PHYSICAL REVIEW D VOLUME 57, NUMBER 7 1 APRIL 1998

Two loops calculation in chiral perturbation theory and the unitarization program of current algebra

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(Received 29 August 1997; published 18 February 1998)

In this paper we compare the two loop chiral perturbation theory calculation of pion-pion scattering with the unitarity second-order correction to the current algebra soft-pion theorem. It is shown that both methods lead to the same analytic structure for the scattering amplitude. [S0556-2821(98)06105-0]

PACS number(s): 12.39.Fe, 11.30.Rd, 13.75.Lb

I. INTRODUCTION

In the early 1960s, many results for low-energy meson physics were derived from the assumptions of a local chiral $SU(2)\times SU(2)$ algebra of vector and axial vector current densities together with the partial conservation of axial vector current (PCAC) hypothesis relating the derivative of the axial vector current to the pion field. By itself, the PCAC relation cannot help one to extend the applicability of the current algebra method to the energy corresponding to meson-meson resonances. Nevertheless, a method was invented to obtain results for a process in which mesons are not ''soft particles.'' We are referring to the hard meson methods of current algebra [1]. Even ignoring the underlying theory, chiral current algebra implies a set of Ward identities and the method consists in solving the system of Ward identities under the general principles of analyticity, crossing, and elastic unitarity.

In 1979, Weinberg suggested that it is possible to summarize these previous results in a phenomenological Lagrangian which incorporates all the constraints coming from chiral symmetry of the underlying theory [2]. In a set of very important and fundamental papers, Gasser and Leutwyler developed chiral perturbation theory (ChPT), which allows one to compute many different Green functions involving low-energy pions [3]. One of the main obstacles for applications of ChPT to high energies lies in the issue of unitarity. Several methods were proposed to extend the range of energies where ChPT could be applied. These methods include the Padé expansion [4], the inverse amplitude method [5], and the introduction of fields describing resonances [6].

Let us remember, however, that, in order to treat the pion-pion scattering amplitude obtained by the hard meson method based on the Ward identity technique, one of us has introduced the constraints of elastic unitarity for partial waves [7]. We will call this approach the *unitarization program of current algebra* (UPCA). It was applied to obtain first-order corrections (QU1) to the soft-pion $\pi\pi$ Weinberg amplitude [8], as well as to calculate second-order corrections (QU2) to the referred amplitude [9].

A ChPT calculation of pion-pion scattering to one loop was performed in Ref. [3] and, only recently, a two loops calculation appeared in the literature [10]. Our aim is to compare UPCA and ChPT calculations. In a previous paper, we showed that one loop ChPT is equivalent to QU1 [9] and in the present paper, we will compare the QU2 amplitude with the ChPT Lagrangian the two loops calculation performed in Ref. [10]. We conclude that, as conjectured by one of us [9], the two approaches are equivalent. In Sec. II we will recall the comparison at the one loop level, and in Sec. III this comparison is extended to the two loops case. In Sec. IV we present the conclusions.

II. FIRST-ORDER-CORRECTED UPCA AMPLITUDE AND ONE LOOP ChPT

Let us remember the main points of the UPCA. The starting point in our derivation was an exact hard-pion expression for the correlation function of four currents, with the quantum numbers of the pion, in terms of three- and two-point functions. From this expression, by using vertex and propagator estimates, we could reobtain the so-called soft-pion Weinberg amplitude: namely,

$$A^{\mathrm{CA}}(s,t,u)=\frac{1}{F_\pi^2}(s-m^2). \tag{1}$$

The remainder of the amplitude reflects the difference between soft- and hard-meson results. In the UPCA one estimates the behavior of form factors and propagators at low energies and assumes that, for instance, for small values of its argument, the scalar pion form factor F_I is of the same order of magnitude as current algebra amplitudes near threshold. This can be obtained by setting, for $x\simeq m^2$, $F_I(x)\simeq 1+f_I^{(1)}(x)+O(\epsilon^2)$ for $I=0$ and 2, where the superscript (1) denotes the order $\epsilon=m^2/M^2$, M being of order of magnitude of vector meson masses.

