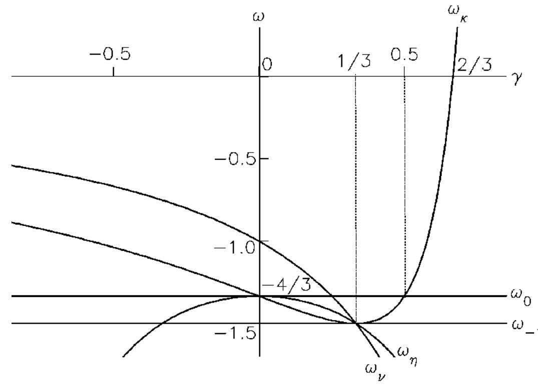

FIG. 3. In four dimensions ($D=4$), all functions ($\omega_\kappa, \omega_\nu, \ldots$), defined by the relation between $a(\tau)$ and τ, are presented on a parameter space of γ and ω.

or

$$\left(\sqrt{1+\omega\frac{D-2}{D-1}}\right)^2 > [1+\omega(1-\gamma)]^2. \quad (53)$$

The first inequality in Eq. (53) is equivalent to

$$\omega < \omega_\nu, \quad (54)$$

where ω_ν was written in Eq. (12). As one can see in Fig. 3, $\omega_\kappa < \omega_\nu$ if $\gamma < 1/(D-1)$ and $\omega_\kappa > \omega_\nu$ if $\gamma > 1/(D-1)$. Together with $\omega < \omega_\kappa$, the first inequality condition gives the region of ω satisfying $H_- > 0$

$$\omega_{-1} < \omega < \omega_\kappa \quad \text{for} \quad \gamma < \frac{1}{(D-1)}. \quad (55)$$

The second inequality in Eq. (53) is rewritten as

$$\omega(\omega - \omega_\kappa) < 0. \quad (56)$$

Notice that $\omega_\kappa < 0$ if $\gamma < D/2(D-1)$ and $\omega_\kappa > 0$ if $\gamma > D/2(D-1)$. Since $\omega < \omega_\kappa$, we can rewrite Eq. (56) as

$$0 < \omega < \omega_\kappa \quad \text{for} \quad \gamma > \frac{D}{2(D-1)}. \quad (57)$$

As a result, Eqs. (55) and (57) are the regions of ω satisfying the condition $H_- > 0$.

Now we consider the condition $H_+ > 0$:

$$\sqrt{1+\omega\frac{D-2}{D-1}} > -[1+\omega(1-\gamma)]. \quad (58)$$

Like the case $H_- > 0$, this inequality is divided into two inequalities

$$1+\omega(1-\gamma) > 0$$

or

$$\left(\sqrt{1+\omega\frac{D-2}{D-1}}\right)^2 > [1+\omega(1-\gamma)]^2. \quad (59)$$

The first inequality gives the region of ω satisfying the condition $H_+ > 0$:

$$\omega > \omega_\nu . \tag{60}$$

Using $\omega > \omega_{-1}$ and $\kappa < 0$, Eq. (60) is rewritten as the following:

$$\omega_{-1} < \omega < \omega_\kappa \quad \text{for} \quad \gamma > \frac{1}{D-1}. \tag{61}$$

The second inequality in Eq. (59) has the same region of ω that appeared in Eq. (57). Because the region of ω in Eq. (61) contains the region of ω in Eq. (57), Eq. (61) is the solution satisfying the condition $H_+ > 0$.

B. $\kappa > 0$ case

The exponent of the scale factor $\alpha(\tau)$ is given by

$$\alpha(\tau) = \frac{2}{|\kappa|}\left[\sqrt{1+\omega\frac{D-2}{D-1}}c\tau - \{1+\omega(1-\gamma)\}\ln\left\{\frac{q}{c}|\sinh(c\tau)|\right\}\right] + B. \tag{62}$$

1. $\eta < 1$ case

In this case, the scale factor $a(\tau)$ has no singular behavior. So we investigate the behavior of a at $\tau \to \pm\infty$.

