Power counting in dimensionally regularized nonrelativistic QCD

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We present a scheme for calculating in nonrelativistic QCD (NRQCD) with consistent power counting in the heavy quark velocity v. As an example, we perform the systematic matching of an external current onto NRQCD at subleading order in v, a calculation relevant for the process $e^+e^-\rightarrow$hadrons near threshold. Consistent velocity power counting in dimensional regularization is achieved by including two distinct gluon fields, one corresponding to gluon radiation and one corresponding to an instantaneous potential. In this scheme power counting is manifest in any gauge, and also holds for nongauge interactions. The matching conditions for an external vector current in NRQCD are calculated to $O(g^2v^2)$ and the cancellation of infrared divergences in the matching conditions is shown to require both gluon fields. Some subtleties arising in the matching conditions at subleading order are addressed.
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I. INTRODUCTION

Nonrelativistic QCD (NRQCD) [1] is a powerful tool for analyzing the dynamics of systems with two or more heavy quarks at momentum transfers much less than their mass. Such systems are more complicated than single-heavy-quark systems described by the heavy quark effective theory (HQET) [2] because the quarks scatter via the QCD potential. This introduces infrared divergences in heavy quark scattering near threshold in HQET which must be regulated by resumming an infinite number of insertions of the kinetic energy operator. Since this operator is subleading in $1/m_Q$, this violates HQET power counting. This kinematic regime of QCD is of interest for a number of physical systems, including quarkonium, $e^+e^-\rightarrow$hadrons near threshold and nonrelativistic QCD sum rules [3]. Similar techniques are also of interest for other nonrelativistic systems, such as positronium [4] and low-energy nucleon-nucleon scattering [5].

A concrete example where HQET power counting fails is provided by an external current in QCD coupled to $\overline{q}\Gamma q$, where Γ is some Dirac matrix. In processes with a single incoming and outgoing heavy quark (such as semileptonic $b\rightarrow c$ decay) there is no potential scattering and HQET is the appropriate low-energy theory. Loop graphs such as Fig. 1(a) are well defined in HQET, and the matching conditions for this current in HQET are currently known to $O(\alpha_s,1/m_Q^2)$ [6]. On the other hand, the one-loop correction to quark-antiquark production by the same current near threshold cannot be correctly described in HQET; the one-loop graph in Fig. 1(b) is infinite when the four-velocities of the quarks are the same, and gives rise to the well-known infinite complex anomalous dimension for the current [7]. The appropriate low-energy theory for the second process is NRQCD, which treats potential scattering near threshold properly.

In this paper we consider the general problem of matching an external fermion-antifermion production current in a nonrelativistic theory. We pay particular attention to power counting in the NRQCD expansion parameter v, the relative three-velocity of the heavy quarks. Power counting in NRQCD is less transparent than in HQET, for several reasons. First, since v is not a parameter in the Lagrangian, power counting is not manifest in the NRQCD Lagrangian, although there are simple rules for determining the v scaling of an operator [8]. Second, the power counting for on-shell gluons differs from that of virtual gluons contributing to potential scattering [9–11] and in order to have simple v counting this distinction must be implemented at the level of the Lagrangian. Also, in order to retain simple v counting beyond tree level the theory must be regulated with a mass-independent regulator such as dimensional regularization, instead of the usual momentum cutoff (otherwise divergent loop integrals change the power counting of Feynman graphs in the effective theory).

The paper is organized as follows. In Sec. II we discuss velocity power counting in NRQCD, and the relation between the results of Refs. [10,11]. In order to maintain manifest v power counting in dimensionally regulated NRQCD

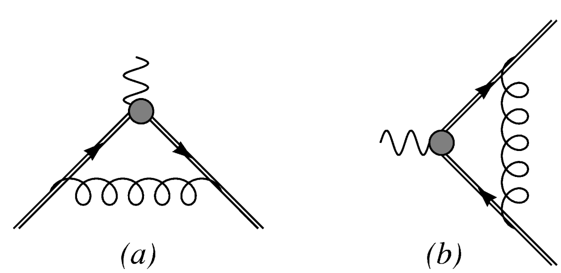

FIG. 1. Two of the one-loop contributions to the matching of an external current in HQET.

we introduce two sets of gluon fields, corresponding to propagating gluons and an instantaneous potential. In Sec. III we discuss the matching of an external current in nonrelativistic Yukawa theory (NRY), and show that the dependence on the infrared regulator vanishes in the matching when both gluon fields are included in the low-energy theory. We consider this theory both because it is simpler than QCD, as well as to stress that manifest velocity power counting does not depend on working in any particular gauge, such as Coulomb gauge. In Sec. IV we match an external current in NRQCD to $O(g^2v^2)$, and explicitly show that the low-energy theory reproduces the nonanalytic behavior of QCD to this order. Finally, in Sec. V we present our conclusions.

II. VELOCITY POWER COUNTING

In NRQCD, the power counting of terms in the Lagrangian is different from HQET, allowing potential scattering near threshold to be correctly described. Operators are classified according to how they scale with the three-velocity v instead of $1/m_Q$ [8]. Since the kinetic energy of a nonrelativistic state is proportional to v^2 while the momentum is proportional to v, space and time derivatives scale differently with v, and power counting is not manifest in the NRQCD Lagrangian:

$$\mathcal{L}=\psi_h^\dagger\left(iD_0+\frac{1}{2m_Q}\mathbf{D}^2\right)\psi_h-\frac{1}{4}G^{\mu\nu a}G_{\mu\nu a}+\mathcal{L}_{\rm gf}+\dots. \tag{2.1}$$

Nonrelativistic fields are distinguished here from fields in the full theory by the subscript h. $\mathcal{L}_{\rm gf}$ is the gauge-fixing term, $D_\mu=\partial_\mu+ig_sA_\mu$ is the gauge-covariant derivative ($A_\mu\equiv A_\mu^aT^a$) and the dots denote higher dimension operators whose matrix elements are suppressed by powers of v.

In Ref. [10] a rescaling of the fields and coordinates was introduced to make the v counting of operators manifest at the level of the Lagrangian. The NRQCD Lagrangian was written in terms of new coordinates $\mathbf{X}$ and T and new fields Ψ_h and $\mathcal{A}$, where

$$\mathbf{x}=\lambda_x\mathbf{X},\quad t=\lambda_tT,\quad \psi_h=\lambda_Q\Psi_h, \qquad A^0=\lambda_{A^0}\mathcal{A}^0,\quad A^i=\lambda_{\mathbf{A}}\mathcal{A}^i, \tag{2.2}$$

and where $\lambda_x=1/m_Qv$, $\lambda_t=1/m_Qv^2$, $\lambda_Q=\lambda_x^{-3/2}$, and $\lambda_A=\lambda_{A^0}=(m_Q\lambda_x^3)^{-1/2}$. In this form the v scaling of operators is manifest in the Lagrangian. In the Lorentz gauge,

$$\begin{aligned}\mathcal{L}^R=&\Psi_h^\dagger\left(i\partial_0-\frac{g}{\sqrt{v}}\mathcal{A}_0\right)\Psi_h-\frac{1}{2}\Psi_h^\dagger(i\boldsymbol{\nabla}-g\sqrt{v}\mathcal{A})^2\Psi_h-\frac{1}{4}(\partial_i\mathcal{A}_j^a-\partial_j\mathcal{A}_i^a-g\sqrt{v}f_{abc}\mathcal{A}_i^b\mathcal{A}_j^c)^2\\&+\frac{1}{2}(\partial_i\mathcal{A}_0^a-v\partial_0\mathcal{A}_i^a-g\sqrt{v}f_{abc}\mathcal{A}_i^b\mathcal{A}_0^c)^2-\frac{1}{2\alpha}(v\partial^0\mathcal{A}_0^a+\partial^i\mathcal{A}_i^a)^2\\=&\Psi_h^\dagger\left(i\partial_0+\frac{\boldsymbol{\nabla}^2}{2}-\frac{g}{\sqrt{v}}\mathcal{A}_0\right)\Psi_h-\frac{1}{4}(\partial_i\mathcal{A}_j^a-\partial_j\mathcal{A}_i^a)^2+\frac{1}{2}(\partial_i\mathcal{A}_0^a)^2-\frac{1}{2\alpha}(\partial^i\mathcal{A}_i^a)^2+O(v,g\sqrt{v}).\end{aligned} \tag{2.3}$$

