

## Chiral anomaly and $\eta$ - $\eta'$ mixing

E. P. Venugopal and Barry R. Holstein

*Department of Physics and Astronomy, University of Massachusetts, Amherst, Massachusetts 01003*

(Received 29 October 1997; published 19 February 1998)

We determine the  $\eta$ - $\eta'$  mixing angle via a procedure relatively independent of theoretical assumptions by simultaneously fitting  $\eta, \eta'$  reactions involving the anomaly  $\eta, \eta' \rightarrow \gamma\gamma, \pi^+\pi^-\gamma$ . We extract reasonably precise renormalized values of the octet and singlet pseudoscalar decay constants  $F_8, F_0$ , as well as the mixing angle  $\theta$ . [S0556-2821(98)02707-6]

PACS number(s): 14.40.Aq, 11.30.Rd, 13.40.Hq

### I. INTRODUCTION

From a strictly theoretical perspective, there exists a significant difference between the octet pseudoscalar mesons  $\pi, K, \eta_8$  and their singlet counterpart  $\eta_0$  [1]. The former are legitimate pseudo-Goldstone bosons whose masses vanish in the chiral limit while the latter is not due to the anomalous breaking of axial U(3) symmetry down to SU(3). However, in the real world, this does not seem to make much difference. Indeed, the physical eigenstates  $\eta, \eta'$  are mixtures of octet and singlet components,

$$\begin{aligned} \eta &= \eta_8 \cos \theta - \eta_0 \sin \theta \\ \eta' &= \eta_8 \sin \theta + \eta_0 \cos \theta \end{aligned} \quad (1)$$

and the mixing angle  $\theta$  is an important quantity in confronting theoretical calculations with experimental results in these systems [2].

The mixing angle can be evaluated in various ways but a standard procedure involves the diagonalization of the  $\eta, \eta'$  mass matrix, which at lowest order yields a mixing angle  $\theta \approx -10^\circ$ . This can easily be seen by writing the mass matrix as

$$m^2 = \begin{pmatrix} m_8^2 & m_{08}^2 \\ m_{08}^2 & m_0^2 \end{pmatrix}. \quad (2)$$

Employing the Gell-Mann–Okubo (GMO) relation to fix  $m_8^2$  as [3]

$$m_8^2 = \frac{1}{3}(4m_K^2 - m_\pi^2) \quad (3)$$

and diagonalizing, one can determine  $m_{08}^2, m_0^2$  and  $\theta$ , in terms of  $m_\eta, m_{\eta'}$ , yielding

$$\theta = -9.4^\circ, \quad m_{08}^2 = -0.44m_K^2, \quad m_{\eta_0} = 0.95 \text{ GeV}. \quad (4)$$

However, the GMO relation, Eq. (3), is valid only at lowest order,  $\mathcal{O}(p^2)$ , in chiral perturbation theory. The inclusion of  $\mathcal{O}(p^4)$  corrections to the relation results in significant changes. Characterizing these via

$$m_8^2 = \frac{1}{3}(4m_K^2 - m_\pi^2)(1 + \delta) \quad (5)$$

a leading logarithm estimate [4]

$$\delta \approx \frac{-2 \frac{m_K^4}{(4\pi F_\pi)^2} \ln \frac{m_K^2}{(4\pi F_\pi)^2}}{4m_K^2 - m_\pi^2} \quad (6)$$

yields  $\delta \approx 0.16$ , which yields

$$\theta \approx -20^\circ, \quad m_{08}^2 = -0.81m_K^2, \quad m_0 = 0.90 \text{ GeV} \quad (7)$$

suggesting a doubling of the mixing angle. A full one-loop chiral perturbation theory calculation confirms this finding [5]. It is also interesting that this solution is consistent with the assumptions of simple U(3) symmetry wherein  $\eta_8, \eta_0$  have the same wave function, leading to

$$m_{08}^2 \approx \frac{2\sqrt{2}}{3} \left( \frac{\hat{m} - m_s}{\hat{m} + m_s} \right) \approx -0.9m_K^2. \quad (8)$$

Note that this result is strongly dependent upon chiral symmetry breaking effects on the GMO prediction.

An alternative and independent approach to the problem involves the use of the chiral anomaly, which is responsible for the well-known  $\pi, \eta, \eta' \rightarrow \gamma\gamma$  decays [6]. In the case of  $\pi^0 \rightarrow \gamma\gamma$ , at lowest order,  $\mathcal{O}(p^4)$ , the anomalous (Wess-Zumino-Witten) chiral Lagrangian predicts [7]

$$A_{\pi^0 \rightarrow \gamma\gamma} = \frac{\alpha N_c}{3\pi\bar{F}} \epsilon^{\mu\nu\alpha\beta} \epsilon_{1\mu} \epsilon_{2\nu} k_{1\alpha} k_{2\beta} \quad (9)$$

where  $\epsilon_i, k_i$  are the polarization, momenta of the outgoing photons, and  $\bar{F}$  is the pion decay constant in the chiral limit. A leading log calculation of the chiral corrections reveals that the dominant effect is simply to replace  $\bar{F}$  by its physical value  $F_\pi = 92.4 \text{ MeV}$  [8]. The resulting amplitude is guaranteed by general theorems to remain unchanged in higher chiral orders [9]. One then finds that the predicted amplitude

$$F_{\pi\gamma\gamma}(0) = \frac{\alpha N_c}{3\pi F_\pi} = 0.025 \text{ GeV}^{-1} \quad (10)$$

is in excellent agreement with the corresponding experimental value [10]

$$F_{\pi\gamma\gamma}(0) = (0.024 \pm 0.001) \text{ GeV}^{-1} \quad (11)$$

thus providing the confidence that one may be able to analyze the  $\eta, \eta'$  decays with a similar precision.