In the limit $\tau \to \pm\infty$, $a(\tau)$ is given by

$$a(\tau) \approx e^{H_\pm \tau}, \tag{63}$$

where $H_\pm$ is defined as

$$H_\pm = \frac{2c}{|\kappa|}\left[\sqrt{1+\omega\frac{D-2}{D-1}} \mp \{1+\omega(1-\gamma)\}\right].$$

The condition $H_- > 0$ is exactly equal to Eq. (58) due to the sign of κ. When we solve Eq. (58) under the condition $\kappa > 0$, $\omega > \omega_\kappa$ instead of $\omega < \omega_\kappa$ must be applied to the solution. Then we obtain the region of ω satisfying the condition $H_- > 0$:

$$\omega > \omega_\kappa \quad \text{for} \quad \text{all} \quad \gamma. \tag{64}$$

The condition $H_+ > 0$ described by Eq. (52) and $\omega > \omega_\kappa$ gives the region of ω:

$$\omega_\kappa < \omega < 0 \quad \text{for} \quad \gamma < \frac{D}{2(D-1)}. \tag{65}$$

2. $\eta > 1$ case

In this case, we need to investigate the behavior of $a(\tau)$ at $\tau \to 0$ because $a(\tau)$ has a singular behavior at $\tau = 0$. In the limit $\tau \to 0$, $a(\tau)$ is written as

$$a(\tau) \approx e^B (q|\tau|)^{-2(1-\gamma)(\omega-\omega_\nu)/|\kappa|}. \tag{66}$$

For $\omega > \omega_\nu$, where $-(1-\gamma)(\omega-\omega_\nu)$ is negative, $a(\tau)$ goes to infinite at $\tau \to 0$. And for $\omega < \omega_\nu$, $a(\tau)$ goes to zero at $\tau \to 0$. The behavior of $a(\tau)$ at $\tau \to \pm\infty$ was described already when we discussed the case $\eta < 1$.

From these studies, we classify the behavior of $a(\tau)$ on the parameter space of γ and ω. This is shown in Fig. 4.

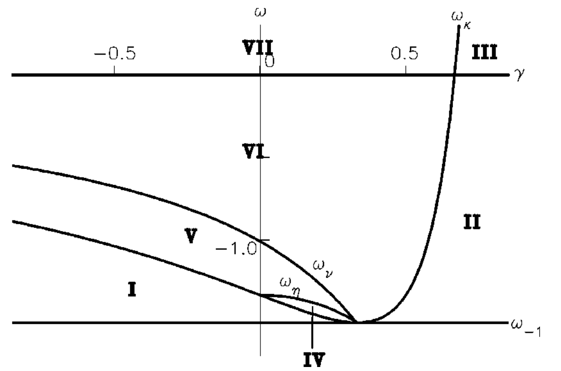

FIG. 4. In four dimensions ($D = 4$), the parameter space is classified by the behavior of $a(\tau)$.

As shown in Fig. 4, we summarize the behavior of $a(\tau)$ as the following.

(1) In region I, $H_- > 0$ and $H_+ < 0$ ($\omega < \omega_\kappa$). In the limit $\tau \to \pm\infty$, $a(\tau)$ goes to a zero size.

(2) In region II, $H_- < 0$ and $H_+ > 0$ ($\omega < \omega_\kappa$). In the limit $\tau \to \pm\infty$, $a(\tau)$ goes to an infinite size.

(3) In region III, $H_- > 0$ and $H_+ > 0$ ($\omega < \omega_\kappa$). In the limit $\tau \to -\infty$, $a(\tau)$ goes to a zero size. And in the limit $\tau \to \infty$, $a(\tau)$ goes to an infinite size.

(4) In region IV, $H_- > 0$ and $H_+ > 0$ ($\omega > \omega_\kappa$ and $\omega < \omega_\eta$). In this region, the behavior of t is not singular, so we need not consider the behavior of $a(\tau)$ at $\tau = 0$. In the limit $\tau \to -\infty$, $a(\tau)$ goes to a zero size. And in the limit $\tau \to \infty$, $a(\tau)$ goes to an infinite size.

(5) In region V, $H_- > 0$ and $H_+ > 0$ ($\omega > \omega_\kappa$, $\omega > \omega_\eta$ and $\omega < \omega_\nu$). In this region, because t has a singular behavior at $\tau = 0$, we interpret the behavior of $a(\tau)$ as the following: For $-\infty < \tau < 0$, $a(\tau)$ goes to a zero size at $\tau \to -\infty$ and $\tau \to 0$. For $0 < \tau < \infty$, $a(\tau)$ goes to a zero size at $\tau \to 0$ and goes to an infinite size at $\tau \to \infty$.

