Worldline approach to eikonals for QED and linearized quantum gravity and their off mass shell extensions

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We construct the worldline expression pertaining to a four-point process involving the scattering of two spin-1/2 particles via photon exchange. Restricting our attention to the case of forward scattering at extremely high energies, we show how to formulate the corresponding eikonal version of the four-point Green's function. We proceed to distinguish between the on and off mass shell cases within the framework of our description. For the on mass shell situation we recover the well-known result for the QED eikonal which corresponds to the infinite Coulomb phase. The (slightly) off mass shell case is confronted next. We produce a relevant expression for the eikonal phase in analytic form. Finally, we extend our considerations to a linearized quantum gravitational model and recover, via a series of elementary steps, the 0th order eikonal result for Planckian scattering (both for on and off mass shell). [S0556-2821(98)05004-8]

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I. INTRODUCTION

The eikonal approximation in quantum field theory offers the most powerful methodological tool for the study of particle collisions at very high energies. Perhaps the greatest asset of the eikonal approach is its capacity to account for the unitarity-induced behavior of amplitudes and/or cross sections as the center of mass energy of the collisions reaches asymptotically large values. With respect to gauge theories (Abelian or non-Abelian), at least, the seminal work of Cheng and Wu [1,2] exemplifies the validity of the above claim at a basic calculational level.

More recent advances in the application of eikonal techniques in field theory, QCD in particular, have been promoted by Lipatov [3] in connection with multiparticle production at high energies. This corresponds well with the traditional view that the eikonal approach applies more directly to the forward scattering amplitude, at high energies, the imaginary part of which relates, via unitarity, to the total cross section.

A second aspect of eikonal modeling in field theory is, for QED at least, its natural association with the infrared (IR) structure of the full theory. In a general context, this can be understood on the basis of the no-recoil assumption entailed by the eikonal approximation. Indeed, for such a regime the matter particles appear too heavy for the, emitted or absorbed, live gauge field degrees of freedom.

A unified treatment of the above and other important aspects of the eikonal methodology, encompassing many branches of physics, have been pursued by Fried, with a comprehensive account of relevant results to be found in a monograph [4] which contains original references. With particular reference to quantum field theory, QED especially, Fried's approach to eikonal physics adopts the functional language of Schwinger [5] and its subsequent recasting, in a particle-based representation, by Fradkin [6]. Nowadays, Fradkin's path integral casting of field systems has been reformulated and goes by the name of ''worldline approach,'' having widened its scope of applications [7–9].

The origins of our own involvement with the subject [10] can be traced to our attempt to understand the field theoretical basis of Polyakov's work [11] which discusses geometrical aspects of particle path propagation in a given (Euclidean) space-time background.[1] Given the worldline casting of field systems, our present attempt focuses on its application to a four-point process, in the eikonal approximation. In particular, we shall consider the forward, very high energy, scattering among two spin-1/2 matter particles in an Abelian gauge field theory as well as for a linearized version of quantum gravity.

A systematic program which advocates Fradkin's representation of Green's functions, both in potential theory and quantum field theory, for generalized eikonal approximations to particle scattering is currently being pursued by Fried and Gambellini [12,13]. The computational techniques employed by these authors rely on Schwinger's functional methodology [5], as opposed to particle-based path integrals. It has already been successful on two important fronts: (1) A well-defined strategy has been produced which leads to bonafied corrections to the eikonal approximation of a nonperturbative nature [12] and (2) a partial, nonperturbative confrontation of the non-Abelian eikonal problem (for noninteracting gluons) has been attained, which leads to an effective ''Reggeization'' of the exchanged bosons [13].

Our current undertaking is a natural continuation of recent work which has applied the, geometrically based, worldline formalism to infrared physics, where nonperturbative considerations invariably enter. So far we have considered cases involving two- and three-point functions [14–16]. Our inaugural efforts, aiming at the study of a four-point process, have been chosen to apply to cases where nonperturbative results in the infrared can be arrived at without too many complications. The present work not only serves to illustrate the viability of our approach toward the confrontation of

[1]Special emphasis, in Polyakov's scheme, is placed on the ''geometrical accommodation'' of the particle's spin.

processes involving a pair of *open* fermionic lines but also to demonstrate its ability to produce a relevant, (slightly) off mass shell eikonal expression. Entering the non-Abelian [2,13] and—to a fuller degree than presently attempted—quantum gravity [17–21] domains is a task that will be left to future investigations.

Let us close our introductory discussion with a brief sketch of the worldline philosophy. The basic idea is that one translates the description of a system, originally given in terms of field degrees of freedom, into particle-based ones. Explicitly, one goes from a functional to a path integral casting of the system:

$$\int \mathcal{D}\bar{\psi}\mathcal{D}\psi \, \exp\{S_{field}[\bar{\psi},\psi]\} \rightarrow \int [dx(\tau)][dp(\tau)]\exp\{S_{particle}[x(\tau),p(\tau)]\}.$$

We have implicitly used notation pertaining to spin-1/2 fields since this is the main case of interest in worldline applications. The reason is that, for renormalizable theories at least, the integration over fermionic fields, in a Euclidean space-time, is Gaussian and therefore lends itself to a particle-based casting without the involvement of some approximation. As far as the (dynamical) contribution from the gauge field sector is concerned, one aims at extracting from the corresponding functional integration an expression of the form $I[x(\tau),(p(\tau)]$ which communicates directly with the path integral. How this is accomplished in practice will become amply evident in the course of our analysis.

The paper is organized as follows. In the next section we sketch the procedure that leads to the derivation of the worldline expression for the four-point Green's function appropriate to the physical situation under study. Section III tends to matters of an organizational nature which pertain to a convenient kinematical decomposition and to factorization issues between ''soft'' and ''hard'' contributions to the Green's function. The on mass shell eikonal description of the dynamical piece of the amplitude is taken up in Sec. IV, while an off mass shell extension is considered in Sec. V. The case of Planckian scattering, where the exchanged field quanta are gravitons, is taken up in Sec. VI within a linearized gravity setting [16]. Technical aspects entering the analysis of the last three sections are dealt with in three Appendixes. Finally, in Sec. VII we present an outlook that stems from our work.

II. WORLDLINE CASTING OF THE FERMION FOUR-POINT FUNCTION IN THE EIKONAL APPROXIMATION

In this section we establish the basic worldline expression which will become the object of calculational interest in the present paper. It refers to a four-point process in QED pertaining to elastic scattering between two spin-1/2 matter particles under conditions which favor an eikonal mode of description.

Consider the generating functional for an Abelian gauge theory with Dirac, spin-1/2 matter fields in a Euclidean space-time setting. Formal integration over the fermionic fields leads to the following expression for the partition function:

$$Z[\bar{\eta},\eta.J_\mu]=\int \mathcal{D}A \, \det(\gamma\cdot D+m)\exp\left\{\int d^dx[\bar{\eta}(x)(\gamma\cdot D+m)^{-1}\eta(x)]\right\}e^{-S_J[A,\xi]}, \tag{1}$$

where $\bar{\eta},\eta$ are sources for Dirac fields ψ, $\bar{\psi}$, respectively, and where $S_J[A,\xi]\equiv S[A,\xi]-\int J_\mu A_\mu \; d^dx$, with $S[A,\xi]$ standing for the Maxwell action in a covariant gauge, while J_μ is a source for the gauge potential A_μ. We shall adopt the Feynman gauge throughout, corresponding to the choice $\xi=0$.

