Decay amplitudes in two-dimensional QCD

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(Received 31 March 1997; published 17 February 1998)

Decay amplitudes for mesons in two-dimensional QCD are discussed. We show that in spite of an infinite number of conserved charges, particle production is not entirely suppressed. This phenomenon is explained in terms of quantum corrections to the combined algebra of higher-conserved and spectrum-generating currents. We predict the qualitative form of particle production probabilities and verify that they are in agreement with numerical data. We also discuss four-dimensional self-dual Yang-Mills theory in the light of our results. [S0556-2821(98)01704-4]

PACS number(s): 12.38.Aw, 11.10.Kk

I. INTRODUCTION

Major advances in two-dimensional QCD have been made since the pioneering work of 't Hooft [1], who found the mesonic spectrum of the theory in the limit of a large number of colors. In subsequent works, an exact expression for the decay amplitudes of the mesons was found [2,3]. However, both the 't Hooft equation and the decay amplitudes could only be solved numerically. The numerical solution of the theory has been enterprised by many authors [4,5]. The numerical calculations are difficult to converge for massless fermions. A strongly convergent algorithm for very reliable numerical computations is also available in the literature [14]. These new results open challenging ways of testing various aspects of the theory, specifically those concerning decay amplitudes of the theory.

It has been suggested [7] that massless two-dimensional QCD is an integrable system. It is believed that integrability of a theory implies stability of its bound states. One could, therefore, expect vanishing decay amplitudes for the mesonic states of the theory. The numerical results indicate otherwise. The decay amplitudes, however, are small relative to those obtained for the massive theory.

The main purpose of this article is to resolve this apparent conflict by relating diverse properties of two-dimensional QCD, as well as understand more about the dynamical structure of two-dimensional QCD. Specifically, we show that the 't Hooft sector and the integrable sector of the theory are decoupled, in the sense that the conservation laws found before do not commute with the spectrum generating algebra due to quantum corrections. We also believe that the theory is not integrable in the 't Hooft sector, that is to say that the integrability properties are spoiled by quantum corrections. However, we think that a strong simplification occurs in the massless case as compared to the massive fermion theory, allowing us to draw special attention to such a difference in terms of the conservation laws and the anomalies, which presumably have a mild contribution semi-classically.

We review the known bosonisation procedure in Sec. II and show how the theory can be re-expressed in a fermionic language, by using gauge-invariant chiral fermions. The final re-fermionised action provides a convenient starting point for the analysis of the mesonic spectrum. The significant outcome of this procedure is that the Poisson brackets receive quantum corrections. The conserved Noether currents then satisfy the Kac-Moody algebra only on taking the contribution from the anomalous Poisson bracket into account. In Sec. III, we combine the algebra of these conserved currents with that of the fermionic bilinears, which have been shown to generate a W_∞ algebra. We show that, except for the pion, the mesons generated by the above W_∞ algebra are not eigenstates of the Sugawara operator. Conservation of the Sugawara operator indicates the existence of an infinite number of conservation laws. If such conservation laws survive quantization and physical states are eigenstates of the Sugawara operator then the dynamics of the theory is severely constrained. On the other hand, the breakdown of conservation laws by anomalous terms permits the decay of higher states. This explains the small decay rate numerically observed in the massless theory. Based on these results, we make detailed predictions, in Sec. IV, about various features of the spectrum. In Sec. V, we use the recent numerical results to verify our predictions and hence the accuracy of our calculational results.

II. BOSONISATION VS FERMIONISATION

We start with the Lagrangian

$$\mathcal{L}=-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+i\bar{\psi}\gamma^{\mu}D_{\mu}\psi, \tag{1}$$

which describes QCD with massless fermions. In two dimensions, the theory can be bosonised by computing the fermionic determinant which arises in performing the path integration. In the bosonisation procedure one first defines left and right components of the gauge field in terms of the matrix-valued fields U and V, i.e.,

$$A_{+}=\frac{i}{e}U^{-1}\partial_{+}U \quad \text{and} \quad A_{-}=\frac{i}{e}V\partial_{-}V^{-1}. \tag{2}$$

It is well-known that the Jacobian resulting from the above change of variables together with the effective action can be expressed in terms of the Wess-Zumino-Witten (WZW) functional [8]. That is to say,

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$$\det i\not{D} \equiv e^{iW[A]} = \int \mathcal{D}g e^{iS_F[A,g]}. \tag{3}$$

The bosonic action $S_F[A,g]$ may be obtained by the repeated application of the Polyakov-Wiegmann identity [9,6] and is given by

$$S_F[A,g] = \Gamma[g] + \frac{1}{4\pi}\int d^2x[e^2A^2 - e^2A_+gA_-g^{-1} - ieA_+g\partial_-g^{-1} - ieA_-g^{-1}\partial_+g] = \Gamma[UgV] - \Gamma[UV], \tag{4}$$

where Γ is the WZW action functional. Subsequent inclusion of the Faddeev-Popov ghost action $S[gh]$, replacement of the variables UV by Σ and UgV by $\tilde{g}$–using systematically the invariance of the Haar measure, and finally the introduction of a scalar field E, to disentangle the $F_{\mu\nu}F^{\mu\nu}$ interaction, yields the following form of the partition function [6,7]:

$$\mathcal{Z} = \int \mathcal{D}\tilde{g} e^{i\Gamma[\tilde{g}]}\mathcal{D}[gh]e^{iS[gh]}\mathcal{D}\Sigma\mathcal{D}E e^{-i(c_V+1)\Gamma[\Sigma]} \times \exp\left[-(c_V+1)\mathrm{tr}\int d^2x\partial_+E\Sigma\partial_-\Sigma^{-1} - 2e^2(c_V+1)^2\mathrm{tr}\int d^2xE^2\right]. \tag{5}$$

