Updated estimate of running quark masses

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Stimulated by the recent development of the calculation methods of the running quark masses $m_q(\mu)$ and renewal of the input data, for the purpose of making a standard table of $m_q(\mu)$ for the convenience of particle physicists, the values of $m_q(\mu)$ at various energy scales μ ($\mu=1$ GeV, $\mu=m_c$, $\mu=m_b$, $\mu=m_t$ and so on), especially at $\mu=m_Z$, are systematically evaluated by using the mass renormalization equations and taking into consideration a matching condition at the quark threshold. [S0556-2821(98)01207-7]

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I. INTRODUCTION

It is very important to have reliable values of the quark masses m_q not only for hadron physicists who intend to evaluate observable quantities on the basis of an effective theory, but also for quark-lepton physicists who intend to build a model for quark and lepton unification. For such a purpose, for example, a review article [1] in 1982 by Gasser and Leutwyler has provided us with useful information on the running quark masses $m_q(\mu)$. However, during the 15 years since the Gasser-Leutwyler review article, there have been some developments in the input data and calculation methods: The QCD parameter $\Lambda_{\overline{MS}}^{(n)}$ has been revised [2], the top-quark mass m_t has been observed [3–5], the three-loop diagrams have been evaluated for the pole mass M_q^{pole} [6] and for the running quark mass $m_q(\mu)$ [7], and an alternative treatment of the matching condition at the quark threshold has been proposed [8]. On the other hand, so far, there are few articles that review the masses of all quarks systematically, although there have been some reestimates [9–18] for specific quark masses. For a recent work on a systematic study of all quark masses, for example, see Ref. [19] by Rodrigo. We will give further systematic studies on the basis of recent data and obtain a renewed table of the running quark mass values.

The purpose of the present paper is to provide a useful table of the running quark masses $m_q(\mu)$ to hadron physicists and quark-lepton physicists. In Sec. IV, using the mass renormalization equation (4.1), we will evaluate the value of $m_q(\mu)$ at various energy scales μ, e.g., $\mu=1$ GeV, $\mu=m_q$ $(q=c,b,t)$, $\mu=M_q^{pole}$, $\mu=m_Z$, $\mu=\Lambda_W$, where M_q^{pole} is a "pole" mass of the quark q and Λ_W is the symmetry-breaking energy scale of the electroweak gauge symmetry $SU(2)_L \times U(1)_Y$:

$$\Lambda_W \equiv \langle \phi^0 \rangle = (\sqrt{2}G_F)^{-1/2}/\sqrt{2} = 174.1 \ \text{GeV}. \tag{1.1}$$

In Sec. II, we review the light quark masses $m_u(\mu)$, $m_d(\mu)$, and $m_s(\mu)$ at $\mu=1$ GeV. In Sec. III, we review the pole mass values of the heavy quark masses M_c^{pole}, M_b^{pole}, and M_t^{pole}. In Sec. IV, running quark masses $m_q(\mu)$ are evaluated for various energy scales μ below $\mu=\Lambda_W=174.1$ GeV. In Sec. V, we comment on the reliability of the perturbative calculations of the running quark masses $m_q(\mu)$ $(\mu\leqslant\Lambda_W)$. In Sec. VI, we summarize our numerical results of the running quark mass values $m_q(\mu)$, the charged lepton masses $m_l(\mu)$, the Cabibbo-Kobayashi-Maskawa (CKM) [20] matrix $V_{CKM}(\mu)$, and the $SU(3)_c\times SU(2)_L\times U(1)_Y$ gauge coupling constants $g_i(\mu)$ $(i=1,2,3)$ at $\mu=m_Z$. In Sec. VII, for reference, the evolution of the Yukawa coupling constants is estimated at energy scales higher than $\mu=\Lambda_W$ for the cases of the standard model with one Higgs boson (Sec. VII A) and the minimal SUSY model (Sec. VII B). Finally, Sec. VIII is devoted to a summary and discussion.

II. LIGHT QUARK MASSES AT $\mu=1$ GeV

Gasser and Leutwyler [1] had concluded in their review article of 1982 that the light quark masses $m_u(\mu)$, $m_d(\mu)$, and $m_s(\mu)$ at $\mu=1$ GeV are

$$m_u(1 \ \text{GeV}) = 5.1\pm 1.5 \ \text{MeV},$$
$$m_d(1 \ \text{GeV}) = 8.9\pm 2.6 \ \text{MeV}, \tag{2.1}$$
$$m_s(1 \ \text{GeV}) = 175\pm 55 \ \text{MeV}$$

from QCD sum rules. In 1987, Dominguez and de Rafael [9] had reestimated those values from the QCD finite energy sum rules. They obtained the same ratios of the light quark masses as those estimated by Gasser and Leutwyler, but they used a different value of m_u+m_d at $\mu=1$ GeV,

$$(m_u+m_d)_{\mu=1 \ \text{GeV}} = 15.5\pm 2.0 \ \text{MeV}, \tag{2.2}$$

instead of the Gasser-Leutwyler value $(m_u+m_d)_{\mu=1 \ \text{GeV}}=14\pm 3$ MeV. Therefore, Dominguez and de Rafael concluded that

$$m_u(1 \ \text{GeV}) = 5.6\pm 1.1 \ \text{MeV},$$

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$$m_d(1\ \text{GeV}) = 9.9 \pm 1.1\ \text{MeV}, \tag{2.3}$$
$$m_s(1\ \text{GeV}) = 199 \pm 33\ \text{MeV}.$$

Recently, by simulating τ-like inclusive processes for the old Das-Mathur-Okubo sum rule relating e^+e^- to the $I=0$ and $I=1$ hadron total cross-section data, Narison [10] has obtained the values

$$m_u(1\ \text{GeV}) = 4 \pm 1\ \text{MeV},$$
$$m_d(1\ \text{GeV}) = 10 \pm 1\ \text{MeV}, \tag{2.4}$$
$$m_s(1\ \text{GeV}) = 197 \pm 29\ \text{MeV},$$

which are roughly in agreement with Eqs. (2.3).

On the other hand, by combining various pieces of the information on the quark mass ratios, Leutwyler [11] has recently reestimated the ratios

$$m_u/m_d = 0.553 \pm 0.043,$$
$$m_s/m_d = 18.9 \pm 0.8 \tag{2.5}$$

and has obtained

$$m_u(1\ \text{GeV}) = 5.1 \pm 0.9\ \text{MeV},$$
$$m_d(1\ \text{GeV}) = 9.3 \pm 1.4\ \text{MeV}, \tag{2.6}$$
$$m_s(1\ \text{GeV}) = 175 \pm 25\ \text{MeV}.$$

The values (2.6) are in agreement with Eqs. (2.1), (2.3), and (2.4).

There is not much discrepancy among these estimates as far as m_u and m_d are concerned, except for the estimates by Donoghue, Holstein, and Wyler [12], who have obtained

$$m_d/m_u = 3.49, \quad m_s/m_d = 20.7 \tag{2.7}$$

from the constraints of chiral symmetry treated to next to leading order. Eletsky and Ioffe [13] and Adami, Drukarev, and Ioffe [14] have obtained

$$(m_d - m_u)_{\mu=0.5\ \text{GeV}} = 3 \pm 1\ \text{MeV}, \tag{2.8}$$

from the QCD sum rules on the isospin-violating effects for D and D^* and for N, Σ, and Ξ, respectively. The value (2.8) is consistent with Eqs. (2.3) and (2.6). The value

$$(m_u + m_d)_{\mu=1\ \text{GeV}} = (12 \pm 2.5)\ \text{MeV}, \tag{2.9}$$

obtained from QCD finite energy sum rules and Laplace sum rules by Bijnens, Prades, and de Rafael [15], is consistent with Eq. (2.2).

On the contrary, for the strange quark mass m_s, two different values, $m_s \simeq 175$ MeV [Eqs. (2.1) and (2.6)] and $m_s \simeq 200$ MeV [Eqs. (2.3) and (2.4)], have been reported. Recently, Chetyrkin *et al.* [16] have estimated

$$m_s(1\ \text{GeV}) = 205.5 \pm 19.1\ \text{MeV} \tag{2.10}$$

by an order-α_s^3 determination from the QCD sum rules. The value (2.10) is consistent with Eq. (2.3). (Of course, if we take their errors into consideration, these values are consistent.) Hereafter, for the light quark masses at $\mu=1$ GeV we will use the following values, which are weighted averages of the values (2.3), (2.4), (2.6), and (2.10):

$$m_u(1\ \text{GeV}) = 4.88 \pm 0.57\ \text{MeV},$$
$$m_d(1\ \text{GeV}) = 9.81 \pm 0.65\ \text{MeV}, \tag{2.11}$$
$$m_s(1\ \text{GeV}) = 195.4 \pm 12.5\ \text{MeV}.$$

III. HEAVY QUARK MASSES

A. Charm and bottom quark masses

Gasser and Leutwyler [1] have estimated charm and bottom quark masses m_c and m_b from the QCD sum rules as

$$m_c(m_c) = 1.27 \pm 0.05\ \text{GeV},$$
$$m_b(m_b) = 4.25 \pm 0.10\ \text{GeV}. \tag{3.1}$$

Titard and Yndurán [17] have reestimated the charm and bottom quark masses by using the three-level QCD and the full one-loop potential. They have concluded that

$$M_c^{pole} = 1.570 \pm 0.019 \mp 0.007\ \text{GeV},$$
$$M_b^{pole} = 4.906^{+0.069}_{-0.051} \mp 0.004^{+0.011}_{-0.040}\ \text{GeV}, \tag{3.2}$$

$$m_c(m_c) = 1.306^{+0.021}_{-0.034} \pm 0.006\ \text{GeV},$$
$$m_b(m_b) = 4.397^{+0.007-0.003+0.016}_{-0.002+0.004-0.032}\ \text{GeV}, \tag{3.3}$$

where the first and second errors come from the use of the QCD parameter $\Lambda_{\overline{MS}}^{(4)} = 0.20^{+0.08}_{-0.06}$ GeV and the gluon condensate value $\langle \alpha_s G^2 \rangle = 0.042 \pm 0.020\ \text{GeV}^4$, respectively, and the third error denotes a systematic error.

On the other hand, from the QCD spectral sum rules to two loops for ψ and Υ, Narison [18] has estimated the running quark masses

$$m_c(M_c^{PT2}) = 1.23^{+0.02}_{-0.04} \pm 0.03\ \text{GeV},$$
$$m_b(M_b^{PT2}) = 4.23^{+0.03}_{-0.04} \pm 0.02\ \text{GeV}, \tag{3.4}$$

corresponding to the short-distance perturbative pole masses to two loops

$$M_c^{PT2} = 1.42 \pm 0.03\ \text{GeV},$$
$$M_b^{PT2} = 4.62 \pm 0.02\ \text{GeV} \tag{3.5}$$

and the three-loop values of the short-distance pole masses

$$M_c^{PT3} = 1.64^{+0.10}_{-0.07} \pm 0.03\ \text{GeV},$$
$$M_b^{PT3} = 4.87 \pm 0.05 \pm 0.02\ \text{GeV}. \tag{3.6}$$

The values (3.6) are in agreement with the values (3.2) estimated by Titard, and Yndurán while the values (3.5) are not. Narison asserts that one should not use M_q^{PT3} because the hadronic correlators are only known to two-loop accuracy. Although we must keep Narison's statement in mind, since we use the three-loop formula (4.5) for the running quark masses $m_q(\mu)$ for all quarks $q=u,d,\ldots,t$, hereafter, we adopt the following weighted averages of Eqs. (3.2) and (3.6):

$$M_c^{pole}=1.59\pm0.02 \text{ GeV},$$
$$M_b^{pole}=4.89\pm0.05 \text{ GeV} \tag{3.7}$$

as the pole mass values.