In case of the  $\eta, \eta' \rightarrow \gamma\gamma$  decays, one must include both mixing and renormalization of the octet, singlet couplings, yielding the predicted amplitudes<sup>1</sup>

$$F_{\eta\gamma\gamma}(0) = \frac{\alpha N_c}{3\sqrt{3}\pi F_\pi} \left( \frac{F_\pi}{F_8} \cos \theta - 2\sqrt{2} \frac{F_\pi}{F_0} \sin \theta \right)$$

$$F_{\eta'\gamma\gamma}(0) = \frac{\alpha N_c}{3\sqrt{3}\pi F_\pi} \left( \frac{F_\pi}{F_8} \sin \theta + 2\sqrt{2} \frac{F_\pi}{F_0} \cos \theta \right). \quad (12)$$

From the Particle Data Group (PDG) one extracts experimental values [10]

$$F_{\eta\gamma\gamma}(0) = 0.0249 \pm 0.0010 \text{ GeV}^{-1}$$

$$F_{\eta'\gamma\gamma}(0) = 0.0328 \pm 0.0024 \text{ GeV}^{-1}. \quad (13)$$

In order to solve the system, however, we require an additional assumption since there are three unknowns  $F_8, F_0, \theta$  and only two pieces of data. The usual approach in this case is to use the leading log prediction of one-loop chiral perturbation theory to predict [5]

$$\frac{F_8}{F_\pi} = \left[ 1 - \frac{m_K^2}{(4\pi F_\pi)^2} \ln \frac{m_K^2}{(4\pi F_\pi)^2} + \frac{m_\pi^2}{(4\pi F_\pi)^2} \right] \approx 1.30 \quad (14)$$

and then solve for  $F_0, \theta$ , yielding

$$\frac{F_0}{F_\pi} = 1.04, \quad \theta = -20^\circ. \quad (15)$$

It is interesting to note that these results are quite consistent with those obtained from the one-loop analysis of the mass matrix—i.e.  $\theta \approx -20^\circ$  and  $F_0/F_\pi$  consistent with the value of unity which one would have if the singlet state and the pion were to have the same wave function.

While this agreement is satisfying, the extraction of these mixing parameters requires certain theoretical inputs, either Eq. (14) or Eq. (6), and it is interesting to inquire whether one can predict the mixing angle purely phenomenologically. As we shall show, the answer is yes, provided one utilizes the additional information available from the anomalous decays  $\eta, \eta' \rightarrow \pi^+ \pi^- \gamma$  [11].

In the next section then we show how the decays  $\eta, \eta' \rightarrow \pi^+ \pi^- \gamma$  can be analyzed in order to isolate the chiral anomaly. This involves a careful study of final state interactions and unitarity constraints in order to realistically extrapolate to zero four-momenta, as required by the anomaly.

<sup>1</sup>Note that we implicitly assume here, as do other workers, that all  $\mathcal{O}(m_\eta^2, m_{\eta'}^2, \Lambda_\chi^2)$  effects, where  $\Lambda_\chi \sim 4\pi F_\pi$  is the chiral scale, are included in the renormalization of the pseudoscalar couplings  $F_8, F_0$  and in the mixing angle  $\theta$ . This does not have to be the case, but appears to be borne out by the consistency of the results obtained from treatments of differing manifestations of the anomaly, as we show below and as others have found.

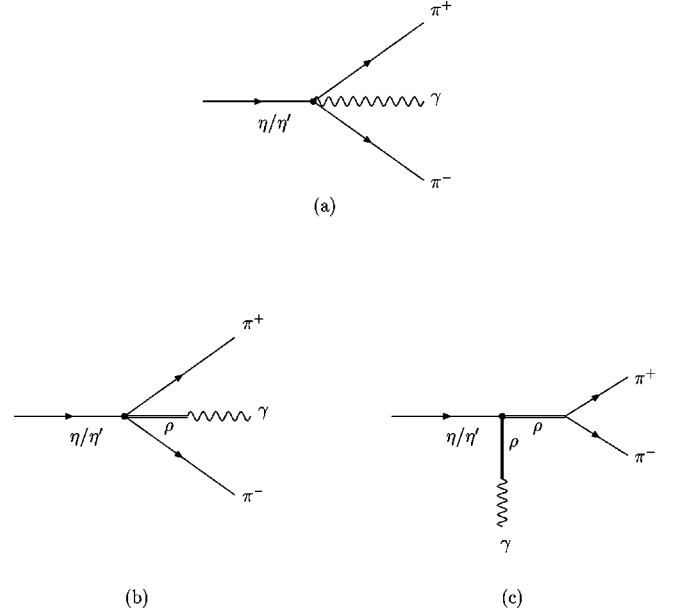


FIG. 1. Shown are contact (a) and vector meson dominance (VMD) (b), (c) contributions to  $\eta, \eta' \rightarrow \pi^+ \pi^- \gamma$  decay.