Of course, there is no physics in a simple rescaling. However, in order to have an effective theory in which the v power counting is manifest, the additional prescription that terms which are subleading in v be treated as operator insertions must be added (otherwise, loop graphs evaluated in dimensional regularization will mix powers of v). As was noted in [10], once this prescription is added, the rescaling in Eq. (2.3) misses important physics. While it provides the correct description of virtual gluon exchange corresponding to an instantaneous potential, it fails to correctly describe on-shell gluons. The problem is that the pole in the full gluon propagator occurs at $k_0^2=\mathbf{k}^2$, or, in terms of rescaled variables, $v^2K_0^2=\mathbf{K}^2$. Unless the time derivative in the gluon kinetic term is treated exactly instead of as an insertion (which would violate manifest v counting), amplitudes in the effective theory do not have the correct branch cut corresponding to physical gluon propagation, so the effective theory cannot describe on-shell gluons.

In Ref. [11] it was demonstrated that this problem is avoided by a further rescaling (in Coulomb gauge) of the space coordinates of only the transverse components of the gauge fields. In the language of Ref. [10],[1] this corresponds to the rescaling

$$\mathbf{x}=\lambda_t\mathbf{Y},\quad t=\lambda_tT,\quad A^i=\lambda_t^{-1}\widetilde{\mathcal{A}}^i. \tag{2.4}$$

The kinetic term for the $\widetilde{\mathcal{A}}_i$'s is canonically normalized,

$$L_{\rm kin}=-\int d^3Y\,dT\left[\frac{1}{4}(\partial_i\widetilde{\mathcal{A}}_j^a-\partial_j\widetilde{\mathcal{A}}_i^a-gf_{abc}\widetilde{\mathcal{A}}_i^b\widetilde{\mathcal{A}}_j^c)^2+\frac{1}{2}(\partial_0\widetilde{\mathcal{A}}_j^a)^2\right], \tag{2.5}$$

[1]The Lagrangian in [11] was written as an expansion in powers of $1/c$ instead of v; however, the two descriptions are equivalent. Note that as $c\to\infty$, $\alpha_s=g^2/4\pi c\to 0$, whereas as $v\to 0$, α_s remains constant, so factors of α_s scale differently with $1/c$ than with v.

while the transverse gluon-quark interaction may be expanded in terms of multipoles[2]

$$\mathcal{L}_{\rm int} = -\frac{i}{2} g v \Psi_h^\dagger(\mathbf{X},T) \overleftrightarrow{\boldsymbol{\nabla}}_i \Psi_h(\mathbf{X},T) \widetilde{\mathcal{A}}_i(v\mathbf{X},T)$$
$$= -\frac{i}{2} g v \Psi_h^\dagger(\mathbf{X},T) \overleftrightarrow{\boldsymbol{\nabla}}_i \Psi_h(\mathbf{X},T) \times [1 + vX\cdot\boldsymbol{\nabla} + \dots] \widetilde{\mathcal{A}}_i(0,T). \quad (2.6)$$

Note that three-momentum is not conserved at the multipole interaction vertex, since the theory breaks translational invariance, although energy is conserved. Once the multipole expansion is performed, loop integrals in dimensional regularization do not change the v scaling of a graph determined by the vertices.

There is, however, a subtlety arising from the multipole expansion. Since the three-momentum is not conserved at the vertices, transverse gluons cannot alter the three-momenta of the heavy quarks. Potential scattering via transverse gluon exchange therefore does not occur in the low energy theory. Since the amplitude for potential scattering is not analytic in the external momenta, it cannot be reproduced in NRQCD by the addition of local operators. Both potential scattering and real gluon emission are long-distance effects, so in order to correctly describe the infrared physics of QCD in the nonrelativistic limit the instantaneous potential due to transverse gluon exchange must be added to the effective theory. This may either be done by explicitly including spatially nonlocal operators in the nonrelativistic theory (the need for which was discussed in [11]), or by reintroducing a second gluon field which couples according to Eq. (2.3).

The need for two distinct gluon fields is understood by comparing the energy and momentum of radiated gluons compared to those involved in potential scattering. In the nonrelativistic regime of QCD there are nonanalytic contributions to scattering amplitudes arising from gluons in two separate kinematic regions, both with energy of order mv^2. The gluons with spatial momenta $\sim mv$ are far off shell and contribute to potential scattering. In contrast, the gluons with spatial momenta $\sim mv^2$ may be on shell, describing real radiation, but do not contribute to the scattering of quarks with three-momenta of order mv (in the limit $v\to 0$). Each of the rescalings discussed above treats one of these kinematic regimes correctly, while missing the physics of the other. As argued in Ref. [10], it is not possible to describe both regimes via any single rescaling. In order to correctly describe the infrared physics of QCD in the nonrelativistic limit two separate gluon fields must be included, which will be referred to in this work as ''potential'' and ''radiation'' gluons, corresponding to the two different kinematic regimes described above.

In this approach the heavy quark Lagrangian in NRQCD in Lorentz gauge is therefore, in standard (unrescaled) units,

$$\mathcal{L} = \psi_h^\dagger \left(i\partial_0 + \frac{\boldsymbol{\nabla}^2}{2m} - gA_P^0 - gA_R^0(0,t) \right) \psi_h - \frac{1}{4}(\boldsymbol{\nabla}^i \mathbf{A}_P^j - \boldsymbol{\nabla}^j \mathbf{A}_P^i)^2 + \frac{1}{2}(\boldsymbol{\nabla} A_P^0)^2 - \frac{1}{4} G_R^{\mu\nu} G_{\mu\nu R} - \frac{1}{2\alpha}(\boldsymbol{\nabla}\cdot\mathbf{A}_P)^2 - \frac{1}{2\alpha}(\partial_\mu A_R^\mu)^2 + O(v, g\sqrt{v}), \quad (2.7)$$

where the subscripts P and R denote potential and radiation gluons, $G_R^{\mu\nu}$ is the field strength tensor for radiation gluons, and α is the gauge parameter. For practical calculations, this version of the Lagrangian is much more convenient than the rescaled version. The rescaled theory simply guarantees that loop graphs computed in the unrescaled theory, with the appropriate terms treated as operator insertions, will have v scaling determined by the vertices.

III. YUKAWA THEORY

Coulomb gauge is usually used in NRQCD because in this gauge A_0 exchange corresponds to an instantaneous potential. Once v counting is performed as in the previous section, potential gluon exchange is instantaneous in all gauges for both transverse and longitudinal gluons. More generally, potential scattering proceeds via an instantaneous interaction even in theories with no gauge freedom. To illustrate this in a theory which is simpler than QCD, in this section we consider a nonrelativistic Yukawa theory (NRY) of a massive Fermi field ψ coupled to a massless scalar φ.