B. Top quark mass

The explicit value of the top quark mass has been reported by the CDF Collaboration [3] from the data of $p\bar{p}$ collisions at $\sqrt{s}=1.8$ TeV:

$$m_t=174\pm10^{+13}_{-12} \text{ GeV}. \tag{3.8}$$

They [4] have also reported an updated value

$$m_t=176\pm8\pm10 \text{ GeV}. \tag{3.9}$$

On the other hand, the D0 Collaboration [5] has reported the value

$$m_t=199^{+19}_{-21}\pm22 \text{ GeV}. \tag{3.10}$$

The Particle Data Group (PDG) [21] has quoted the value

$$m_t=180\pm12 \text{ GeV} \tag{3.11}$$

as the top quark mass from direct observations of top quark events. Hereafter, we use the value (3.11) as the pole mass of the top quark.

C. Mass values $m_q(\mu)$ at $\mu=M_q^{pole}$

The relation between the pole mass M_q^{pole} and the running quark mass $m_q(M_q^{pole})$ at $\mu=M_q^{pole}$ has been calculated by Gray *et al.* [6]:

$$m_q(M_q^{pole})=M_q^{pole}\left[1+\frac{4}{3}\,\frac{\alpha_s(M_q^{pole})}{\pi}+K_q\left(\frac{\alpha_s(M_q^{pole})}{\pi}\right)^2+O(\alpha_s^3)\right]^{-1}, \tag{3.12}$$

where $K_c=14.5$, $K_b=12.9$, and $K_t=11.0$. The definition of K_q and their estimates are given in Appendix A. The values of $\alpha_s(\mu)$ at various values of μ and errors are given in Table VII in Appendix B. By using Eq. (3.12), from Eqs. (3.7) and (3.11) we obtain

$$\begin{aligned} m_c(M_c^{pole})&=1.213\pm0.018^{-0.040}_{+0.034} \text{ GeV},\\ m_b(M_b^{pole})&=4.248\pm0.046^{-0.040}_{+0.036} \text{ GeV},\\ m_t(M_t^{pole})&=170.1\pm11.4\mp0.3 \text{ GeV},\end{aligned} \tag{3.13}$$

where the first and second errors come from $\pm\Delta M_q^{pole}$ and $\pm\Delta\Lambda_{\overline{MS}}^{(5)}$, respectively.

IV. BEHAVIORS OF $m_q(\mu)$ AT THE QUARK THRESHOLDS

The scale dependence of a running quark mass $m_q(\mu)$ is governed by the equation [7]

$$\mu\frac{d}{d\mu}m_q(\mu)=-\gamma(\alpha_s)m_q(\mu), \tag{4.1}$$

where

$$\gamma(\alpha_s)=\gamma_0\frac{\alpha_s}{\pi}+\gamma_1\left(\frac{\alpha_s}{\pi}\right)^2+\gamma_2\left(\frac{\alpha_s}{\pi}\right)^3+O(\alpha_s^4), \tag{4.2}$$

$$\gamma_0=2,\quad \gamma_1=\frac{101}{12}-\frac{5}{18}n_q,$$

$$\gamma_2=\frac{1}{32}\left[1249-\left(\frac{2216}{27}+\frac{160}{3}\zeta(3)\right)n_q-\frac{140}{81}n_q^2\right]. \tag{4.3}$$

Then $m_q(\mu)$ is given by

$$m_q(\mu)=R(\alpha_s(\mu))\hat{m}_q, \tag{4.4}$$

TABLE I. Running quark mass values $m_q(\mu)$ at $\mu=m_q$. Input values $m_q(1\ \text{GeV})$ for $q=u,d,s$ and $m_q(M_q^{pole})$ for $q=c,b,t$ are used. The first and second errors come from $\pm\Delta m_q$ (or $\pm\Delta M_q^{pole}$) and $\pm\Delta\Lambda_{\overline{MS}}^{(5)}$, respectively. The values with an asterisk should not be taken rigidly because these values have been calculated in the region with a large $\alpha_s(\mu)$.

	Input $m_q(1\ \text{GeV})$ or $m_q(M_q^{pole})$	Output $m_q(m_q)$
u	4.88 ± 0.57 MeV	$^*0.436^{+0.001+0.058}_{-0.002-0.052}$ GeV
d	9.81 ± 0.65 MeV	$^*0.448\pm0.001^{+0.059}_{-0.053}$ GeV
s	195.4 ± 12.5 MeV	* $0.553\pm0.005^{+0.058}_{-0.052}$ GeV
c	$1.213\pm0.018^{-0.040}_{+0.034}$ GeV	$1.302\pm0.018^{-0.020}_{+0.019}$ GeV
b	$4.248\pm0.046^{-0.040}_{+0.036}$ GeV	$4.339\pm0.046^{-0.029}_{+0.027}$ GeV
t	$170.1\pm11.4\mp0.3$ GeV	$170.8\pm11.5\mp0.2$ GeV

$$R(\alpha_s)=\left(\frac{\beta_0}{2}\frac{\alpha_s}{\pi}\right)^{2\gamma_0/\beta_0}\left\{1+\left(2\frac{\gamma_1}{\beta_0}-\frac{\beta_1\gamma_0}{\beta_0^2}\right)\frac{\alpha_s}{\pi}+\frac{1}{2}\left[\left(2\frac{\gamma_1}{\beta_0}-\frac{\beta_1\gamma_0}{\beta_0^2}\right)^2+2\frac{\gamma_2}{\beta_0}-\frac{\beta_1\gamma_1}{\beta_0^2}-\frac{\beta_2\gamma_0}{16\beta_0^2}+\frac{\beta_1^2\gamma_0}{2\beta_0^3}\right]\left(\frac{\alpha_s}{\pi}\right)^2+O(\alpha_s^3)\right\}, \quad (4.5)$$

where $\hat{m}_q$ is the renormalization-group invariant mass that is independent of $\ln(\mu^2/\Lambda^2)$, $\alpha_s(\mu)$ is given by Eq. (B4), and β_i $(i=0,1,2)$ are also defined by Eq. (B3). By using Eq. (4.5) and $\Lambda_{\overline{MS}}^{(n)}$ obtained in Appendix B, we can evaluate $R^{(n)}(\mu)$ for $\mu<\mu_{n+1}$, where μ_n is the nth quark flavor threshold and we take $\mu_n=m_{qn}(m_{qn})$.

Quite recently, the four-loop β function and quark mass anomalous dimension have been obtained by several authors [22]. In this paper, we evaluate the running quark masses by using the three-loop results (4.1)–(4.5). The effects of the four-loop results to the three-loop results will be discussed in Sec. V.

We can evaluate the values of $m_q(m_q)$ $(q=c,b,t)$ by using the values of M_q^{pole} given in Sec. III and the relation

$$m_{qn}(\mu)=[R^{(n)}(\mu)/R^{(n)}(M_{qn}^{pole})]m_{qn}(M_{qn}^{pole}) \quad (\mu<\mu_{n+1}). \quad (4.6)$$

Similarly, we evaluate the light quark masses $m_q(m_q)$ $(q=u,d,s)$ using the relation

$$m_q(\mu)=[R^{(3)}(\mu)/R^{(3)}(1\ \text{GeV})]m_q(1\ \text{GeV}) \quad (\mu<\mu_4) \quad (4.7)$$

and the values $m_q(1\ \text{GeV})$ given in Eq. (2.11). The results are summarized in Table I. The values of $m_u(m_u)$, $m_d(m_d)$, and $m_s(m_s)$ should not be taken rigidly because the perturbative calculation is not reliable for such a region in which $\alpha_s(\mu)$ takes a large value (see Sec. V).

Exactly speaking, the estimates of $\Lambda_{\overline{MS}}^{(n)}$ in Table VII in Appendix B are dependent on the choices of the quark threshold $\mu_n=m_{qn}(m_{qn})$. The values in Tables VII and I have been obtained by iterating the evaluation of $\Lambda_{\overline{MS}}^{(n)}$ and $m_q(m_q)$.

Running quark mass values $m_{qn}(\mu)$ at $\mu\geq\mu_{n+1}$ cannot be evaluated by formula (4.4) straightforwardly because of the quark threshold effects. As seen in Fig. 1, the behavior of $R(\mu)$ is discontinuous at $\mu=\mu_n\equiv m_{qn}(m_{qn})$.

The behavior of the nth quark mass $m_{qn}^{(N)}$ $(n<N)$ at $\mu_N\leq\mu<\mu_{N+1}$ are given by the matching condition [8]

$$m_{qn}^{(N)}(\mu)=m_{qn}^{(N-1)}(\mu)\left[1+\frac{1}{12}\left(x_N^2+\frac{5}{3}x_N+\frac{89}{36}\right)\times\left(\frac{\alpha_s^{(N)}(\mu)}{\pi}\right)^2\right]^{-1}, \quad (4.8)$$

where

$$x_N=\ln\{[m_{qN}^{(N)}(\mu)]^2/\mu^2\}. \quad (4.9)$$

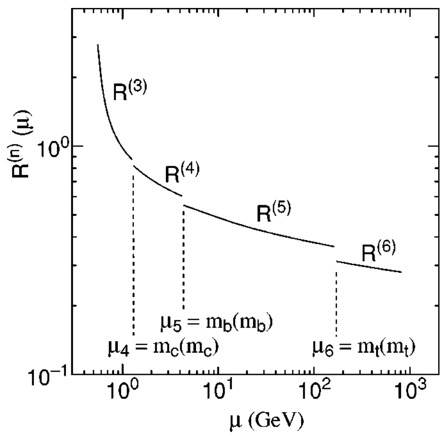

FIG. 1. Threshold behavior of $R^{(n)}(\mu)$ versus μ.

For example, the behavior of $m_c(\mu)$ at $\mu<\mu_5$, $m_c^{(4)}(\mu)$, can be evaluated by using Eq. (4.6), while those at $\mu_5\leq\mu<\mu_6$ and $\mu_6\leq\mu$ must be evaluated by using Eq. (4.8) with $m_c^{(4)}(\mu)$ and $x_5=\ln\{[m_b^{(5)}(\mu)]^2/\mu^2\}$ and with $m_c^{(5)}(\mu)$ and $x_6=\ln\{[m_t^{(6)}(\mu)]^2/\mu^2\}$, respectively. In Fig. 2, we illustrate the μ dependence of the light quark masses $m_q(\mu)$ $(q=u,d,s)$, where we have taken the matching condition (4.8) into account. We can see that the discontinuity that was seen in Fig. 1 disappears in Fig. 2.