In the concluding section, we apply these results to evaluate  $\theta, F_8, F_0$  in an essentially model independent fashion.

## II. ANALYSIS OF $\eta, \eta' \rightarrow \pi^+ \pi^- \gamma$ DECAYS

Our goal is to use the experimental data on  $\eta, \eta' \rightarrow \pi^+ \pi^- \gamma$  in order to isolate the value of the anomaly in these decays. The resulting numbers can then be fit to the appropriate theoretical expressions, thus allowing extraction of the renormalized mixing angle and coupling constants. To this end, we define

$$A_{\eta, \eta' \rightarrow \pi\pi\gamma} = B_{\eta, \eta'}(s, s_{\pi\pi}) \epsilon^{\mu\nu\alpha\beta} \epsilon_\mu k_\nu p_{+\alpha} p_{-\beta} \quad (16)$$

where  $p_\pm, k$  are the outgoing 4-momenta,  $\epsilon_\mu$  is the photon polarization vector,  $s = m_{\eta, \eta'}^2$  and  $s_{\pi\pi} = (p_+ + p_-)^2$ . The chiral anomaly [cf. Fig. 1(a)] requires

$$B_\eta(0,0) = \frac{eN_c}{12\sqrt{3}\pi^2 F_\pi^3} \left( \frac{F_\pi}{F_8} \cos \theta - \sqrt{2} \frac{F_\pi}{F_0} \sin \theta \right)$$

$$B_{\eta'}(0,0) = \frac{eN_c}{12\sqrt{3}\pi^2 F_\pi^3} \left( \frac{F_\pi}{F_8} \sin \theta + \sqrt{2} \frac{F_\pi}{F_0} \cos \theta \right). \quad (17)$$

Note, however, that the chiral anomaly, strictly speaking, only constrains the form factors  $B_{\eta, \eta'}(s, s_{\pi\pi})$  at zero four-momentum,  $B_{\eta, \eta'}(0,0)$ , while the experimental input occurs at  $s = m_{\eta, \eta'}^2, s_{\pi\pi} \geq 4m_\pi^2$ . One indication of this fact is that, using the simple energy-independent form given in Eq. (17) to calculate the decay rate, one obtains for the  $\eta$ -channel the value  $\Gamma_{\eta \rightarrow \pi\pi\gamma} = 35.7 \text{ eV}$  compared to the experimental rate of  $\Gamma_{\eta \rightarrow \pi\pi\gamma}^{\text{expt}} = 64 \pm 6 \text{ eV}$ . For the  $\eta'$  channel things are even worse,  $\Gamma_{\eta' \rightarrow \pi\pi\gamma} = 61 \pm 5 \text{ keV}$ , while the theoretical value arising from use of the simple anomaly amplitude is, amazingly, a factor of 20 less. We conclude that in order to extract values for the anomaly in these transitions, it is absolutely

essential to correctly model the energy dependence of the amplitude in the physical region.

This problem is not unique, of course, to the  $\eta, \eta' \rightarrow \pi\pi\gamma$  system and has been addressed previously in the extraction of the anomaly in  $\gamma \rightarrow 3\pi$  from the Primakoff-effect data of Antipov *et al.* [12]. In this case, one also has a clear prediction of the chiral anomaly [13]. Writing

$$A_{\gamma \rightarrow 3\pi} = F_{3\pi}(s, t, u) \epsilon^{\mu\nu\alpha\beta} \epsilon_{\mu p_+ \nu p_- \alpha p_0 \beta} \quad (18)$$

where  $s = (p_+ + p_-)^2$ ,  $t = (p_+ + p_0)^2$ ,  $u = (p_- + p_0)^2$ , the chiral stricture demands

$$F_{3\pi}(0,0,0) = \frac{eN_c}{12\pi F_\pi^3} = 9.7 \text{ GeV}^{-3}. \quad (19)$$

The effects of p-wave pi-pi interactions at low and moderate energies are known to be reasonably described by vector

dominance [14]. In particular, the models described in Ref. [15], which incorporate vector dominance and chiral symmetry, when applied to the  $\gamma \rightarrow 3\pi$  reaction, provide a form

$$F_{3\pi}(s, t, u) = -\frac{1}{2} F_{3\pi}(0,0,0) \left[ 1 - \frac{m_\rho^2}{m_\rho^2 - s} - \frac{m_\rho^2}{m_\rho^2 - t} - \frac{m_\rho^2}{m_\rho^2 - u} \right] \quad (20)$$

which matches the anomaly at zero four-momentum but also offers a plausible extension into the physical region. This form must be modified, of course, in order to confront real data since unitarity demands the presence of branch cuts and consequent imaginary components in the form factor. This is clear from a one-loop chiral perturbation theory calculation, which yields [16]