As a warmup for our calculation of the matching conditions for $e^+e^-\to$hadrons in QCD, we consider here the matching conditions for an external current coupling to $\overline{\psi}\gamma^\mu\psi$ in NRY. In processes with a single incoming and outgoing fermion there is no potential scattering and the analog of HQET for a Yukawa interaction is the appropriate effective theory. However, as has been discussed, the $1/v$ behavior of potential scattering near threshold cannot be correctly described in HQET. In NRY, the $1/v$ behavior is reproduced by potential scalar exchange. There are also infrared divergences in the full theory due to soft scalar bremsstrahlung, which are reproduced in NRY by radiation scalars. In this section this is demonstrated explicitly. The theory is regulated in the ultraviolet by working in $d=4-\epsilon$ dimensions, and in the infrared by introducing a small scalar mass m_φ. (The theory could be regulated in the infrared with dimensional regularization, as at the end of this section, but this obscures the distinction between the infrared and ultraviolet divergences.)

The Lagrangian in the full theory is

$$\mathcal{L} = \overline{\psi}(i\partial\!\!\!/ - m)\psi + \frac{1}{2}\partial^\mu\varphi\partial_\mu\varphi - g\overline{\psi}\psi\varphi, \quad (3.1)$$

while the nonrelativistic Yukawa theory (NRY), to leading order in the three-velocity v, is

[2]At the level of Feynman diagrams, the observation that on-shell gauge fields couple via the multipole expansion was made in Ref. [9].

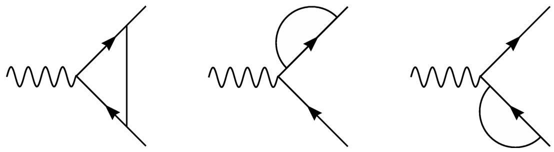

FIG. 2. One-loop diagrams in Yukawa theory.

$$\mathcal{L}_{\rm NR}=\psi_h^\dagger\left(i\partial_0+\frac{\boldsymbol{\nabla}^2}{2m}\right)\psi_h+\frac{1}{2}\partial^\mu\varphi_R\partial_\mu\varphi_R-\frac{1}{2}(\boldsymbol{\nabla}\varphi_P)^2 -g_1\psi_h^\dagger\psi_h\varphi_P-g_2\psi_h^\dagger\psi_h\varphi_R(0,t), \tag{3.2}$$

where φ_P and φ_R are the potential and radiation scalars, respectively. There is a similar kinetic term for the antifermion field χ_h. (Note that ψ_h annihilates incoming particles, while χ_h creates outgoing antiparticles). At tree level $g_1=g_2=g$.

The external vector current in the full theory,

$$J_\mu\overline{\psi}\gamma^\mu\psi, \tag{3.3}$$

matches onto a number of terms in the low energy theory,[3]

$$J_i(\chi_h^\dagger\boldsymbol{\sigma}^i\psi_h+\psi_h^\dagger\boldsymbol{\sigma}^i\chi_h)+J_0(\psi_h^\dagger\psi_h+\chi_h^\dagger\chi_h)+O(g^2,v^2). \tag{3.4}$$

The one-loop matrix element of the current in the full theory is given by the diagrams in Fig. 2. Performing the loop integration, the infrared divergent part of the vertex graph is

$$i\mathcal{A}^V=4ig^2m^2J_i\overline{u(p_1)}\gamma^iv(p_2) \times\int\frac{d^dk}{(2\pi)^d}\frac{1}{(k^2-m_\varphi^2)(k^2+2p_1\cdot k)(k^2-2p_2\cdot k)} +\cdots, \tag{3.5}$$

where terms finite as $m_\varphi\to 0$ have been neglected. Combining denominators in the usual way and performing the loop and Feynman parameter integrals, the infrared divergent term is found to be

$$i\mathcal{A}^V=\frac{g^2}{4\pi^2}J_i\overline{u}\gamma^iv\,\frac{(1-\beta^2)}{\beta}\text{arctanh}\left(\frac{1}{\beta}\right)\ln m_\varphi+\cdots, \tag{3.6}$$

where

$$\beta=\sqrt{1-\frac{4m^2}{s}} \tag{3.7}$$

[3]Spinors in the full theory are normalized such that $\overline{u}_r(\mathbf{p})u_s(\mathbf{p})=-\overline{v}_r(\mathbf{p})v_s(\mathbf{p})=\delta_{rs}$. The two-component spinors are $u_{h_1}=v_{h_2}=\binom{1}{0}$, $u_{h_2}=v_{h_1}=\binom{0}{1}$.

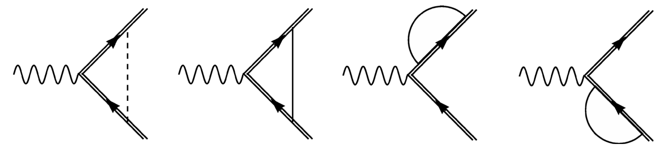

FIG. 3. Infrared divergent one-loop diagrams in NRY. The dashed line corresponds to the potential scalar while the solid line is the radiation scalar.

is the magnitude of the three-velocity of each fermion, and $s=(p_1+p_2)^2$ is the invariant mass of the fermion-antifermion pair. The infrared divergent piece of the wave function renormalization is

$$i\mathcal{A}^W=\frac{g^2}{4\pi^2}\;J_i\overline{u}\gamma^iv\;\;\ln m_\varphi+\cdots \tag{3.8}$$

and therefore the infrared divergence in the full theory amplitude is

$$\begin{aligned} i\mathcal{A}_{\rm IR}&=\frac{g^2}{4\pi^2}J_i\overline{u}\gamma^iv\left[\frac{1-\beta^2}{\beta}\text{arctanh}\left(\frac{1}{\beta}\right)+1\right]\ln m_\varphi+\cdots\\ &=\left[-\frac{ig^2}{8\pi\beta}+\frac{g^2}{2\pi^2}+O(\beta)\right]J_i\overline{u}\gamma^iv\;\ln m_\varphi+\cdots\\ &=\left[-\frac{ig^2}{8\pi\beta}+\frac{g^2}{2\pi^2}+O(\beta)\right]J_iu_h^\dagger\boldsymbol{\sigma}^iv_h\;\ln m_\varphi+\cdots, \end{aligned} \tag{3.9}$$

where u_h and v_h are two-component spinors. Note that Eq. (3.9) may be written

$$i\mathcal{A}_{\rm IR}=\frac{g^2}{4\pi^2}J_i\overline{u}\gamma^iv[r(w)-1]\ln m_\varphi+\cdots, \tag{3.10}$$

where

$$w=\frac{1+\beta^2}{-1+\beta^2}, \tag{3.11}$$

and the function $r(w)$ is given by

$$r(w)=\frac{1}{\sqrt{w^2-1}}\ln[w+\sqrt{w^2-1}]. \tag{3.12}$$

This is the analytic continuation to the production region (negative $w=v\cdot v'<0$) of the infrared divergence encountered in HQET (for scalar exchange) [12]. The first term in Eq. (3.9) is singular as $\beta\to 0$, corresponding to the infinite complex anomalous dimension found in HQET at threshold. Since it is imaginary, it does not contribute to the decay rate at $O(g^2)$. The second term in Eq. (3.9) cancels in physical matrix elements with scalar bremsstrahlung.