We introduce the four-point function

$$G^4(x_1,x_2,y_1,y_2)=\frac{\delta^4}{\delta\bar{\eta}(x_1)\,\delta\bar{\eta}(x_2)\,\delta\eta(y_1)\,\delta\eta(y_2)} Z/_{\bar{\eta}=\eta=J_\mu=0}, \tag{2}$$

which, via the use of Eq. (1), assumes the form

$$G^4(x_1,x_2,y_1,y_2)=\int \mathcal{D}A \, \det(\gamma\cdot D+m)e^{-S_J[A,\xi]}\langle x_1|(\gamma\cdot D+m)^{-1}|y_1\rangle\langle x_2|(\gamma\cdot D+m)^{-1}|y_2\rangle. \tag{3}$$

Employing the Schwinger proper time representation for the matrix elements of the inverse Dirac propagator, entering Eq. (3), we achieve a worldline form for the fermionic Green's function $G^{(2)}(x,y|A)$ $[\equiv\langle x|1/(\gamma\cdot D+m)|y\rangle]$ in the presence of a background gauge field, which reads as follows:

$$\begin{aligned}G^{(2)}(x,y|A)=&\int_{0^+}^{\infty}dT\int_{\substack{x(0)=x\\x(T)=y}}[dx(\tau)]\int [dp(\tau)]\\&\times\exp\left\{+i\int_0^T d\tau p(\tau)\cdot\dot{x}(\tau)\right\}\\&\times\mathcal{P}\exp\left\{-\int_0^T d\tau[i\gamma\cdot p(\tau)+m]\right\}\\&\times\exp\left\{ig\int_0^T d\tau\dot{x}(\tau)\cdot A(x(\tau))\right\}.\end{aligned} \tag{4}$$

A well-defined procedure which leads to the above result has been given in [9]. The main point is that the functional measures entering Eq. (4) can be carefully defined so that the worldline casting of the field system can support both perturbative and nonperturbative considerations [7–16].

Substituting into Eq. (3), we obtain the worldline expression for the full four-point function which reads

$$G^4(x_1,x_2,y_1,y_2)=\int_0^\infty dT_1\int_0^\infty dT_2\int_{\substack{x_1(0)=x_1\\x_1(T_1)=y_1}}[dx_1(\tau)]\int[dp_1(\tau)]\int_{\substack{x_2(0)=x_2\\x_2(T_2)=y_2}}[dx_2(\tau)]\int[dp_2(\tau)]$$
$$\times\exp\left\{+i\int_0^{T_1}d\tau p_1(\tau)\cdot\dot{x}_1(\tau)\right\}\exp\left\{+i\int_0^{T_2}d\tau p_2(\tau)\cdot\dot{x}_2(\tau)\right\}\mathcal{P}\exp\left\{-\int_0^{T_1}d\tau[i\gamma\cdot p_1(\tau)+m]\right\}$$
$$\times\mathcal{P}\exp\left\{-\int_0^{T_2}d\tau[i\gamma\cdot p_2(\tau)+m]\right\}\left\langle\exp\left\{ig\int_0^{T_1}d\tau\dot{x}_1(\tau)\cdot A(x_1(\tau))+ig\int_0^{T_2}d\tau\dot{x}_2(\tau)\cdot A(x_2(\tau))\right\}\right\rangle_A, \tag{5}$$

where we have abbreviated $\langle\mathcal{O}\rangle_A\equiv\int\mathcal{D}Ae^{-S_J[A,\xi]}\mathcal{O}\det(\gamma\cdot D+m)$.

In a diagrammatic language the above expression incorporates all Feynman diagrams with four external fermions and an arbitrary number of loops. In the strict context of the eikonal approximation we admit only that subclass which comprises the so-called exchange diagrams. Their features are (1) no closed fermion loops make their entrance and (2) the photon propagators cannot attach to the same fermion line.

Our next step is to align the general expression (5) with the requirements (1), (2) above. The first one guides us to the constraint (quenched approximation)

$$\det(\gamma\cdot D+m)=1. \tag{6}$$

Next, to enforce the second requirement, we write

$$\left\langle\exp\left\{ig\int_0^{T_1}d\tau\dot{x}_1(\tau)\cdot A(x_1(\tau))+ig\int_0^{T_2}d\tau\dot{x}_2(\tau)\cdot A(x_2(\tau))\right\}\right\rangle_A$$
$$=\sum_{k=0}^{\infty}\sum_{n=0}^{2k}\frac{(ig)^{2k}}{(2k)!}\binom{2k}{n}\int\mathcal{D}Ae^{-S_J[A,\xi]}\left(\int_0^{T_1}d\tau\dot{x}_1(\tau)\cdot A(x_1(\tau))\right)^{2k-n}\left(\int_0^{T_2}d\tau\dot{x}_2(\tau)\cdot A(x_2(\tau))\right)^{n}. \tag{7}$$

With reference to the above relation, we demand that the summation over n be restricted only to terms for which $n=k$. It can easily be seen that this particular choice obliges a photon propagator to attach itself on each of the two spin-1/2 particle lines.

It follows that the exchange graphs lead to the specification

$$\left\langle\exp\left\{ig\int_0^{T_1}d\tau\dot{x}_1(\tau)\cdot A(x_1(\tau))+ig\int_0^{T_2}d\tau\dot{x}_2(\tau)\cdot A(x_2(\tau))\right\}\right\rangle_A\to\exp\left\{-g^2\int_0^{T_1}d\tau_1\int_0^{T_2}d\tau_2\dot{x}_{1\mu}(\tau_1)\dot{x}_{2\nu}(\tau_2)\right.$$
$$\left.\times\langle A_\mu(x_1(\tau_1))A_\nu(x_2(\tau_2))\rangle_A\right\}, \tag{8}$$

where $\langle A_\mu(x_1(\tau_1))A_\nu(x_2(\tau_2))\rangle_A$ denotes the free photon correlator whose well-known expression, in the Feynman gauge, is

$$\langle A_\mu(x_1(\tau_1))A_\nu(x_2(\tau_2))\rangle_A=\delta_{\mu\nu}\frac{1}{(2\pi)^4}\int d^4k\frac{1}{k^2}e^{ik\cdot[x_2(\tau_2)-x_1(\tau_1)]}. \tag{9}$$

Putting everything together, our worldline expression for the exchange-type diagrams becomes

$$G^4_{exch}(x_1,x_2,y_1,y_2)=\int_0^\infty dT_1\int_0^\infty dT_2\int_{\substack{x_1(0)=x_1\\x_1(T_1)=y_1}}[dx_1(\tau)]\int[dp_1(\tau)]\int_{\substack{x_2(0)=x_2\\x_2(T_2)=y_2}}[dx_2(\tau)]\int[dp_2(\tau)]$$
$$\times\exp\left\{i\int_0^{T_1}d\tau p_1(\tau)\cdot\dot{x}_1(\tau)\right\}\exp\left\{i\int_0^{T_2}d\tau p_2(\tau)\cdot\dot{x}_2(\tau)\right\}\mathcal{P}\exp\left\{-\int_0^{T_1}d\tau[i\gamma\cdot p_1(\tau)+m]\right\}$$
$$\times\mathcal{P}\exp\left\{-\int_0^{T_2}d\tau[i\gamma\cdot p_2(\tau)+m]\right\}\exp\left\{-g^2\int_0^{T_1}d\tau_1\int_0^{T_2}d\tau_2\dot{x}_{1\mu}(\tau_1)\dot{x}_{2\nu}(\tau_2)\delta_{\mu\nu}\frac{1}{(2\pi)^4}\right.$$
$$\left.\times\int d^4k\frac{1}{k^2}e^{ik\cdot[x_2(\tau_2)-x_1(\tau_1)]}\right\}. \tag{10}$$

According to the eikonal setting, the fermions travel with large momenta while the exchanged photons are soft, in comparison. In a worldline context, this means that the absorption and/or emission of virtual photons does not induce recoil of the spin-1/2 matter particles. We view the latter as travelling practically on mass shell with almost constant energy and spatial direction, i.e., with fixed four-velocities. This allows us to introduce two constant four-velocities in Eq. (10), one for each fermion, and cast the exponential associated with the gauge sector in the form

$$E_{exch}(u_1,u_2)\equiv \exp\left\{-g^2 u_1\cdot u_2\int_0^{T_1}d\tau_1\int_0^{T_2}d\tau_2\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2}e^{ik\cdot(u_2\tau_2-u_1\tau_1+x_2-x_1)}\right\}. \tag{11}$$

The four-velocities u_1,u_2 entering the above expression should, first of all, obey the (near) mass shell condition $|u_1|=|u_2|\simeq 1$. Second, because of the fact that the eikonal approximation is used to describe only forward scattering, we demand that $\vec{u}_1//\vec{u}_2$ hold true. Note that $u_1\cdot u_2\neq 1$ due to the fact that the two fermions have different energies.