We also illustrate the behavior of the heavy quark masses $m_q(\mu)$ $(q=c,b,t)$ in Fig. 3. Exactly speaking, the phrase "the running mass value $m_Q(\mu)$" of a heavy quark Q at a lower-energy scale μ than $\mu=m_Q(m_Q)$ loses its meaning. For example, the effective quark flavor number n_q is 3 at $\mu=1$ GeV, so that the value of $m_t(\mu)$ at $\mu=1$ GeV does not have meaning. However, for reference, in Fig. 3, we have calculated the value of $m_Q(\mu)$ $(Q=q_N)$ at $\mu_n\leq\mu<\mu_{n+1}$ $(n<N)$ by using the relation $m_Q(\mu)=\hat{m}_QR^{(N)}(\mu)$ [not $m_Q(\mu)=\hat{m}_QR^{(n)}(\mu)$].

The numerical results are summarized in Table II. As stressed by Vermaseren and co-workers [22], the invariant mass $\hat{m}_q$ is a good reference mass for the accurate evolution of the $\overline{MS}$ quark masses to the necessary scale μ in phenomenological applications. The values of $\hat{m}_q$ are also listed in Table II.

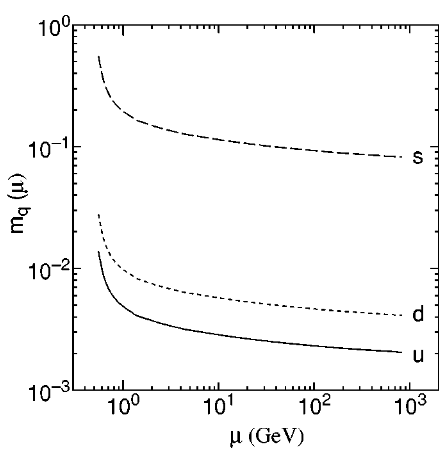

FIG. 2. Light quark masses $m_q(\mu)$ $(q=u,d,s)$ versus μ.

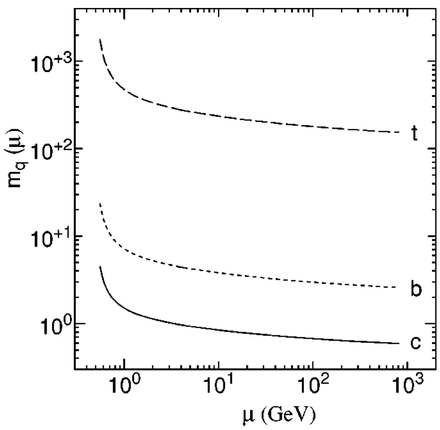

FIG. 3. Heavy quark masses $m_q(\mu)$ $(q=c,b,t)$ versus μ.

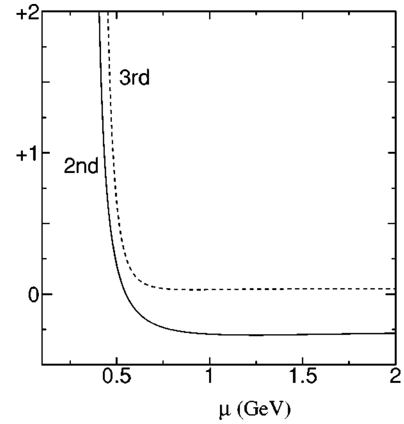

FIG. 4. Reliability of the perturbative calculation of $\alpha_s^{(n)}(\mu)$. The curves show the behaviors of the second and third terms in curly brackets in Eq. (B4).

V. RELIABILITY OF THE PERTURBATIVE CALCULATION BELOW $\mu\sim 1$ GeV

As we noted already, the values of the light quark masses $m_q(m_q)$ $(q=u,d,s)$ should not be taken rigidly because the perturbative calculation below $\mu\sim 1$ GeV seems to be unreliable. In order to see the reliability of the calculation of $\alpha_s(\mu)$, in Fig. 4 we illustrate the values of the second and third terms in curly brackets in Eq. (B4) separately. The values of the second and third terms exceed one at $\mu\simeq 0.42$ GeV and $\mu\simeq 0.47$ GeV, respectively. Also, in Fig. 5 we illustrate the values of the second and third terms in curly brackets in Eq. (4.5) separately. The values of the second and third terms exceed one at $\mu\simeq 0.58$ GeV and $\mu\simeq 0.53$ GeV, respectively. This means that the perturbative calculation is not reliable below $\mu\simeq 0.6$ GeV. Therefore, the values with asterisks in Tables I, II, and VI should not be taken strictly.

These situations are not improved even if we take the four-loop correction into consideration. For example, for $n_q=3$, $d(\alpha_s/\pi)/d\ln\mu$ is given by [22]

$$\frac{d(\alpha_s/\pi)}{d\ln\mu}=-\frac{9}{2}\left(\frac{\alpha_s}{\pi}\right)^2\left[1+1.79\frac{\alpha_s}{\pi}+4.47\left(\frac{\alpha_s}{\pi}\right)^2+21.0\left(\frac{\alpha_s}{\pi}\right)^3+\cdots\right]. \tag{5.1}$$

TABLE II. Running quark masses $m_q(\mu)$ and invariant masses $\hat{m}_q$ (in units of GeV). The values with an asterisk should not be taken strictly because the perturbative calculation is not reliable in the region with a large $\alpha_s(\mu)$.

	$q=u$	$q=d$	$q=s$	$q=c$	$q=b$	$q=t$
M_q^{pole}	* 0.501 $^{+0.068}_{-0.061}$	* 0.517 $^{+0.068}_{-0.062}$	* 0.687 $^{+0.074}_{-0.067}$	1.59 ± 0.02	4.89 ± 0.05	180 ± 12
$m_q(M_q^{pole})$	* 0.0307 $^{+0.0022}_{-0.0026}$	* 0.0445 $^{+0.0018}_{-0.0023}$	* 0.283 $^{+0.013}_{-0.016}$	1.213 $^{+0.052}_{-0.058}$	4.248 $^{+0.082}_{-0.086}$	170 ± 12
$m_q(m_q)$	* 0.436 $^{+0.059}_{-0.054}$	* 0.448 $^{+0.060}_{-0.054}$	* 0.553 $^{+0.064}_{-0.057}$	1.302 $^{+0.037}_{-0.038}$	4.339 $^{+0.073}_{-0.076}$	171 ± 12
$m_q(1\ \mathrm{GeV})$	0.00488 ± 0.00057	0.00981 ± 0.00065	0.1954 ± 0.0125	1.506 $^{+0.048}_{-0.037}$	7.18 $^{+0.59}_{-0.44}$	475 $^{+86}_{-71}$
$m_q(m_c)$ $m_c=1.302$	0.00418 $^{+0.00056}_{-0.00060}$	0.00840 $^{+0.00071}_{-0.00077}$	0.1672 $^{+0.0137}_{-0.0150}$	1.302 $^{+0.037}_{-0.038}$	6.12 $^{+0.32}_{-0.25}$	399 $^{+58}_{-51}$
$m_q(m_b)$ $m_b=4.339$	0.00317 $^{+0.00052}_{-0.00056}$	0.00637 $^{+0.00073}_{-0.00081}$	0.1268 $^{+0.0142}_{-0.0159}$	0.949 $^{+0.063}_{-0.070}$	4.34 $^{+0.07}_{-0.08}$	272 $^{+26}_{-25}$
$m_q(m_W)$ $m_W=80.33$	0.00235 $^{+0.00042}_{-0.00045}$	0.00473 $^{+0.00061}_{-0.00067}$	0.0942 $^{+0.0119}_{-0.0131}$	0.684 $^{+0.056}_{-0.061}$	3.03 ± 0.11	183 ± 13
$m_q(m_Z)$ $m_Z=91.187$	0.00233 $^{+0.00042}_{-0.00045}$	0.00469 $^{+0.00060}_{-0.00066}$	0.0934 $^{+0.0118}_{-0.0130}$	0.677 $^{+0.056}_{-0.061}$	3.00 ± 0.11	181 ± 13
$m_q(m_t)$ $m_t=170.8$	0.00223 $^{+0.00040}_{-0.00043}$	0.00449 $^{+0.00058}_{-0.00064}$	0.0894 $^{+0.0114}_{-0.0125}$	0.646 $^{+0.054}_{-0.059}$	2.85 ± 0.11	171 ± 12
$m_q(\Lambda_W)$ $\Lambda_W=174.1$	0.00223 $^{+0.00040}_{-0.00043}$	0.00448 $^{+0.00058}_{-0.00064}$	0.0893 $^{+0.0114}_{-0.0125}$	0.645 $^{+0.054}_{-0.059}$	2.84 ± 0.11	171 ± 13
$\hat{m}_q$	0.00496 $^{+0.00095}_{-0.00101}$	0.00998 $^{+0.00141}_{-0.00153}$	0.199 $^{+0.028}_{-0.030}$	1.59 $^{+0.15}_{-0.16}$	7.87 $^{+0.40}_{-0.41}$	546 ± 49

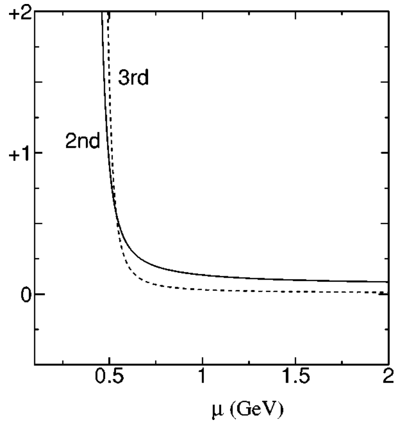

FIG. 5. Reliability of the perturbative calculation of $m_q(\mu)$. The curves show the behaviors of the second and third terms in curly brackets in Eq. (4.5).

Since the value of α_s/π is $\alpha_s/\pi \simeq 0.16$ at $\mu \simeq 1$ GeV, the numerical values of the right-hand side of Eq. (5.1) become

$$\frac{d(\alpha_s/\pi)}{d\ln\mu} = -\frac{9}{2}\left(\frac{\alpha_s}{\pi}\right)^2 [1+0.28+0.11+0.085+\cdots], \tag{5.2}$$

so that the fourth term is not negligible compared to the third term. This suggests that the fifth term, which is of the order of $(\alpha_s/\pi)^6$, will also not be negligible below $\mu \sim 1$ GeV. However, we consider that the evolution of $m_q(\mu)$ above $\mu \sim 1$ GeV (from $\mu \simeq 1$ GeV to $\mu \sim m_Z$) is reliable in spite of the large error of $\alpha_s(\mu)$ at $\mu \sim 1$ GeV.