To construct a unitarized amplitude, we observed that current algebra gives real partial waves. The unitarization method must provide an imaginary part to the corrected partial wave. Thus, at the first order of the calculation, by the optical theorem, one must have

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$$\mathrm{Im}\, t_{\ell I}^{(1)}(s)=\frac{1}{32\pi}\sigma(s)t_{\ell I}^{\mathrm{CA}}(s)^2,$$

where $t_{\ell I}^{\mathrm{CA}}$ is the soft-pion ℓ partial wave, isospin I, Weinberg amplitude obtained from Eq. (1) and

$$\sigma(s)=\sqrt{\frac{s-4m^2}{s}}.$$

In the program, we work with exact total amplitude expression which follows from the Ward identities, and we use the implications of elastic unitarity for form factors and propagators in a peculiar way. For instance, from the relations valid for the scalar form factor and scalar propagator: namely,

$$\mathrm{Im}\, F_I(s)=\frac{1}{32\pi}\sigma(s)T_I^*(s)F_I(s) \quad \text{and}$$

$$\mathrm{Im}\, \Delta_I(s)=\frac{1}{32\pi}\sigma(s)|F_I(s)|^2, \quad I=0 \text{ and } 2, \tag{2}$$

we obtain, within the first order of the approximation,

$$\mathrm{Im}\, f_0^{(1)}(s)=\frac{1}{32\pi}\sigma(s)t_{00}^{\mathrm{CA}}(s) \quad \text{and } \mathrm{Im}\delta_I^{(1)}(s)=\frac{1}{32\pi}\sigma(s),$$

where t_{00}^{CA} is the current algebra isospin zero S-wave $\pi\pi$ amplitude,

$$t_{00}^{\mathrm{CA}}(s)=\frac{1}{F_\pi^2}(2s-m^2).$$

Considering the known imaginary part of each function entering into the amplitude, the method consists in obtaining their real parts by the dispersion relation technique. To converge, dispersion integrals need subtraction which are model-free parameters. They can be fixed by fitting experimental data.

In this way, the first-order-corrected amplitude QU1, derived in the context of the UPCA [7], can be written in the following form:

$$F_\pi^4 A_{\mathrm{QU1}}^{(1)}(s,t,u)=\tfrac{1}{3}(2s-m^2)\Phi_0^{(1)}(s)-\tfrac{1}{3}(2m^2-s)\Phi_2^{(1)}(s)$$
$$+\tfrac{1}{2}\xi_1(s-2M^2)^2+[\tfrac{1}{2}(2m^2-t)\Phi_2^{(1)}(t)+(s$$
$$-u)\Phi_1^{(1)}(t)-\tfrac{1}{4}\xi_2(t-2M^2)^2+(t\leftrightarrow u)],$$

with

$$\Phi_0^{(1)}(x)=(2x-m^2)[g(x)+\alpha_0],$$

$$\Phi_2^{(1)}(x)=(2m^2-x)[g(x)+\alpha_2],$$

$$\Phi_1^{(1)}(x)=\tfrac{1}{3}(x-4m^2)g(x)-\tfrac{1}{3}[2x\alpha_1-4m^2g(0)],$$

where

$$32\pi^2 g(s)=(s-4m^2)\int_{4m^2}^{\infty}dx\frac{\sigma(x)}{(x-4m^2)(x-s)}$$
$$=\sigma(s)\ln\frac{\sigma(s)-1}{\sigma(s)+1}. \tag{3}$$

On the other hand, the one loop elastic pion scattering obtained from the ChPT Lagrangian [3] is

$$F_\pi^4 A_{\mathrm{ChPT}}^{(1)}(s,t,u)=\tfrac{1}{2}(s^2-m^4)\bar{J}(s)+\tfrac{1}{6}\{[t(t-u)-2m^2t+4m^2u-2m^4]\bar{J}(t)+(t\leftrightarrow u)\}+[2(\bar{\ell}_1-4/3)(s-2m^2)^2$$
$$+(\bar{\ell}_2-5/6)(s^2+(t-u)^2)+12m^2s(\bar{\ell}_4-1)-3(\bar{\ell}_3+4\bar{\ell}_4-5)m^4]/96\pi^2. \tag{4}$$