In summary, our worldline approach for the eikonal four-fermion Green's function is encoded in the formula

$$\begin{aligned}G^4_{exch}(x_1,x_2,y_1,y_2)=&\int_0^\infty dT_1\int_0^\infty dT_2\int_{\substack{x_1(0)=x_1\\x_1(T_1)=y_1}}[dx_1(\tau)]\int[dp_1(\tau)]\int_{\substack{x_2(0)=x_2\\x_2(T_2)=y_2}}[dx_2(\tau)]\\&\times\int[dp_2(\tau)]\exp\left\{i\int_0^{T_1}d\tau p_1(\tau)\cdot\dot{x}_1(\tau)\right\}\exp\left\{i\int_0^{T_2}d\tau p_2(\tau)\cdot\dot{x}_2(\tau)\right\}\mathcal{P}\exp\left\{-\int_0^{T_1}d\tau[i\gamma\cdot p_1(\tau)+m]\right\}\\&\times\mathcal{P}\exp\left\{-\int_0^{T_2}d\tau[i\gamma\cdot p_2(\tau)+m]\right\}\exp\left\{-g^2u_1\cdot u_2\int_0^{T_1}d\tau_1\int_0^{T_2}d\tau_2\right.\\&\left.\times\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2}e^{ik\cdot(u_2\tau_2-u_1\tau_1+x_2-x_1)}\right\}.\end{aligned}\tag{12}$$

The extraction of specific results from the above expression will be the objective of our work in this paper.

III. KINEMATICAL DECOMPOSITION AND SOFT FACTORIZATION

The most important feature of the expression for the exchange part of the four-point function arrived at in the previous section and on which our eikonal considerations will be based is the Wilson line operator (expectation value of). It carries all the dynamical aspects of any given calculation that addresses itself to G_{exch}. For the Abelian case in hand and in the quenched approximation we are adopting, it gives rise to the nonperturbative expression furnished, in the Feynman gauge, by Eq. (11). In the present section we shall make some general assessments related to this quantity which will facilitate our subsequent analysis.

Let us begin by writing

$$E_{exch}=\exp\{-g^2I(u_1,u_2)\}. \tag{13}$$

Allowing for the possibility of being forced to regularize at some point let us revert to d dimensions and focus our attention on

$$I(u_1,u_2)=\int\frac{d^dk}{(2\pi)^d}\frac{e^{ik\cdot x}}{k^2}\frac{e^{-ik\cdot u_1T_1}-1}{-ik\cdot u_1}\frac{e^{ik\cdot u_2T_2}-1}{ik\cdot u_2}, \tag{14}$$

where T_1, T_2 are the limits of the parametric integrations over τ_1 and τ_2, respectively.

We observe that

$$I(u_1,u_2)\to\int\frac{k^{d-1}dk}{(2\pi)^d}\frac{1}{k^2},$$

as $k\to 0$, and therefore remains finite for $d\to 4$ as long as T_1 and T_2 do not go to infinity. In other words, T_1 and T_2 offer protection against IR divergences via an off mass shellness of the matter particles and this we understand on the basis that a Wilson line of finite extent does not take into account the full gauge field cloud with which they interact; hence they remain somewhat off shell. We shall, of course, assume that $T_1\sim T_2(=T)$ are long enough so that the off shellness can be as sufficiently small as we wish. If, on the other hand, we let $T_1,T_2\to\infty$, then we are creating an on mass shell description with an IR divergence that must be controlled by a small photon mass λ or, alternatively, by dimensionally regularizing to $d>4$.

On the ultraviolet (UV) end we determine, as $d\to 4$, that $\lim_{\epsilon\to 0}\int dk e^{ik\cdot x}/k^{1+\epsilon}$ remains finite due to the very quick oscillations of the term $e^{ik\cdot x}$, as long as x does not go to zero. A further comment on this point will be made at the end of this section.

Consider, now, the following decomposition of the four-vector $\tilde{k}$ entering Eq. (14):

$$\tilde{k}=\vec{k}_\perp+k_1\hat{u}_1+k_2\hat{e}, \tag{15}$$

with

$$\vec{k}_\perp\cdot\hat{u}_1=\vec{k}_\perp\cdot\hat{e}=\hat{u}_1\cdot\hat{e}=0, \tag{16}$$

where $\hat{u}_1$ is the unit vector along $\tilde{u}_1$, i.e., $\tilde{u}_1=|\tilde{u}_1|\hat{u}_1$.

We write

$$\tilde{k}\cdot\hat{u}_1=k_1,\quad \tilde{k}\cdot\tilde{u}_2=k_1\hat{u}_1\cdot\hat{u}_2+k_2\hat{e}\cdot\hat{u}_2=k_1\cos\gamma+k_2\sin\gamma,$$

$$\tilde{k}^2=k_\perp^2+k_1^2+k_2^2 \tag{17}$$

and determine that

$$I(u_1,u_2)=\frac{u_1\cdot u_2}{4\pi}\int\frac{d^{d-2}k_\perp}{(2\pi)^{d-2}}\int_0^\infty\frac{dz}{z}e^{-zk_\perp^2+i\vec{k}_\perp\cdot\vec{x}_\perp}\tilde{I}, \tag{18}$$

where the quantity $\tilde{I}$ is given by

$$\tilde{I}=\int_0^T d\tau_1\int_0^T d\tau_2 e^{i(k_1\cos\gamma+k_2\sin\gamma)\tau_2-ik_1\tau_1}. \tag{19}$$

Our first observation is that, in such a kinematical analysis, the UV protection is taken over by $\vec{x}_\perp$ which registers as the impact parameter for the four-point process. Second, we notice that for the on shell case ($T_1,T_2\to\infty$) $\tilde{I}$ comes out of the τ_i integrals, $i=1,2$. For the off-shell case, on the other hand, calculational complications *do* present themselves and they will occupy our attention later (Sec. V).

Our final remarks pertain to the factorization of the ''soft'' behavior at the Green's-function level. To this end, we shall use renormalization properties of the field system as a guide. Let us begin by forming a physical picture associated with our requirement that the matter particles propagate along straight lines. As already mentioned, the no-recoil basis on which emission and absorption of gauge field quanta occurs implies that the corresponding ''live'' degrees of freedom entering the computation of the four-point Green's function are ''soft.'' Clearly, the meaning of softness here is relative to the c.m. energy of the colliding fermions and includes all photons that cannot cause appreciable derailment from their propagation paths. It follows that our restriction to straight line path propagation corresponds to isolating a sector of the full theory that involves only ''soft'' photon exchanges. We shall refer to this subsystem as the soft sector of the full theory.

We have already demonstrated, in a number of previous studies [15,16,22], that a subtheory factorized in this way has its own high and low energy domains, the former of which calls for a renormalization treatment. Furthermore, straight line path propagation will invariably identify any multiplicative renormalization factor common to all contours joining given initial and final points in space-time (which pertain exclusively to the Wilson line operators) and hence factorizes itself from the overall result. Of course, non-straight-line propagation in the path integral induces extra contributions to the full n-point Green's function. This allows us to write, in general,

$$G=G_{ren}^{(s)}G^{(h)}, \tag{20}$$

where $G_{ren}^{(s)}$ is the contribution to the Green's function from the soft sector, as defined by our worldline approach. It incorporates the renormalization factor which is picked up by the straight line configurations and will be the object of study in what follows. In an eikonal setting, wherein $s\to\infty$, $t/s\to 0$, it contains the *total* contribution to G. On the other hand, $G^{(h)}$ represents what the rest of the theory (hard sector) contributes to G in the generic case.

For an Abelian system $G_{ren}^{(s)}$ registers directly in a nonperturbative form given the relation

$$\left\langle \exp\left\{-ig\int_0^T d\tau\dot{x}(\tau)\cdot A(x(\tau))\right\}\right\rangle_A$$

$$=\exp\left\{-g^2\int_0^T d\tau_1\int_0^T d\tau_2\dot{x}_\mu(\tau_1)\dot{x}_\nu(\tau_2)\times\langle A_\mu(x(\tau_1))A_\nu(x(\tau_2))\rangle_A\right\}, \tag{21}$$

whereas for a non-Abelian system Eq. (20) holds true in the sense of a perturbative expansion.

Our practice, throughout this paper, will be to bypass renormalization issues in the soft sector by keeping a finite impact parameter. In effect, then, we shall be dealing exclusively with renormalized quantities. Therefore, we shall omit the indication ''ren'' from hereon.