From a further replacement of the variable E by a field β, satisfying the relation $\partial_+E = (i/4\pi)\beta^{-1}\partial_+\beta$, the factorized form of the partition function,

$$\mathcal{Z} = \mathcal{Z}[g,1]\mathcal{Z}[\tilde{\Sigma}, -(c_V+1)]\mathcal{Z}_{gh}\mathcal{Z}_\beta, \tag{6}$$

is obtained. In the above expression, $\mathcal{Z}[M,n]$ is the partition function of a WZW field M with central charge n, $\mathcal{Z}_{gh}$ is the ghost contribution, the non-trivial coupling-constant-dependent part of the partition function is

$$\mathcal{Z}_\beta \equiv \int \mathcal{D}\beta\mathcal{D}C_- e^{i\Gamma[\beta]+\mathrm{tr}\int d^2x[\frac{1}{2}(\partial_+C_-)^2 + ieC_-\beta^{-1}\partial_+\beta]}, \tag{7}$$

and the auxiliary field C_- is introduced to make the action local.[1] The equations of motion are equivalent to the conservation law $\partial_+I_- = 0$, where

$$I_-(x) = 4\pi e^2\beta^{-1}\partial_+\beta - 2\Pi'_- + i(4\pi)^2e^3C_- - (4\pi e)^2[C_-,\Pi_-], \tag{8}$$

$\Pi_- = \partial_+C_-$ and $\Pi_\beta = (1/4\pi)\partial_+\beta^{-1} + ieC_-\beta^{-1}$ are canonical momenta conjugate to C_- and β respectively. Canonical commutation relations lead to a Kac-Moody algebra obeyed by the current I_-. As in the case of conformally invariant theories, the conservation law $\partial_+I_- = 0$ implies that there is an infinite number of conserved charges, $Q^{(n)} = \int I_-(x^-)(x^-)^n dx^-$. These act on asymptotic β fields as multiplications by powers of the (negative) component of the momentum times right-SU(n) transformation, i.e.,

$$Q^{(n)^{ij}}|\beta^{kl}(p)\rangle \sim p_-^n\delta^{jl}U^{jn}|\beta^{kn}\rangle. \tag{9}$$

This puts a stringent requirement on particle scattering, forbidding particle production or decay.[2]

Recently, there have been suggestions [14] that the stability of the spectrum can be tested in the framework of 't Hooft's mesons decay amplitudes. The numerical computations of the amplitudes provide a detailed check of the claims presented here, namely that the infinite number of conservation laws does not imply absence of particle production or stability of the meson and zero decay amplitude.

As mesons are fermionic bound states, the relevance of the above conserved quantities to the decay of mesons can be conveniently studied by writing the β sector of the theory, defined in Eq. (7), in terms of fermions. In the fermionisation procedure, the WZW term is equivalent to an action for massless fermions and the interactions are treated in perturbation theory by using the adiabatic principle of form invariance [15].

Using these techniques, we obtain the following fermionized form of the action:

$$S = \int d^2x\left[\psi_+^{\dagger i}i(\delta^{ji}\partial_- - ieC_-^{ij})\psi_+^j + \frac{1}{2}\mathrm{tr}(\partial_+C_-)^2 + i\psi_-^{\dagger i}\partial_+\psi_-^i\right]. \tag{10}$$

The 't Hooft spectrum can be algebraically generated by using the spectrum-generating algebra of gauge-invariant fermion bilinears. Since the β fields in Eq. (7), as well as their fermionic replacements in Eq. (10), are gauge-invariant objects, the above action represents a chiral theory which has to be quantized using anomalous Poisson brackets [16] (see also chapters 13 and 14 of [15]). Indeed, the left-moving Noether currents,

$$I_-^f = \psi^\dagger\psi - \frac{2}{e}\Pi'_- - \frac{e}{4\pi}C_- + i[C_-,\Pi_-], \tag{11}$$

generate a Kac-Moody algebra only if the anomalous Poisson brackets (APB),

$$\{\Pi_-(x), \psi^\dagger\psi(y)\}_{\mathrm{APB}} = \frac{e}{4\pi}\delta(x-y), \tag{12}$$

are used.[3]

[1]The factorized form of the partition function is actually elusive, due to several Becchi-Rouet-Stora-Tyutin (BRST) constraints remaining from the gauge condition and change of variables. For the definition of vacuum and physical states the reader is referred to [10–12].

[2]A detailed discussion of the action of conserved currents on β field, and the consequences for β-scattering have been discussed in Ref. [13], where an exact S-matrix has been conjectured.

[3]The bosonic formulation is advantageous since it contains information of order $\hbar$ already at the classical level.

The operator (11) [or (8)] contains color indices. However, in 't Hooft's formulation one deals exclusively with colorless states and, therefore, it is natural to consider the Sugawara operators

$$L(x) = :I_{ij}(x) I_{ji}(x):, \tag{13}$$

where the color indices are contracted and the divergent terms in the Wilson expansion (corresponding to the Kac-Moody algebra) are subtracted by means of a normal-ordering prescription.

III. CLASSIFICATION OF THE MESONIC STATES

In this section, we review the action of the W_∞ algebra on the spectrum. The spectrum-generating algebra, as obtained in [17], arises from the bilinears $M_{\alpha\gamma}(x,y) = \psi^\dagger_\alpha(x) e^{ie\int_x^y A_\mu(\xi) d\xi^\mu} \psi_\gamma(y)$, where α and γ are the chirality indices $(+,-)$. In the massive case, these bilinears are related by the equations of motion. In the massless case, however, the mixed term, M_{+-}, corresponds to the integrable-β sector, while the right-right term, M_{--}, corresponds to the usual meson bound states of 't Hooft. Therefore, we take

$$M_{--}(x^-, y^-; x^+) = \psi^{i\dagger}_-(x^-, x^+) \psi^i_-(y^-; x^+) \tag{14}$$

as the spectrum-generating current. In the momentum space of the x^- variables, a classical solution of the equations of motion obeying the Gauss constraint is

$$(M_{--}(k_-, k'_-; x^+))_{\text{class}} = \delta(k_- - k'_-)\, \theta(k_-). \tag{15}$$

Expanding M_{--} around such a solution in the large N limit [17] and using

$$M_{--} = e^{i/\sqrt{N}\, W} (M_{--})_{\text{class}} e^{-i/\sqrt{N}\, W} \tag{16}$$

one finds that the Fourier modes of W, which represent quantum fluctuation around classical solution, obey the 't Hooft equation. Thus, the W_∞ algebra found for the bilinears in [17] is a spectrum-generating algebra.

We emphasize that in the massive theory, the full content of the theory is preserved in the individual chiral sectors, M_{++} and M_{--}, which are related to each other by the fermion equation of motion. In the massless case, however, we study the bilinears constructed from ψ_+. There is a mixing of the sectors and it does not suffice to study one sector on its own.