(6) In region VI, $H_- > 0$ and $H_+ > 0$ ($\omega > \omega_\kappa$, $\omega > \omega_\eta$ and $\omega > \omega_\nu$). For $-\infty < \tau < 0$, $a(\tau)$ goes to a zero size at $\tau \to -\infty$ and goes to an infinite size at $\tau \to 0$. For $0 < \tau < \infty$, $a(\tau)$ goes to an infinite size at $\tau \to 0$ and at $\tau \to \infty$.

(7) In region VII, $H_- > 0$ and $H_+ < 0$ ($\omega > \omega_\kappa$, $\omega > \omega_\eta$ and $\omega > \omega_\nu$). For $-\infty < \tau < 0$, $a(\tau)$ goes to a zero size at $\tau \to -\infty$ and goes to an infinite size at $\tau \to 0$. For $0 < \tau < \infty$, $a(\tau)$ goes to an infinite size at $\tau \to 0$ and goes to a zero size at $\tau \to \infty$.

VI. THE PHASES OF COSMOLOGY

Using all the results obtained from Secs. IV and V, we now classify the parameter space of γ and ω into several phases and find the behavior of $a(t)$. These phases are characterized according to the behavior of $a(t)$.

Using Eqs. (19) and (50) in the limit $\tau \to \pm\infty$, $a(t)$ is written as

$$a(t) \approx [T_-(t-t_i)]^{H_-/T_-} \quad \text{at} \quad \tau \to -\infty,$$

$$a(t) \approx [T_+(t-t_f)]^{H_+/T_+} \quad \text{at} \quad \tau \to \infty, \tag{67}$$

where t_i and t_f, which were defined in Sec. IV, are real constant. Notice that t_i (t_f) becomes the starting point (the

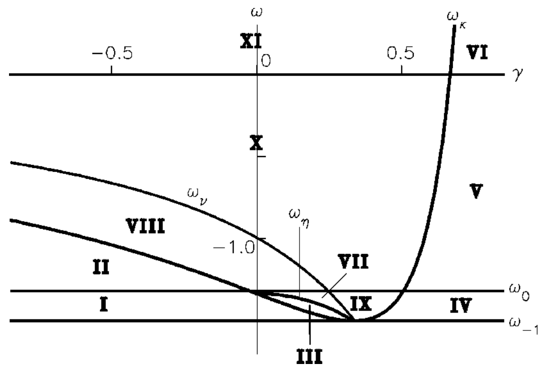

FIG. 5. In four dimensions $(D=4)$, the parameter space is classified by the behavior of $a(t)$.

ending point) in the case $T_{-}>0$ $(T_{+}<0)$ and that t_i (t_f) can be neglected in the case $T_{-}<0$ $(T_{+}>0)$ because $t\to\pm\infty$ as $\tau\to\pm\infty$.

Now, we describe two examples: (i) For $T_{-}<0$ and $H_{-}/T_{-}>0$, $T_{-}(t-t_i)$ is positive and $a(t)$ goes to positive infinite at $t\to-\infty$; (ii) for $T_{-}>0$ and $H_{-}/T_{-}>0$, t can be defined in the region $t>t_i$ only. So $t-t_i$ is positive and $a(t)$ goes to zero at $t\to t_i$. Other cases can be analyzed using the same method.

In the region $\omega>\omega_\kappa$ and $\eta>1$, we must investigate the behavior of $a(t)$ at $\tau\to 0$. From Eqs. (47), (48), and (66), $a(t)$ is obtained as

$$a(t)\approx E\times|t|^{2(1-\gamma)(\omega-\omega_\nu)/(\eta-1)|\kappa|}, \tag{68}$$

where

$$E=[q(\eta-1)]^{2(1-\gamma)(\omega-\omega_\nu)/(\eta-1)|\kappa|} \times e^{B[1-2(D-1)\gamma(1-\gamma)(\omega-\omega_\nu)/(\eta-1)|\kappa|]}$$

is a positive value because q and $(1-\gamma)$ are positive in the previous definition. $a(t)$ goes to zero at $t\to\pm\infty$ $(\tau\to\pm 0)$ in the case $\omega<\omega_\nu$ and $a(t)$ goes to infinite at $t\to\pm\infty$ in the case $\omega>\omega_\nu$.

As shown in Fig. 5, using the sign of $T_\pm$ and $H_\pm$ with the consideration of the behavior of $a(t)$ at $\tau\to 0$, the behavior of $a(t)$ in each region is characterized as the following.