We can identify the function $\bar{J}(x)$ with $2[g(x)-g(0)]$, and we have verified that the polynomial coefficients of these functions are the same. We then conclude that the above amplitude has the same analytical structure as QU1. Each approach has its free parameters: the model-free parameters of QU1 are ξ's and α's linear combinations, and the free parameters of ChPT are $\bar{\ell}$'s linear combinations. In the UPCA the free parameters are subtraction constants, inherent to the dispersion relation technique, and in ChPT they come from tadpole graphs and tree graphs of order O(p^4). From this comparison we showed, in a recent letter, that one-loop ChPT amplitude can fit experimental S and P waves up to the resonance region by adjusting only two parameters, namely $\bar{\ell}_1$ and $\bar{\ell}_2$ [11].

In Sec. III, we will compare the second-order-corrected UPCA amplitude, QU2, with the two loops calculation [10]. To do this, we will need a one loop partial wave corresponding to the ChPT amplitude given above. For this, one expands the combinations with definite isospin in the s channel into partial waves:

$$T_I(s,t)=32\pi\sum_{\ell=0}^{\infty}(2\ell+1)P_\ell(\cos\theta)t_{\ell I}^{(1)}(s),$$

$$I=0,1, \quad \text{and } 2.$$

Using Eq. (4), the resulting one loop P-wave amplitude is

$$t_{11}^{(1)}(s)=\frac{1}{18F_\pi^4}(s-4m^2)^2\bar{J}(s)+\left\{\frac{m^4}{8}\left(\frac{s^2}{2}-\frac{13}{16}sm^2-m^4\right)\frac{L^2(s)}{(s-4m^2)^2}+\left(\frac{s^3}{288}-\frac{1}{18}s^2m^2+\frac{1}{4}sm^4-\frac{m^6}{8}\right)\frac{L(s)}{\sigma(s-4m^2)}\right.$$
$$\left.-\frac{1}{864(s-4m^2)}(s^3+37s^2m^2-149sm^4+120m^6)+\frac{s-4m^2}{288}[(2\bar{\ell}_2-2\bar{\ell}_1-1)s+8m^2]\right\}\frac{1}{F_\pi^4\pi^2},$$

the isospin $I=0$ one loop S wave is

$$t_{00}^{(1)}(s)=\frac{1}{2F_\pi^4}(2s-m^2)^2\bar{J}(s)+\frac{1}{F_\pi^4\pi^2}\left\{\frac{m^4}{8}\left(s-\frac{25}{6}m^2\right)\frac{L^2(s)}{(s-4m^2)}-\left(\frac{7}{144}s^2-\frac{5}{18}sm^2+\frac{25}{48}m^4\right)\frac{L(s)}{\sigma}+\left(\frac{11}{144}\bar{\ell}_1+\frac{7}{72}\bar{\ell}_2-\frac{7}{96}\right)s^2\right.$$
$$\left.-\left(\frac{5}{18}(\bar{\ell}_1+\bar{\ell}_2)-\frac{1}{4}\bar{\ell}_4-\frac{481}{432}\right)sm^2+\frac{m^4}{18}\left(\frac{11}{2}\bar{\ell}_1+7\bar{\ell}_2-\frac{45}{16}\bar{\ell}_3-\frac{9}{4}\bar{\ell}_4+\frac{355}{16}\right)\right\},$$