IV. THE COMBINED ALGEBRA

In this section, we study the interplay between the W_∞-spectrum-generating algebra and the conserved current (13). We identify the bilinear $\psi^\dagger_+ \psi_+$ as the pion and $\psi^\dagger_+ D^n \psi_+$ as the higher states obtained from the fermion bilinear M_{++}.

The pion is an eigenstate of $L(x)$;[4]

$$[L(x), \psi^\dagger(y)\psi(y)] = \frac{e^2}{2(x-y)^2} \psi^\dagger \psi(y) + \frac{e^2}{2(x-y)} \partial(\psi^\dagger \psi(y)) + \text{regular terms}. \tag{17}$$

The higher-state fermion bilinears, however, due to various anomalous terms, are not eigenstates;

$$\begin{aligned}[L(x), \partial\psi^\dagger\psi(y) - \psi^\dagger(y)\partial\psi(y)] = &-\frac{e^2}{4}\left[\frac{2}{(x-y)^2}(\partial\psi^\dagger\psi(y) - \psi^\dagger\partial\psi(y)) - \frac{1}{x-y}\partial(\partial\psi^\dagger\psi(y) - \psi^\dagger\partial\psi(y)) - \frac{2}{(x-y)^4}\right] \\ &+ e^2\left[\frac{1}{(x-y)^2}(\psi^{i\dagger}\psi^j\psi^{j\dagger}\psi^i) - \frac{1}{x-y}(\psi^{i\dagger}\psi^j\partial(\psi^{j\dagger}\psi^i))\right] + \left[-\frac{ie^2}{2}[\Pi^{kj}C^{ki} - C^{jk}\Pi^{ik}] \right. \\ &\left. - 16\frac{e^3}{\pi}C^{ji} + e\,\partial_1\Pi^{ij}\right]\left[-\frac{1}{x-y}\partial(\psi^{j\dagger}\psi^i) + \frac{1}{(x-y)^2}\psi^{j\dagger}\psi^i\right] \\ &+ \text{anomalous Poisson Brackets terms} + \text{regular terms}. \end{aligned} \tag{18}$$

Using the above two equations, we can write down the general form of the algebra obtained from the action of the Sugawara operator on the physical states, that is,

$$L \cdot M_n \approx \frac{n+1}{(x-y)^2} M_n + \frac{1}{x-y}\partial M_n + \text{anomalous terms}. \tag{19}$$

Thus we see that, in the absence of anomalies, the nth state is an eigenstate of L with eigenvalue $n+1$. Taking this into account, we find that the action of L on the in and out states is given by

$$\begin{aligned}(n - n_1 - n_2 - 1)\langle M_{n_1} M_{n_2} | M_{n_j}\rangle &= \int dx \langle | L_{out} - L_{in} | \rangle \\ &= \int dx \int_{-\infty}^{\infty} dt \frac{d}{dt} \langle M_{n_1} M_{n_2} | L | M_n \rangle = \text{anomalies}, \end{aligned} \tag{20}$$

which shows the existence of an infinite number of conservation equations. In the absence of the anomalous terms, the

[4]Recall that in the present fermionic formulation we have to use the anomalous Poisson brackets.

decay amplitudes vanish. However, when anomalous terms are present unphysical (non-mesonic) operators are introduced in the right-hand side. Therefore, the amplitude is non-vanishing. If one of the out states is a pion, then we will have the following set of recursive relations:

$$(n-n_1-1)\langle M_0 M_{n_1}|M_n\rangle = \sum_X \langle M_0 X|M_n\rangle + \sum_Y \langle M_0 M_{n_1}|Y\rangle \qquad (21)$$

where X,Y are bound states of lower states and a pion state is always present on the right-hand side. The solution to the above relations is found by using

$$\langle M_0 M_0|Y\rangle = 0, \qquad (22)$$

which is obtained from the 1/N expansion of the decay amplitude and is valid up to second order. These solutions require the vanishing of the decays involving pions.[5]

From Eq. (18), we see that there are further corrections which come from both higher terms in the Wilson expansion and anomalous Poisson bracket quantization of the chiral fermions. These terms cannot cancel one another, due to their different functional form (as an example, higher Π derivatives never arise from anomalous Poisson brackets). There are further higher terms when one considers the gauge-covariantized current.

The decay amplitudes in the massless theory (which is integrable in the absence of anomalies) are suppressed as compared to those of the massive theory (non-integrable). This is entirely due to the quantum corrections. Since the anomalous terms are of order $\hbar$, they disappear for quasi-classical decays. In the massive theory, in addition to quantum corrections, there are mass terms which spoil integrability. These terms do not vanish in the quasi-classical approximation. In the massless theory, this approximation is reliable for decays of highly-placed states (e.g. near the continuum limit) to large-momentum states. On the contrary, decays into small-momentum states, marked by interference terms, are highly influenced by quantum corrections which spoil quasiclassical approximation.

On the basis of the preceding results, we predict the following features:

(1) The pion decouples in the massless fermion theory. Such a decoupling is valid in all orders of the $1/N$ expansion.

(2) For massless fermions, the decay of large-mass states into states with large momenta is severely suppressed.

(3) The probabilities for further decays, although reduced, are not very small. For small N, the amplitudes are much smaller than the corresponding ones in the massive case, due to the enhanced significance of the larger-order terms in the $1/N$ expansion.[6]

(4) Finally, the decay amplitudes of very massive mesons vanish, unexceptionally, in the massless fermion case.

V. NUMERICAL BACK-UP AND LARGE N BEHAVIOUR

In this section, we verify these predictions by means of the numerical computation of the amplitudes (for predictions 1, 2 and 3), as well as by using large N analysis (for prediction 4).