In order to be able to match onto NRY at one loop, both of these divergences must be reproduced in the low energy theory (the singularities higher order in β will only be reproduced when operators higher order in the velocity are included in the effective theory). Diagrams with both potential φ_P and radiation φ_R scalars contribute to the amplitude in NRY (Fig. 3). The wave function graphs with φ_P exchange vanish, while the vertex graph gives

$$i\mathcal{A}_P^V = ig^2 J_i u_h^\dagger \boldsymbol{\sigma}^i v_h \int \frac{d^dk}{(2\pi)^d} \frac{1}{(T+k_0-(\mathbf{k}+\mathbf{p})^2/2m+i\varepsilon)(T-k_0-(\mathbf{k}+\mathbf{p})^2/2m+i\varepsilon)(\mathbf{k}^2+m_\varphi^2-i\varepsilon)}, \tag{3.13}$$

where $T=E-m$ is the fermion kinetic energy. Closing the k_0 integral in the upper half plane, using the leading order equation of motion $T=p^2/2m$ and picking out the corresponding pole leaves a $(d-1)$-dimensional Euclidean integral, which gives

$$i\mathcal{A}_P^V = g^2 m J_i u_h^\dagger \boldsymbol{\sigma}^i v_h \frac{\Gamma[2-(d-1)/2]}{(4\pi)^{d-1}} \int_0^1 dx [m_\varphi^2(1-x)-x^2\mathbf{p}^2-i\varepsilon]^{(d-1)/2-2} = -\frac{ig^2}{8\pi\beta} J_i u_h^\dagger \boldsymbol{\sigma}^i v_h \ln m_\varphi + \cdots \tag{3.14}$$

and reproduces the first term in Eq. (3.9). The radiation scalar vertex correction is

$$i\mathcal{A}_R^V = -ig^2 J_i u_h^\dagger \boldsymbol{\sigma}^i v_h \int \frac{d^dk}{(2\pi)^d} \frac{1}{(k^2-m_\varphi^2+i\varepsilon)k_0^2}, \tag{3.15}$$

where the $+i\varepsilon$'s in the fermion propagators have been dropped because these poles do not contribute to physical matrix elements. One can see by working with off-shell states that the poles from the fermion propagators only give contributions proportional to powers of $E-\mathbf{p}^2/2m$ which vanish by the lowest order equations of motion of NRY. Using the standard HQET trick of combining denominators with a dimensionful parameter, the vertex graph becomes

$$i\mathcal{A}_R^V = 8ig^2 J_i u_h^\dagger \boldsymbol{\sigma}^i v_h \int_0^\infty \lambda\, d\lambda \int \frac{d^dk}{(2\pi)^d} \frac{1}{(-k^2-2\lambda k_0+m_\varphi^2)^3}$$
$$= \frac{g^2}{4\pi^2} J_i u_h^\dagger \boldsymbol{\sigma}^i v_h \ln m_\varphi + \cdots. \tag{3.16}$$

The wave function graphs give an identical contribution, so the sum of radiation scalar graphs becomes

$$i\mathcal{A}_R = \frac{g^2}{2\pi^2} J_i u_h^\dagger \boldsymbol{\sigma}^i v_h \ln m_\varphi + \cdots, \tag{3.17}$$

reproducing the second term in Eq. (3.9).

This illustrates that both φ_P and φ_R are required for the difference between the matrix elements of the external current in the full and effective theories to be infrared finite. Having demonstrated this, it is easier to calculate the matching conditions by regulating both the infrared and ultraviolet divergences in the full and effective theories with dimensional regularization. The matrix element in the full theory is found to be

$$i\mathcal{A}_{\rm full} = J_i u_h^\dagger \boldsymbol{\sigma}^i v_h \left(1 - \frac{g^2}{4\pi^2}\left[\frac{2}{d-4} + \gamma_E + \ln\frac{m^2}{4\pi\mu^2}\right] + i\frac{g^2}{16\pi\beta}\left[\frac{2}{d-4} - i\pi + \gamma_E + \ln\frac{m^2\beta^2}{\pi\mu^2}\right]\right) + O(v). \tag{3.18}$$

Regulating the theory in both the ultraviolet and infrared with dimensional regularization has the advantage that one-loop graphs in the nonrelativistic theory containing radiation scalars vanish. This is due to a cancellation of infrared and ultraviolet divergences, since one-loop integrals containing radiation scalars are of the form

$$\int \frac{d^dk}{(2\pi)^d} f(k_0,k^2), \tag{3.19}$$

which has no mass scale and so vanishes in dimensional regularization. Thus, radiation scalars do not contribute to the one-loop matching conditions. This does not mean, however, that radiation scalars are irrelevant. Integrals of the form (3.19) are both ultraviolet and infrared divergent, with the divergent terms having the same magnitude but opposite signs. In the difference between the full and effective theories the infrared divergences in the two theories cancel, leaving an ultraviolet divergence in the effective theory. Unlike the infrared divergence, the ultraviolet divergence is cancelled in the low-energy theory by a local counterterm.

The matrix element of the current in the effective theory with the tree-level matching is, in dimensional regularization,

$$i\mathcal{A}_{\rm NRY} = J_i u_h^\dagger \boldsymbol{\sigma}^i v_h \left(1 + i\frac{g^2}{16\pi\beta}\left[\frac{2}{d-4} - i\pi + \gamma_E + \ln\frac{m^2\beta^2}{\pi\mu^2}\right]\right) + O(v). \tag{3.20}$$

The difference between the two matrix elements (3.18) and (3.20) is analytic in the external momenta, as it must be to be absorbed into the coefficients of local operators in NRY. The matching condition for the current at one loop at a renormalization scale μ is therefore

$$J_\mu \bar{\psi}\gamma^\mu\psi \to \left(1 - \frac{g^2}{4\pi^2}\left[\frac{2}{d-4} + \gamma_E + \ln\frac{m^2}{4\pi\mu^2}\right]\right) \times [J_i \psi_h^\dagger \boldsymbol{\sigma}^i \chi_h]_0 + \cdots$$
$$= \left(1 - \frac{g^2}{4\pi^2}\ln\frac{m^2}{\mu^2}\right) J_i \psi_h^\dagger \boldsymbol{\sigma}^i \chi_h, \tag{3.21}$$

where we have denoted the bare operator by the subscript 0, and the effective theory is renormalized using the modified minimal subtraction scheme ($\overline{\rm MS}$).

IV. NRQCD

The matching of an external vector current in NRQCD, relevant for $e^+e^- \to$ hadrons near threshold, proceeds in much the same way as in the Yukawa theory of the previous section. Infrared divergences odd in v are reproduced in the nonrelativistic theory by potential gluon exchange, while infrared divergences even in v are reproduced by radiation gluon exchange. In this section the matching conditions for a vector current in NRQCD to order v^2 are calculated.[4] The theory is regulated in both the infrared and ultraviolet in dimensional regularization. Cancellation in the matching conditions of terms which are not analytic in the external momenta is rather nontrivial at subleading order and provides a nice demonstration of the consistency of this approach.

When working at subleading orders in v, there are a few subtleties which must be taken into account. First of all, since the three-momentum in the effective theory is

$$|\mathbf{p}|=m\gamma\beta=m\frac{\beta}{\sqrt{1-\beta^2}}, \tag{4.1}$$

derivatives acting on operators in the nonrelativistic theory give factors of $\gamma\beta$, rather than β. It is therefore more convenient to treat $\gamma\beta$ as the nonrelativistic expansion parameter. In the rest of this paper terms of order $|\mathbf{p}|^n/m^n=\gamma^n\beta^n$ will be referred to as being of order v^n.