and the isospin $I=2$ one loop S wave is

$$t_{02}^{(1)}=\frac{1}{2F_\pi^4}(2m^2-s)^2\bar{J}(s)+\frac{1}{F_\pi^4\pi^2}\left\{-\frac{m^4}{16}\left(s+\frac{m^2}{3}\right)\frac{L^2(s)}{(s-4m^2)}-\left(\frac{11}{288}s^2-\frac{1}{9}sm^2+\frac{m^4}{48}\right)\frac{L(s)}{\sigma}-\left(\frac{1}{72}\bar{\ell}_1+\frac{1}{18}\bar{\ell}_2+\frac{5}{192}\right)s^2\right.$$
$$\left.-\left(\frac{1}{36}\bar{\ell}_1+\frac{7}{36}\bar{\ell}_2+\frac{1}{8}\bar{\ell}_4-\frac{527}{864}\right)sm^2+\left(\frac{1}{18}\bar{\ell}_1+\frac{2}{9}\bar{\ell}_2-\frac{1}{16}\bar{\ell}_3+\frac{1}{4}\bar{\ell}_4-\frac{1}{9}\right)m^4\right\}. \quad (5)$$

In these expressions we have included the contributions from $\bar{J}(4m^2)$, which is lacking in the expressions of partial waves given in Sec. 2 of Ref. [11], and we have corrected an overall sign.

III. SECOND-ORDER-CORRECTED UPCA AMPLITUDE AND TWO LOOPS ChPT

We have shown that the first-order correction to the soft-pion amplitude (QU1) is equivalent to one-loop ChPT scattering amplitude and, in addition, have given the tools for constructing the next order unitarity corrections (QU2) [9]. Formula (3.10) of Ref. [9] can be written as

$$F_\pi^6A_{\mathrm{QU2}}^{(2)}(s,t,u)=\tfrac{1}{3}(2s-m^2)\Phi_0^{(2)}(s)$$
$$-\tfrac{1}{3}(2m^2-s)\Phi_2^{(2)}(s)[\tfrac{1}{2}(2m^2-t)\Phi_2^{(2)}(t)$$
$$+(s-u)\Phi_1^{(2)}(t)+(t\leftrightarrow u)].$$

We can relate the above expression with formula (3) obtained as a consequence of the Goldstone nature of the pion [12]: namely,

$$F_\pi^4W_I(s)=\frac{1}{32\pi}t_I^{\mathrm{CA}}(s)\Phi_I^{(2)}(s)\qquad \text{for } I=0,2$$

$$\text{and}\quad F_\pi^4W_1(s)=\frac{1}{48\pi}\Phi_1^{(2)}(s).$$

The functions $W_I(s)$ are analytic except for a cut singularity at $s\geqslant 4m^2$. Their discontinuities are directly related to the discontinuities of the functions $\Phi_I(s)$.

We would like to emphasize that the general structure of the UPCA solution comes from the Ward identity method, and the hard meson technique implies that the amplitude is written in terms of form factors and propagators. We stress that the UPCA is based on the *implications of elastic unitarity relations for form factors and propagators, and not for partial waves themselves*.

The consequences of using Eq. (2), for instance, for scalar form factors and propagators, to the second order of the approximation, are

$$\mathrm{Im}\ f_I^{(2)}(s)=\frac{1}{32\pi}\sigma(s)[\mathrm{Re}\ t_I^{(1)}(s)+t^{\mathrm{CA}}(s)\ \mathrm{Re}\ f_I^{(1)}(s)],$$

$$\mathrm{Im}\ \delta_I^{(2)}(s)=\frac{1}{32\pi}2\ \mathrm{Re}\ f_I^{(1)}(s),$$

with $I=0$ and 2. The vector form factor and vector propagator are obtained in a similar way. The functions $\Phi_I^{(2)}(s)$, constructed from form factors and propagators, are then discontinued on the right-hand cut as follows:

$$\mathrm{Im}\Phi_I^{(2)}(x)=\frac{1}{32\pi}\sigma(x)2\ \mathrm{Re}\ t_I^{(1)}(x),$$

$$\text{for}\quad I=0,\ 1,\text{ and }\ 2,$$

and Re $t^{(1)}$ stands for the real parts of the functions in Eq. (5). Using dispersion relation technique, we obtain