IV. ON MASS SHELL CASE: THE INFINITE COULOMB PHASE

The computation of E_{exch} for the strictly on mass shell case arises in the limit $T_1,T_2\to\infty$. Under this circumstance the parametric integrations eventually factor out, leaving behind a purely kinematical expression. We direct our efforts on the computation of the quantity

$$E_{exch}=\exp\left\{-2g^2u_1\cdot u_2\int_0^\infty d\tau_1\int_0^\infty d\tau_2\times\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2}e^{ik\cdot(u_2\tau_2-u_1\tau_1+x_2-x_1)}\right\}, \tag{22}$$

where the factor of 2 in the exponent mirrors the necessary ordering of τ_1,τ_2 during the integration course or, on physical grounds, our inability to distinguish the two electrons.

As already pointed out, the above quantity exhibits no ultraviolet divergences as long as the starting points of the fermion lines do not coincide, but does possess infrared ones. We shall confront their regularization in two different ways. First, we shall set $d=4$ and introduce a small photon mass while, second, we shall employ the dimensional technique by analytically continuing to a Euclidean space-time of $d=4+\epsilon$ dimensions.

An immediate issue we have to face is the obligation to take into account all the ways of avoiding the pole at $k=0$ in the k integration. This is reflected in the two possible relative orientations of the four-velocities u_1 and u_2 as the parameters τ_1 and τ_2 follow their course from 0 to ∞. This subtlety is taken care of by letting the $\tau_i,i=1,2$, run along the positive axis and compute the following quantity:

$$E_{exch}=\exp\Bigg\{-2g^2u_1\cdot u_2\int_0^\infty d\tau_1\int_0^\infty d\tau_2 \times\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2}e^{ik\cdot(u_2\tau_2-u_1\tau_1+x_2-x_1)} -2g^2u_1\cdot u_2\int_0^\infty d\tau_1\int_0^\infty d\tau_2 \times\int\frac{d^4k}{(2\pi)^4}\frac{1}{k^2}e^{ik\cdot(u_2\tau_2+u_1\tau_1+x_2-x_1)}\Bigg\}. \tag{23}$$

The second term in the exponent corresponds to a time-ordered propagation of the fermions while the first one allows for an arbitrary ‘‘coexistence’’ of the two electrons (fermions). Intuitively speaking, in the second term propagation of the ‘‘second’’ electron takes place when the propagation of the ‘‘first’’ has already been completed. We expect a contribution that gives rise only to bremsstrahlung radiation owing to the instantaneous acceleration or deceleration of the two particles. As far as the first term is concerned, it obviously allows for the above mentioned but in addition the two fermions now have the possibility of travelling together. In this case we can also view one of the electrons as providing a Coulomb field source inside which the second one is moving. We would, consequently, expect the emergence of the well-known divergent Coulomb phase which attaches itself to the Green’s function.

Adopting the kinematical decomposition of the previous section we write

$$E_{exch}=\exp\Bigg\{-2g^2u_1\cdot u_2\int_0^\infty d\tau_1\int_0^\infty d\tau_2\int\frac{d^4k}{(2\pi)^4} \times\int_0^\infty dz e^{-zk^2+ik_\perp\cdot x_\perp}[e^{i(k_1\cos\gamma+k_2\sin\gamma)\tau_2-ik_1\tau_1} +e^{i(k_1\cos\gamma+k_2\sin\gamma)\tau_2+ik_1\tau_1}]\Bigg\}. \tag{24}$$

Completing the squares and performing the Gaussian integrations over the first two components of k we find

$$E_{exch}=\exp\Bigg\{-\frac{2}{\pi}g^2u_1\cdot u_2[I_1+I_2]\int\frac{d^2k_\perp}{(2\pi)^2} \times\int_0^\infty dz e^{-zk^2{}_\perp+ik_\perp\cdot x_\perp}\Bigg\}, \tag{25}$$

with

$$I_1=\int_0^\infty dt_1\int_0^\infty dt_2 e^{-(t_1^2+t_2^2-2t_1t_2w)},$$

$$I_2=\int_0^\infty dt_1\int_0^\infty dt_2 e^{-(t_1^2+t_2^2+2t_1t_2w)}, \tag{26}$$

where $w\equiv u_1\cdot u_2$.

Relegating the details of the respective calculations to Appendix A we quote the final results:

$$I_1=\frac{\pi}{2\sqrt{1-w^2}}-\frac{\arctan|\tilde{w}|}{2\sqrt{1-w^2}},\qquad I_2=\frac{\arctan|\tilde{w}|}{2\sqrt{1-w^2}}=\frac{w\gamma}{\sin2\gamma},$$

$$\tilde{w}^2=\frac{1}{w^2}-1. \tag{27}$$

Wick rotating back to Minkowski space involves the substitution $\gamma\to i\gamma$, which leads to the relation

$$w(I_1+I_2)=-i\frac{\pi}{2}\coth\gamma. \tag{28}$$

With respect to the remaining integration over $d^2k_\perp$ we initially observe that ultraviolet protection, which allows us to think in terms of renormalized quantities, now resides in the term $e^{ik_\perp\cdot x_\perp}$. As already mentioned, $x_\perp$ has the concrete physical meaning of furnishing the impact parameter for the four-point process.

In contrast, there are infrared divergences and we proceed to regulate them via the introduction of a fictitious mass term for the photon. We get

$$E_{exch}=\exp\Bigg\{ig^2\coth\gamma\int\frac{d^2k_\perp}{(2\pi)^2}\frac{e^{ik_\perp\cdot x_\perp}}{k_\perp^2+\mu^2}\Bigg\}, \tag{29}$$

which is in agreement with the known eikonal result [23].

Employing, instead, dimensional regularization for treating the infrared divergences we determine

$$\begin{aligned}\int\frac{d^{d-2}k_\perp}{(2\pi)^{d-2}}\frac{e^{ik_\perp\cdot x_\perp}}{k_\perp^2}&=\int_0^\infty dz\int\frac{d^{d-2}k_\perp}{(2\pi)^{d-2}}e^{-zk_\perp^2+ik_\perp\cdot x_\perp}\\&=\frac{1}{(2\sqrt{\pi})^{d-2}}\frac{1}{\Gamma\left(\frac{d-2}{2}\right)}\times\int_0^\infty dz e^{-x_\perp^2/4z}\int_0^\infty dx x^{(d-4)/2}e^{-zx}\\&=\frac{1}{4\pi}\frac{1}{(\sqrt{\pi}|x_\perp|)^{d-4}}\Gamma\left(\frac{d-4}{2}\right)\\&=\frac{1}{2\pi}\frac{1}{d-4}+\mathrm{FT},\end{aligned} \tag{30}$$

where FT stands for ‘‘finite terms.’’

Referring to Eq. (28) we conclude

$$E_{exch}=\exp\Bigg\{\frac{i}{2\pi\beta}\frac{g^2}{n-4}\Bigg\}, \tag{31}$$

with $\beta=\tanh\gamma$.

We have thereby obtained the well-known, infinite Coulomb phase which is a trademark of infrared physics in QED. From an intuitive viewpoint it is, indeed, satisfying that the simple setting of straight line paths in the worldline integral has given rise to this result. From the practical side, on the other hand, this should be viewed as just a first step on the way to fuller expressions that bring in correction terms. The question is whether and how the worldline approach leads us to such terms. One source of corrections has already been identified [4] and concerns contributions from an expansion of the Dirac determinant, i.e., going beyond the quenched approximation. They correspond to what is yielded by the unitarity diagrams of Cheng and Wu [1,2]. More to the nature of the worldline scheme are corrections from cusped lines, corresponding to sizable momentum transfers, as well as small deformations of straight line contours coming from velocity expansions [15]. We shall not, in this paper, venture into such directions. What we *shall* do instead, in the next section, is consider the subtleties entering the off mass shell behavior.

