Duality and massive gauge-invariant theories

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It is shown that "$B^\wedge F$" theory in 3+1 dimensions with a Kalb-Ramond field is related by Buscher's duality transformation to two different versions of massive gauge-invariant spin-one theories of Stückelberg-type, one involving vector and scalar fields and the other antisymmetric tensor and vector fields. A similar construction for massive spin-zero theory is also shown. The implication of this equivalence to the five-dimensional theories from which these theories can be obtained is discussed. [S0556-2821(98)01406-4]

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Massive gauge invariant spin-one theories have been studied for a long time with two principle procedures: the Schwinger mechanism in 1+1 dimensions [1] of two-dimensional (2D) quantum electrodynamics of massless Fermion, yielding a massive gauge field, through the axial anomaly and the Higgs mechanism. In 3+1 dimensions, massive spin-one theories with gauge invariance are generally considered following one of two procedures: one by the Stückelberg formulation [2] which is the more familiar Higgs mechanism in its simplest form and the other by using a Kalb-Ramond field [3] (rank two antisymmetric tensor gauge field) in a Chern-Simons-like formulation known as $B^\wedge F$ theory [4]. The latter is well studied in different contexts, including a realization of certain condensed matter systems [5], as an alternate to the Higgs mechanism [6] and as a realization of the bosonized Schwinger model in 3+1 dimensions by Aurilia and Takahashi [7]. Antisymmetric tensor fields also appear naturally in string theories and play an important role in realizing duality among string theories. On the other hand, the Stückelberg formulation of spin-one (and also for higher spin fields) [8] has been studied in various contexts, such as for consistency problems in higher spin fields [9] and in string field theory as a description of massive modes [10] and shown to arise as Kaluza-Klein dimensional reduction of five-dimensional massless theories. Though they appear as different constructions for maintaining gauge invariance in the presence of mass terms, in this paper we show that these theories are related by the duality transformation.

First we note that in $B^\wedge F$ theory the current due to local gauge symmetry is conserved as an algebraic identity, such as that for topological currents and in the case of Stückelberg formulations, it is conserved due to the equation of motion of the Stückelberg field, as it happens for Noether current. Since these theories describe massive spin-one particles and this interchange between topological and Noether current generally takes place under duality transformation, it is natural to enquire if these theories are related by duality transformation. This is demonstrated in this paper by the well known Buscher's duality [11–16] procedure.

An equivalence between non-Abelian $B^\wedge F$ theory and massive Yang-Mills theory has been shown earlier by Freedman and Townsend [19]. But here we show that the Abelian $B^\wedge F$ theory is (i) equivalent to two different Stückelberg formulations of massive spin-one theory (one of them involves vector and scalar fields and the other antisymmetric tensor and vector fields) and (ii) they are related through duality. This equivalence, apart from its intrinsic interest, is also relevant, in the case of compact gauge fields, for three-dimensional Josephson junction arrays, for which $B^\wedge F$ has been shown to be an effective field theory [5] and hence can possibly be used to map its phase structure. Also since these Stückelberg-type theories for the case of compact fields have recently been shown [20] to describe the condensed phase of dual topological defects, it may be of relevance for a unified description of such phases. We use the metric $g_{\mu\nu}=\mathrm{diag}(1,-1,-1,-1)$ and $\epsilon_{0123}=1$.

The topologically massive spin-one theory involving Kalb-Romand field $B_{\mu\nu}$ and a vector field A_μ which is known as $B^\wedge F$ theory, is given by

$$L=-\frac{1}{4}F_{\mu\nu}^2+\frac{1}{2(3!)}H_{\mu\nu\lambda}^2-\frac{m}{3!}H_{\mu\nu\lambda}\epsilon^{\mu\nu\lambda\rho}A_\rho, \qquad (1)$$

where $H_{\mu\nu\lambda}=\partial_\mu B_{\nu\lambda}+\partial_\nu B_{\lambda\mu}+\partial_\lambda B_{\nu\mu}$. This Lagrangian has local invariance under

$$A_\mu \to A_\mu+\partial_\mu\Lambda, \qquad (2)$$

$$B_{\mu\nu}\to B_{\mu\nu}+(\partial_\mu\Lambda_\nu-\partial_\nu\Lambda_\mu). \qquad (3)$$