VI. OBSERVABLE QUANTITIES $m_q(\mu)$, $V_{CKM}(\mu)$, AND $\alpha_i(\mu)$ AT $\mu = m_Z$

For quark mass matrix phenomenology, values of $m_q(\mu)$ at $\mu = m_Z$ are useful because the observed CKM matrix parameters $|V_{ij}|$ are given at $\mu = m_Z$. We summarize the quark and charged lepton masses at $\mu = m_Z$:

$$m_u = 2.33^{+0.42}_{-0.45}\ \text{MeV}, \quad m_c = 677^{+56}_{-61}\ \text{MeV},$$

$$m_t = 181 \pm 13\ \text{GeV}, \quad m_d = 4.69^{+0.60}_{-0.66}\ \text{MeV},$$

$$m_s = 93.4^{+11.8}_{-13.0}\ \text{MeV}, \quad m_b = 3.00 \pm 0.11\ \text{GeV}, \tag{6.1}$$

$$m_e = 0.486\,847\,27 \pm 0.000\,000\,14\ \text{MeV},$$

$$m_\mu = 102.751\,38 \pm 0.000\,33\ \text{MeV}$$

$$m_\tau = 1.746\,69^{+0.000\,30}_{-0.000\,27}\ \text{GeV},$$

where the running charged lepton masses $m_l(\mu)$ have been evaluated from the relation for the physical (pole) masses M_l [23]:

$$m_l(\mu) = M_l \left[1 - \frac{\alpha(\mu)}{\pi}\left(1 + \frac{3}{4}\ln\frac{\mu^2}{m_l^2}\right)\right]. \tag{6.2}$$

The value of $m_b(m_Z)$ in Eq. (6.1) is in good agreement with the value [24]

$$m_b(m_Z) = 2.67 \pm 0.25 \pm 0.27 \pm 0.34\ \text{GeV}\ , \tag{6.3}$$

which has been extracted recently from CERN LEP data.

On the other hand, the standard expression [25] of the CKM matrix V is given by

$$V = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta_{13}} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta_{13}} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta_{13}} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta_{13}} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta_{13}} & c_{23}c_{13} \end{pmatrix}. \tag{6.4}$$

The observed values $|V_{us}|$, $|V_{ub}|$ [21] and $|V_{cb}|$ [26,27] are

$$|V_{us}| = 0.2205 \pm 0.0018,$$

$$|V_{cb}| = 0.0373 \pm 0.0018, \tag{6.5}$$

$$|V_{ub}/V_{cb}| = 0.08 \pm 0.02,$$

where the value of $|V_{cb}|$ has been obtained by combining the OPAL value [26] $|V_{cb}| = 0.0360 \pm 0.0021 \pm 0.0024 \pm 0.0012$ and the ALEPH value [27] $|V_{cb}| = 0.0344 \pm 0.0016 \pm 0.0023 \pm 0.0014$ with the PDG value $|V_{cb}| = 0.041 \pm 0.003$. Because of the hierarchical structure $|V_{us}|^2 \gg |V_{cb}|^2 \gg |V_{ub}|^2$, the following expression of V will also be useful:

$$V \simeq \begin{pmatrix} 1 - \frac{1}{2}\lambda^2 & \lambda & \sigma e^{-i\delta} \\ -\lambda & 1 - \frac{1}{2}\lambda^2 & \rho \\ \lambda\rho - \sigma e^{i\delta} & -\rho & 1 - \frac{1}{2}\rho^2 \end{pmatrix}, \tag{6.6}$$

where $\lambda = |V_{us}|$, $\rho = |V_{cb}|$, and $\sigma = |V_{ub}|$. Hereafter, we will use the observed values (6.5) as the values of $|V_{ij}(\mu)|$ at $\mu = m_Z$. Then, from expression (6.4) [not the approximate expression (6.6)], we obtain the numerical expression of $V(\mu)$ at $\mu = m_Z$,

$$V(m_Z)=\begin{pmatrix} 0.9754 & 0.2205 & 0.0030e^{-i\delta} \\ -0.2203-0.0001e^{i\delta} & 0.9747 & 0.0373 \\ 0.0082-0.0029e^{i\delta} & -0.0364-0.0007e^{i\delta} & 0.9993 \end{pmatrix}. \qquad (6.7)$$

Since we already know the numerical values of $D_u = \mathrm{diag}(m_u, m_c, m_t)$, $D_d = \mathrm{diag}(m_d, m_s, m_b)$, and V_{ij} (except for the parameter δ) at $\mu = m_Z$, by using the relations

$$U_L^u M_u U_R^{u\dagger} = D_u, \quad U_L^d M_d U_R^{d\dagger} = D_d, \quad V = U_L^u U_L^{d\dagger}, \qquad (6.8)$$

we can determine the numerical structures of the squared mass matrices H_u and H_d, which are defined by

$$H_u = M_u M_u^\dagger, \quad H_d = M_d M_d^\dagger. \qquad (6.9)$$

In particular, at a special quark-family basis on which the up-quark mass matrix takes a diagonal form D_u, we can readily obtain the matrix form H_u and H_d:

$$H_u = D_u^2 = m_t^2 \begin{pmatrix} m_u^2/m_t^2 & 0 & 0 \\ 0 & m_c^2/m_t^2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad (6.10)$$

$$H_d = V D_d^2 V^\dagger \simeq m_b^2 \begin{pmatrix} \sigma^2(1+x^2) & \rho\sigma(y+e^{-i\delta}) & \sigma e^{-i\delta} \\ \rho\sigma(y+e^{i\delta}) & \rho^2(1+y^2/x^2) & \rho \\ \sigma e^{i\delta} & \rho & 1 \end{pmatrix}, \qquad (6.11)$$

where

$$x = \frac{\lambda}{\sigma}\frac{m_s}{m_b}, \quad y = \frac{\lambda}{\rho\sigma}\left(\frac{m_s}{m_b}\right)^2. \qquad (6.12)$$

Numerically, by using Eq. (6.7), but without using the approximate expression (6.11), we obtain

$$H_u(m_Z) = m_t^2(m_Z) \begin{pmatrix} 1.66\times 10^{-10} & 0 & 0 \\ 0 & 1.40\times 10^{-5} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad (6.13)$$

$$H_d(m_Z) = m_b^2(m_Z) \begin{pmatrix} 5.84\times 10^{-5} & (2.08+1.11e^{-i\delta})\times 10^{-4} & 2.98\times 10^{-3}e^{-i\delta} \\ (2.08+1.11e^{i\delta})\times 10^{-4} & 2.31\times 10^{-3} & 3.72\times 10^{-2} \\ 2.98\times 10^{-3}e^{i\delta} & 3.72\times 10^{-2} & 0.9986 \end{pmatrix}, \qquad (6.14)$$

where $m_t^2(m_Z) = 3.24\times 10^4\ \mathrm{GeV}^2$ and $m_b^2(m_Z) = 9.00\ \mathrm{GeV}^2$. In the standard model [not $SU(2)_L\times SU(2)_R\times U(1)_Y$, but $SU(2)_L\times U(1)_Y$], by a suitable transformation of the right-handed fields, we can always make quark mass matrices (M_u, M_d) Hermitian. Furthermore, in the quark-family basis where $M_u = D_u$, the quark mass matrices are given by

$$M_u = D_u = m_t \begin{pmatrix} m_u/m_t & 0 & 0 \\ 0 & m_c/m_t & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad (6.15)$$

$$M_d = V D_d V^\dagger \simeq m_b \begin{pmatrix} \frac{m_s}{m_b}\left(\frac{m_d}{m_s}+\lambda^2\right) & \lambda\frac{m_s}{m_b} & \sigma e^{-i\delta} \\ \lambda\frac{m_s}{m_b} & \frac{m_s}{m_b} & \rho \\ \sigma e^{i\delta} & \rho & 1 \end{pmatrix}. \qquad (6.16)$$

It is well known that if we assume $(M_d)_{11}=0$, we obtain the relation [28]

$$\lambda \equiv |V_{us}| \simeq \sqrt{-m_d/m_s}. \qquad (6.17)$$

Then we obtain a simpler expression of M_d

$$M_d \simeq m_b \begin{pmatrix} 0 & \sqrt{-\frac{m_d m_s}{m_b^2}} & \sigma e^{-i\delta} \\ \sqrt{-\frac{m_d m_s}{m_b^2}} & \frac{m_s}{m_b} & \rho \\ \sigma e^{i\delta} & \rho & 1 \end{pmatrix}. \qquad (6.18)$$

Numerically, by using Eq. (6.7), we obtain

$$M_u(m_Z)=m_t(m_Z)\begin{pmatrix}1.29\times10^{-5} & 0 & 0\\ 0 & 3.75\times10^{-3} & 0\\ 0 & 0 & 1\end{pmatrix}, \tag{6.19}$$

$$M_d(m_Z)=m_b(m_Z)\begin{pmatrix}3.01\times10^{-3} & (6.36+0.11e^{-i\delta})\times10^{-3} & (-0.24+2.97e^{-i\delta})\times10^{-3}\\ (6.36+0.11e^{i\delta})\times10^{-3} & 0.0310 & 0.0362\\ (-0.24+2.97e^{i\delta})\times10^{-3} & 0.0362 & 0.9986\end{pmatrix}, \tag{6.20}$$

where $m_t(m_Z)=180$ GeV and $m_b(m_Z)=3.00$ GeV. For the case of $m_s<0$, instead of Eq. (6.20), we obtain

$$M_d(m_Z)=m_b(m_Z)\begin{pmatrix}-1.8\times10^{-5} & (-7.03+0.11e^{-i\delta})\times10^{-3} & (0.26+2.98e^{-i\delta})\times10^{-3}\\ (-7.03+0.11e^{i\delta})\times10^{-3} & -0.0281 & 0.0384\\ (0.26+2.98e^{i\delta})\times10^{-3} & 0.0384 & 0.9986\end{pmatrix}. \tag{6.21}$$

We can obtain quark mass matrix forms on an arbitrary quark-family basis by the unitary transformations $H'_u=UH_uU^\dagger$ and $H'_d=UH_dU^\dagger$ for Eqs. (6.10) and (6.11), respectively [and also $M'_u=UM_uU^\dagger$ and $M'_d=UM_dU^\dagger$ for Eqs. (6.15) and (6.16), respectively]. Explicit mass matrix forms on another special quark-family basis are, for example, given in Refs. [29,30].

By starting from the numerical expressions of the mass matrices H_u and H_d at $\mu=m_Z$ [Eqs. (6.10) and (6.11)], we can also obtain the mass matrix form M_q $(q=u,d)$ (in other words, the Yukawa coupling constants) for an arbitrary energy scale μ that is larger than the electroweak scale Λ_W. In the next section, we discuss the evolution of the Yukawa coupling constants. Then we will use the following values of the $\mathrm{SU}(3)_c\times\mathrm{SU}(2)_L\times\mathrm{U}(1)_Y$ gauge coupling constants at $\mu=m_Z$:

$$\alpha_1(m_Z)=0.016\,829\pm0.000\,017,$$
$$\alpha_2(m_Z)=0.033\,493^{+0.000\,060}_{-0.000\,058}, \tag{6.22}$$
$$\alpha_3(m_Z)=0.118\pm0.003,$$

which are derived from [31]

$$\alpha(m_Z)=(128.89\pm0.09)^{-1},$$
$$\sin^2\theta_W=0.231\,65\pm0.000\,024, \tag{6.23}$$

and $\Lambda^{(n)}_{\overline{MS}}=209^{+39}_{-33}$ MeV [2]. Here the coupling constants of $\mathrm{U}(1)_Y$, $\mathrm{SU}(2)_L$, and $\mathrm{SU}(3)_c$ gauge bosons, g_1, g_2, and g_3, are defined as they satisfy the relation

$$\frac{1}{e^2}=\frac{5}{3}\frac{1}{g_1^2}+\frac{1}{g_2^2} \tag{6.24}$$

and the relation in the SU(5) grand unified theory [32] limit

$$g_1=g_2=g_3. \tag{6.25}$$

VII. EVOLUTION OF YUKAWA COUPLING CONSTANTS

So far we have evaluated values of the running quark masses $m_q(\mu)$ at energy scales that are below the electroweak symmetry-breaking energy scale Λ_W using formula (4.1). However, for the quark masses at an extremely high energy scale far from Λ_W, we must use ''evolution'' equations of Yukawa coupling constants y^a_{ij} $(a=u,d;\ i,j=1,2,3)$. The numerical results of the Yukawa coupling constants already have been given in many works. Since our interest in the present paper is in the updated values of the quark masses $m_q(\mu)$ (i.e., the Yukawa coupling constants y_q), we give only a short review of the evolution of the Yukawa coupling constants and do not give a systematic study of the numerical results.