V. OFF MASS SHELL CONTINUATION OF EIKONAL BEHAVIOR

In this section we shall treat the off mass shell case, for the fermions entering the scattering process. This situation is attained by putting a finite upper limit on the $d\tau_i$ integrations. We focus, as previously, on the term

$$E_{exch}=\exp\{-2g^2w[I_1(u_1,u_2)+I_2(u_1,u_2)]\}, \tag{32}$$

where, now,

$$I_1(u_1,u_2)=\int\frac{d^4k}{(2\pi)^4}\frac{e^{ik_\perp\cdot x_\perp}}{k^2}\frac{e^{-ik\cdot u_1T_1}-1}{-ik\cdot u_1}\frac{e^{ik\cdot u_2T_2}-1}{ik\cdot u_2} \tag{33}$$

and

$$I_2(u_1,u_2)=\int\frac{d^4k}{(2\pi)^4}\frac{e^{ik_\perp\cdot x_\perp}}{k^2}\frac{e^{ik\cdot u_1T_1}-1}{ik\cdot u_1}\frac{e^{ik\cdot u_2T_2}-1}{ik\cdot u_2}, \tag{34}$$

corresponding to the two possible relative orientations of the four-velocities u_1 and u_2.

Using the identities

$$\frac{1}{AB}=\int_0^1dx\frac{1}{(xA+(1-x)B)^2},$$

$$\frac{1}{A^{(2)}}=\int_0^\infty dz(z)e^{-zA},$$

$A>0$, we cast $I_1(u_1,u_2)$ in the form

$$I_1(u_1,u_2)=\int\frac{d^4k}{(2\pi)^4}\int_0^\infty dze^{-zk^2+ik_\perp\cdot x_\perp}\times\int_0^1dy\int_0^\infty dwwe^{-wik\cdot[yu_1-(1-y)u_2]}\times(e^{-ik\cdot u_1T_1}-1)(e^{ik\cdot u_2T_2}-1). \tag{35}$$

We proceed to split the above integral into three parts by setting $I_1=I_{11}+I_{12}-I_{13}$, where

$$I_{11}=\int\frac{d^dk}{(2\pi)^d}\int_0^\infty dze^{-zk^2+ik_\perp\cdot x_\perp}\times\int_0^1dy\int_0^\infty dwwe^{-wik\cdot[yu_1-(1-y)u_2]}, \tag{36}$$

$$I_{12}=\int\frac{d^dk}{(2\pi)^d}\int_0^\infty dze^{-zk^2+ik_\perp\cdot x_\perp}\times\int_0^1dy\int_0^\infty dwwe^{-wik\cdot[yu_1-(1-y)u_2]+ik\cdot(u_2T_2-u_1T_1)}, \tag{37}$$

and

$$I_{13}=\int\frac{d^dk}{(2\pi)^d}\int_0^\infty dze^{-zk^2+ik_\perp\cdot x_\perp}\int_0^1dy\times\int_0^\infty dwwe^{-wik\cdot[yu_1-(1-y)u_2]}\times(e^{ik\cdot u_2T_2}+e^{-ik\cdot u_1T_1}). \tag{38}$$

Notice that we have reverted to d dimensions as each of the quantities I_{1i}, $i=1,2,3$, above possesses, on its own, infrared divergences, even though the full quantity I_1 does not.

The computation of I_{11} is quite obvious and is accomplished by first performing the quadratic k integration and subsequently the one with respect to dw. We obtain

$$I_{11}=\frac{2}{(2\sqrt{\pi})^d}\int_0^1dy\frac{1}{y^2+(1-y)^2-2y(1-y)w}\times\int_0^\infty dzz^{1-d/2}e^{-x_\perp^2/4z}. \tag{39}$$

The remaining z integration is straightforward after the substitution $z\to1/z$. Our final answer is

$$I_{11}=\frac{4}{(2\sqrt{\pi})^d}\left(\frac{4}{x_\perp^2}\right)^{d/2-2}\frac{1}{\sqrt{1-w^2}}\arctan\left(\frac{1-w}{1+w}\right)^{-1/2}\times\Gamma\left(\frac{d}{2}-2\right). \tag{40}$$

We immediately recognize the presence of an infrared pole in the limit $d\to4$. Of course, this pole will disappear, as we have already mentioned, in the full expression for I_1.

The computation of I_{12} is much more involved. For the details of the calculation the reader is referred to Appendix B. The final result is

$$I_{12}=\frac{1}{4\pi^{d/2}}\Gamma\left(\frac{d}{2}-2\right)\frac{1}{\sqrt{1-w^2}}\arctan\left(\frac{1-w}{1+w}\right)^{-1/2}(x_\perp^2+T_1^2+T_2^2-2T_1T_2w)^{2-d/2}-\frac{4}{2^d\pi^{d/2}}\frac{\Gamma\left[\frac{(d-2)}{2}\right]}{d-3}$$
$$\times\int_0^1 dy\frac{1}{f_{11}^2}[-2\sqrt{f_{11}^2}]^{d-4}f_{12}^{4-d}F\left(\frac{d-3}{2},\frac{d-2}{2};\frac{d-1}{2};\frac{-f_{13}f_{11}^2}{f_{12}^2}\right), \tag{41}$$

with $f_{11}(y)=yu_1-(1-y)u_2$, $f_{12}(y)=f_1f_{11}(y)$, $f_{13}(y)=x_\perp^2+f_1^2-f_{12}(y)^2/f_{11}(y)^2$, $f_1=u_2T_2-u_1T_1$, while $F(a,b;\gamma;z)$ denotes a hypergeometric function.

We observe that the second term is finite as $d\to4$ and so we write

$$I_{12}=\frac{1}{4\pi^{d/2}}\Gamma\left(\frac{d}{2}-2\right)\frac{1}{\sqrt{1-w^2}}\arctan\left(\frac{1-w}{1+w}\right)^{-1/2}(x_\perp^2+T_1^2+T_2^2-2T_1T_2w)^{2-d/2}-\frac{1}{4\pi^2}\int_0^1 dy\frac{1}{f_{11}^2}F\left(\frac{1}{2},1;\frac{3}{2};\frac{-f_{13}f_{11}^2}{f_{12}^2}\right). \tag{42}$$

The integral I_{13} is performed in the same fashion and we obtain, as a final answer for I_1, the following:

$$I_1(u_1,u_2)=\frac{1}{4\pi^{d/2}}\Gamma\left(\frac{d}{2}-2\right)\frac{1}{\sqrt{1-w^2}}\arctan\left(\frac{1-w}{1+w}\right)^{-1/2}\{(x_\perp^2)^{2-d/2}+(x_\perp^2+T_1^2+T_2^2-2T_1T_2w)^{2-d/2}-(x_\perp^2+T_1^2)^{2-d/2}$$
$$-(x_\perp^2+T_2^2)^{2-d/2}\}-\frac{1}{4\pi^2}\int_0^1 dy\frac{1}{f_{11}^2}\left\{F\left(\frac{1}{2},1;\frac{3}{2};\frac{-f_{13}f_{11}^2}{f_{12}^2}\right)-F\left(\frac{1}{2},1;\frac{3}{2};\frac{-\tilde f_{12}f_{11}^2}{\tilde f_{11}^2}\right)-F\left(\frac{1}{2},1;\frac{3}{2};\frac{f_{12}'^2(y)f_{11}}{f_{11}'^2}\right)\right\}, \tag{43}$$

with $\tilde f_{11}(y)=f_{11}(y)\tilde f_1$, $\tilde f_{12}(y)=x_\perp^2+\tilde f_1^2-\tilde f_{11}^2/f_{11}^2$, $\tilde f_1=u_2T_2$ and $f_{11}'(y)=f_{11}(y)f_1'$, $f_{12}'(y)=x_\perp^2+f_1'^2-f_{11}'^2/f_{11}^2$, $f_1'=-u_1T_1$. The (last) term containing the hypergeometric functions will be denoted by $\tilde I_1(u_1,u_2)$ in what follows.