$$\Phi_I^{(2)}(s)=p_I(s)Z(s)+q_I(s)G(s)+r_I(s)g(s)+P_I(s)$$

$$\text{for}\quad I=0,\ 1,\text{ and }\ 2. \qquad (6)$$

The polynomials $p_I(s)$, $q_I(s)$, $r_I(s)$, and $P_I(s)$, for each value of total isospin I, are given in the Appendix. The function $g(s)$ is given in Eq. (3), and

$$32\pi^2 G(s)=(s-4m^2)\int_{4m^2}^{\infty}dx\frac{\sigma(x)\ \mathrm{Re}\ g(x)}{(x-4m^2)(x-s)}$$

$$=\frac{1}{64\pi^2}\sigma^2(s)L^2(s)+\frac{\pi^2}{3}\sigma^2(s),$$

$$(32\pi^2)^2 Z(s)=(s-4m^2)\int_{4m^2}^{\infty}dx\frac{\sigma(x)}{(x-4m^2)(x-s)}$$

$$\times[L(x)+i\pi]^2=\frac{1}{3}L(s)[L^2(s)+\pi^2].$$

where

$$L(s)=\ln\frac{\sigma(s)-1}{\sigma(s)+1}.$$

The amplitude $A^{(2)}_{\mathrm{QU2}}$ is then written in terms of powers of $L(s)$ and $L(t)$, and contains the free parameters $\overline{\ell}_1$ and $\overline{\ell}_2$ and the values of the subtraction constants (Φ_I and its derivatives at $s=4m^2$).

On the other hand, the ChPT amplitude calculated at the two-loop level [10] with $m_\pi=1$ is

$$F_\pi^6 A^{(2)}_{\mathrm{ChPT}}(s,t,u)=F^{(2)}(s)+G^{(2)}(s,t)+G^{(2)}(s,u),$$

with

$$F^{(2)}(s)=\overline{J}(s)\left\{\frac{1}{16\pi^2}\left(\frac{503}{108}s^3-\frac{929}{54}s^2+\frac{887}{27}s-\frac{140}{9}\right)+b_1(4s-3)+b_2(s^2+4s-4)+\frac{1}{3}b_3(8s^3-21s^2+48s-32)\right.$$
$$\left.+\frac{1}{3}b_4(16s^3-71s^2+112s-48)\right\}+\frac{1}{18}K_1(s)\left[20s^3-19s^2+210s-135-\frac{9}{16}\pi^2(s-4)\right]+\frac{1}{32}K_2(s)(s\pi^2-24)$$
$$+\frac{1}{9}K_3(s)(3s^2-17s+9), \tag{7a}$$

$$G^{(2)}(s,t)=\overline{J}(t)\left\{\frac{1}{16\pi^2}\left[\frac{412}{27}-\frac{s}{54}(t^2+5t+159)-t\left(\frac{267}{216}t^2-\frac{727}{108}t+\frac{1571}{108}\right)\right]+b_1(2-t)+\frac{1}{3}b_2(t-4)(t^2+s-5)\right.$$
$$\left.-\frac{1}{6}b_3(t-4)^2(3t+2s-8)+\frac{1}{6}b_4[2s(3t-4)(t-4)-32t+40t^2-11t^3]\right\}$$
$$+\frac{1}{36}K_1(t)\left[174+8s-10t^3+72t^2-185t-\frac{1}{16}\pi^2(t-4)(3s-8)\right]+\frac{1}{9}K_2(t)\left[1+4s+\frac{1}{64}\pi^2 t(3s-8)\right]$$
$$+\frac{1}{9}K_3(t)(1+3st-s+3t^2-9t)+\frac{5}{3}K_4(t)(4-2s-t). \tag{7b}$$

In this expression $\overline{J}(s)=2[g(s)-g(0)]$, and

$$K_1=\frac{L^2(s)}{(16\pi^2)^2},\quad (16\pi^2)^2K_2=\sigma^2L^2(s)-4,$$

$$(16\pi^2)^2K_3=\frac{1}{s\sigma}L^3(s)+\frac{\pi^2}{s\sigma}L(s)-\frac{\pi^2}{2},$$

$$K_4=\frac{1}{s\sigma^2}\left(\frac{1}{2}K_1+\frac{1}{3}K_3+\frac{1}{16\pi^2}\overline{J}+s\frac{\pi^2-6}{192\pi^2}\right).$$