The field equations following from this are

$$\partial_\mu F^{\mu\nu}=J^\nu, \qquad (4)$$

$$\partial_\mu H^{\mu\nu\lambda}=J^{\nu\lambda}. \qquad (5)$$

where

$$J^\mu=\frac{m}{3!}\epsilon^{\mu\nu\lambda\rho}H_{\nu\lambda\rho}$$

and

$$j^{\mu\nu}=\frac{m}{3!}\epsilon^{\mu\nu\lambda\rho}F_{\lambda\rho},$$

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are the currents associated with local gauge symmetry (2), (3).

Note that both currents are conserved as an algebraic identity, such as that of topological current. The fact that this describes massive spin-one theory can be shown easily by solving the coupled differential equations [7].

Next we consider the Stückelberg formulation of massive vector theory whose Lagrangian is

$$L=-\frac{1}{4}F_{\mu\nu}^2-\frac{1}{2}(\partial_\mu\Phi-mA_\mu)^2. \tag{6}$$

This Lagrangian has invariance under

$$A_\mu\to A_\mu+\partial_\mu\Lambda, \tag{7}$$

$$\Phi\to\Phi+m\Lambda. \tag{8}$$

The equations of motion following from this Lagrangian (6) for A_μ and Φ are

$$\partial_\mu F^{\mu\nu}+K^\nu=0, \tag{9}$$

$$\partial^\mu(\partial_\mu\Phi-mA_\mu)=0, \tag{10}$$

where

$$K^\nu\equiv m(\partial^\nu\Phi-mA^\nu).$$

Now note that the current associated with A_μ field is conserved due to the equation of motion of the Φ field, such as that for Noether current. The fact that this describes massive spin-one theory can be seen by using the gauge invariance (8) and fixing the field Φ to be zero. This in the usual Higgs mechanism means that the massless vector field ''eats'' up the spin-zero Goldstone particle to become massive.

Thus we have two (apparently) different formulations of spin-one theory. But the nature of currents in the two theories and physical equivalence of the system they describe, viz., massive spin-one particle, forces one to enquire if both these formulations are related by duality transformation. We next show, indeed that is the case.

The dual theory is obtained by the procedure of gauging the global symmetry in the model by a gauge field and constraining its dual field strength to be zero by means of a Lagrange multiplier, and by integrating the original and the gauge field and expressing the theory in terms of the multiplier field. The global symmetry, in question, in model (1) is

$$\delta B_{\mu\nu}=\epsilon_{\mu\nu}, \tag{11}$$

$$\delta A_\mu=0. \tag{12}$$

(Note by dropping a surface term, the global symmetry is on the vector field. This is discussed later.) This symmetry is gauged by introducing a three form gauge potential, G in the Lagrangian (1). The dual field strength of $G_{\mu\nu\lambda}$ is gauge invariant under $\delta G_{\mu\nu\lambda}=\partial_\mu\eta_{\nu\lambda}$. By adding a scalar field as a Lagrange multiplier, the dual field strength is constrained to be flat. Thus the Lagrangian, invariant under Eq. (11), is

$$L=-\frac{1}{4}F_{\mu\nu}^2+\frac{1}{2(3!)}(H_{\mu\nu\lambda}-G_{\mu\nu\lambda})^2-\frac{m}{3!}(H_{\mu\nu\lambda}-G_{\mu\nu\lambda})\epsilon^{\mu\nu\lambda\rho}A_\rho+\frac{1}{3!}\Phi\epsilon^{\mu\nu\lambda\rho}\partial_\mu G_{\nu\lambda\rho}. \tag{13}$$

Note that the original gauge invariance of the vector field $\delta A_\mu=\partial_\mu\Lambda$ is recovered only when, under A_μ gauge transformation, the scalar field also transforms as

$$\Phi\to\Phi+m\Lambda. \tag{14}$$

This transformation is the same as that of Stückelberg formulation of the theory (8). Indeed, by integrating over $B_{\mu\nu}$ and A_μ fields, which appear as the Gaussian, Stückelberg theory (6) result.