One witnesses the (anticipated) cancellation of the individual infrared poles, as $d\to4$, leaving us with the infrared finite expression

$$I_1(u_1,u_2)=\frac{1}{4\pi^2}\frac{1}{\sqrt{1-w^2}}\arctan\left(\frac{1-w}{1+w}\right)^{-1/2}\{-\ln x_\perp^2-\ln(x_\perp^2+T_1^2+T_2^2-2T_1T_2w)+\ln(x_\perp^2+T_1^2)+\ln(x_\perp^2+T_2^2)\}+\tilde I_1(u_1,u_2). \tag{44}$$

Treating in a similar fashion the integral $I_2(u_1,u_2)$ we find

$$I_2(u_1,u_2)=\frac{1}{4\pi^2}\frac{1}{\sqrt{1-w^2}}\arctan\left(\frac{1-w}{1+w}\right)^{1/2}\{-\ln x_\perp^2-\ln(x_\perp^2+T_1^2+T_2^2+2T_1T_2w)+\ln(x_\perp^2+T_1^2)+\ln(x_\perp^2+T_2^2)\}+\tilde I_2(u_1,u_2). \tag{45}$$

We expect to obtain the results of the previous section as the mass shell is approached. Indeed, taking first the limit $T\equiv T_1\sim T_2\gg x_\perp$, using, next, the identity

$$\arctan x+\arctan\frac{1}{x}=\frac{\pi}{2},\quad x>0, \tag{46}$$

and Wick rotating back to Minkowski space we get

$$E_{exch}=\exp\left\{\frac{g^2}{2\pi^2\beta}i\pi\ln\frac{T}{|x_\perp|}-\frac{g^2}{4\pi^2\beta}i\pi\ln(2-2w)-\frac{1}{4\pi^2}g^2\frac{1}{2\beta}\ln\frac{1+\beta}{1-\beta}\ln\frac{1-w}{1+w}+R\right\}, \tag{47}$$

with

$$R\equiv\frac{g^2w}{2\pi^2}\int_0^1 dy\frac{1}{f_{11}^2}\left\{F\left(\frac{1}{2},1;\frac{3}{2};\frac{-f_{13}f_{11}^2}{f_{12}^2}\right)-F\left(\frac{1}{2},1;\frac{3}{2};\frac{-\tilde f_{12}f_{11}^2}{\tilde f_{11}^2}\right)-F\left(\frac{1}{2},1;\frac{3}{2};\frac{-f_{12}'^2f_{11}}{f_{11}'^2}\right)\right\}+\frac{g^2w}{2\pi^2}\int_0^1 dy\frac{1}{f_{21}^2(y)}$$
$$\times\left\{F\left(\frac{1}{2},1;\frac{3}{2};\frac{-f_{23}f_{21}^2}{f_{22}^2}\right)-F\left(\frac{1}{2},1;\frac{3}{2};\frac{-\tilde f_{22}f_{21}^2}{\tilde f_{21}^2}\right)-F\left(\frac{1}{2},1;\frac{3}{2};\frac{-f_{22}'f_{21}^2}{f_{21}'^2}\right)\right\}, \tag{48}$$

where the quantities f_{2j}, $j=1,2,3$, etc. are similarly defined as their counterparts f_{1j}, $j=1,2,3$, etc. (see also Appendix C).

We immediately recognize the emergence of the divergent Coulomb phase in the limit $T\to\infty$. We also see that the ultraviolet divergent behavior, in the limit $|x_\perp|\to 0$, is in a one-to-one correspondence with the infrared one. Concentrating on the near-mass-shell case, in the sense that the lengths of the particle paths are finite but very large with respect to the impact parameter, the explicit computation of R becomes our next objective. The relevant work is carried, always in the limit $T\gg|x_\perp|$, in Appendix C. As it turns out, R is independent of both the ultraviolet and infrared cutoffs. In the asymptotic limit $s\gg m^2$ we obtain the following final answer:

$$E_{exch}=\exp\left\{\frac{ig^2}{2\pi}\ln\frac{T}{|x_\perp|}+\frac{g^2}{\pi^2}\frac{m^2}{s}\ln\frac{s}{m^2}+\frac{ig^2}{4\pi}\ln\frac{s}{m^2}+\mathcal{O}\left(\frac{m^2}{s}\right)\right\}. \tag{49}$$

The above off mass shell eikonal result exhibits a phase readjustment, with respect to the on mass shell expression. The extra piece has an imaginary part which vanishes in the limit $s\to\infty$ and a real part which contributes to the eikonal function. An alternative way to view the final expression is by introducing an UV cutoff and off mass shellness via the correspondences $x_\perp\leftrightarrow 1/\Lambda$ and $T\leftrightarrow 1/(\widetilde{p}^2-m^2)^{1/2}$, respectively, where $|\widetilde{p}|\sim(\sqrt{s}/2)(1+\cosh\gamma)$ is a characteristic four-momentum for the system. Then, the two logarithms can be recombined so that the phase is exhibited in the form

$$\alpha\left[\ln\frac{\Lambda^2}{m^2}-\ln\frac{m^2-\widetilde{p}^2}{s}\right].$$

The appearance of this expression is typical of that which enters unrenormalized Green's functions whose regularization against infrared divergences has been effected by going off shell. The form registered in Eq. (49), on the other hand, possesses the meaning of an off mass shell extended eikonal function.

One final point to be made here concerns the $E_{nonexch}$-containing piece of the four-point Green's function which we have systematically neglected up to now and which enters the T-matrix element in the eikonal description, providing the unit term in the expression $(e^{i\chi}-1)$. It corresponds to the configuration of two noninteracting fermionic lines. As is well known [24], Wilson operators defined on open lines of finite extent are subject to renormalization effects which can be absorbed via a redefinition of the wave function. The relevant divergences can be viewed as collinear bremsstrahlung radiation emitted from the end points of the line where the matter particle suddenly accelerates and/or decelerates.

VI. APPLICATION TO PLANCKIAN SCATTERING

The methodology developed in this paper is transferable to Planckian scattering. This we shall show in the present section by resorting to a linearized version of gravity which will make it possible for us to apply the same procedure we have used for QED. In fact, Jackiw *et al.* [25] *have* worked the opposite way by transferring the gravitational work of [18] to QED. Moreover, Fabbrichesi *et al.* [26] *have* already discovered the feasibility of employing Fradkin's formalism [6] to Planckian scattering descriptions, *albeit* in a different framework from the one presently adopted.

Confining ourselves to the dynamical part of the calculation pertaining to the scattering amplitude, i.e., the part referring to E_{exch}, let us consider the ramifications brought about by the adoption of the following linearized action for gravity, resulting by setting $g_{\mu\nu}=\eta_{\mu\nu}+h_{\mu\nu}$ ($\eta_{\mu\nu}$ the Minkowski metric tensor) and keeping the lowest order terms in the expansion. The relevant action functional reads

$$S=\int d^4x\frac{1}{16\pi G}\left\{\frac{1}{8}h_{\alpha\beta}[\eta^{\alpha\gamma}\eta^{\beta\gamma}-\eta^{\alpha\beta}\eta^{\gamma\delta}]\Box h_{\gamma\delta}+\frac{1}{2}h_{\mu\nu}J^{\mu\nu}\right\}, \tag{50}$$

where $J^{\mu\nu}$ is a matter field current which we shall continue to view as being composed of a bilinear in spin-1/2 fields, but can be assigned to other types of matter fields, e.g., scalar.

The worldline analysis goes through as in QED once the Wilson line replacement

$$\exp\left\{ig\int_0^T d\tau\dot{x}(\tau)\cdot A(x(\tau))\right\}\to\exp\left\{im\int_0^T d\tau\dot{x}^\mu(\tau)\dot{x}^\nu(\tau)h_{\mu\nu}(x(\tau))\right\}$$

is made. The corresponding recasting of the exponential associated with the gauge sector in the expression for the four-point function is

$$\exp\left\{-g^2\int_0^{T_1}d\tau_1\int_0^{T_2}d\tau_2\dot{x}_{1\mu}(\tau_1)\dot{x}_{2\nu}(\tau_2)\times\langle A_\mu(x_1(\tau_1))A_\nu(x_2(\tau_2))\rangle_A\right\}$$
$$\to\exp\left\{-m^2\int_0^{T_1}d\tau_1\int_0^{T_2}d\tau_2\dot{x}_{2\alpha}(\tau_1)\dot{x}_{1\beta}(\tau_1)\dot{x}_{2\gamma}(\tau_2)\times\dot{x}_{2\delta}(\tau_2)\langle h^{\alpha\beta}(x(\tau_1))h^{\gamma\delta}(x(\tau_2))\rangle_h\right\}. \tag{51}$$

In the so-called De Donder gauge, specified by the condition

$$\partial_\nu h^\nu_\mu-\frac{1}{2}\partial_\mu h^\nu_\nu=0, \tag{52}$$

the h-field correlator becomes [21]

$$\langle h_{\alpha\beta}(x_1(\tau_1))h_{\gamma\delta}(x_2(\tau_2))\rangle_h$$
$$=16\pi G(\eta^{\alpha\gamma}\eta^{\beta\delta}+\eta^{\alpha\delta}\eta^{\beta\gamma}-\eta^{\alpha\beta}\eta^{\gamma\delta})$$
$$\times\int\frac{d^4k}{k^2}e^{ik\cdot[x_2(\tau_2)-x_1(\tau_1)]}. \tag{53}$$