Our strategy to compare $A^{(2)}_{\mathrm{QU2}}$ with $A^{(2)}_{\mathrm{ChPT}}$ was to expand them in terms of $L(s)$ and $L(t)$, and then to confront their coefficients. *We have checked that they are the same*. With respect to the polynomials, we also realize that the structures are the same, but clearly the coefficients are different because they have different origins. These polynomials include the model-free parameters to be used in order to fit the available experimental data [13].

IV. CONCLUSIONS

Our aim is to compare chiral perturbation theory (ChPT) calculations with the unitarization program of current algebra (UPCA). In previous works we have compared one loop ChPT with the first order corrected by UPCA results for pion-pion [9] and for kaon-pion scattering [14], and we have concluded that they lead to *the same analytical structure for the amplitudes*.

The two loops calculation of pion-pion scattering only recently appeared in the literature [10]. However, the tools for constructing second-order corrections for the soft-pion current algebra result were presented more than ten years ago [9]. In the present paper we compared these results and we showed that, as conjectured [9], they have *the same analyti-*

cal structure. We will shortly present the tools for constructing a next-order UPCA correction to kaon-pion scattering. ChPT to two loops for $K\pi$ are not yet available, but we expect that they will also be equivalent to the UPCA result.

In fact, the equivalence between the two approaches was expected. In the hard-meson method one starts from the chiral symmetric Ward identity *exact* result for the correlation function of four currents carrying pion quantum numbers. On the other hand, ChPT describes the low-energy dynamics of fields realizing nonlinearly chiral symmetry. Our unitarization program is based in the principles of analyticity, crossing, and elastic unitarity, which in turn are inherent to a field theory such as ChPT.

In the framework of the generalized chiral perturbation theory, it was shown how to implement elastic unitarity starting from the one loop partial waves. That procedure leads to an equivalent $O(p^6)$ ChPT amplitude [12]. However, the main difference from our unitarization procedure is that it uses the consequences of elastic unitarity for the form factors and propagators *rather than for the amplitudes themselves*.

Despite the fact that ChPT is a well-established low-energy effective theory for meson processes, we have shown here that the UPCA is a suitable alternative. One obstacle related to these two approaches is how to fix the free parameters which, in principle, are related to the parameters of the fundamental theory. In the ChPT context, we have shown that the one loop parameters $\bar{\ell}_1$ and $\bar{\ell}_2$ can be fixed by fitting S- and P-wave $\pi\pi$ phase shifts [11]. The final expression of the pion-pion ChPT amplitude up to two loops diagrams has six parameters. However, the D-wave amplitude from $A^{(2)}$ picks up an imaginary part for $s \geqslant 4m^2$, and we claim that the global fit of S-, P- and D-wave phase shifts will allow one to fix the new free parameters [15].

APPENDIX

The polynomials multiplying the functions $g(s)$, $G(s)$, and $Z(s)$ in Eq. (6) of second-order-corrected UPCA amplitudes are

$$\pi^2 p_0(s) = \left(\frac{11}{72}\bar{\ell}_1 + \frac{7}{36}\bar{\ell}_2 + \frac{17}{48}\right)s^2 - \left(\frac{5}{9}\bar{\ell}_1 + \frac{5}{9}\bar{\ell}_2 - \frac{1}{2}\bar{\ell}_4 - \frac{95}{72}\right)s + \frac{11}{18}\bar{\ell}_1 + \frac{7}{9}\bar{\ell}_2 - \frac{45}{8}\bar{\ell}_3 - \frac{9}{2}\bar{\ell}_4 + \frac{373}{144},$$