(1) In region I, $T_{-}>0$, $T_{+}<0$, $H_{-}>0$, and $H_{+}<0$. The universe evolves from a zero size at finite initial time t_i to a zero size at finite final time t_f.

(2) In region II, $T_{-}<0$, $T_{+}<0$, $H_{-}>0$, and $H_{+}<0$. The universe evolves from a zero size at negative infinity to a zero size at finite final time t_f.

(3) In region III, $T_{-}>0$, $T_{+}>0$, $H_{-}>0$, and $H_{+}>0$. The universe evolves from a zero size at finite initial time t_i to an infinite size at positive infinity.

(4) In region IV, $T_{-}<0$, $T_{+}>0$, $H_{-}<0$, and $H_{+}>0$. The universe evolves from an infinite size at negative infinity to an infinite size at positive infinity.

(5) In region V, $T_{-}>0$, $T_{+}>0$, $H_{-}<0$, and $H_{+}>0$. The universe evolves from an infinite size at finite initial time t_i to an infinite size at positive infinity.

(6) In region VI, $T_{-}>0$, $T_{+}>0$, $H_{-}>0$, and $H_{+}>0$. The universe evolves from a zero size at finite initial time t_i to an infinite size at positive infinity.

From (7) to (11) below, we consider the case $\eta>1$ and $\omega>\omega_\kappa$ in which t has a singular behavior at $\tau\to 0$. In these cases, we can divide the region of τ into $-\infty<\tau<0$ and $0<\tau<\infty$. Therefore we obtain two branches of $a(t)$ having different behaviors in each region of τ.

(7) In region VII, $T_{-}>0$, $T_{+}>0$, $H_{-}>0$, and $H_{+}>0$. In the region $-\infty<\tau<0$, the universe evolves from a zero size at finite initial time t_i to a zero size at positive infinity. In the region $0<\tau<\infty$, the universe evolves from a zero size at negative infinity to an infinite size at positive infinity.

(8) In region VIII, $T_{-}>0$, $T_{+}<0$, $H_{-}>0$, and $H_{+}>0$. In the region $-\infty<\tau<0$, the universe evolves from a zero size at finite initial time t_i to a zero size at positive infinity. In the region $0<\tau<\infty$, the universe evolves from a zero size at negative infinity to an infinite size at finite final time t_f.

(9) In region IX, $T_{-}>0$, $T_{+}>0$, $H_{-}>0$, and $H_{+}>0$. In the region $-\infty<\tau<0$, the universe evolves from a zero size at finite initial time t_i to an infinite size at positive infinity. In the region $0<\tau<\infty$, the universe evolves from an infinite size at negative infinity to an infinite size at positive infinity.

(10) In region X, $T_{-}>0$, $T_{+}<0$, $H_{-}>0$, and $H_{+}>0$. In the region $-\infty<\tau<0$, the universe evolves from a zero size at finite initial time t_i to an infinite size at positive infinity. In the region $0<\tau<\infty$, the universe evolves from an infinite size at negative infinity to an infinite size at finite final time t_f.

(11) In region XI, $T_{-}>0$, $T_{+}<0$, $H_{-}>0$, and $H_{+}<0$. In the region $-\infty<\tau<0$, the universe evolves from a zero size at finite initial time t_i to an infinite size at positive infinity. In the region $0<\tau<\infty$, the universe evolves from an infinite size at negative infinity to a zero size at finite final time t_f.

VII. DISCUSSION AND CONCLUSION

In this paper we studied the effect of the gas of a solitonic p-brane by treating them as a perfect fluid in the Brans-Dicke theory. We found exact cosmological solutions for any Brans-Dicke parameter ω and for general constant γ and classified the cosmology of the solutions according to the parameters involved. We assumed that the universe is dominated by one kind of p-brane and they are treated as a perfect fluid. We found the analytic solution which is singularity free for some γ and ω. It is very interesting that $a(t)$ has no initial and final singularities at finite initial and final cosmic time in regions IV and VII. $a(t)$ has also an inflation behavior in region VII. So we need to study more intensively the behavior of $a(t)$ in these regions.

Presumably the value of γ as well as ω should be fixed once p is fixed. Without knowing the value of γ for a given p, the classification was the best thing we could do. It would be very interesting to determine the parameter γ for the given p. Also we need more rigorous justification of our basis for the p-brane cosmology. If what we took as basis goes wrong, then what we have done is just Brans-Dicke cosmology in the presence of some perfect fluid type matter. We wish that more study of the effect of the solitons in the string cosmology be done in the future.

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