Aside from the longer tensor structure and the appearance of dimensionful parameters in the above expressions, the computational procedure matches step for step the QED one. Setting $\dot{x}_1(\tau)\equiv u_1$ and $\dot{x}_2(\tau)\equiv u_2$ we find, in place of Eq. (25), for the on mass shell case,

$$E^{gr}_{exch}=\exp\Bigg\{-\frac{2}{\pi}m^2[2(u_1\cdot u_2)^2-(u_1)^2(u_2)^2]$$
$$\times[I_1+I_2]16\pi G\int\frac{d^2k_\perp}{(2\pi)^2}\int_0^\infty dz e^{-k_\perp^2+ik_\perp\cdot x_\perp}\Bigg\}. \tag{54}$$

But

$$m^2[2(u_1\cdot u_2)^2-(u_1)^2(u_2)^2]=2p_1\cdot p-2)(u_1\cdot u_2)-m^2 \tag{55}$$

and since $2p_1\cdot p_2=s/4-2m^2$, we obtain, upon going to the limit $s\gg m^2$ as well as using Eq. (28),

$$E^{gr}_{exch}=\exp\left\{i\coth\gamma 4\pi Gs\int\frac{d^2k_\perp}{(2\pi)^2}\frac{e^{ik_\perp\cdot x_\perp}}{k_\perp^2+\mu^2}\right\} \tag{56}$$

or ($\mu|x_\perp|\ll 1$, coth$\sim$1)

$$E^{gr}_{exch}=\exp(-2iGs\ \ln\mu x_\perp), \tag{57}$$

which is the well-known, 0th order, result for Planckian scattering [17–21].

The off mass shell extension follows the analysis of Sec. V in a straightforward manner. The resulting expression for E_{exch} is ($s\gg m^2$)

$$E^{gr}_{exch}=\exp\left[2iGs\ \ln\frac{T}{x_\perp}+iGs\ \ln\frac{s}{m^2}\right]. \tag{58}$$

VII. CONCLUSIONS AND OUTLOOK

Our major attempt in this paper has been to establish the applicability of the worldline approach [4,6–10] to a nontrivial (scattering) process which involves open fermionic lines. As in a series of previous papers [14–16,22], we have focused our efforts on a situation where straight lines dominate the path integral. Such a restriction corresponds to instances where the ‘‘live’’ gauge field degrees of freedom are soft and is tantamount to employing an eikonal mode of description.

Besides reproducing the leading eikonal for QED in a two spin-1/2 particle collision process, we were able to derive a corresponding expression for an off mass shell situation. This occurrence encourages us to think beyond the leading approximation as well as consider extensions of our approach to other cases of interest. Already, Fried and Gambellini [12] *have* discussed a strategy for correcting eikonal results which, in our approach, amounts to small distortions of straight line paths. Another direction towards which our particle-based language could be tested is when a *cusped* line enters the four-point configuration [16], in which case a non-negligible momentum transfer takes place. Even though this is not an exact eikonal situation, it *would* serve as an approach to momentum transfer corrections in a high energy scattering process involving four fermions.

With respect to recent string-based work on quantum gravity, Planckian scattering in particular [17–21], we have been able to turn things around and circumvent string theory (in favor of worldline) input in the analysis of extremely high energy gravitational scattering, to leading order at least. This is the analogue of the string-inspired idea for closed fermion line calculations [27] being replaced by the worldline philosophy [7]. It is our conviction that, as long as one can effectively ensure UV protection, such as the one provided by the string, the worldline approach always offers an alternative framework for physical descriptions. Moreover, our present work has explicitly exemplified its capability for factoring out the soft sector of a given field theory—which is by far a more demanding task than that of securing UV immunity.

As far as applications to non-Abelian gauge systems are concerned, the, apparent, inevitability of a perturbative expansion of the Wilson line operator has already been circumvented, *albeit* under restricted conditions, by the breakthrough work of Ref. [13]. This offers solid hope that even more general approaches to the non-Abelian eikonal can be attempted. The true challenge, of course, is how to incorporate gluon interaction vertices into the worldline scheme. We find very promising, in this respect, the work of Di Vecchia *et al.* [28] which, in combination with the methodology employed for the computation of eikonals in gravity theory, especially those of Refs. [17,20], could point the way towards confronting non-Abelian eikonals in a fully nonlinear context.

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APPENDIX A

The goal of this appendix will be the computation of the integrals I_1, I_2 defined in Eq. (26) of the text. Referring first to I_1 we write

$$I_1=\int_0^\infty dt_1\int_0^\infty dt_2 e^{-(t_1^2+t_2^2-2t_1t_2w)}$$
$$=\int_0^\infty dt_1 e^{-t_1^2(1-w^2)}\int_{-t_1w}^\infty dt_2 e^{-t_2^2}. \tag{A1}$$

According to our conventions $w>0$, whereupon we obtain

$$I_1=\int_0^\infty dt_1 e^{-t_1^2(1-w^2)}\left[\int_{-t_1w}^0 dt_2 e^{-t_2^2}+\frac{\sqrt{\pi}}{2}\right]=\frac{\pi}{4\sqrt{1-w^2}}+\int_0^\infty dt_1 e^{-t_1^2(1-w^2)}\frac{\sqrt{\pi}}{2}\Phi(t_1w) \tag{A2}$$

with

$$\Phi(x)=\frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt. \tag{A3}$$

Using the tabulated integral [28]

$$\int_0^\infty [1-\Phi(x)]e^{-\mu^2x^2}dx=\frac{\arctan\mu}{\sqrt{\pi}\mu}\quad \mathrm{Re}\mu>0 \tag{A4}$$

we find

$$\int_0^\infty \Phi(x)e^{-\mu^2x^2}dx=\frac{\sqrt{\pi}}{2\mu}-\frac{\arctan\mu}{\sqrt{\pi}\mu}\quad \mathrm{Re}\mu>0 \tag{A5}$$

and consequently

$$I_1=\frac{\pi}{2\sqrt{1-w^2}}-\frac{\arctan\widetilde{w}}{2w\widetilde{w}}, \tag{A6}$$

with $\widetilde{w}^2=1/w^2-1$. Making use of the ''angle'' parameter γ we obtain the alternative expression

$$I_1=\frac{\pi-\gamma}{2\sin\gamma}. \tag{A7}$$

Wick rotation back to Minkowski space involves the substitution $\gamma\to i\gamma$. Denoting $\beta=\tanh\gamma$ we find

$$\gamma=\frac{1}{2}\ln\frac{1+\beta}{1-\beta}$$

and so

$$wI_1=-\frac{1}{2\beta}\left(\frac{1}{2}\ln\frac{1+\beta}{1-\beta}+i\pi\right). \tag{A8}$$

We now turn our attention to the integral I_2 given by

$$I_2=\int_0^\infty dt_1\int_0^\infty dt_2 e^{-(t_1^2+t_2^2+2t_1t_2w)}=\int_0^\infty dt_1 e^{-t_1^2(1-w^2)}\int_{t_1w}^\infty dt_2 e^{-t_2^2}. \tag{A9}$$

For $w>0$ we determine ($t_2^2\equiv x$)

$$I_2=\frac{1}{2}\int_0^\infty dt_1 e^{-t_1^2(1-w^2)}\int_{t_1^2w^2}^\infty dx x^{-1/2}e^{-x}=\frac{1}{2}\int_0^\infty dt_1 e^{-t_1^2(1-w^2)}\Gamma\left(\frac{1}{2},t_1^2w^2\right). \tag{A10}$$

Substituting now $t_1^2\to x$ we get

$$I_2=\frac{1}{4}\int_0^\infty dx x^{-1/2}e^{-x(1-w^2)}\Gamma\left(\frac{1}{2},xw^2\right)=\frac{|w|}{2}F\left(1,1;\frac{3}{2};\sin^2\gamma\right)=\frac{\gamma}{2\sin\gamma}. \tag{A11}$$

The corresponding Minkowskian expression is

$$wI_2=\frac{1}{2\beta}\frac{1}{2}\ln\frac{1+\beta}{1-\beta}. \tag{A12}$$

APPENDIX B

In this appendix we dispose of the calculation of $I_{12}(u_1,u_2)$ given by Eq. (37) in the text. Performing the quadratic k integration we obtain

$$I_{12}(u_1,u_2)=\int_0^\infty dz\int_0^1 dy\int_0^\infty dww\frac{1}{(2\pi)^d}\left(\frac{\pi}{z}\right)^{d/2}\times e^{-[x_\perp-wf_{11}(y)+f_1]^2/4z}, \tag{B1}$$

with $f_{11}(y)=yu_1-(1-y)u_2, f_1=u_2T_2-u_1T_1$.