We define the Yukawa coupling constants y^a_{ij} as

$$H_{mass}=\sum_{a=u,d}\sum_{i=1}^{3}\sum_{j=1}^{3}y^a_{ij}\overline{\psi}_{Lai}\psi_{Raj}\phi^0_a+\text{H.c.}, \tag{7.1}$$

where ϕ^0_a are the vacuum expectation values of the neutral components of the Higgs bosons ϕ_a that couple with fermions ψ_a and they mean ϕ^0_u and ϕ^0_d for the minimal SUSY model (model B), while they mean a single Higgs boson $\phi^0_u=\phi^0_d=\phi^0$ for the standard model with one Higgs boson (model A). The quark mass matrices M_u and M_d at $\mu=\Lambda_W$ are given by

$$M_a(\mu)=\frac{1}{\sqrt{2}}Y_a(\mu)v_a, \tag{7.2}$$

where Y_a denotes a matrix $(Y_a)_{ij}=y^a_{ij}$ and v_a are the vacuum expectation values of ϕ^0_a, $v_a=\sqrt{2}\langle\phi^0_a\rangle$, and $v_u=v_d=\sqrt{2}\Lambda_W$ for model A and $\sqrt{v_u^2+v_d^2}=\sqrt{2}\Lambda_W$ for model B.

The evolution of the coupling constants $Y_a(\mu)$ from $Y_a(\Lambda_W)$ is given by the equations [33]

$$\frac{dY_a}{dt}=\left[\frac{1}{16\pi^2}\beta_a^{(1)}+\frac{1}{(16\pi^2)^2}\beta_a^{(2)}\right]Y_a,\quad (a=u,d,e), \tag{7.3}$$

TABLE III. Coefficients $\beta_a^{(1)}$ in the evolution equations of Yukawa coupling constants Y_a.

Model A Standard single Higgs	Model B SUSY
$G_u=\frac{17}{20}g_1^2+\frac{9}{4}g_2^2+8g_3^2$	$G_u=\frac{13}{15}g_1^2+3g_2^2+\frac{16}{3}g_3^2$
$G_d=\frac{1}{4}g_1^2+\frac{9}{4}g_2^2+8g_3^2$	$G_d=\frac{7}{15}g_1^2+3g_2^2+\frac{16}{3}g_3^2$
$G_e=\frac{9}{4}g_1^2+\frac{9}{4}g_2^2$	$G_e=\frac{9}{5}g_1^2+3g_2^2$
$T_u=T_d=T_e$	$T_u=3\,\mathrm{Tr}H_u$
$=3\,\mathrm{Tr}(H_u+H_d)+\mathrm{Tr}H_e$	$T_d=T_e$
	$=3\,\mathrm{Tr}H_d+\mathrm{Tr}H_e$
$a_u^u=a_d^d=+3/2$	$a_u^u=a_d^d=+3$
$a_u^d=a_d^u=-3/2$	$a_u^d=a_d^u=+1$
$a_e^e=+3/2$	$a_e^e=+3$

$$t=\ln(\mu/\Lambda_W), \tag{7.4}$$

$$\beta_a^{(1)}=c_a^{(1)}\mathbf{1}+\sum_b a_a^b H_b, \tag{7.5}$$

$$\beta_a^{(2)}=c_a^{(2)}\mathbf{1}+\sum_b b_a^b H_b+\sum_{b,c} b_a^{bc} H_b H_c, \tag{7.6}$$

$$H_a=Y_a Y_a^\dagger, \tag{7.7}$$

where, for convenience, we have changed the definition of the Hermitian matrix H_a from Eq. (6.8) to Eq. (7.7). The coefficients $c_a^{(1)}$ and a_a^b in the one-loop contributions $\beta_a^{(1)}$ are given in Table III according to models A and B, where

$$c_a^{(1)}=T_a-G_a. \tag{7.8}$$

The coefficients $c_a^{(2)}$, b_a^b and b_a^{bc} in the two-loop contributions $\beta_a^{(2)}$ are given in Appendix C, because their expressions are too long. The evolution of the gauge coupling constants $g_i(\mu)$ is given in Appendix D.

By using the information of $V_{ij}(\mu)$ at $\mu=m_Z$ in Sec. VI, we can obtain the knowledge not only of $M_q(m_Z)$, but also of $H_q(m_Z)$, i.e., $H_u=D_u^2$ and $H_d=VD_d^2V^\dagger$, where D_q ($q=u,d$) are the diagonalized matrices of Y_q. Then the expression for $H_a(\mu)$

$$\frac{d}{dt}H_a=\left[\frac{1}{16\pi^2}\beta_a^{(1)}+\frac{1}{(16\pi^2)^2}\beta_a^{(2)}\right]H_a+H_a\left[\frac{1}{16\pi^2}\beta_a^{(1)\dagger}+\frac{1}{(16\pi^2)^2}\beta_a^{(2)\dagger}\right], \tag{7.9}$$

is more useful rather than Eq. (7.3), which is the expression for Y_a. Hereafter, for simplicity, we calculate the evolution not from $\mu=\Lambda_W$, but from $\mu=m_Z$ because most of the input values at $\mu=m_Z$ have been given already in Sec. VI. Since the numerical results are insensitive to the value of the phase parameter δ_{13} ($\pi/3<\delta_{13}<2\pi/3$) in the CKM matrix V [Eq. (6.4)], we will use the value $\delta\equiv\delta_{13}=\pi/2$ below. For model A (the standard model with one Higgs boson), we must assume the value of the Higgs boson mass m_H. We will take a typical value $m_H=\sqrt{2}\Lambda_W=246.2$ GeV (see the later discussion). For model B (the minimal SUSY model), we must assume the value of $\tan\beta=v_u/v_d$. We will take a typical value $\tan\beta=10$. The numerical results of y_q are given below. Here the values y_{ii}^a are obtained by diagonalizing the matrix H_a, which does not mean $\sqrt{(H_a)_{ii}}$.

A. Standard model with one Higgs boson

As seen in Appendix A, in the calculation of the two-loop contributions, the evolution of the Yukawa coupling constants y_q depends on the coupling constant λ_H of the Higgs boson ϕ, which is related to the Higgs boson mass m_H as

$$\lambda_H=m_H^2/v^2. \tag{7.10}$$

We find [34] that the input value of $m_H(m_Z)$ that is less than 2.2×10^2 GeV leads to a negative λ_H at a unification scale $\mu=M_X$, while that which is larger than 2.6×10^2 GeV leads to the burst of λ_H at the unification scale. Therefore, if we establish an ansatz that nature accepts only the parameter

TABLE IV. Evolution of the Yukawa coupling constants y_a in the standard model with one Higgs boson (model A). For convenience, instead of $y_a(\mu)$, the values of $m_a(\mu)=y_a(\mu)v/\sqrt{2}$ are listed, where $v=\sqrt{2}\Lambda_W=246.2$ GeV. The errors $\pm\Delta m$ at $\mu=10^9$ GeV and $\mu=m_X$ denote only those from $\pm\Delta m$ at $\mu=m_Z$.

	$\mu=m_Z$		$\mu=10^9$ GeV		$\mu=M_X$	
$m_u(\mu)$	$2.33^{+0.42}_{-0.45}$	MeV	$1.28^{+0.23}_{-0.25}$	MeV	$0.94^{+0.17}_{-0.18}$	MeV
$m_c(\mu)$	677^{+56}_{-61}	MeV	371^{+31}_{-33}	MeV	272^{+22}_{-24}	MeV
$m_t(\mu)$	181 ± 13	GeV	109^{+16}_{-13}	GeV	84^{+18}_{-13}	GeV
$m_d(\mu)$	$4.69^{+0.60}_{-0.66}$	MeV	$2.60^{+0.33}_{-0.37}$	MeV	$1.94^{+0.25}_{-0.28}$	MeV
$m_s(\mu)$	$93.4^{+11.8}_{-13.0}$	MeV	$51.9^{+6.5}_{-7.2}$	MeV	$38.7^{+4.9}_{-5.4}$	MeV
$m_b(\mu)$	3.00 ± 0.11	GeV	$1.51^{+0.05}_{-0.06}$	GeV	1.07 ± 0.04	GeV
$m_e(\mu)$	0.48684727	MeV	0.51541746	MeV	0.49348567	MeV
	±0.00000014		±0.00000015		±0.00000014	
$m_\mu(\mu)$	102.75138	MeV	108.78126	MeV	104.15246	MeV
	±0.00033		±0.00035		±0.00033	
$m_\tau(\mu)$	1746.7 ± 0.3	MeV	1849.2 ± 0.3	MeV	1770.6 ± 0.3	MeV

TABLE V. Evolution of the Yukawa coupling constants y_a in the minimal SUSY model (model B). For convenience, instead of $y_a(\mu)$, the values of $m_a(\mu)=y_a(\mu)v\sin\beta/\sqrt{2}$ for up-quark sector and $m_a(\mu)=y_a(\mu)v\cos\beta/\sqrt{2}$ for the down-quark sector are listed, where $v=\sqrt{2}\Lambda_W$. The errors $\pm\Delta m$ at $\mu=10^9$ GeV and $\mu=M_X$ denote only those from $\pm\Delta m$ at $\mu=m_Z$.

	$\mu=m_Z$		$\mu=10^9$ GeV		$\mu=M_X$	
$m_u(\mu)$	$2.33^{+0.42}_{-0.45}$	MeV	$1.47^{+0.26}_{-0.28}$	MeV	$1.04^{+0.19}_{-0.20}$	MeV
$m_c(\mu)$	677^{+56}_{-61}	MeV	427^{+35}_{-38}	MeV	302^{+25}_{-27}	MeV
$m_t(\mu)$	181 ± 13	GeV	149^{+40}_{-26}	GeV	129^{+196}_{-40}	GeV
$m_d(\mu)$	$4.69^{+0.60}_{-0.66}$	MeV	$2.28^{+0.29}_{-0.32}$	MeV	$1.33^{+0.17}_{-0.19}$	MeV
$m_s(\mu)$	$93.4^{+11.8}_{-13.0}$	MeV	$45.3^{+5.7}_{-6.3}$	MeV	$26.5^{+3.3}_{-3.7}$	MeV
$m_b(\mu)$	3.00 ± 0.11	GeV	1.60 ± 0.06	GeV	1.00 ± 0.04	GeV
$m_e(\mu)$	0.48684727	MeV	0.40850306	MeV	0.32502032	MeV
	±0.00000014		±0.00000012		±0.00000009	
$m_\mu(\mu)$	102.75138	MeV	86.21727	MeV	68.59813	MeV
	±0.00033		±0.00028		±0.00022	
$m_\tau(\mu)$	1746.7 ± 0.3	MeV	$1469.5^{+0.3}_{-0.2}$	MeV	1171.4 ± 0.2	MeV

regions in which the perturbative calculations are valid, we can conclude that the Higgs boson mass m_H in the standard model must be in

$$220\ \text{GeV}<m_H(m_Z)<260\ \text{GeV}. \tag{7.11}$$

In Table IV, we list the numerical results of $m_q(\mu)=y_q(\mu)v/\sqrt{2}$ at the typical energy scales $\mu=m_Z$, $\mu=10^9$ GeV, and $\mu=M_X$. For the comparison with the SUSY model (model B) later, the values $m_q(\mu)$ at $\mu=M_X$ are listed, where M_X is a unification scale of SUSY, $M_X=2\times10^{16}$ GeV. Here we have tentatively taken a value $m_H=\sqrt{2}\Lambda_W=246.2$ GeV (i.e., $\lambda_H=1$) as the input value of $m_H(m_Z)$.