$$\pi^2 p_1(s) = \frac{1}{432}\frac{1}{s-4}\{(6\bar{\ell}_1 - 6\bar{\ell}_2 - 2)s^3 - (48\bar{\ell}_1 - 48\bar{\ell}_2 - 61)s^2 + (96\bar{\ell}_1 - 96\bar{\ell}_2 - 197)s + 120\},$$

$$\pi^2 p_2(s) = \left(\frac{1}{36}\bar{\ell}_1 + \frac{1}{9}\bar{\ell}_2 + \frac{17}{96}\right)s^2 - \left(\frac{1}{18}\bar{\ell}_1 + \frac{7}{18}\bar{\ell}_2 + \frac{1}{4}\bar{\ell}_4 + \frac{61}{144}\right)s + \frac{1}{9}\bar{\ell}_1 + \frac{4}{9}\bar{\ell}_2 - \frac{1}{8}\bar{\ell}_3 + \frac{1}{2}\bar{\ell}_4 + \frac{5}{18},$$

$$q_0(s) = -\frac{2}{9(s-4)}(50s^3 - 260s^2 + 303s - 36),$$

$$q_1(s) = -\frac{8}{9(s-4)^2}(6s^2 - 55s + 64),$$

$$q_2(s) = -\frac{4}{9(s-4)}(10s^3 - 52s^2 + 93s - 72),$$

$$r_0(s) = \frac{4}{3(s-4)}(6s - 25),$$

$$r_1(s) = \frac{4}{3(s-4)^2}(3s^2 - 13s - 6),$$

$$r_2(s) = -\frac{4}{3(s-4)}(3s + 1).$$

The polynomial part of QU2, P_I, in Eq. (6) is written as

$$P_I(s) = A_I s^2 + B_I s + C_I \quad \text{for } I = 0 \text{ and } 2,$$

$$(s-4)P_1 = A_1 s^3 + B_1 s^2 + C_1 s + D_1,$$

with

$$\pi^4 A_0 = \frac{19}{2304}\bar{\ell}_1 + \frac{13}{1152}\bar{\ell}_2 + \frac{15}{1024}\bar{\ell}_3 + \frac{11}{768}\bar{\ell}_4 - \frac{21}{4096} + \frac{25\pi^2}{576} - \frac{3083\pi^4}{34560} + \frac{\pi^4}{2}\Phi_0''(4),$$

$$\pi^4 A_1 = -\frac{1}{1152}\bar{\ell}_1 + \frac{1}{1152}\bar{\ell}_2 - \frac{13}{55296} - \frac{\pi^2}{452} - \frac{13\pi^4}{30240} + \frac{\pi^4}{2}\Phi_1''(4),$$

$$\pi^4 A_2 = \frac{1}{576}\bar{\ell}_1 + \frac{7}{1152}\bar{\ell}_2 + \frac{1}{3072}\bar{\ell}_3 - \frac{1}{384}\bar{\ell}_4 + \frac{\pi^2}{576} - \frac{131\pi^4}{8640} + \frac{\pi^4}{2}\Phi_2''(4),$$

$$\pi^4 B_0 = -\frac{61}{1152}\bar{\ell}_1 - \frac{37}{576}\bar{\ell}_2 - \frac{105}{512}\bar{\ell}_3 - \frac{59}{384}\bar{\ell}_4 - \frac{17\pi^2}{48} + \frac{4331\pi^4}{8640} + \pi^4\Phi_0'(4) - 4\pi^4\Phi_0''(4),$$

$$\pi^4 B_1 = \frac{1}{96}\bar{\ell}_1 - \frac{1}{96}\bar{\ell}_2 - \frac{29}{4608} + \frac{35\pi^2}{288} + \frac{331\pi^4}{30240} + \pi^4\Phi_1''(4) - 6\pi^4\Phi_1'''(4),$$

$$\pi^4 B_2 = -\frac{5}{576}\bar{\ell}_1 - \frac{11}{288}\bar{\ell}_2 - \frac{7}{1536}\bar{\ell}_3 + \frac{5}{384}\bar{\ell}_4 - \frac{5\pi^2}{48} + \frac{409\pi^4}{4320} - \pi^4\Phi_2'(4) - 4\pi^4\Phi_2''(4),$$