$$F_{3\pi}(s, t, u) = F_{3\pi}(0,0,0) \left[ 1 + \frac{3m_\pi^2}{2m_\rho^2} + \frac{m_\pi^2}{24\pi^2 F_\pi^2} \left( \frac{3}{4} \ln \frac{m_\rho^2}{m_\pi^2} + F(s) + F(t) + F(u) \right) \right] \quad (21)$$

where

$$F(s) = \begin{cases} \left( 1 - \frac{s}{4m_\pi^2} \right) \sqrt{\frac{s-4m_\pi^2}{s}} \ln \frac{1 + \sqrt{\frac{s-4m_\pi^2}{s}}}{-1 + \sqrt{\frac{s-4m_\pi^2}{s}}} - 2, & s > 4m_\pi^2, \\ 2 \left( 1 - \frac{s}{4m_\pi^2} \right) \sqrt{\frac{4m_\pi^2-s}{s}} \tan^{-1} \sqrt{\frac{s}{4m_\pi^2-s}} - 2, & s \leq 4m_\pi^2 \end{cases} \quad (22)$$

Here we note that the imaginary component of the function  $F(s)$  is given in terms of the energy-dependent width of the rho meson via

$$\frac{m_\pi^2}{24\pi^2 F_\pi^2} \text{Im} F(s) = \frac{1}{m_\rho} \Gamma_\rho(s) \quad (23)$$

where

$$\Gamma_\rho(s) = \frac{g_{\rho\pi\pi}^2 s}{48\pi m_\rho} \left( 1 - \frac{4m_\pi^2}{s} \right)^{3/2}. \quad (24)$$

This one-loop form is no doubt appropriate in the near threshold region. However, once  $s_{\pi\pi} \geq 10m_\pi^2$  or so, one does not anticipate that a simple one-loop description will be adequate. In order to address these difficulties, the use of an N/D form has been suggested [17]. In this approach, one utilizes the Omnes function, which encodes information concerning the pi-pi interaction [18]

$$D_1(s) = \exp \left( -\frac{s}{\pi} \int_{4m_\pi^2}^{\infty} \frac{ds' \delta_1(s')}{s'(s'-s-i\epsilon)} \right) \quad (25)$$

where  $\delta_1(s)$  is the ( $l=1$ ) p-wave pi-pi phase shift at center of mass energy  $\sqrt{s}$ . There are two ways to proceed at this point.

(i) One can use the experimental phase shifts, with some assumptions made about their asymptotic form. In our case we took the values quoted by Froggatt and Peterson [19], which are given up to  $\sqrt{s} = 1 \text{ GeV}$ , and assumed a constant value after that. We label the Omnes function obtained in this way as  $D_1^{\text{expt}}(s)$ .

(ii) One can employ a simple analytic form [20]

$$D_1(s) = 1 - \frac{s}{m_\rho^2} - \frac{s}{96\pi^2 F_\pi^2} \ln \frac{m_\rho^2}{m_\pi^2} - \frac{m_\pi^2}{24\pi^2 F_\pi^2} F(s) \quad (26)$$

which has been shown to provide an approximate description of the empirical  $\pi\pi$  p-wave phase shifts in the low energy region [21]. We label the Omnes function obtained via this procedure as  $D_1^{\text{anal}}(s)$ .

Using either of these forms, and postulating an N/D form of the  $\gamma \rightarrow 3\pi$  amplitude as

$$F_{3\pi}(s,t,u) = -\frac{1}{2}A_{3\pi}(0) \left[ 1 - \left( \frac{m_\rho^2}{m_\rho^2 - s} + \frac{m_\rho^2}{m_\rho^2 - t} + \frac{m_\rho^2}{m_\rho^2 - u} \right) \right] \\ \times \left( \frac{1 - \frac{s}{m_\rho^2}}{D_1(s)} \right) \left( \frac{1 - \frac{t}{m_\rho^2}}{D_1(t)} \right) \left( \frac{1 - \frac{u}{m_\rho^2}}{D_1(u)} \right) \quad (27)$$

one can see that Eq. (27) matches the one-loop chiral result in the limit of low energies *and* the vector dominance form when unitarity inspired logarithms are dropped.

Whether such an N/D form accurately describes the data for  $\gamma \rightarrow 3\pi$  awaits the arrival of sufficiently precise information from CEBAF and CERN. However, it certainly appears to satisfy the various criteria which nature demands, and suggests the treatment of the related  $\eta, \eta' \rightarrow \pi^+ \pi^- \gamma$  decay amplitudes in a parallel fashion.