Similarly an alternative spin-one theory is given by

$$L=\frac{1}{2(3!)}H_{\mu\nu\lambda}^2+(mB_{\mu\nu}-\Phi_{\mu\nu})^2, \tag{15}$$

where $\Phi_{\mu\nu}=(\partial_\mu\Phi_\nu-\partial_\nu\Phi_\mu)$. This Lagrangian has invariance under

$$\delta B_{\mu\nu}=(\partial_\mu\epsilon_\nu-\partial_\nu\epsilon_\mu), \tag{16}$$

$$\delta\Phi_\mu=m\epsilon_\mu+\partial_\mu\chi. \tag{17}$$

This Stückelberg-type action for the two-form field [14] was constructed and studied earlier by Aurilia and Takahashi.

Here the current associated with $B_{\mu\nu}$ behaves as a Noether current. By fixing the gauge $\Phi_\mu=0$, it is obvious that it describes massive spin-one theory. But in contrast to the usual Higgs mechanism, here it is the massless spin-zero field, described by $B_{\mu\nu}$ eats up Φ_μ to become massive. In the case of compact fields this theory describes condensation of dual topological object. Next we show that this formulation also results from Eq. (1).

Instead of considering the global symmetry in the two-form B field, one could, after omitting a surface term in the Lagrangian (1), consider a global symmetry in A_μ field of the form $\delta A_\mu=\epsilon_\mu$ and $\delta B_{\mu\nu}=0$. Gauging this symmetry, one gets

$$L=-\frac{1}{4}(F_{\mu\nu}-G_{\mu\nu})^2+\frac{1}{3!}H_{\mu\nu\lambda}^2+mB_{\mu\nu}\epsilon^{\mu\nu\lambda\rho}(F_{\lambda\rho}-G_{\lambda\rho})+\Phi_\mu\epsilon^{\mu\nu\lambda\rho}\partial_\nu G_{\lambda\rho}. \tag{18}$$

Here $G_{\mu\nu}$ is a two form gauge field, with transformation $\delta G_{\mu\nu}=\partial_\mu\epsilon_\nu-\partial_\nu\epsilon_\mu$. Note, as earlier, this gauge transformation is maintained, only if Φ_μ undergoes a compensating transformation ϵ_μ.

Interestingly in both cases (6),(18) the transformation property of the multiplier field (Φ and Φ_μ, respectively), as in a Stückelberg field, comes naturally, due to the requirement of the gauge symmetry associated with A_μ and $B_{\mu\nu}$, respectively.

A similar construction to describe topologically massive spin-zero field is given by the Lagrangian

$$L=\frac{1}{(4!)}H^2_{\mu\nu\lambda\rho}+\frac{1}{(2)}(\partial_\mu\Phi)^2+\frac{1}{(4!)}\Phi\epsilon^{\mu\nu\lambda\rho}H_{\mu\nu\lambda\rho}, \tag{19}$$

where $H_{\mu\nu\lambda\rho}=\partial_\mu B_{\nu\lambda\rho}-\partial_\nu B_{\lambda\rho\mu}-\partial_\lambda B_{\rho\nu\mu}+\partial_\rho B_{\nu\mu\lambda}$. The fact that this Lagrangian describes spin zero can be easily seen by solving the two coupled linear equations. Also, it shares the same feature as in the case of spin-one theory (1) of having a topological current. The corresponding dual theory is obtained, as earlier, by considering the global symmetry