$$\pi^4 C_0 = \frac{23}{288}\bar{\ell}_1 + \frac{11}{144}\bar{\ell}_2 + \frac{45}{128}\bar{\ell}_3 + \frac{37}{96}\bar{\ell}_4 + \frac{13\pi^2}{18} - \frac{421\pi^4}{720} + \pi^4\Phi_0(4) - 4\pi^4\Phi_0'(4) + 8\pi^4\Phi_0''(4),$$

$$\pi^4 C_1 = -\frac{1}{24}\bar{\ell}_1 + \frac{1}{24}\bar{\ell}_2 - \frac{67}{1152} - \frac{67\pi^2}{72} + \frac{53\pi^4}{2520} + \pi^4\Phi_1'(4) - 8\pi^2\Phi_1''(4) + 24\pi^4\Phi_1'''(4),$$

$$\pi^4 C_2 = \frac{1}{144}\bar{\ell}_1 + \frac{1}{18}\bar{\ell}_2 + \frac{5}{384}\bar{\ell}_3 - \frac{1}{96}\bar{\ell}_4 + \frac{7\pi^2}{18} - \frac{163\pi^4}{720} + \pi^4\Phi_2(4) - 4\Phi_2'(4) + 8\pi^4\Phi_2''(4),$$

$$\pi^4 D_1 = \frac{1}{18}\bar{\ell}_1 - \frac{1}{18}\bar{\ell}_2 - \frac{157}{1728} + \frac{52\pi^2}{27} - \frac{253\pi^4}{840}.$$

In order to compare the two approaches we have not included the dependence on $1/F_\pi^8$ in the parameters b_i. In this way the quantities b_i that we used in Eqs. (7) stand for

$$b_1 = 8\bar{\ell}_1 + 2\bar{\ell}_3 - 2\bar{\ell}_4 + \frac{1}{48\pi^2}\left(7\ln\frac{m_\pi}{\mu} + \frac{13}{6}\right),$$

$$b_2 = -8\bar{\ell}_1 + 2\bar{\ell}_4 - \frac{1}{12\pi^2}\left(\ln\frac{m_\pi}{\mu} + \frac{1}{6}\right),$$

$$b_3 = 2\bar{\ell}_1 + \frac{1}{2}\bar{\ell}_2 - \frac{1}{16\pi^2}\left(\ln\frac{m_\pi}{\mu} + \frac{7}{12}\right),$$

$$b_4 = \frac{1}{2}\bar{\ell}_2 - \frac{1}{48\pi^2}\left(\ln\frac{m_\pi}{\mu} + \frac{5}{12}\right),$$

$$b_5 = \frac{1}{16\pi^2}\left[-\frac{31}{6}\bar{\ell}_1 - \frac{145}{36}\bar{\ell}_2 + \frac{7}{864} + \frac{1}{16\pi^2}\left(\frac{625}{144}\ln\frac{m_\pi}{\mu} - \frac{66029}{20736}\right)\right] - \frac{21}{16}k_1 - \frac{107}{96}k_2 + r_5^r,$$

$$b_6 = \frac{1}{16\pi^2}\left[-\frac{7}{18}\bar{\ell}_1 - \frac{35}{36}\bar{\ell}_2 + \frac{1}{432} + \frac{1}{16\pi^2}\left(\frac{257}{432}\ln\frac{m_\pi}{\mu} - \frac{11375}{20736}\right)\right] - \frac{5}{48}k_1 - \frac{25}{96}k_2 + r_6^r,$$

where

$$k_1 = \frac{1}{192\pi^4}\ln\frac{m_\pi}{\mu}\left(\bar{\ell}_1 + \ln\frac{m_\pi}{\mu}\right),$$

$$k_2 = \frac{1}{96\pi^4}\ln\frac{m_\pi}{\mu}\left(\bar{\ell}_2 + \ln\frac{m_\pi}{\mu}\right),$$

and μ is the renormalization mass scale.

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