Second, Feynman diagrams in the full theory yield S matrix elements evaluated between relativistically normalized states, satisfying $\langle k'|k\rangle=2E_k\,(2\pi)^3\delta^{(3)}(\vec{k}-\vec{k}')$. However, in the nonrelativistic theory defined such that the residue of the pole in the propagator is i, Feynman diagrams instead give S matrix elements between nonrelativistically normalized states, defined such that $\langle \vec{k}'|k\rangle=2m(2\pi)^3\delta^{(3)}(\vec{k}-\vec{k}')$. To demonstrate this, consider the two-point functions in the full and effective theories. Expanding the relativistic propagator for quarks in terms of the low energy variables gives

$$\begin{aligned}\frac{i(\not p+m)}{p^2-m^2}&=\frac{2i\,m}{(m+T)^2-\mathbf{p}^2-m^2}\\&=\frac{2i\,m}{2mT+T^2-\mathbf{p}^2}\\&=\frac{i(1-\mathbf{p}^2/2m^2)}{T-\mathbf{p}^2/2m}+\frac{i}{T-\mathbf{p}^2/2m}\left[\frac{i\mathbf{p}^4}{8m^3}\right]\\&\quad\times\frac{i}{T-\mathbf{p}^2/2m}+O(\mathbf{p}^6),\end{aligned} \tag{4.2}$$

where an irrelevant constant term has been dropped. This two-point function is reproduced in the nonrelativistic theory by a Lagrangian

$$\begin{aligned}\mathcal{L}'=\psi_h'^{\dagger}\left(i\partial_0+\frac{\boldsymbol{\nabla}^2}{2m}\right)\psi_h'-\psi_h'^{\dagger}\frac{\boldsymbol{\nabla}^2}{2m^2}\left(i\partial_0+\frac{\boldsymbol{\nabla}^2}{2m}\right)\psi_h'\\+\psi_h'^{\dagger}\frac{\boldsymbol{\nabla}^4}{8m^3}\psi_h',\end{aligned} \tag{4.3}$$

[4]See also Ref. [13], where the matching of an external electromagnetic current to nonrelativistic quantum mechanics (regulated in position space) was discussed.

where the field operator ψ_h' ($\psi_h'^{\dagger}$) annihilates (creates) a nonrelativistic particle. However, the residue of the pole in $T-\mathbf{p}^2/2m$ in this theory is not i, but $i(1-\mathbf{p}^2/2m+\cdots)=im/E$. While this is perfectly consistent, it is preferable to remove this extra factor of m/E. This may be easily done, since the operator

$$-\psi_h'^{\dagger}\frac{\boldsymbol{\nabla}^2}{2m^2}\left(i\partial_0+\frac{\boldsymbol{\nabla}^2}{2m}\right)\psi_h' \tag{4.4}$$

is proportional to the equations of motion, and may therefore be removed by the field redefinition

$$\psi_h'\to\psi_h=\left(1-\frac{\boldsymbol{\nabla}^2}{4m^2}\right)\psi_h'=\sqrt{\frac{E}{m}}\;\psi_h'+O(v^4). \tag{4.5}$$

However, because of this rescaling, an additional Feynman rule of $\sqrt{E/m}$ for each external leg must be included in NRQCD, when evaluating matrix elements between relativistically normalized states. If this term is omitted, Feynman diagrams in NRQCD correspond to matrix elements between nonrelativistically normalized states. [A more careful analysis using the Lehmann-Symanzik-Zimmermann (LSZ) reduction formula in the nonrelativistic theory instead of rescaling arguments reproduces this result.] In the rest of the discussion the ψ_h fields will be used, and matching conditions are calculated using nonrelativistically normalized states in the full and effective theories (this is the origin of the factors of $\sqrt{m/E}$ in the matching conditions presented in [14]).

The kinetic term for the nonrelativistic fields therefore takes the usual form

$$\begin{aligned}\mathcal{L}_h=\psi_h^{\dagger}\left(i\partial_0+\frac{\boldsymbol{\nabla}^2}{2m}\right)\psi_h+\chi_h^{\dagger}\left(i\partial_0-\frac{\boldsymbol{\nabla}^2}{2m}\right)\chi_h\\+\frac{1}{8m^3}(\psi_h^{\dagger}\boldsymbol{\nabla}^4\psi_h-\chi_h^{\dagger}\boldsymbol{\nabla}^4\chi_h)+O(v^4).\end{aligned} \tag{4.6}$$

The low energy theory contains both potential ($A_P^{\mu}\equiv A_P^{\mu a}T^a$) and radiation ($A_R^{\mu}\equiv A_R^{\mu a}T^a$) gluons. The kinetic term for the gauge fields is, including the gauge-fixing terms,

$$\begin{aligned}\mathcal{L}_g=-\frac{1}{4}(\boldsymbol{\nabla}^i\mathbf{A}_P^j-\boldsymbol{\nabla}^j\mathbf{A}_P^i)^2+\frac{1}{2}(\boldsymbol{\nabla}A_P^0)^2-\frac{1}{2\alpha}(\boldsymbol{\nabla}\cdot\mathbf{A}_P)^2\\-\frac{1}{4}G_R^{\mu\nu}G_{\mu\nu R}-\frac{1}{2\alpha}(\partial_iA_R^i)^2+O(g\sqrt{v}),\end{aligned} \tag{4.7}$$

where Coulomb gauge corresponds to the limit $\alpha\to0$. The $O(g\sqrt{v})$ terms correspond to the triple-potential gluon vertex and will not be required for the one-loop matching we consider in this section. Also note that in Coulomb gauge there is no $O(v^2)$ correction to the A_P^0 propagator. In Lorentz gauge, the gauge-fixing term for the radiation fields is instead $-(1/2\alpha)(\partial_\mu A_R^{\mu})^2$, and there are additional terms bilinear in the A_P's suppressed by powers of v coming from the expansion of the gauge-fixing term.

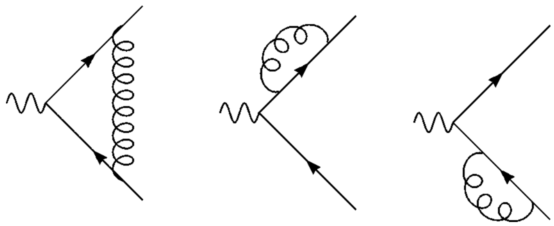

FIG. 4. One-loop contributions to quark-antiquark production in QCD.

Working in the center of mass frame $\mathbf{p}_1 = -\mathbf{p}_2 \equiv \mathbf{p}$, and using the relation (easily verified with on-shell bispinors)

$$\overline{u(p_1)}\gamma^i v(p_2) = u_h^\dagger\left(1+\frac{\mathbf{p}^2}{2m^2}\right)\boldsymbol{\sigma}^i h - \frac{1}{2m^2}h^\dagger \mathbf{p}\cdot\boldsymbol{\sigma}\mathbf{p}^i v_h + O(v^4)$$
$$= \frac{E}{m}u_h^\dagger\left(\boldsymbol{\sigma}^i - \frac{1}{2m^2}\mathbf{p}\cdot\boldsymbol{\sigma}\mathbf{p}^i\right)v_h + O(v^4), \tag{4.8}$$

the tree-level matching conditions for an external vector current $J(x)$ are found to be

$$J_\mu \overline{\psi}\gamma^\mu\psi \to c_1\mathbf{O}_1 + c_2\mathbf{O}_2 + c_3\mathbf{O}_3 + \dots + O(v^4),$$
$$c_1 = c_2 = 1 + O(g^2),$$
$$c_3 = O(g^2), \tag{4.9}$$

where

$$\mathbf{O}_1 = J_i \psi_h^\dagger \boldsymbol{\sigma}^i \chi_h,$$
$$\mathbf{O}_2 = \frac{1}{4m^2}J_i[\psi_h^\dagger(\vec{\boldsymbol{\nabla}}\cdot\boldsymbol{\sigma}\vec{\boldsymbol{\nabla}}^i + \overleftarrow{\boldsymbol{\nabla}}\cdot\boldsymbol{\sigma}\overleftarrow{\boldsymbol{\nabla}}^i)\chi_h],$$
$$\mathbf{O}_3 = \frac{1}{2m^2}J_i[\psi_h^\dagger\boldsymbol{\sigma}^i(\vec{\boldsymbol{\nabla}}^2 + \overleftarrow{\boldsymbol{\nabla}}^2)\chi_h], \tag{4.10}$$

and only the terms contributing to quark-antiquark production have been included. The operators in Eq. (4.10) are renormalized in the $\overline{\mathrm{MS}}$.