$$\Phi\to\Phi+\epsilon, \tag{20}$$

and making it local by introducing a gauge field A_μ. The Lagrange multiplier enforcing the constraint that the dual field strength of gauge field A_μ is zero is now $\Phi_{\mu\nu}$. This gives, on integrating Φ and A_μ,

$$L=\frac{1}{(4!)}H^2_{\mu\nu\lambda\rho}+\frac{1}{(3!)}(mA_{\mu\nu\lambda}-F_{\mu\nu\lambda})^2, \tag{21}$$

where

$$F_{\mu\nu\lambda}=\partial_\mu\Phi_{\nu\lambda}+\partial_\nu\Phi_{\lambda\mu}+\partial_\lambda\Phi_{\mu\nu},$$

and having a transformation

$$\Phi_{\mu\nu}\to\Phi_{\mu\nu}+(\partial_\mu\epsilon_\nu-\partial_\nu\epsilon_\mu)+\epsilon_{\mu\nu}. \tag{22}$$

The massless three-form B field has no dynamics, but on coupling with $\Phi_{\mu\nu}$ field, it acquires dynamics and describes massive spin-zero field. This formulation also appears in Polyakov's description of confining strings. This is understood as the $B_{\mu\nu\lambda}$ field having no degrees of freedom, eating up massless spin-zero field, and becoming a massive field, with the mass scale set by the coupling between the two fields. One cannot have a massive spin-zero particle, by gauging the global symmetry associated with shifting the field $B_{\mu\nu\lambda}$, as the field strength associated with this symmetry is identically zero.

In this paper, we have shown that topologically massive $B^\wedge F$ theory describing spin-one particle is dually equivalent to two different formulations of Stückelberg-type spin-one theories. These two Stückelberg-type formulations are called the dual Higgs mechanism in [20]. These two formulations were shown to describe the phases of the condensation of dual geometric objects, for compact gauge fields. Thus it appears, a unified formalism describing these phases may be possible, in terms of $B^\wedge F$ theory. A generalization to describe a spin-zero particle by this procedure was also shown.

It is curious to note that these three massive gauge invariant spin-one theories [(1), (2), (15)], can be obtained from five dimensions by dimensional reduction. $B^\wedge F$ theory for spin-one can be obtained by the dimensional reduction of five-dimensional, topologically massive Kalb-Ramond theory, described by

$$L=F_{\mu\nu\lambda}F^{\mu\nu\lambda *}+im\epsilon_{\mu\nu\lambda\rho\sigma}(H^{\mu\nu\lambda}A^{\rho\sigma *}+cc), \tag{23}$$

by keeping only zero mode (note that complex fields have to be used or else the Chern-Simon term will be a total derivative). The dimensional reduction of massless Maxwell action [8] and Kalb-Ramond [17] action, keeping nonzero modes gives Eqs. (1) and (6), respectively, with their masses inversely proportional to size of compact dimension. Note that since we consider noninteracting theories, the infinite number of massive modes are uncoupled and any one of them can be considered. Dimensional reduction of the zero mode of Eq. (23) gives Eq. (1), which was shown here to be dual to the nonzero mode of five-dimensional Maxwell and Kalb-Ramond theory, on dimensional reduction [18]. Since these four-dimensional theories are shown to be equivalent, it should be interesting to see if these five-dimensional theories bear any relationship. Of course, Kalb-Ramond theory is equivalent to Maxwell action in 4+1 dimensions (as it is equivalent to scalar theory in 3+1 dimensions). Also, it should be interesting to see if the *nonzero* mode of Eq. (23) has any relationship with the *zero mode* of Maxwell and Kalb-Ramond theories from five dimensions.

We have shown the duality equivalence between the three different forms of massive gauge invariant spin-one theories, only at the local level and not at the global level. The duality equivalence between massless scalar fields and antisymmetric tensor gauge theories is broken at the quantum level, when coupled with gravity [21]. Hence it should be interesting to see if the equivalence shown here between free massive gauge theories survives quantization and interactions.

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