In the case of the  $\eta, \eta' \rightarrow \pi^+ \pi^- \gamma$  decays we can proceed similarly. In this case, the one-loop chiral perturbation theory calculation gives

$$B_\eta^{1\text{-loop}}(s, s_{\pi\pi}) = B_\eta(0,0) \left[ 1 + \frac{1}{32\pi^2 F_\pi^2} \left( \left( -4m_\pi^2 + \frac{1}{3}s_{\pi\pi} \right) \ln \frac{m_\pi^2}{m_\rho^2} + \frac{4}{3}F(s_{\pi\pi}) - \frac{20}{3}m_\pi^2 + \frac{3}{2m_\rho^2}s_{\pi\pi} \right) \right], \quad (28)$$

while vector dominance [cf. Figs. 1(b,c)] yields

$$B_{\eta, \eta'}(s, s_{\pi\pi}) = B_{\eta, \eta'}(0,0) \left[ 1 + \frac{3}{2} \frac{s_{\pi\pi}}{m_\rho^2 - s_{\pi\pi}} \right]. \quad (29)$$

Certainly, in order to treat the decay of the  $\eta'$ , one must go further and include unitarity effects via final state interactions. One very simple approach is to include the (energy-dependent) width of the rho meson in the propagator via

$$\frac{s_{\pi\pi}}{m_\rho^2 - s_{\pi\pi}} \rightarrow \frac{s_{\pi\pi}}{m_\rho^2 - s_{\pi\pi} - im_\rho \Gamma_\rho(s_{\pi\pi})}. \quad (30)$$

This use of vector width-modified vector dominance already makes an important difference from the simple anomaly—tree level—results (especially in the case of the  $\eta'$ ), changing the predicted decay widths from the values 35 eV and 3 keV quoted above to the much more realistic numbers

$$\Gamma_{\eta \rightarrow \pi\pi\gamma} = 62.3 \text{ eV}, \quad \Gamma_{\eta' \rightarrow \pi\pi\gamma} = 67.5 \text{ keV}, \quad (31)$$

if the parameters

$$F_8/F_\pi = 1.3, \quad F_0/F_\pi = 1.04, \quad \theta = -20^\circ \quad (32)$$

are employed. However, this approach does not reproduce the one-loop chiral form in the low energy limit.

In order to determine a form for the final state interactions which matches both the one-loop chiral correction *and* the vector dominance result in the appropriate limits, we postulate an N/D structure, as in the related  $\gamma \rightarrow 3\pi$  case:

$$B_{\eta \rightarrow \pi\pi\gamma}(s, s_{\pi\pi}) = B_{\eta \rightarrow \pi\pi\gamma}(0,0) \left[ 1 - c + c \frac{1 + as_{\pi\pi}}{D_1(s_{\pi\pi})} \right] \quad (33)$$

where for the Omnes function we use one of the two forms itemized above and  $a, c$  are free parameters to be determined. In order to reproduce the coefficient of the  $F(s_{\pi\pi})$  function, which contains the rho width, we require  $c=1$ . On the other hand, matching the VMD result at  $\mathcal{O}(p^6)$  can be achieved by the choice  $a=1/2m_\rho^2$ . Thus in the case of the  $\eta$  the form is completely determined. Since the  $\eta'$  spectrum is closely related and is dominated by the presence of the rho we shall assume an identical form for the  $\eta'$  case. Using these forms we can then calculate the decay widths assuming the theoretical values for the anomaly. Using the parameters given in Eq. (32) one finds, for example,

$$(i) D_1^{\text{expt}}(s) \quad \Gamma_{\eta \rightarrow \pi\pi\gamma} = 65.7 \text{ eV}, \quad \Gamma_{\eta' \rightarrow \pi\pi\gamma} = 66.2 \text{ keV}$$

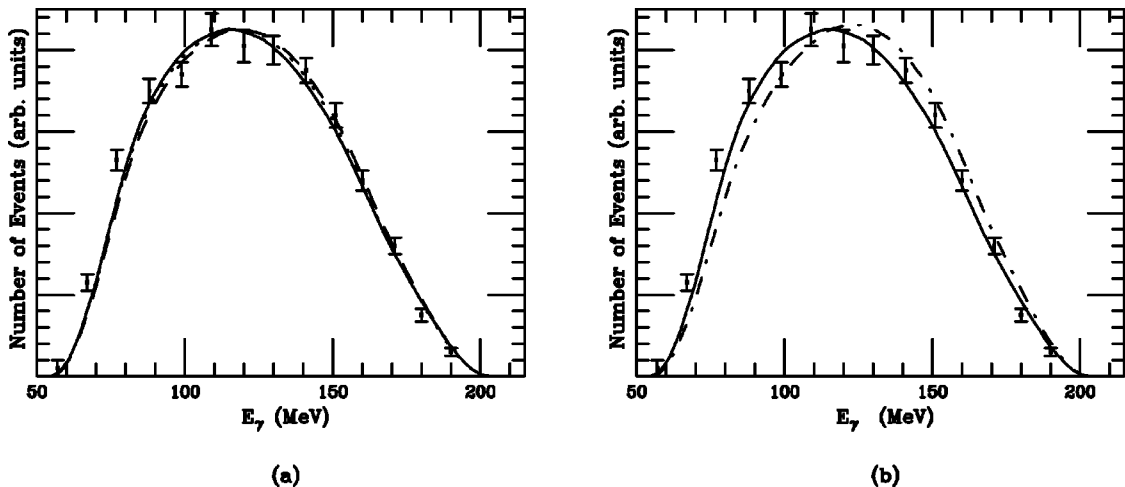


FIG. 2. Shown is the photon spectrum in  $\eta \rightarrow \pi^+ \pi^- \gamma$  from Gormley *et al.* [26] as well as various theoretical fits. In the first figure, the dashed line represents the (width-modified) VMD model. The (hardly visible) dotted line and the solid line represent the final state interaction ansatz, Eq. (33), with use of the analytic and experimental versions of the Omnes function, respectively. The second figure shows the experimental Omnes function result (solid line) compared with the one-loop result (dot-dashed line).