We also obtain the numerical expression of the CKM matrix $V(\mu)$ at $\mu=M_X$,

$$V(M_X)=\begin{pmatrix} 0.9754 & 0.2206 & -0.0035i \\ -0.2203 & 0.9745 & 0.0433 \\ 0.0101e^{-19°i} & -0.0422e^{+1.0°i} & 0.9991 \end{pmatrix}, \tag{7.12}$$

correspondingly to Eq. (6.7) at $\mu=m_Z$, where we have taken $\delta=90°$ tentatively. We also obtain the numerical result of (M_u,M_d) at $\mu=M_X$ correspondingly to Eqs. (6.19), (6.20), and (6.21):

$$M_u(M_X)=m_t(M_X)\begin{pmatrix} 1.11\times10^{-5} & 0 & 0 \\ 0 & 3.23\times10^{-3} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{7.13}$$

$$M_d(M_X)=m_b(M_X)\begin{pmatrix} 0.0035 & 0.0074e^{-1.2°i} & 0.0035e^{-95.3°i} \\ 0.0074e^{+1.2°i} & 0.0363 & 0.0418e^{+0.03°i} \\ 0.0035e^{+95.3°i} & 0.0418e^{-0.03°i} & 0.9982 \end{pmatrix}, \tag{7.14}$$

and

$$M_d(M_X)=m_b(M_X)\begin{pmatrix} -1.9\times10^{-5} & -0.0082e^{+1.1°i} & 0.0035e^{-84.1°i} \\ -0.0082e^{-1.1°i} & -0.0324 & 0.0447e^{-0.04°i} \\ 0.0035e^{+84.1°i} & 0.0447e^{+0.04°i} & 0.9980 \end{pmatrix}, \tag{7.15}$$

where $m_t(M_X)=84.2$ GeV and $m_b(M_X)=1.071$ GeV.

B. Minimal SUSY model

The scale of the SUSY symmetry breaking m_{SUSY} is usually taken as $m_{SUSY}\simeq m_t$ or $m_{SUSY}\simeq1$ TeV. For simplicity, we take $m_{SUSY}=m_Z$ in the present numerical study because the numerical results of $y_q(\mu)$ are not sensitive to the value of m_{SUSY}.

The values of $m_q(\mu)=y_q(\mu)v/\sqrt{2}$ $(q=u,d)$ are sensitive to the value of $\tan\beta=v_u/v_d$. A large value of $\tan\beta$, $\tan\beta\simeq60$, leads to the burst of $m_b(\mu)$ at the unification scale $\mu=M_X\simeq2\times10^{16}$ GeV. On the other hand, a small value of

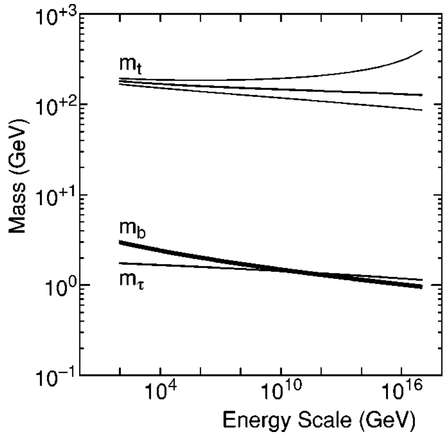

FIG. 6. Behavior of the Yukawa coupling constants $y_t(\mu)$, $y_b(\mu)$, and $y_\tau(\mu)$ in the minimal SUSY model. For convenience, the values are illustrated by the form $m_t(\mu)=y_t(\mu)v\sin\beta/\sqrt{2}$, $m_b(\mu)=y_b(\mu)v\cos\beta/\sqrt{2}$, and $m_\tau(\mu)=y_\tau(\mu)v\cos\beta/\sqrt{2}$.

$\tan\beta$, $\tan\beta\simeq 1.5$, leads to the burst of $m_t(\mu)$ at the unification scale. The values of $m_q(\mu)$ are insensitive to the value of $\tan\beta$ in the region from $\tan\beta\simeq 5$ to $\tan\beta\simeq 30$ [35]. In Table V, we list the numerical results of $m_q(\mu)$ at the typical energy scales, $\mu=m_Z$, $\mu=10^9$ GeV, and $\mu=M_X$. Here we have tentatively taken a value $\tan\beta=10$ as the input value of $\tan\beta$.

In Fig. 6, for reference, we illustrate the behavior of $m_t(\mu)$, $m_b(\mu)$, and $m_\tau(\mu)$. The value of $m_t(M_X)$ is highly dependent on the input value of $m_t(m_Z)$. Therefore, the value of $m_t(M_X)$ in Table V should not be taken strictly. Also, the energy scale μ_X at which $m_b(\mu_X)=m_\tau(\mu_X)$ is highly dependent on the input value of $m_b(m_Z)$. Therefore, the value μ_X should also not be taken strictly.

As seen in Fig. 6, it is very interesting that the observed top quark mass value is given by almost the upper value, which gives $m_q(\Lambda_W)\leqslant m_q(M_X)$. However, since the purpose of the present paper is not to investigate the evolution of the Yukawa coupling constants in the SUSY model under some postulation [e.g., $m_b(\mu)=m_\tau(\mu)$ at $\mu=M_X$], we do not go further. Several such studies will be found in Refs. [35,36].

We also obtain the numerical expression of the CKM matrix $V(\mu)$ at $\mu=M_X$,

$$V(M_X)=\begin{pmatrix} 0.9754 & 0.2205 & -0.0026i \\ -0.2203e^{+0.03°i} & 0.9749 & 0.0318 \\ 0.0075e^{-19°i} & -0.0311e^{+1.0°i} & 0.9995 \end{pmatrix}, \tag{7.16}$$

correspondingly to Eq. (6.7) at $\mu=m_Z$, where we have taken $\delta=90°$ tentatively. We also obtain the numerical result of (M_u,M_d) at $\mu=M_X$ correspondingly to Eqs. (6.19), (6.20), and (6.21):

$$M_u(M_X)=m_t(M_X)\begin{pmatrix} 8.0\times10^{-6} & 0 & 0 \\ 0 & 2.33\times10^{-3} & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{7.17}$$

$$M_d(M_X)=m_b(M_X)\begin{pmatrix} 0.0026 & 0.0054e^{-0.9°i} & 0.0025e^{-93.9°i} \\ 0.0054e^{+0.9°i} & 0.0263 & 0.0310e^{+0.03°i} \\ 0.0025e^{+93.9°i} & 0.0310e^{-0.03°i} & 0.9990 \end{pmatrix}, \tag{7.18}$$

and

$$M_d(M_X)=m_b(M_X)\begin{pmatrix} -1.6\times10^{-5} & -0.0060e^{+0.8°i} & 0.0026e^{-85.8°i} \\ -0.0060e^{-0.8°i} & -0.0241 & 0.0326e^{-0.03°i} \\ 0.0026e^{+85.8°i} & 0.0326e^{+0.03°i} & 0.9990 \end{pmatrix}, \tag{7.19}$$

where $m_t(M_X)=129.3$ GeV and $m_b(M_X)=0.997$ GeV.

VIII. SUMMARY

In conclusion, we have evaluated the running quark mass values $m_q(\mu)$ $(q=u,d,s,c,b,t)$ at various energy scales μ ($\mu=1$ GeV, $\mu=m_q$, $\mu=m_Z$, and so on). The values of $m_q(m_q)$ given in Table II in Sec. IV will be convenient for hadron physicists who want to calculate hadronic matrix elements on the bases of the quark-parton model, heavy-quark effective theory, and so on. Also, the values of $m_q(\mu)$, $m_l(\mu)$, $|V_{ij}(\mu)|$, and $\alpha_i(\mu)$ at $\mu=m_Z$ given in Sec. VI will be convenient for quark and lepton mass-matrix model builders. In quark mass-matrix phenomenology, the values of $m_q(\mu)$ at $\mu=1$ GeV conventionally have been used. However, we recommend the use of the values $m_q(m_Z)$ rather than $m_q(1\text{ GeV})$ because we can use the observed values of $|V_{ij}|$ as the values $|V_{ij}(m_Z)|$ straightforwardly and, exactly speaking, the value of $m_t(1\text{ GeV})$ does not have meaning.

Although, in Sec. VII, we have given the values of $m_q(\mu)$ at $\mu=M_X$, i.e., the evolution of the Yukawa coupling constants $y_q(\mu)$, the study was not systematic, in contrast to the study for $\mu\leqslant\Lambda_W$. The values of $y_q(\mu)$ in the standard model with one Higgs boson depend on the input value of the boson mass $m_H(m_Z)$. The values of $y_q(\mu)$ in the minimal SUSY model depend on the values of the parameters m_{SUSY} and $\tan\beta$. Therefore, the values $m_q(M_X)$ given in Tables IV and V in Sec. VII should be taken only for reference.

We hope that most of the present results, Table II in Sec. IV and Eqs. (6.1), (6.7), (6.13), and (6.14) in Sec. VI, are made useful by particle physicists.

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APPENDIX A: RELATION BETWEEN $m_q(m_q)$ AND M_q^{pole}

The pole mass $M_q^{pole}(p^2=m_q^2)$ is a gauge-invariant, infrared-finite, renormalization-scheme-independent quantity. Generally, the mass function $M(p^2)$, which is defined by [1]

$$S(p)=Z(p^2)/[M(p^2)-\not{p}], \tag{A1}$$

$$Z(p^2)=1-\frac{\alpha_s}{3\pi}\left(a-3b+\frac{2}{3}\right)\lambda+O(\alpha_s^2), \tag{A2}$$

is related to

$$M(p^2)=m(\mu)\left[1+\frac{\alpha_s}{\pi}(a+\lambda b)+O(\alpha_s^2)\right], \tag{A3}$$

$$a=\frac{4}{3}-\ln\frac{m^2}{\mu^2}+\frac{m^2-p^2}{p^2}\ln\frac{m^2-p^2}{m^2}, \tag{A4}$$

$$b=-\frac{m^2-p^2}{3p^2}\left(1+\frac{m^2}{p^2}\ln\frac{m^2-p^2}{m^2}\right), \tag{A5}$$

where λ is given by $\lambda=0$ in the Landau gauge and by $\lambda=1$ in the Feynman gauge. For $p^2=m^2$, we obtain $a=4/3$ and $b=0$, so that we obtain the relation

$$M_q^{pole}(p^2=m_q^2)=m_q(m_q)\left(1+\frac{4}{3}\frac{\alpha_s}{\pi}+O(\alpha_s^2)\right). \tag{A6}$$

TABLE VI. Pole masses M_q^{pole} and the related quantities. The values with an asterisk should not be taken rigidly because these values have been calculated in the region with a large $\alpha_s(\mu)$.