The A_P^0 coupling to heavy fermions, giving the Coulomb potential, is $O(v^{-1/2})$,

$$\mathcal{L}_C = -g(\psi_h^\dagger A_P^0 \psi_h + \chi_h^\dagger A_P^0 \chi_h) + O(g^3). \tag{4.11}$$

The leading corrections to this are the Darwin and spin-orbit couplings, arising at $O(v^{3/2})$,

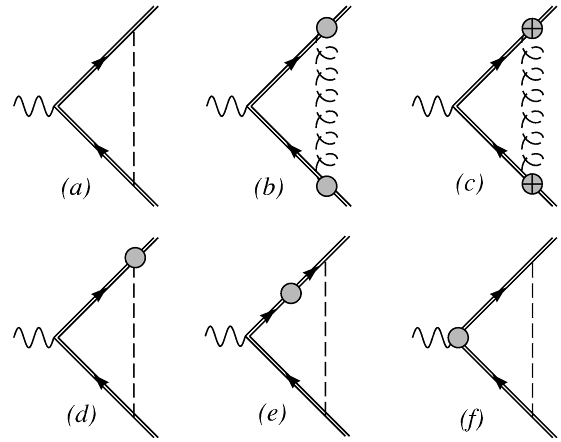

FIG. 5. One-loop contributions to quark-antiquark production in NRQCD. The dashed line corresponds to a potential A_0 gluon, the dashed gluon line to a potential A_i gluon. The shaded circles represent (b) the $\mathbf{p}\cdot\mathbf{A}$ vertex, (c) the Fermi vertex, (d) the Darwin vertex, (e) the relativistic kinematic correction to the fermion leg, and (f) $\mathbf{O}_2$. Implicit in both (d) and (e) are graphs with the same operator insertion on the antiquark line. The wave function graphs vanish.

$$\mathcal{L}_{D,\mathrm{SO}} = \frac{g}{8m^2}(\psi_h^\dagger T^a \psi_h + \chi_h^\dagger T^a \chi_h)\boldsymbol{\nabla}^2 A_P^{0a}$$
$$+ i\frac{g}{4m^2}\epsilon^{ijk}(\psi_h^\dagger T^a \boldsymbol{\sigma}^i \boldsymbol{\nabla}^j \psi_h + \chi_h^\dagger T^a \boldsymbol{\sigma}^i \boldsymbol{\nabla}^j \chi_h)\boldsymbol{\nabla}^k A_P^{0a}$$
$$+ O(g^3). \tag{4.12}$$

Transverse potential gluons couple through the $\mathbf{p}\cdot\mathbf{A}$ and Fermi (chromomagnetic dipole moment) terms at $O(v^{1/2})$,

$$\mathcal{L}_{p\cdot A,F} = \frac{g}{2m}[\psi_h^\dagger(\mathbf{A}_P\cdot\boldsymbol{\nabla} + \boldsymbol{\nabla}\cdot\mathbf{A}_P)\psi_h$$
$$- \chi_h^\dagger(\mathbf{A}_P\cdot\boldsymbol{\nabla} + \boldsymbol{\nabla}\cdot\mathbf{A}_P)\chi_h]$$
$$- \frac{g}{2m}[\psi_h^\dagger\boldsymbol{\sigma}\cdot(\boldsymbol{\nabla}\times\mathbf{A}_P)\psi_h - \chi_h^\dagger\boldsymbol{\sigma}\cdot(\boldsymbol{\nabla}\times\mathbf{A}_P)\chi_h]$$
$$+ O(g^3) \tag{4.13}$$

and so transverse gluon exchange and the leading relativistic corrections to Coulomb exchange both contribute to potential scattering at $O(g^2 v)$, as expected. This is also the same order as the correction to Coulomb scattering due to the $\boldsymbol{\nabla}^4$ correction to the fermion legs. This agrees with the power counting of Ref. [8] in which the matrix element of each of these terms is given as $O(v^2)$, since in quarkonium $v \sim g^2$. Note also that the Fermi, Darwin, spin-orbit and relativistic kinematic corrections in the previous equations are only the leading pieces of the usual form of these terms [1],

$$\delta\mathcal{L}_{\rm bilinear}=\frac{1}{8m^3}[\psi_h^\dagger(\mathbf{D}^2)^2\psi_h-\chi_h^\dagger(\mathbf{D}^2)^2\chi_h]+\frac{1}{8m^2}[\psi_h^\dagger(\mathbf{D}\cdot g\mathbf{E}-g\mathbf{E}\cdot\mathbf{D})\psi_h+\chi_h^\dagger(\mathbf{D}\cdot g\mathbf{E}-g\mathbf{E}\cdot\mathbf{D})\chi_h]+\frac{1}{8m^2}[\psi_h^\dagger(i\mathbf{D}\times g\mathbf{E}$$

$$-g\mathbf{E}\times i\mathbf{D})\cdot\boldsymbol{\sigma}\psi_h+\chi_h^\dagger(i\mathbf{D}\times g\mathbf{E}-g\mathbf{E}\times i\mathbf{D})\cdot\boldsymbol{\sigma}\chi_h]+\frac{1}{2m}[\psi_h^\dagger(g\mathbf{B}\cdot\boldsymbol{\sigma})\psi_h-\chi_h^\dagger(g\mathbf{B}\cdot\boldsymbol{\sigma})\chi_h]+O(g^3), \tag{4.14}$$

since covariant derivatives and $\mathbf{E}$ both consist of two terms of differing orders in v.

Since radiation gluons do not contribute to one-loop graphs, as was discussed in the previous section, their couplings are not presented here.

Using the relations (4.8) and

$$\overline{u(p_1)}v(p_2)=-\frac{1}{m}u_h^\dagger\ \ \mathbf{p}\cdot\boldsymbol{\sigma}v_h+O(\mathbf{p}^3) \tag{4.15}$$

(still working in the center of mass frame), the amplitude for quark-antiquark production in QCD from the diagrams in Fig. 4 may be expanded in powers of $\mathbf{p}/m$:

$$i\mathcal{A}_{\rm QCD}=u_h^\dagger\boldsymbol{\sigma}^i v_h\left(1-\frac{2g^2}{3\pi^2}\right)-\frac{1}{2m^2}u_h^\dagger\mathbf{p}\cdot\boldsymbol{\sigma}\mathbf{p}^i v_h\left(1-\frac{g^2}{3\pi^2}\right)+\frac{g^2}{12\pi^2}u_h^\dagger\boldsymbol{\sigma}^i v_h\left\{\frac{m}{|\mathbf{p}|}\left[\pi^2+i\pi\left(\gamma_E+\frac{2}{d-4}+\ln\frac{\mathbf{p}^2}{\pi\mu^2}\right)\right]\right.$$
$$+\frac{3|\mathbf{p}|}{2m}\left[\pi^2+i\pi\left(\gamma_E-2+\frac{2}{d-4}+\ln\frac{\mathbf{p}^2}{\pi\mu^2}\right)\right]+\left.\frac{\mathbf{p}^2}{3m^2}\left(\frac{2}{3}-8\gamma_E-\frac{16}{d-4}-8\ln\frac{m^2}{4\pi\mu^2}\right)\right\}$$
$$+\frac{g^2}{12\pi^2}\frac{u_h^\dagger\mathbf{p}\cdot\boldsymbol{\sigma}\mathbf{p}^i v_h}{2m^2}\left\{\frac{m}{|\mathbf{p}|}\left[-\pi^2-i\pi\left(\gamma_E-2+\frac{2}{d-4}+\ln\frac{\mathbf{p}^2}{\pi\mu^2}\right)\right]\right\}+O(v^3) \tag{4.16}$$

(where an overall factor of E/m has been divided out, to convert to nonrelativistically normalized states). The amplitude has the expected $1/v$ singularity from Coulomb scattering, signalling the failure of perturbation theory close to threshold.