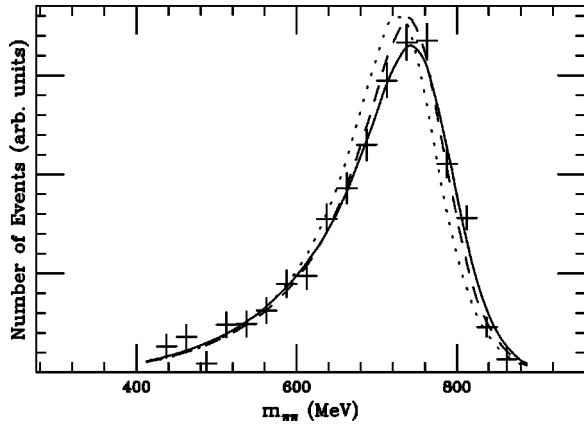


FIG. 3. Shown is the photon spectrum in  $\eta' \rightarrow \pi^+ \pi^- \gamma$  from Abele *et al.* [24] as well as various theoretical fits. As in Fig. 2, the dashed line represents the (width-modified) VMD model. The dotted and solid lines represent the final state interaction ansatz, Eq. (33), with use of the analytic and experimental versions of the Omnes function, respectively. Here the curves have been normalized to the same number of events.

$$(ii) D_1^{\text{anal}}(s) \quad \Gamma_{\eta \rightarrow \pi\pi\gamma} = 69.7 \text{ eV}, \quad \Gamma_{\eta' \rightarrow \pi\pi\gamma} = 77.8 \text{ keV}. \quad (34)$$

There is a tendency then for the numbers obtained via the analytic form of the Omnes function to be somewhat too high.

We can also compare the predicted spectra with the corresponding experimentally determined values. As shown in Fig. 2, we observe that the experimental spectra are well fit in the  $\eta$  case in terms of both the N/D or the VMD forms, but that the one-loop chiral expression does not provide an adequate representation of the data [22]. In the case of the corresponding  $\eta'$  decay the results are shown in Fig. 3, wherein we observe that either the unitarized VMD or the use of  $N/D_1^{\text{expt}}$  provides a reasonable fit to the data (we get  $\chi^2/N_{\text{DF}} = 32/17$  and  $20/17$ , respectively), while the use of the analytic form for the Omnes function yields a predicted spectrum ( $\chi^2/N_{\text{DF}} = 104/17$ ) which is slightly too low on the high energy end. However, for both  $\eta$  and  $\eta'$  we see that our simple ansatz, Eq. (33), provides a very satisfactory representation of the decay spectrum.

### III. EVALUATION OF $\eta$ - $\eta'$ MIXING PARAMETERS

Our conclusion in the last section was that if the mixing angle and pseudoscalar coupling constants were given values consistent with present theoretical and experimental leanings, then the predicted widths and spectra of both  $\eta, \eta' \rightarrow \pi^+ \pi^- \gamma$  are basically consistent with experimental values. Our goal in this section is to go the other way, however. That is, using the assumed N/D forms for the decay amplitude, and treating the pseudoscalar decay constants  $F_8, F_0$  as well as the  $\eta$ - $\eta'$  mixing angle  $\theta$  as free parameters, we wish to inquire as to how well they can be constrained purely from the experimental data on  $\eta, \eta' \rightarrow \gamma\gamma$  and  $\eta, \eta' \rightarrow \pi^+ \pi^- \gamma$  decays, with reasonable assumptions made about the final state interaction effects in these two channels.

On theoretical grounds, one is somewhat more confident about the extraction of the threshold amplitude in the case of

TABLE I. Values of the renormalized pseudoscalar coupling constants and the  $\eta$ - $\eta'$  mixing angle using the  $\eta, \eta' \rightarrow \gamma\gamma$  and  $\eta, \eta' \rightarrow \pi\pi\gamma$  amplitudes in a three parameter fit.

	$F_8/F_\pi$	$F_0/F_\pi$	$\theta$
VMD	$1.28 \pm 0.24$	$1.07 \pm 0.48$	$-20.3^\circ \pm 9.0^\circ$
$N/D_1^{\text{anal}}$	$1.49 \pm 0.29$	$1.02 \pm 0.42$	$-22.6^\circ \pm 9.6^\circ$
$N/D_1^{\text{expt}}$	$1.37 \pm 0.26$	$1.02 \pm 0.45$	$-21.2^\circ \pm 9.3^\circ$

the lower energy  $\eta \rightarrow \pi^+ \pi^- \gamma$  system. Indeed, in this case the physical region extends only slightly into the tail of the rho unlike the related  $\eta'$  decay wherein the spectrum extends completely over the resonance so that there exists considerable sensitivity to details of the shape. Thus a first approach might be to utilize only the two-photon decays together with the  $\eta \rightarrow \pi^+ \pi^- \gamma$  width in order to determine the three desired parameters. In this fashion one finds the results shown in Table I. We observe that the results are in agreement, both with each other and with the chiral symmetry expectations,  $F_8/F_\pi \sim 1.3$ ,  $F_0/F_\pi \sim 1$ , and  $\theta \sim -20^\circ$ . However, the uncertainties obtained in this way are uncomfortably high.