Amplitudes for meson decay were initially derived in the framework of $1/N$-expansion in [2], and in more detail, including higher order corrections, in [3]. The $1/N$ corrections vanish in the massless case.[7] This can be explained by studying the expression for the decay amplitude [3],

$$\begin{aligned}\mathcal{A} = &(1-\mathcal{C})\frac{1}{1-\omega}\int_0^\omega \mathrm{d}x\,\phi_n(x)\,\phi_p\!\left(\frac{x}{\omega}\right)\Phi_q\!\left(\frac{x-\omega}{1-\omega}\right)\\ &-(1-\mathcal{C})\frac{1}{\omega}\,\mathrm{d}x\int_\omega^1 \mathrm{d}x\,\phi_n(x)\,\phi_q\!\left(\frac{x-\omega}{1-\omega}\right)\Phi_p\!\left(\frac{x}{\omega}\right)\\ &+\frac{1}{N}(1-\mathcal{C})\frac{f_q}{1-\omega}\int_0^\omega \mathrm{d}x\,\phi_n(x)\,\phi_p\!\left(\frac{x}{\omega}\right),\end{aligned} \qquad (23)$$

where $\omega = k_+^p/k_+^n$, $\Phi_n(x) = \int_0^1 \mathrm{d}y[\phi_n(y)/(x-y)^2]$ and $\mathcal{C}$ denotes the interchange of final states.

We observe that the higher-order corrections are always multiplied by the factor

$$f_n = \int_0^1 \mathrm{d}x\,\phi_n(x), \qquad (24)$$

where $\phi_n(x)$ is 't Hooft's wave function of the decaying state. It can be verified using 't Hooft's wave equation, that f_n vanishes for massless fermions [2]. Moreover, the authors of Ref. [3] claim that, as higher-order corrections amount to a redefinition of constants in the massless case, only the very first term in the $1/N$ series survives. Because this does not hold in the massive case, corrections to all order exist. We restrict our analysis to the large-N limit where amplitude for both massless and massive fermions may be non-vanishing.

In order to fully understand the behavior of the leading-order term [i.e. the first term in expression (23)] we need to solve the 't Hooft equation numerically. The solution is then inserted back into Eq. (23) and the numerical integration is carried out. We are then in a position to compare the results for the massless and the massive cases.

The method which employs a MATHEMATICA program is unsatisfactory because of its bad-convergence behavior. The numerical results are unreliable although they are compatible with the predicted vanishing decay rate in the massless case,

[5] It is worth mentioning that the algebra is independent of the number of colors, and the anomalous term only makes a first-order contribution in the $1/N$ expansion.

[6] We recall again that $1/N$ corrections are very important in the massive case which strengthens these predictions for groups such as SU(2).

[7] The lowest-order term, however, seems to give a non-vanishing contribution. This is discussed later in this section.

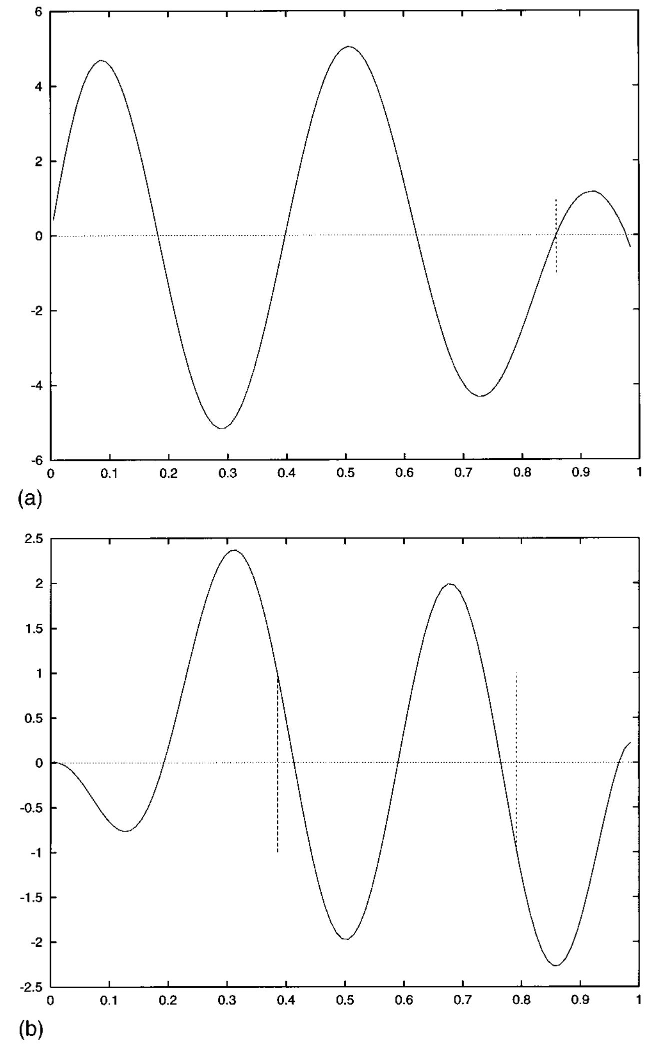

FIG. 1. Decay rate of the 5th excited state into the 1st and ground (pion) states is plotted as a function of the outgoing momenta. The first diagram corresponds to massless and the second to massive fermions. The vertical bars mark the on-mass-shell values.

and the non-vanishing decay rate in the massive case. Therefore, a more sophisticated method of numerical computation is needed.[8]

A more elaborate method [14] confirms that particles do not decay into states which contain the pion whereas in the massive case, the decay amplitudes are shown to be significantly large (see Fig. 1).

For the massless case, several decays of the type $n \to 1+1$, $2+2$, where 1 and 2 denote the first and second excited states of 't Hooft's series, have been computed (see Figs. 2 and 3). For higher values of n the amplitude approaches zero rapidly. For the massive case, the amplitudes vary randomly (see Figure 4).

The overall results can be summarized as follows:

(i). In the massless case, for a general decay $k \to l+p$ the amplitude can start from a high value (see specifically the example $14 \to 3+3$ in Ref. [14]) but decreases rapidly with increasing k. These results are also compatible with the earlier results of Ref. [3]. Although the results for the massless case are very precise, those for the massive case require further refinement. However, all the results obtained are precise