The matching conditions for the current are given by the difference between Eq. (4.16) and the graphs in Fig. 5 computed on shell in NRQCD [15]. The graph in Fig. 5(a) corresponds to Coulomb A_P^0 exchange. The only subtlety in evaluating this graph is that, to the order in which we are working, the on-shell condition in the effective theory is

$$T=\frac{\mathbf{p}^2}{2m}-\frac{\mathbf{p}^4}{8m^3}, \tag{4.17}$$

instead of the leading order relation $T=\mathbf{p}^2/2m$, and this extra term must be treated correctly as a perturbation so as not to violate power counting. Evaluating the graph in Fig. 5(a) for arbitrary off-shell spinors,

$$\frac{2mT}{\mathbf{p}^2}=1+\delta, \tag{4.18}$$

gives

$$(a)=c_1\frac{g^2m^{d-6}\mathbf{p}^2}{3\cdot 2^{d-3}\pi^{(d-1)/2}}u_h^\dagger\boldsymbol{\sigma}^i v_h\left(-\frac{\mathbf{p}^2}{m^2}+i\epsilon\right)^{(d-7)/2}\delta^{(d-5)/2}\Gamma\left(\frac{3-d}{2}\right){}_2F_1\left(\frac{d-3}{2},\frac{5-d}{2},\frac{d-1}{2},-\frac{1}{\delta}\right). \tag{4.19}$$

The hypergeometric function ${}_2F_1(a,b,c;\xi)$ may be expanded in powers of δ, giving

$$(a)=c_1\frac{(d-3)g^2m^{d-2}}{3(d-4)2^{d-2}\mathbf{p}^2\pi^{(d-1)/2}}u_h^\dagger\boldsymbol{\sigma}^i v_h\left(-\frac{\mathbf{p}^2}{m}+i\epsilon\right)^{(d-3)/2}\Gamma\left(\frac{3-d}{2}\right)\left[1-\left(2-\frac{d}{2}\right)\delta\right.$$
$$\left.+\frac{(d-4)\Gamma(4-d)\Gamma[(d-3)/2]}{\Gamma[5-d)/2}\right]\delta^{d-4}+O(\delta^2). \tag{4.20}$$

Evaluating this graph with the leading order on-shell condition, $\delta=0$, the $O(\delta^{d-4})$ term vanishes as long as $\mathrm{Re}(d)>4$. In order to have this result remain as the leading term in the expansion away from $\delta=0$ so that power counting is retained, the

δ^{d-4} term must be evaluated in dimensional regularization as a formal power series, $f(\delta)=f(0)+\delta f'(0)+\cdots$. In this case, each term in the expansion vanishes for sufficiently large $\mathrm{Re}(d)$, so the entire series vanishes in dimensional regularization. The final result for this graph near $d=4$ is, therefore,

$$(a)=c_1\frac{g^2}{12\pi^2}u_h^\dagger\boldsymbol{\sigma}^i v_h\frac{m}{|\mathbf{p}|}\left[\pi^2+i\pi\left(\gamma_E+\frac{2}{d-4}+\ln\frac{\mathbf{p}^2}{\pi\mu^2}+i\delta\right)+O(\delta^2)\right], \tag{4.21}$$

which, for $\delta=-\mathbf{p}^2/4m^2$, gives

$$(a)=c_1\frac{g^2}{12\pi^2}u_h^\dagger\boldsymbol{\sigma}^i v_h\left\{\frac{m}{|\mathbf{p}|}\left[\pi^2+i\pi\left(\gamma_E+\frac{2}{d-4}+\ln\frac{\mathbf{p}^2}{\pi\mu^2}\right)\right]-\frac{i\pi}{4}\frac{|\mathbf{p}|}{m}\right\}. \tag{4.22}$$

This reproduces the $O(1/v)$ term in the full amplitude.

As discussed above, there are no graphs at $O(g^2v^{2n})$ in NRQCD from radiation gluon loops. At $O(g^2v)$ there are contributions from the leading relativistic corrections to Coulomb scattering. In addition, since Coulomb exchange scales as v^{-1}, the dressing of $\mathbf{O}_2$ with a single A_P^0 exchange also contributes at $O(g^2v)$. A_P^i exchange contributes both via the $\mathbf{p}\cdot\mathbf{A}$ coupling [Fig. 5(b)]

$$(b)=c_1\frac{g^2}{12\pi^2}u_h^\dagger\boldsymbol{\sigma}^i v_h\frac{|\mathbf{p}|}{m}\left[\pi^2+i\pi\left(\gamma_E-1+\frac{2}{d-4}+\ln\frac{\mathbf{p}^2}{\pi\mu^2}\right)\right] \tag{4.23}$$

and the Fermi coupling

$$(c)=c_1\frac{g^2}{12\pi^2}\left(u_h^\dagger\boldsymbol{\sigma}^i v_h\frac{|\mathbf{p}|}{m}+\frac{m}{|\mathbf{p}|}\frac{u_h^\dagger\mathbf{p}\cdot\boldsymbol{\sigma}\mathbf{p}^i v_h}{m^2}\right)\left(-\frac{i\pi}{2}\right). \tag{4.24}$$

Coulomb exchange is corrected by the Darwin vertex,

$$(d)=c_1\frac{g^2}{12\pi^2}u_h^\dagger\boldsymbol{\sigma}^i v_h\frac{|\mathbf{p}|}{m}(-i\pi), \tag{4.25}$$

while the spin-orbit coupling does not contribute. The relativistic corrections to the quark and antiquark propagators give

$$(e)=c_1\frac{g^2}{12\pi^2}u_h^\dagger\boldsymbol{\sigma}^i v_h\frac{|\mathbf{p}|}{2m}\left[\pi^2+i\pi\left(\gamma_E+\frac{1}{2}+\frac{2}{d-4}+\ln\frac{\mathbf{p}^2}{\pi\mu^2}\right)\right], \tag{4.26}$$

and finally, the one-loop correction to $\mathbf{O}_2$ in Fig. 5(f) gives

$$(f)=-c_2\frac{g^2}{12\pi^2}\frac{u_h^\dagger\mathbf{p}\cdot\boldsymbol{\sigma}\mathbf{p}^i v_h}{m^2}\frac{m}{2|\mathbf{p}|}\left[\pi^2+i\pi\left(\gamma_E-3+\frac{2}{d-4}+\ln\frac{\mathbf{p}^2}{\pi\mu^2}\right)\right]-c_2\frac{g^2}{12\pi^2}u_h^\dagger\boldsymbol{\sigma}^i v_h\frac{|\mathbf{p}|}{m}\left(\frac{i\pi}{2}\right). \tag{4.27}$$

Combining these results gives

$$\begin{aligned}i\mathcal{A}_{\mathrm{NRQCD}}=&\,c_1u_h^\dagger\boldsymbol{\sigma}^i v_h-\frac{c_2}{2m^2}u_h^\dagger\mathbf{p}\cdot\boldsymbol{\sigma}\mathbf{p}^i v_h+\frac{g^2}{12\pi^2}u_h^\dagger\boldsymbol{\sigma}^i v_h\left(c_1\frac{m}{|\mathbf{p}|}\left[\pi^2+i\pi\left(\gamma_E+\frac{2}{d-4}+\ln\frac{\mathbf{p}^2}{\pi\mu^2}\right)\right]\right.\\&\left.+\frac{3|\mathbf{p}|}{2m}\left\{c_1\left[\pi^2+i\pi\left(\gamma_E-\frac{5}{3}+\frac{2}{d-4}+\ln\frac{\mathbf{p}^2}{\pi\mu^2}\right)\right]-\frac{i\pi}{3}c_2\right\}\right)+\frac{g^2}{12\pi^2}\frac{u_h^\dagger\mathbf{p}\cdot\boldsymbol{\sigma}\mathbf{p}^i v_h}{2m^2}\\&\times\left(\frac{m}{|\mathbf{p}|}\left\{-c_2\left[\pi^2+i\pi\left(\gamma_E-3+\frac{2}{d-4}+\ln\frac{\mathbf{p}^2}{\pi\mu^2}\right)\right]-i\pi c_1\right\}\right).\end{aligned} \tag{4.28}$$

As required, all the nonanalytic dependence on the external momentum cancels in the matching. This result is also gauge independent.