In order to ameliorate this problem, we have also done a maximum likelihood fit including the  $\eta' \rightarrow \pi\pi\gamma$  decay rate, yielding the results shown in Table II. We observe that the central values stay fixed but that the error bars are somewhat reduced. The conclusions are the same, however—substantial renormalization for  $F_8 \sim 1.3F_\pi$ , almost none for  $F_0 \sim F_\pi$ , and a mixing angle  $\theta \sim -20^\circ$ . These numbers appear nearly invariant, regardless of the approach.

### IV. CONCLUSION

Before summarizing the results of our above analysis, it should certainly be emphasized that we are not the first to undertake the program of isolating the anomaly from the  $\eta, \eta' \rightarrow \pi\pi\gamma$  data. Indeed, there has been considerable work in this regard, both on the theoretical side [11,23] as well as experimentally, including the most recently published  $\eta' \rightarrow \pi^+ \pi^- \gamma$  data [24]. The recent analysis of Ref. [23] leads to results quite different from ours in both the mixing angle as well as the renormalization of the pseudoscalar couplings. On the other hand, Ref. [24] (at least for the model labeled  $M_1$ ) finds a somewhat smaller mixing angle ( $\theta \sim -16^\circ$ ) and pseudoscalar renormalization ( $F_8/F_0 \sim 1.1$ ).

However, there is an important difference between these analyses and our own. In Refs. [23] and [24], the decay amplitude is written in terms of a piece due to the anomaly (parametrized by  $E_X$ ,  $X = \eta, \eta'$ ) and a component due to the

TABLE II. Values of the renormalized pseudoscalar coupling constants and of the  $\eta$ - $\eta'$  mixing angle obtained from a maximum likelihood analysis using the  $\eta, \eta' \rightarrow \gamma\gamma$  and  $\eta, \eta' \rightarrow \pi\pi\gamma$  amplitudes.

	$F_8/F_\pi$	$F_0/F_\pi$	$\theta$
VMD	$1.28 \pm 0.20$	$1.07 \pm 0.04$	$-20.8^\circ \pm 3.2^\circ$
$N/D_1^{\text{anal}}$	$1.48 \pm 0.24$	$1.09 \pm 0.03$	$-24.0^\circ \pm 3.0^\circ$
$N/D_1^{\text{expt}}$	$1.38 \pm 0.22$	$1.06 \pm 0.03$	$-22.0^\circ \pm 3.3^\circ$

rho pole (parametrized by  $F_X$ ,  $X = \eta, \eta'$ ). The ratio  $E_X/F_X$  is then fitted, through a minimization procedure, to produce the experimental spectrum and partial widths. In our analysis, the parameters of the two pieces are fixed *a priori* to reproduce the results of one-loop chiral perturbation theory [16] (by fixing  $c=1$ ) and VMD [15] (by fixing  $a=1/2m_\rho^2$ ). (Indeed a recent analysis of other  $I=1$   $\pi^+\pi^-$  processes found that only within a model such as Ref. [15], which links chiral symmetry and VMD, could the data be fit consistently [25].) However, following the picture of Ref. [15], we do not include a non-resonant coupling in  $\gamma\pi^+\pi^-$  (as considered in models  $M_1$  and  $M_3$  of [23]). Our successful fits of the experimental data speak for themselves.

In previous treatments of the  $\eta, \eta'$  system via the anomaly, which have omitted  $\eta, \eta' \rightarrow \pi\pi\gamma$  constraints, the mixing angle  $\theta$  has generally been determined only at the cost of theoretical assumptions about the renormalization of the octet pseudoscalar coupling constant  $F_8$  with respect to  $F_\pi$ . We have in this paper asked whether it is possible to obtain the mixing angle in a fashion relatively independent of such theoretical assumptions by simultaneously fitting  $\eta, \eta' \rightarrow \gamma\gamma$  as well as  $\eta, \eta' \rightarrow \pi^+\pi^-\gamma$  decays. As shown above, the answer is affirmative. However, one must incorporate *some* sort of model for the final state interactions of the pi-pi system in order to extrapolate down to zero four-momentum where the anomaly obtains. We have argued that the  $N/D_1^{\text{expt}}$  form given in Eq. (33) is reasonable both on theoretical grounds—matching both the requirements of VMD and of low energy chiral symmetry—and via success-

ful fitting of the experimental spectra. Using this form and the PDG values for  $\eta, \eta' \rightarrow \gamma\gamma$  and  $\eta, \eta' \rightarrow \pi\pi\gamma$  amplitudes we have obtained values

$$F_8/F_\pi = 1.38 \pm 0.22, \quad F_0/F_\pi = 1.06 \pm 0.03, \\ \theta = -22.0^\circ \pm 3.3^\circ \quad (35)$$