	K_0	$\Delta(M_i/M_n)$	K	M_q^{pole}	$m_q(M_q^{pole})$
u	16.11	0	* 16.11	* 0.501 MeV	* 0.0307 MeV
d	15.07	* 0.838	* 16.19	* 0.517 MeV	* 0.0445 MeV
s	14.03	* 1.364	* 15.85	* 0.687 MeV	* 0.283 MeV
c	12.99	1.114	14.47	1.59 GeV	1.213 GeV
b	11.94	0.746	12.94	4.89 GeV	4.248 GeV
t	10.90	0.0555	10.98	180 GeV	170.1 GeV

Similarly, for the spacelike value of p^2, $p^2=-m_q^2$, we obtain $a=4/3-2\ln2$ and $b=(2/3)(1-\ln2)$, so that we obtain the gauge-dependent ''Euclidean'' masses

$$M_q^{pole}(p^2=-m_q^2)=m_q(m_q)\left[1+\frac{\alpha_s}{\pi}\left(\frac{4}{3}-2\ln2\right)+O(\alpha_s^2)\right]. \tag{A7}$$

The estimate of the pole mass has been given by Gray *et al.* [6] (also see [38]):

$$m_q(M_q^{pole})=M_q^{pole}\Big/\left[1+\frac{4}{3}\frac{\alpha_s(M_q^{pole})}{\pi}+K_q\left(\frac{\alpha_s(M_q^{pole})}{\pi}\right)^2+O(\alpha_s^3)\right], \tag{A8}$$

$$K_q=K_0+\frac{4}{3}\sum_{i=1}^{n-1}\Delta(M_i^{pole}/M_n^{pole}), \tag{A9}$$

$$K_0=\frac{1}{9}\pi^2\ln2+\frac{7}{18}\pi^2-\frac{1}{6}\zeta(3)+\frac{3673}{288}-\left(\frac{1}{18}\pi^2+\frac{71}{144}\right)n, \tag{A10}$$

$$\Delta(r)=\frac{1}{4}\left[\ln^2r+\frac{1}{6}\pi^2-\left(\ln r+\frac{3}{2}\right)r^2\right. \tag{A11}$$

$$\left.-(1+r)(1+r^3)L_+(r)-(1-r)(1-r^3)L_-(r)\right], \tag{A12}$$

$$L_\pm(r)=\int_0^{1/r}dx\frac{\ln x}{x\pm1}. \tag{A13}$$

Here the sum in Eq. (A9) is taken over $n-1$ light quarks with masses M_i^{pole} ($M_i^{pole}<M_n^{pole}\equiv M_q^{pole}$). The numerical results are summarized in Table VI.

In Table VI, the values of M_q^{pole} and $m_q(M_q^{pole})$ for the light quarks $q=u,d,s$ have been obtained by solving the relation (A8) with the help of Eq. (A7) with the input (2.11). These values for the light quarks should not be taken rigidly because the perturbative calculation is unreliable for the region at which $\alpha_s(\mu)$ takes a large value. Fortunately, the values of K_q are not sensitive to the values of M_q^{pole} for the light quarks $q=u,d,s$. Therefore, the values of K_q in Table VI are valid not only for the heavy quarks $q=c,b,t$ but also for the light quarks $q=u,d,s$.

APPENDIX B: ESTIMATE OF $\Lambda_{\overline{MS}}^{(n)}$

The effective QCD coupling $\alpha_s = g_s^2/4\pi$ is governed by the β function:

$$\mu \frac{\partial \alpha_s}{\partial \mu} = \beta(\alpha_s), \tag{B1}$$

where

$$\beta(\alpha_s) = -\frac{\beta_0}{2\pi}\alpha_s^2 - \frac{\beta_1}{4\pi^2}\alpha_s^3 - \frac{\beta_2}{64\pi^3}\alpha_s^4 + O(\alpha_s^5), \tag{B2}$$

$$\beta_0 = 11 - \frac{2}{3}n_q, \quad \beta_1 = 51 - \frac{19}{3}n_q,$$

$$\beta_2 = 2857 - \frac{5033}{9}n_q + \frac{325}{27}n_q^2, \tag{B3}$$

and n_q is the effective number of quark flavors [39]. The solution $\alpha_s(\mu)$ of Eq. (B1) is given by [2]

$$\alpha_s(\mu) = \frac{4\pi}{\beta_0}\frac{1}{L}\left\{1 - \frac{2\beta_1}{\beta_0^2}\frac{\ln L}{L} + \frac{4\beta_1^2}{\beta_0^4 L^2}\left[\left(\ln L - \frac{1}{2}\right)^2 + \frac{\beta_2\beta_0}{8\beta_1^2} - \frac{5}{4}\right]\right\} + O\left(\frac{\ln^2 L}{L^3}\right), \tag{B4}$$

where

$$L = \ln(\mu^2/\Lambda^2). \tag{B5}$$

The value of $\alpha_s(\mu)$ is not continuous at the nth quark threshold μ_n (at which the nth quark flavor channel is opened) because the coefficients β_0, β_1, and β_2 in Eq. (B2) depend on the effective quark flavor number n_q. Therefore, we use the expression $\alpha_s^{(n)}(\mu)$ [Eq. (B3)] with a different $\Lambda_{\overline{MS}}^{(n)}$ for each energy scale range $\mu_n \leqslant \mu < \mu_{n+1}$. The relationship between $\Lambda_{\overline{MS}}^{(n-1)}$ and $\Lambda_{\overline{MS}}^{(n)}$ is fixed at $\mu = m_q^{(n)}$, where $m_q^{(n)}$ is the value of the nth running quark mass $m_q^{(n)} = m_{qn}(m_{qn})$ and is given as [40]

$$2\beta_0^{(n-1)}\ln\left(\frac{\Lambda_{\overline{MS}}^{(n)}}{\Lambda_{\overline{MS}}^{(n-1)}}\right) = (\beta_0^{(n)} - \beta_0^{(n-1)})L_{\overline{MS}}^{(n)} + 2\left(\frac{\beta_1^{(n)}}{\beta_0^{(n)}} - \frac{\beta_1^{(n-1)}}{\beta_0^{(n-1)}}\right)\ln(L_{\overline{MS}}^{(n)}) - \frac{2\beta_1^{(n-1)}}{\beta_0^{(n-1)}}\ln\left(\frac{\beta_0^{(n)}}{\beta_0^{(n-1)}}\right) + \frac{4\beta_1^{(n)}}{(\beta_0^{(n)})^2}\left(\frac{\beta_1^{(n)}}{\beta_0^{(n)}} - \frac{\beta_1^{(n-1)}}{\beta_0^{(n-1)}}\right)\frac{\ln(L_{\overline{MS}}^{(n)})}{L_{\overline{MS}}^{(n)}} + \frac{1}{\beta_0^{(n)}}\left[\left(\frac{2\beta_1^{(n)}}{\beta_0^{(n)}}\right)^2 - \left(\frac{2\beta_1^{(n-1)}}{\beta_0^{(n-1)}}\right)^2 - \frac{\beta_2^{(n)}}{2\beta_0^{(n)}} + \frac{\beta_2^{(n-1)}}{2\beta_0^{(n-1)}} - \frac{22}{9}\right]\frac{1}{L_{\overline{MS}}^{(n)}}, \tag{B6}$$

TABLE VII. Values of $\Lambda_{\overline{MS}}^{(n)}$ in units of GeV and $\alpha_s(\mu_n)$. The underlined values are input values.

n	$\Lambda_{\overline{MS}}^{(n)}$	$\alpha_s^{(n)}(\mu_n)$	μ_n
3	$0.333^{+0.047}_{-0.042}$	$1.69^{+0.38}_{-0.33}$	$\mu_3 = 0.553$ GeV
4	$0.291^{+0.048}_{-0.041}$	$0.379^{+0.048}_{-0.039}$	$\mu_4 = 1.302$ GeV
5	$\underline{0.209^{+0.039}_{-0.033}}$	$0.222^{+0.013}_{-0.012}$	$\mu_5 = 4.339$ GeV
6	$0.0882^{+0.0191}_{-0.0159}$	$0.1078^{+0.0036}_{-0.0035}$	$\mu_6 = 170.8$ GeV

where

$$L_{\overline{MS}}^{(n)} = \ln(m_q^{(n)}/\Lambda_{\overline{MS}}^{(n)})^2. \tag{B7}$$

The Particle Data Group [2] has concluded that the world average of $\Lambda_{\overline{MS}}^{(5)}$ is

$$\Lambda_{\overline{MS}}^{(5)} = 209^{+39}_{-33} \text{ MeV}. \tag{B8}$$

Starting from $\Lambda_{\overline{MS}}^{(5)} = 0.209$ GeV, by using the relation (B6), at $\mu_5 = m_b(m_b) = 4.339$ GeV, $\mu_4 = m_c(m_c) = 1.302$ GeV, and $\mu_6 = m_t(m_t) = 170.8$ GeV, we evaluate the values of $\Lambda_{\overline{MS}}^{(n)}$ for $n = 3,4$, and 6. The results are summarized in Table VII.

We show the threshold behaviors of $\alpha_s^{(n)}(\mu)$ in Fig. 7. We can see that $\alpha_s^{(n-1)}(\mu)$ in $\mu_{n-1} \leqslant \mu < \mu_n$ connects with $\alpha_s^{(n)}(\mu)$ in $\mu_n \leqslant \mu < \mu_{n+1}$ continuously.