Comparing Eqs. (4.16) and (4.28) gives the coefficients c_1-c_3 (regulating the low-energy theory as usual in $\overline{\rm MS}$) to $O(g^2)$:

$$c_1=1-\frac{8\alpha_s}{3\pi}+O(\alpha_s^2),$$

$$c_2=1-\frac{4\alpha_s}{3\pi}+O(\alpha_s^2),$$

$$c_3=-\frac{\alpha_s}{9\pi}\left(\frac{2}{3}-8\ln\frac{m^2}{\mu^2}\right)+O(\alpha_s^2). \tag{4.29}$$

The result for c_1 reproduces the familiar short-distance correction to $e^+e^-\to q\bar{q}$ near threshold [16], whereas c_2 and c_3 generalize this to $O(v^2)$. Note that the bare c_1 is finite while the bare c_3 is divergent. This reflects the fact that there are no infrared or ultraviolet divergences in the amplitude at $O(v^0)$ since the quarks are in a color singlet state, and therefore cannot radiate a gluon at leading order in the multipole expansion.

The major result of this section is that the nonanalytic dependence on the external momenta in the QCD amplitude is exactly reproduced in NRQCD. This provides a nontrivial check of the consistency of this approach beyond leading order. However, only for values of the coupling and external momenta such that $\alpha_s \ll v \ll 1$ does the tree-level matching of $\mathbf{O}_2$ and $\mathbf{O}_3$ dominate the two-loop matching of $\mathbf{O}_1$. For scattering states closer to threshold (as well as for bound states) where $v\lesssim\alpha_s$, ladder graphs containing potential gluons must be summed to all orders via the Schrödinger equation. In this case, graphs containing a single insertion of the tree-level matching of $\mathbf{O}_2$, the two-loop matching of $\mathbf{O}_2$ and the tree-level matching of $\mathbf{O}_1$ combined with a single higher order potential contribution are equally important. In this region, the one-loop matching to $\mathbf{O}_2$ and $\mathbf{O}_3$ that are presented here are the same order as the three loop matching to $\mathbf{O}_1$.

V. CONCLUSIONS

In this paper we have presented a power counting scheme for nonrelativistic effective theories that allows for a systematic calculation of subleading effects. A systematic v counting scheme simplifies the calculation of relativistic corrections to QCD processes such as quarkonium production and decay. As it is usually presented, NRQCD does not have manifest v power counting in the Lagrangian, nor is v power counting preserved by loop graphs either in dimensional regularization or with a momentum cutoff. While there is nothing in principle wrong with this, it makes calculating matching conditions somewhat awkward, since the matching conditions for any given operator will change by $O(1)$ when higher order operators are included in the Lagrangian.

Velocity power counting is only preserved by loop graphs in dimensionally regulated NRQCD when gluons contributing to potential scattering are treated separately from on-shell gluons. This was accomplished in this paper by introducing two distinct gluon fields in the effective theory. Potential gluons propagate instantaneously and give rise to the QCD potential, whereas radiation gluons do not contribute to potential scattering, but correspond to on-shell gluons. The power counting is manifest in the Lagrangian when space, time and the fields are rescaled for the potential fields as discussed in Ref. [10] and for radiation fields as discussed in Ref. [11]. As shown in Ref. [11], under this rescaling radiation fields couple to fermions via the multipole expansion. Separating these gluon modes realizes at the level of the Lagrangian the separation advocated for NRQED in Ref. [9]. Under this rescaling v power counting is manifest in any gauge, not just Coulomb gauge, and also holds for nongauge interactions.

The infrared divergences arising in fermion-antifermion production in Yukawa theory at order v^{-1} and v^0 were shown to be reproduced in the nonrelativistic effective theory only when both potential and radiation scalars were included, and the matching conditions at that order were shown to be analytic in the external momentum. Finally, the matching conditions for quark-antiquark production by an external vector current were computed in NRQCD to $O(g^2v^2)$.

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[1] W. E. Caswell and G. P. Lepage, Phys. Lett. **167B**, 437 (1986); G. T. Bodwin, E. Braaten, and G. P. Lepage, Phys. Rev. D **51**, 1125 (1995); **55**, 5853(E) (1997).

[2] N. Isgur and M. B. Wise, Phys. Lett. B **232**, 113 (1989); **237**, 527 (1990); H. Georgi, *ibid.* **240**, 447 (1990); E. Eichten and B. Hill, *ibid.* **243**, 427 (1990).

[3] V. A. Novikov *et al.*, Phys. Rep. **41**, 1 (1978); M. B. Voloshin, Int. J. Mod. Phys. A **10**, 2865 (1995).

[4] W. E. Caswell and G. P. Lepage, Phys. Rev. A **20**, 36 (1979); P. Labelle, G. P. Lepage, and U. Magnea, Phys. Rev. Lett. **72**, 2006 (1994).

[5] S. Weinberg, Phys. Lett. **51B**, 288 (1990); Nucl. Phys. **B363**, 3 (1991); Phys. Lett. B **295**, 114 (1992); C. Ordonez and U. van Kolck, *ibid.* **291**, 459 (1992): C. Ordonez, L. Ray, and U. van Kolck, Phys. Rev. Lett. **72**, 1982 (1994); Phys. Rev. C **53**, 2086 (1996); U. van Kolck, *ibid.* **49**, 2932 (1994); D. B. Kaplan, M. J. Savage, and M. B. Wise, Nucl. Phys. **B478**, 629 (1996).

[6] A. Falk, H. Georgi, B. Grinstein, and M. B. Wise, Nucl. Phys. **B343**, 1 (1990); A. Falk and B. Grinstein, Phys. Lett. B **249**, 314 (1990); M. Neubert, Nucl. Phys. **B371**, 149 (1992); A. F. Falk and M. Neubert, Phys. Rev. D **47**, 2965 (1993).

[7] B. Grinstein, W. Kilian, T. Mannel, and M. B. Wise, Nucl. Phys. **B363**, 19 (1991).
[8] G. P. Lepage *et al.*, Phys. Rev. D **46**, 4052 (1992).
[9] P. Labelle, McGill Report No. McGill/96-33 1996, hep-ph/9608491.
[10] M. Luke and A. V. Manohar, Phys. Rev. D **55**, 4129 (1997).
[11] B. Grinstein and I. Z. Rothstein, hep-ph/9703298, 1997.
[12] A. Falk, H. Georgi, B. Grinstein, and M. B. Wise, Nucl. Phys. **B343**, 1 (1990); E. Jenkins and M. J. Savage, Phys. Lett. B **281**, 331 (1992); J. Goity, Phys. Rev. D **46**, 3929 (1992).
[13] A. Hoang, hep-ph/9702331 1997; Phys. Rev. D **56**, 5851 (1997).
[14] A. V. Manohar, Phys. Rev. D **56**, 230 (1997).
[15] H. Georgi, Nucl. Phys. **B361**, 339 (1991).
[16] R. Karplus and A. Klein, Phys. Rev. **87**, 848 (1952); R. Barbieri, R. Gatto, R. Kögerler, and Z. Kunzst, Phys. Lett. **57B**, 455 (1975).