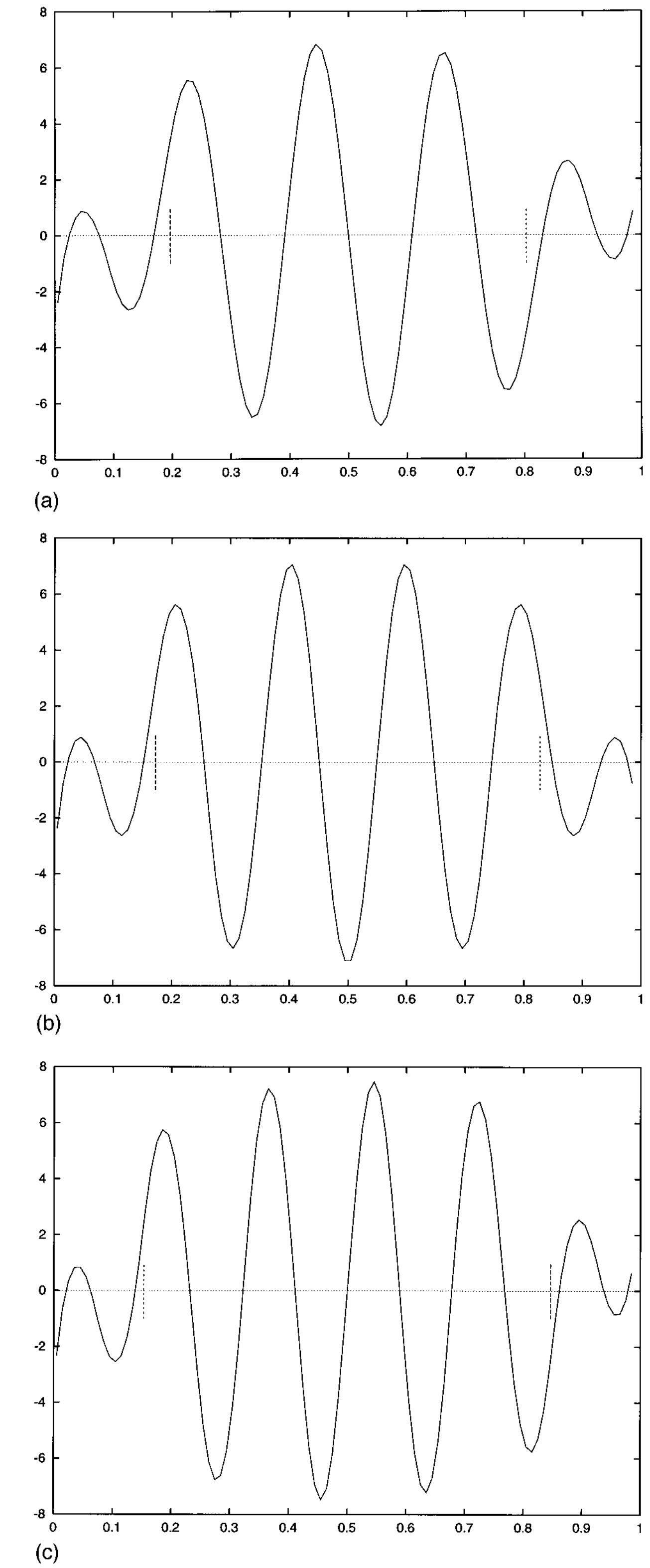

FIG. 2. Decay rates of the 10th, 11th and 12th states into two mesons in the 2nd excited state (for massless fermions) are plotted as functions of the outgoing momenta. The vertical bars mark the on-mass-shell values.

[8]For a survey of these methods using orthogonal eigenfunctions expansion we refer the reader to [3,18,19].

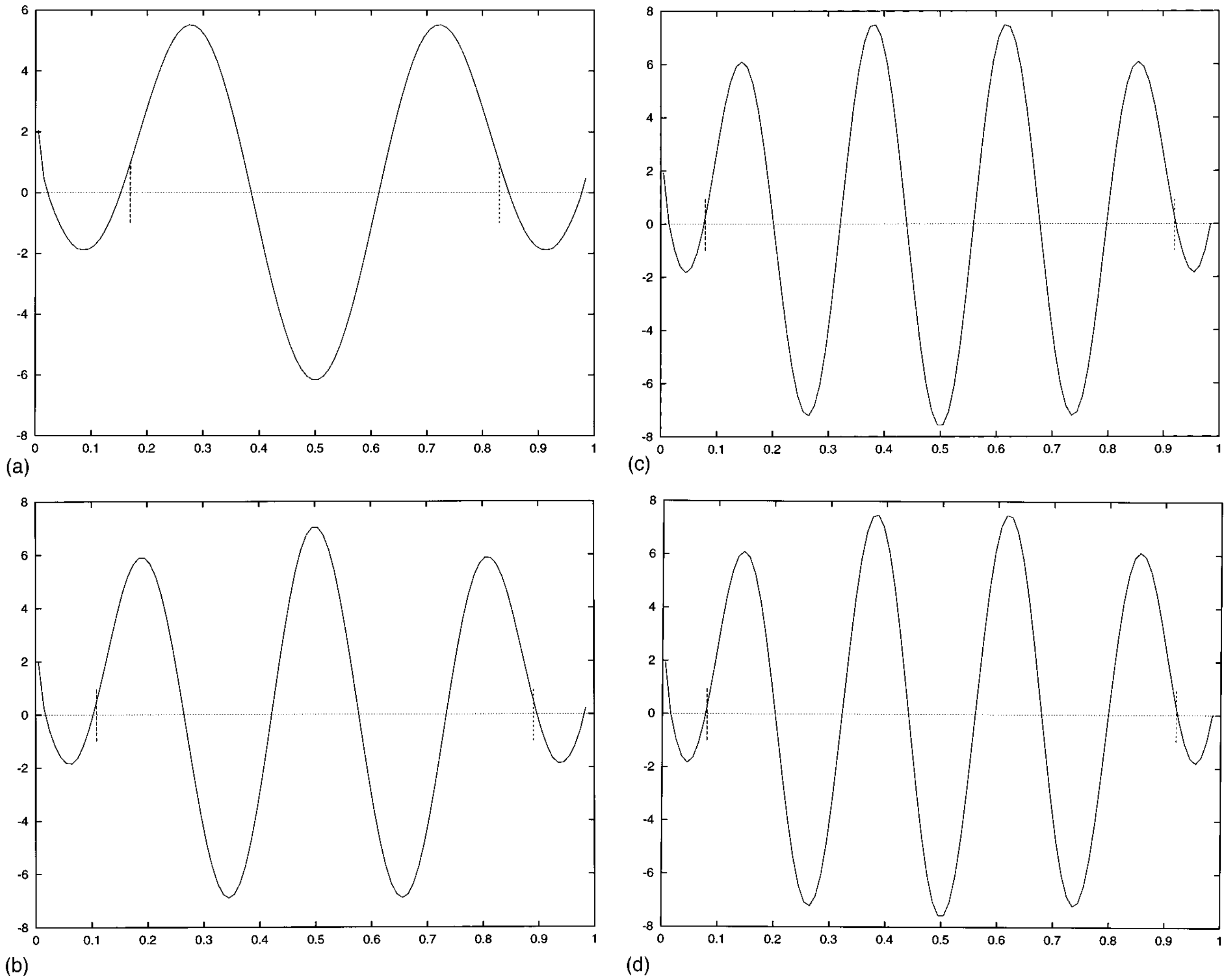

FIG. 3. Decay rates of the 5th, 7th, 9th and 11th excited state into two mesons in the 1st excited state are plotted as functions of the outgoing momenta for massless fermions. The vertical bars mark the on-mass-shell values.