APPENDIX C: EVOLUTION OF THE YUKAWA COUPLING CONSTANTS

The coefficients $c_a^{(2)}$, b_a^b, and b_a^{bc} in the two-loop contributions $\beta_a^{(2)}$ are given as follows. Here T_a $(a = u,d,e)$ are given in Table III in Sec. VII and n_g is the number of generations. For the standard model with one Higgs scalar,

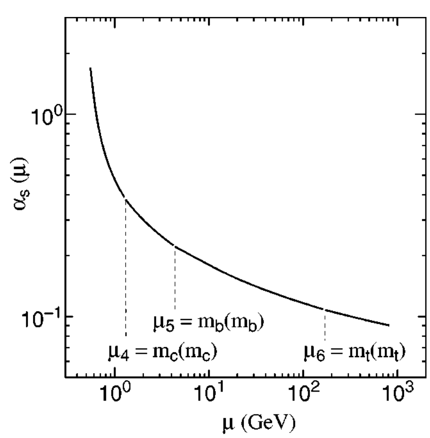

FIG. 7. Threshold behavior of $\alpha_s^{(n)}(\mu)$ versus μ.

$$b_u^{uu}=b_d^{dd}=\frac{3}{2},\quad b_u^{dd}=b_d^{uu}=\frac{11}{4},\quad b_e^{ee}=\frac{3}{2},$$

$$b_u^{ud}=b_d^{du}=-\frac{1}{4},\quad b_u^{du}=b_d^{ud}=-1, \tag{C1}$$

$$b_u^{u}=-\frac{9}{4}T_u-6\lambda_H+\frac{223}{80}g_1^2+\frac{135}{16}g_2^2+16g_3^2,$$

$$b_d^{d}=-\frac{9}{4}T_d-6\lambda_H+\frac{187}{80}g_1^2+\frac{135}{16}g_2^2+16g_3^2,$$

$$b_e^{e}=-\frac{9}{4}T_e-6\lambda_H+\frac{387}{80}g_1^2+\frac{135}{16}g_2^2, \tag{C2}$$

$$b_u^{d}=\frac{5}{4}T_u-2\lambda_H-\left(\frac{43}{80}g_1^2-\frac{9}{16}g_2^2+16g_3^2\right),$$

$$b_d^{u}=\frac{5}{4}T_d-2\lambda_H-\left(\frac{79}{80}g_1^2-\frac{9}{16}g_2^2+16g_3^2\right), \tag{C3}$$

$$c_u^{(2)}=-X_4+\frac{5}{2}Y_4+\frac{3}{2}\lambda_H^2+\left(\frac{9}{200}+\frac{29}{45}n_g\right)g_1^4-\frac{9}{20}g_1^2g_2^2+\frac{19}{15}g_1^2g_3^2-\left(\frac{35}{4}-n_g\right)g_2^4+9g_2^2g_3^2-\left(\frac{404}{3}-\frac{80}{9}n_g\right)g_3^4,$$

$$c_d^{(2)}=-X_4+\frac{5}{2}Y_4+\frac{3}{2}\lambda_H^2-\left(\frac{29}{200}+\frac{1}{45}n_g\right)g_1^4-\frac{27}{20}g_1^2g_2^2+\frac{31}{15}g_1^2g_3^2-\left(\frac{35}{4}-n_g\right)g_2^4+9g_2^2g_3^2-\left(\frac{404}{3}-\frac{80}{9}n_g\right)g_3^4,$$

$$c_e^{(2)}=-X_4+\frac{5}{2}Y_4+\frac{3}{2}\lambda_H^2+\left(\frac{51}{200}+\frac{11}{5}n_g\right)g_1^4+\frac{27}{20}g_1^2g_2^2-\left(\frac{35}{4}-n_g\right)g_2^4, \tag{C4}$$

where

$$X_4=\frac{9}{4}\mathrm{Tr}\left(3H_u^2-\frac{2}{3}H_uH_d+3H_d^2+H_e^2\right), \tag{C5}$$

$$Y_4=\left(\frac{17}{20}g_1^2+\frac{9}{4}g_2^2+8g_3^2\right)\mathrm{Tr}H_u+\left(\frac{1}{4}g_1^2+\frac{9}{4}g_2^2+8g_3^2\right)\mathrm{Tr}H_d+\left(\frac{3}{4}g_1^2+\frac{3}{4}g_2^2\right)\mathrm{Tr}H_e\ , \tag{C6}$$

$$\lambda_H=m_H^2/v^2, \tag{C7}$$

and the evolutions of g_i ($i=1,2,3$) and λ_H are given in Appendix D. For the minimal SUSY model,

$$b_u^{uu}=b_d^{dd}=-4,\quad b_u^{dd}=b_d^{uu}=-2,\quad b_e^{ee}=-4,$$

$$b_u^{ud}=b_d^{du}=-2,\quad b_u^{du}=b_d^{ud}=0, \tag{C8}$$

$$b_u^{u}=-3T_u+\frac{2}{5}g_1^2+6g_2^2,\quad b_d^{d}=-3T_d+\frac{4}{5}g_1^2+6g_2^2,$$

$$b_e^{e}=-3T_e+6g_2^2,$$

$$b_u^{d}=-T_d+\frac{2}{5}g_1^2,\quad b_d^{u}=-T_u+\frac{4}{5}g_1^2, \tag{C9}$$

$$c_u^{(2)}=-3\ \mathrm{Tr}(3H_u^2+H_uH_d)+\left(\frac{4}{5}g_1^2+16g_3^2\right)\mathrm{Tr}H_u+\left(\frac{403}{450}+\frac{26}{15}n_g\right)g_1^4+g_1^2g_2^2+\frac{136}{45}g_1^2g_3^2-\left(\frac{21}{2}-6n_g\right)g_2^4+8g_2^2g_3^2-\left(\frac{304}{9}-\frac{32}{3}n_g\right)g_3^4,$$

$$c_d^{(2)}=-3\ \mathrm{Tr}(3H_d^2+H_uH_d+H_e^2)+\left(-\frac{2}{5}g_1^2+16g_3^2\right)\mathrm{Tr}H_d+\frac{6}{5}g_1^2\mathrm{Tr}H_e+\left(\frac{7}{18}+\frac{14}{15}n_g\right)g_1^4+g_1^2g_2^2+\frac{8}{9}g_1^2g_3^2-\left(\frac{21}{2}-6n_g\right)g_2^4+8g_2^2g_3^2-\left(\frac{304}{9}-\frac{32}{3}n_g\right)g_3^4,$$

$$c_e^{(2)}=-3\ \mathrm{Tr}(3H_d^2+H_uH_d+H_e^2)+\left(-\frac{2}{5}g_1^2+16g_3^2\right)\mathrm{Tr}H_d+\frac{6}{5}g_1^2\mathrm{Tr}H_e+\left(\frac{27}{10}+\frac{18}{5}n_g\right)g_1^4+\frac{9}{5}g_1^2g_2^2-\left(\frac{21}{2}-6n_g\right)g_2^4. \tag{C10}$$

APPENDIX D: EVOLUTION OF THE GAUGE COUPLING CONSTANTS

The evolution of gauge coupling constants is given by

$$\frac{dg_i}{dt}=-b_i\frac{g_i^3}{16\pi^2}-\sum_k\ b_{ik}\frac{g_i^3g_k^2}{(16\pi^2)^2}-\frac{g_i^3}{(16\pi^2)^2}\sum_a\ c_{ia}\mathrm{Tr}H_a\,, \tag{D1}$$

TABLE VIII. Coefficients in the evolution equations of gauge coupling constants.

Model A	Model B
$b_1=-(\frac{1}{10}+\frac{4}{3}n_g)$	$b_1=-(\frac{3}{5}+2n_g)$
$b_2=\frac{43}{6}-\frac{4}{3}n_g$	$b_2=5-2n_g$
$b_3=11-\frac{4}{3}n_g$	$b_3=9-2n_g$
$(b_{ik})=\begin{pmatrix} -\frac{9}{50} & -\frac{9}{10} & 0 \\ -\frac{3}{10} & \frac{259}{6} & 0 \\ 0 & 0 & 102 \end{pmatrix}$	$(b_{ik})=\begin{pmatrix} -\frac{9}{25} & -\frac{9}{5} & 0 \\ -\frac{3}{5} & 17 & 0 \\ 0 & 0 & 54 \end{pmatrix}$
$-n_g\begin{pmatrix} \frac{19}{15} & \frac{3}{5} & \frac{44}{15} \\ \frac{1}{5} & \frac{49}{3} & 4 \\ \frac{11}{30} & \frac{3}{2} & \frac{76}{3} \end{pmatrix}$	$-n_g\begin{pmatrix} \frac{38}{15} & \frac{6}{5} & \frac{88}{15} \\ \frac{2}{5} & 14 & 8 \\ \frac{11}{15} & 3 & \frac{68}{3} \end{pmatrix}$
$(c_{ia})=\begin{pmatrix} \frac{17}{10} & \frac{1}{2} & \frac{3}{2} \\ \frac{3}{2} & \frac{3}{2} & \frac{1}{2} \\ 2 & 2 & 0 \end{pmatrix}$	$(c_{ia})=\begin{pmatrix} \frac{26}{5} & \frac{14}{5} & \frac{18}{5} \\ 6 & 6 & 2 \\ 4 & 4 & 0 \end{pmatrix}$

where the coefficients b_i, b_{ik}, and c_{ia} are given in Table VIII. The evolution of the coupling constants λ_H given in Eq. (C7) is given by

$$\frac{d\lambda_H}{dt}=\frac{1}{16\pi^2}\beta_\lambda^{(1)}+\frac{1}{(16\pi^2)^2}\beta_\lambda^{(2)}, \tag{D2}$$

$$\beta_\lambda^{(1)}=12\lambda_H^2-\left(\frac{9}{5}g_1^2+9g_2^2\right)\lambda_H+\frac{9}{4}\left(\frac{3}{25}g_1^4+\frac{2}{5}g_1^2g_2^2+g_2^4\right)+4\lambda_H\mathrm{Tr}(3H_u+3H_d+H_e)-4\mathrm{Tr}(3H_u^2+3H_d^2+H_e^2), \tag{D3}$$

$$\begin{aligned}\beta_\lambda^{(2)}=&-78\lambda_H^3+18\left(\frac{3}{5}g_1^2+3g_2^2\right)\lambda_H^2-\left[\left(\frac{313}{8}-10n_g\right)g_2^4-\frac{117}{20}g_1^2g_2^2+\frac{9}{25}\left(\frac{229}{24}+\frac{50}{9}n_g\right)g_1^4\right]\lambda_H+\left(\frac{497}{8}\right.\\&\left.-8n_g\right)g_2^6-\frac{3}{5}\left(\frac{97}{24}+\frac{8}{3}n_g\right)g_1^2g_2^4-\frac{9}{25}\left(\frac{239}{24}+\frac{40}{9}n_g\right)g_1^4g_2^2-\frac{27}{125}\left(\frac{59}{24}+\frac{40}{9}n_g\right)g_1^6-64g_3^2\mathrm{Tr}(H_u^2\\&+H_d^2)-\frac{8}{5}g_1^2\mathrm{Tr}(2H_u^2-H_d^2+3H_e^2)-\frac{3}{2}g_2^4\mathrm{Tr}(3H_u+3H_d+H_e)+10\lambda_H\left[\left(\frac{17}{20}g_1^2+\frac{9}{4}g_2^2+8g_3^2\right)\mathrm{Tr}H_u\right.\\&\left.+\left(\frac{1}{4}g_1^2+\frac{9}{4}g_2^2+8g_3^2\right)\mathrm{Tr}H_d+\frac{3}{4}(g_1^2+g_2^2)\mathrm{Tr}H_e\right]+\frac{3}{5}g_1^2\left[\left(-\frac{57}{10}g_1^2+21g_2^2\right)\mathrm{Tr}H_u+\left(\frac{3}{2}g_1^2+9g_2^2\right)\mathrm{Tr}H_d\right.\\&\left.+\left(-\frac{15}{2}g_1^2+11g_2^2\right)\mathrm{Tr}H_e\right]-24\lambda_H^2\mathrm{Tr}(3H_u+3H_d+H_e)-\lambda_H\mathrm{Tr}[3(H_u-H_d)^2+H_e^2]+20\,\mathrm{Tr}(3H_u^3+3H_d^3\\&+H_e^3)-12\,\mathrm{Tr}[H_uH_d(H_u+H_d)].\end{aligned} \tag{D4}$$

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