which are quite consistent with those obtained in previous analyses which required assumptions about chiral symmetry breaking. One can then assess these results in two different ways. Although it is our contention that the assumptions made above concerning pion-pion interactions are relatively model-independent and that the numbers given thereby in Eq. (35) are quite solid, one could also take a contrary view that the forms utilized for final state interactions *do* require critical dynamical assumptions. In this case, however, we would argue that via three quite different routes, (i) mass matrix analysis including GMO breaking, (ii)  $\eta, \eta' \rightarrow \gamma\gamma$  analysis with assumptions made about  $F_8/F_\pi$ , and (iii) simultaneous analysis of  $\eta, \eta' \rightarrow \gamma\gamma$  and  $\eta, \eta' \rightarrow \pi^+\pi^-\gamma$  with (minimal) assumptions concerning final state pi-pi interactions, one finds virtually the same value of the mixing angle,  $\theta \approx -20^\circ$ , and for pseudoscalar couplings,  $F_8/F_\pi \sim 1.3$ ,  $F_0/F_\pi \sim 1.0$ . In any case, we would assert that these values are now strongly (and independently) confirmed from within the chiral anomaly sector.

This research was supported in part by the National Science Foundation.

- 
- [1] See, e.g., J. F. Donoghue, B. R. Holstein and E. Golowich, *Dynamics of the Standard Model* (Cambridge University Press, New York, 1992).
- [2] See, for example, F. J. Gilman and R. Kauffman, Phys. Rev. D **36**, 2761 (1987); P. Ball *et al.*, Phys. Lett. B **365**, 367 (1996); A. Bramon *et al.*, *ibid.* **403**, 339 (1997).
- [3] S. Okubo, Prog. Theor. Phys. **27**, 949 (1962); M. Gell-Mann, R. J. Oakes and B. Renner, Phys. Rev. **175**, 2195 (1968).
- [4] J. F. Donoghue, B. R. Holstein and Y.-C. R. Lin, Phys. Rev. Lett. **55**, 2766 (1985).
- [5] J. Gasser and H. Leutwyler, Ann. Phys. (N.Y.) **158**, 142 (1984); Nucl. Phys. **B250**, 465 (1985).
- [6] S. L. Adler, Phys. Rev. **177**, 2426 (1969); S. L. Glashow, R. Jackiw and S. S. Shei, *ibid.* **187**, 1916 (1969); J. S. Bell and R. Jackiw, Nuovo Cimento **60**, 147 (1969).
- [7] J. Wess and B. Zumino, Phys. Lett. **37B**, 95 (1971); E. Witten, Nucl. Phys. **B223**, 422 (1983).
- [8] B. R. Holstein, Phys. Lett. B **244**, 83 (1990).
- [9] S. Adler and W. A. Bardeen, Phys. Rev. **182**, 1517 (1969).
- [10] Particle Data Group, R. M. Barnett *et al.*, Phys. Rev. D **54**, 1 (1996).
- [11] M. Chanowitz, Phys. Rev. Lett. **35**, 977 (1975); **44**, 59 (1980).
- [12] Yu. M. Antipov *et al.*, Z. Phys. C **27**, 21 (1985); Phys. Rev. D **36**, 21 (1987).
- [13] S. L. Adler *et al.*, Phys. Rev. D **4**, 3497 (1971); R. Aviv and A. Zee, *ibid.* **5**, 2372 (1972); M. V. Terent'ev, Phys. Lett. **38B**, 419 (1972).
- [14] S. Rudaz, Phys. Lett. **145B**, 281 (1984); T. D. Cohen, Phys. Lett. B **233**, 467 (1989).
- [15] M. Bando *et al.*, Phys. Rev. Lett. **54**, 1215 (1985); T. Fujiwara *et al.*, Prog. Theor. Phys. **73**, 926 (1985).
- [16] J. Bijnens *et al.*, Phys. Lett. B **237**, 488 (1990).
- [17] B. R. Holstein, Phys. Rev. D **53**, 4099 (1996).
- [18] R. Omnes, Nuovo Cimento **8**, 316 (1958).
- [19] C. D. Froggatt and J. L. Peterson, Nucl. Phys. **B129**, 89 (1977).
- [20] L. Brown and R. Goble, Phys. Rev. Lett. **20**, 346 (1968); M. A. B. Beg and A. Zepeda, Phys. Rev. D **6**, 2912 (1972).
- [21] S. D. Protopescu *et al.*, Phys. Rev. D **7**, 1279 (1973); P. Estabrooks and A. D. Martin, Nucl. Phys. **B79**, 301 (1974).
- [22] The feature that a vector dominance form provides a good fit to the  $\eta \rightarrow \pi\pi\gamma$  data has already been noted by C. Picciotto, Phys. Rev. D **45**, 1569 (1992).
- [23] M. Benayoun *et al.*, Z. Phys. C **58**, 31 (1993); **65**, 399 (1995).
- [24] A. Abele *et al.*, Phys. Lett. B **402**, 195 (1997); S. I. Bityukov *et al.*, Z. Phys. C **50**, 451 (1991).
- [25] M. Benayoun *et al.*, Adelaide Report No. ADP-97-14/T251, hep-ph/9707509.
- [26] M. Gormley *et al.*, Phys. Rev. D **2**, 501 (1970); J. G. Layter *et al.*, *ibid.* **7**, 2565 (1973).