enough for the conclusions drawn hitherto, and will be verified by other methods.

(ii). An exact computation of the decay rate of the higher-mass states guarantees the correctness of the numerical results obtained so far. Indeed, using the large N techniques of [2] or [20] one finds the required amplitude in the large k limit. In terms of the fermion mass m, a parameter γ satisfying

$$\pi\gamma \cot \pi\gamma = 1 - \frac{\pi m^2}{e^2 N} \tag{25}$$

is defined (the massless case corresponds to $\gamma=0$). In the large n limit, the decay amplitude is found to obey

$$\mathcal{A} \sim \sin \pi\gamma, \tag{26}$$

which is valid up to constants describing the behavior of the 't Hooft wave functions at the origin (see [20] for further details) and which shows a non-vanishing result only for non-zero values of the parameter γ. The same computation may be explicitly carried out in the massless case.

In order to investigate the issue of pair production for large number of colors and high mesonic states, we calculate the amplitude of the decay of a very massive meson of mass m_n into two mesons of masses m_{n_1} and m_{n_2}. Such an amplitude, at lowest order in the inverse number of colors and up to an overall constant, is given by the vertex

$$V_{n_1 n_2 n} = e^2 \frac{p_1}{p_2} \int_0^1 dx \int_0^1 dy \frac{1}{\left(x \frac{p_1}{p_2}\right)^2} \phi_{n_1}(x) \phi_{n_2}(y) \times \left[\phi_n\left(\frac{1-y}{1+p_1/p_2}\right) - \phi_n\left(\frac{1+xp_1/p_2}{1+p_1/p_2}\right)\right], \tag{27}$$

where p_i is the plus-component of the momentum. For decay of very massive mesons, $m_n \to \infty$, energy-momentum conservation implies

$$m_n^2 = m_2^2 \frac{p_1}{p_2}. \tag{28}$$

Thus the main contribution to the integral comes from $x\sim 0$. Next, we make the following substitutions

$$x=\eta\left(\frac{m_2}{e}\right)^2, \tag{29}$$

$$\phi_i(x)\approx C_i x^{\gamma}, \tag{30}$$

where γ is defined in Eq. (25) and x is small, and

$$\phi_n\left[\xi\left(\frac{e}{m_n}\right)^2\right]\approx\phi(\xi)\equiv\sin\left(\frac{\xi}{\pi}+\delta(\xi)\right), \tag{31}$$

where $\delta(\xi)$ is a phase,[9] in Eq. (27) to find

$$V_{n_1 n_2 n}=\left(\frac{m_2}{m_n}\right)^{2\gamma}C_1\int_0^\infty d\eta\int_0^1 dx'\,\eta^{\gamma}\frac{\phi_2(x')}{(\eta+x')^2}\times\left[\phi\left((1-x')\frac{m_2^2}{e^2}\right)-\phi\left((1+\eta)\frac{m_2^2}{e^2}\right)\right]=\left(\frac{m_n}{e}\right)^{-2\gamma}C_1 I_2. \tag{32}$$

Since the vertex is symmetric under the exchange $1\leftrightarrow 2$, the relation $C_i/I_i=r$ must be the same for $i=1,2$ and can be computed in a convenient limit. We thus compute r for $i=2$ and large values of m_2. In this limit, the asymptotic behavior of the wave functions can be used to find the explicit values of C_2 and I_2, i.e.,

$$C_2\approx\left(\frac{m_2}{e}\right)^{-2\gamma},\quad I_2\approx\left(\frac{m_2}{e}\right)^{2\gamma}\pi\sin\pi\gamma. \tag{33}$$

Subsequently, the expression

$$V_{n_1 n_2 n}\approx e^2\left(\frac{m_n}{e}\right)^{2\gamma}\sin\pi\gamma \tag{34}$$

is obtained which shows that the vertex vanishes for $\gamma\rightarrow 0$.

For massless fermions, that is for $\gamma=0$, the η integration in the first term of Eq. (32) can be performed. Moreover, by using 't Hooft's equation in the same term (to replace ϕ/x) we obtain

$$\tilde{I}_2=-\int_0^1 dx\,\phi_2(x)\int_0^\infty d\eta\frac{\phi[(1+\eta)\mu^2]}{(\eta+x)^2}-\int_0^1 dx\,\phi_2(x)\left[\mu^2+\frac{1}{1-x}\right]\phi[(1-x)\mu^2]-\int_0^\infty\frac{d\xi}{\mu^2}\int_0^1 dx\,\phi(\xi)\frac{\phi_2(x)}{\left(x-1+\frac{\xi}{\mu^2}\right)^2} \tag{35}$$

where $\mu=m_2/m_n$. The last two terms cancel [21] due to the identity

$$\int_0^\infty d\zeta\frac{\phi(\zeta)}{(\zeta-\eta)^2}=-\left(\frac{1}{\eta}+1\right)\phi(\eta). \tag{36}$$

In order to confirm these conclusions further, we make a detailed comparison between the amplitudes obtained in the massive and massless cases. This can be conveniently done by considering the table presented below.

Decay series	$\mathcal{A}$	k	$\mathcal{A}/k$
$8\rightarrow 1+1$, $m=0$.	.25	.4	.65
$8\rightarrow 1+1$, $m\neq 0$	.5	.25	2
$9\rightarrow 1+1$, $m=0$	.2	.4	.5
$9\rightarrow 1+1$, $m\neq 0$	.5	.28	1.8
$10\rightarrow 1+1$, $m\neq 0$	.8	.3	2.7
$11\rightarrow 1+1$, $m=0$	$<.05$	.45	$<.1$
$13\rightarrow 1+1$, $m=0$	≈ 0	.45	≈ 0

In this table, the second column represents the momenta of the outgoing particles and the figures in the last column are proportional to the decay probabilities.[10] These tabulated results once again confirm the prediction that the ratio between massless and massive fermion decay rates goes to zero-being smaller than 3% for the 11th state. For other series of decays, this ratio approaches zero more slowly. Nevertheless, one clearly observes that this ratio approaches zero, e.g., for massless series $k\rightarrow 2+2$ (see Fig. 2).

VI. CONCLUSION

Massless QCD contains higher-conservation laws which in general imply integrability. These conservation laws have been derived in the massive (β) sector. The mesons in the 't Hooft sector are built up of fermion bilinears which are dressed with bosonic fields of the massless sector. We have shown here that the spectrum-generating algebra, which defines the 't Hooft sector, does not commute with the higher conservation laws due to quantum corrections to the short-distance expansions. This implies the breakdown of integrability in the meson sector. The quantum nature of these corrections means that they are insignificant for quasi-classical decays. This renders the theory *quasi-integrable* and accounts for the exact decoupling of the pion.

The theory has a complex structure of constraints. The mesonic states, although physical, are not eigenstates of the conserved charges obtained from the Sugawara operators. This is because the massive and the massless sectors of the theory are connected by the constraint equations. The massive sector alone (the β sector) is integrable but does not generate the physical Hilbert space since it is not BRST invariant. The change of variables made to decouple the dynamics of the massive and the massless sectors of the theory [see Eqs. (2), (5) and (6)] leads to non-trivial Jacobian which breaks the BRST invariance [11,12].

Therefore, due to quantum corrections, the integrability and the BRST-invariance properties of the theory fall into

[9]The validity of Eq. (31) is confirmed for massless fermions by using numerical simulations.

[10]We have chosen the units such that the mass of the initial state is unity (the influence of the fermion mass is small in the present cases).

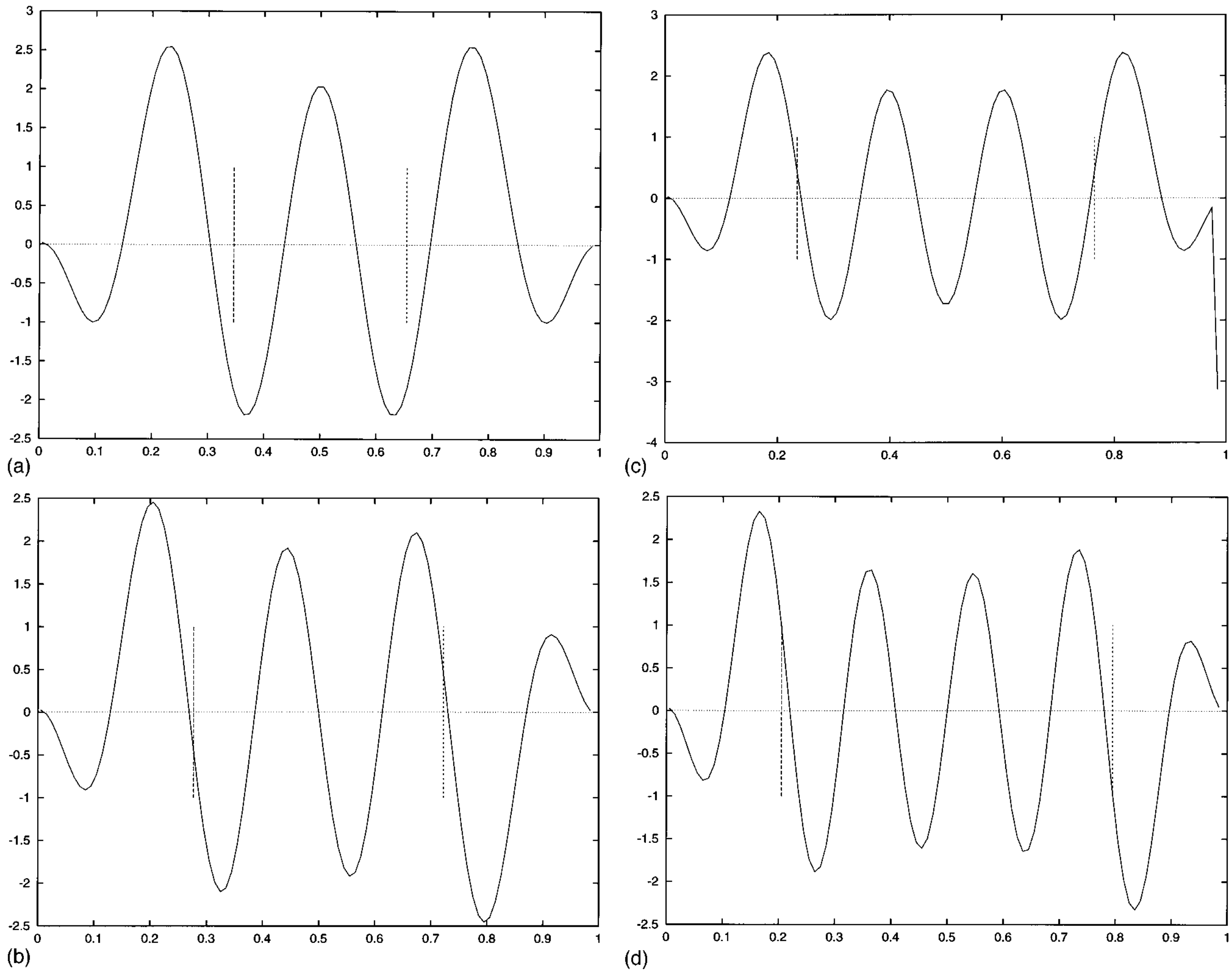

FIG. 4. Decay rates of the 7th, 8th, 9th and 10th states into two mesons in the 1st excited state are plotted for massive fermions.

two different sectors, the former being valid in the massive sector alone and the latter in the meson sector. This can be interpreted as the stability of the unphysical β particles. We have verified this characteristic of the theory in the quasi-classical approximation by numerical methods.

The theory simplifies in the large N limit, and meson wavefunctions and masses can be computed. Large N corrections are well known in the literature, and the numerical problem can be tackled.

Nevertheless, two dimensional QCD is far from trivial. In spite of the large N techniques, the formal aspects of the theory have not been fully understood, and only recently the vacuum structure has been studied, and separated from the description of the massive excitations. Such structures are the core of the understanding of the Schwinger model, and led to very profound consequences in that case. Our aim has been to deepen the understanding of the theory, obtaining results similar to those known to two dimensional QED, the Schwinger model.

However, it is clear that a development in that direction encounters a wall, since the theory is not soluble. Nevertheless, the numerical methods used, permit to obtain new structures otherwise unknown. Moreover, there are indications that the theory has an unexpected simplification in the massless case.

The result of our paper is to show that in spite of the complexity of the situation, and the fact that the model is used to study QCD mesons, the decaying amplitudes are simpler than imagined before, and indicate further structures not known before. The zeroes of the decaying amplitudes are a demonstration of that fact. The theory is not integrable, but the decay amplitudes are nearly vanishing. Our numerical results are an ‘‘experimental’’ demonstration of that fact, and in Sec. III we give a field theoretic argument to support that fact.

Finally, we wish to point out that the methods are not borrowed from techniques invented for integrable systems, but rather well defined and established techniques based on the computation of the exact fermionic determinant, leading to the bosonised version, namely the gauge Wess-Zumino-Witten model and its gauge interaction.

It is of prime importance to generalize the concept of quasi-integrability to higher dimensions. Indeed, Bardeen [22] has recently pointed out that helicity amplitudes in high-energy QCD are very simple at tree level and are described by a self-dual Yang-Mills theory. The classical solution of this theory strongly resembles the Bethe ansatz solution of integrable two-dimensional models. Moreover, the one-loop amplitudes are reminiscent of those corresponding to anomalous conservation laws. It is known that the self-dual Yang-

Mills theory is an integrable theory and is described by very simple actions [23]. On the other hand, integrable Lagrangians with either anomalies [24] or with non-vanishing amplitudes for particle production [25] are known and are well documented in the literature. It remains an interesting open problem to see whether the quasi-integrability idea is the most efficient framework for the description of non-trivial dynamics in theories with higher conservation laws, in general space-time dimensions, in spite of the Coleman-Mandula no-go theorem [26] and its more general version [27].

ACKNOWLEDGMENTS

We wish to thank Chand Devchand for a critical reading of the manuscript and many useful suggestions.

[1] G. 't Hooft, Nucl. Phys. **B75**, 461 (1974).
[2] C. G. Callan, N. Coote and D. J. Gross, Phys. Rev. D **13**, 1649 (1976).
[3] J. C. F. Barbon and K. Demeterfi, Nucl. Phys. **B434**, 109 (1995).
[4] M. Burkardt, Nucl. Phys. **A504**, 762 (1989).
[5] K. Hornbostel, S. J. Brodsky and H.-C. Pauli, Phys. Rev. D **41**, 3814 (1990).
[6] E. Abdalla and M. C. B. Abdalla, Phys. Rep. **265**, 253 (1996).
[7] E. Abdalla and M. C. B. Abdalla, Int. J. Mod. Phys. A **10**, 1611 (1995).
[8] E. Witten, Commun. Math. Phys. **92**, 455 (1984).
[9] A. M. Polyakov and P. B. Wiegmann, Phys. Lett. **131B**, 121 (1983).
[10] E. Abdalla and K. D. Rothe, Phys. Lett. B **363**, 85 (1995).
[11] K. D. Rothe, F. Scholtz and N. Theron, Ann. Phys. (N.Y.) **255**, 97 (1997).
[12] K. D. Rothe and D. Cabra, Phys. Rev. D **55**, 2240 (1997).
[13] E. Abdalla and M. C. B. Abdalla, Phys. Rev. D **52**, 6660 (1995).
[14] W. Krauth and M. Staudacher, Phys. Lett. B **388**, 808 (1996).
[15] E. Abdalla, M. C. Abdalla and K. D. Rothe, *Non-perturbative Methods in Two Dimensional Quantum Field Theory* (World Scientific, Singapore, 1991).
[16] H. J. Rothe and K. D. Rothe, Phys. Rev. D **40**, 545 (1989); F. Otto, H. J. Rothe and K. D. Rothe, Phys. Lett. B **231**, 299 (1989); F. Otto and K. D. Rothe, Phys. Rev. D **42**, 2829 (1990).
[17] K. Kikkawa, Ann. Phys. (N.Y.) **135**, 222 (1981); A. Dhar, G. Mandal and S. Wadia, Mod. Phys. Lett. A **8**, 3557 (1993); Phys. Lett. B **329**, 15 (1994).
[18] H. Hanson, R. D. Peccei and M. K. Prasad, Nucl. Phys. **B121**, 474 (1977).
[19] R. L. Jaffe and P. F. Mende, Nucl. Phys. **B369**, 189 (1992); B. Grinsteins and P. F. Mende, *ibid.* **B425**, 451 (1994).
[20] R. C. Brower, J. Ellis, M. Schmidt and J. H. Weisz, Nucl. Phys. **B128**, 131 (1977); **B128**, 175 (1977).
[21] M. B. Einhorn, Phys. Rev. D **14**, 3451 (1976).
[22] W. A. Bardeen, Prog. Theor. Phys. Suppl. **123**, 1 (1996).
[23] G. Chalmers and W. Siegel, Phys. Rev. D **54**, 7628 (1996); A. N. Leznov and A. Mukhtarov, J. Math. Phys. **28**, 2574 (1987); A. Parkes, Phys. Lett. B **286**, 265 (1992).
[24] E. Abdalla, M. C. B. Abdalla and M. Gomes, Phys. Rev. D **23**, 1800 (1981); E. Abdalla, M. Forger and A. Lima-Santos, Nucl. Phys. **B256**, 145 (1985).
[25] T. L. Curtright and C. Zachos, Phys. Rev. D **49**, 5408 (1994).
[26] S. Coleman and J. Mandula, Phys. Rev. **159**, 1251 (1967).
[27] R. Haag, J. Lopuszanski and M. Sohnius, Nucl. Phys. **B88**, 257 (1975).