Equivalently, we write

$$I_{12}(u_1,u_2)=\frac{1}{(2\sqrt{\pi})^d}\int_0^\infty dz z^{-d/2}e^{-(x_\perp^2+f_1^2)/4z}\times\int_0^1 dy\int_0^\infty dw\ we^{-w^2f_{11}(y)^2/4z+(w/2z)f_{11}(y)f_1}. \tag{B2}$$

Denoting $f_{11}(y)f_1\equiv f_{12}(y)$ and using the tabulated integral [28]

$$\int_0^\infty x^{\nu-1}e^{-\beta x^2-\gamma x}dx=(2\beta)^{-\nu/2}\Gamma(\nu)\exp\left\{\frac{\gamma^2}{8\beta}\right\}D_{-\nu}\left(\frac{\gamma}{\sqrt{2\beta}}\right),\quad \mathrm{Re}\beta>0,\quad \mathrm{Re}\nu>0, \tag{B3}$$

where $D_{-\nu}(z)$ is the parabolic cylinder function of order $-\nu$, we find

$$I_{12}(u_1,u_2)=\frac{2}{(2\sqrt{\pi})^d}\int_0^\infty dz z^{1-d/2}e^{-(x_\perp^2+f_1^2)/4z}\times\int_0^1 dy\frac{1}{f_{11}(y)^2}\exp\left(\frac{f_{12}^2}{8zf_{11}^2}\right)D_{-2}\left(-\frac{f_{12}}{\sqrt{2zf_{11}^2}}\right). \tag{B4}$$

The last function is well known:

$$D_{-2}(z)=e^{-z^2/4}-z\sqrt{\frac{\pi}{2}}e^{z^2/4}\left[1-\Phi\left(\frac{z}{\sqrt{2}}\right)\right]. \tag{B5}$$

With the aid of the above relation we may split the integral $I_{12}(u_1,u_2)$ in the form

$$I_{12}(u_1,u_2)\equiv I_{12}^I(u_1,u_2)+I_{12}^{II}(u_1,u_2). \tag{B6}$$

The first term can be computed along the lines followed for the computation of $I_{11}(u_1,u_2)$, as discussed in the text. The final result is

$$I^{I}_{12}(u_1,u_2)=\frac{4}{(2\sqrt{\pi})^d}\left(\frac{4}{x_\perp^2+f_1^2}\right)^{d/2-2}\times\frac{1}{\sqrt{1-w^2}}\arctan\left(\frac{1+w}{1-w}\right)^{1/2}\Gamma\left(\frac{d}{2}-2\right). \tag{B7}$$

As far as the second term in Eq. (B6) is concerned, after the substitution $z\to 1/z^2$ it can be cast in the form

$$I^{II}_{12}(u_1,u_2)=-\sqrt{2}\int_0^1 dy\,\frac{f_{12}(y)}{f_{11}(y)^2\sqrt{f_{11}(y)^2}}\int_0^\infty dz z^{d-4}\times e^{(-z^2/4)f_{13}(y)}\left[1-\Phi\left(-\frac{f_{12}(y)}{2\sqrt{f_{11}(y)^2}}z\right)\right], \tag{B8}$$

where $f_{13}(y)=x_\perp^2+f_1^2-f_{12}(y)^2/f_{11}(y)^2$.

One more tabulated integral is now needed, namely,

$$\int_0^\infty[1-\Phi(\beta x)]e^{\mu^2x^2}x^{\nu-1}dx=\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\pi}\nu\beta^\nu}F\left(\frac{\nu}{2},\frac{\nu+1}{2};\frac{\nu}{2}+1;\frac{\mu^2}{\beta^2}\right),$$
$$\mathrm{Re}\nu>0,\quad \mathrm{Re}\beta^2>\mathrm{Re}\mu^2, \tag{B9}$$

with the aid of which we obtain

$$I^{II}_{12}(u_1,u_2)=2\sqrt{2}\int_0^1 dy\,\frac{1}{f_{11}^2}\frac{\Gamma\left(\frac{d-2}{2}\right)}{\sqrt{\pi}(d-3)}\left(-\frac{2\sqrt{f_{11}^2}}{f_{12}}\right)^{d-4}\times F\left(\frac{d-3}{2},\frac{d-2}{2};\frac{d-1}{2};-\frac{f_{13}f_{11}^2}{f_{12}^2}\right). \tag{B10}$$

We may freely set $d=4$ and so we are led to the result

$$I^{II}_{12}(u_1,u_2)=\frac{2\sqrt{2}}{\sqrt{\pi}}\int_0^1 dy\,\frac{1}{f_{11}^2}F\left(\frac{1}{2},1;\frac{3}{2};-\frac{f_{13}f_1^2}{f_{12}^2}\right). \tag{B11}$$

In a similar fashion we treat the term I_{13}. Collecting pieces together we finally arrive at expression (43) of the text.

APPENDIX C

Here we perform the calculations referring to the quantity R given Eq. (47) in the text. To this end we must, first of all, specify the arguments entering the various hypergeometric functions appearing in R. Through $I_1(u_1,u_2)$ the following quantities enter:

$$f_{11}(y)=yu_1-(1-y)u_2,$$
$$f_{12}(y)=y(1+w)(T_2-T_1)+wT_1-T_2,$$
$$-\frac{f_{13}(y)f_{11}^2}{f_{12}^2}=-\frac{1+w}{1-w}(1-2y)^2, \tag{C1}$$

$$\tilde{f}_{11}(y)=T_2[y(1+w)-1],$$
$$-\frac{\tilde{f}_{12}(y)f_{11}^2}{\tilde{f}_{11}^2}=-\frac{y^2(1-w^2)}{[y(1+w)-1]^2}, \tag{C2}$$

$$f'_{11}(y)=T_1[w-y(1+w)],$$
$$-\frac{f'_{12}(y)f_{11}^2}{f'^2_{11}}=-\frac{(1-w^2)(1-y)^2}{[y(1+w)-w]^2}. \tag{C3}$$

Through $I_2(u_1,u_2)$ we gain the quantities

$$f_{21}(y)=yu_1+(1-y)u_2,$$
$$f_{22}(y)=y(w-1)(T_2-T_1)+wT_1+T_2,$$
$$-\frac{f_{23}(y)f_{21}^2}{f_{22}^2}=-\frac{1-w}{1+w}(1-2y)^2, \tag{C4}$$

$$\tilde{f}_{21}(y)=T_2(yw-y+1),$$
$$-\frac{\tilde{f}_{22}(y)f_{21}^2}{\tilde{f}_{21}^2}=-\frac{y^2(1-w^2)}{[1-y(1-w)]^2}, \tag{C5}$$

$$f'_{21}(y)=T_1[y+(1-y)w],$$
$$-\frac{f'_{22}(y)f_{21}^2}{f'^2_{21}}=-\frac{(1-w^2)(1-y)^2}{[w+y(1-w)]^2}. \tag{C6}$$

We mention that the above arguments of the hypergeometric functions are given in the limit $T_1\sim T_2\gg|x_\perp|$ and we see explicitly the claim introduced in the text, namely, that in this limit the quantity R becomes independent of T and $|x_\perp|$.

Subsequently we use the analytic expression

$$F\left(\frac{1}{2},1;\frac{3}{2};-z^2\right)=\frac{\arctan z}{z}, \tag{C7}$$

and after performing simple integrations we arrive at the result

$$R=\frac{g^2w}{2\pi^2}\Bigg\{-\frac{1}{\sqrt{1-w^2}}\ln\frac{1-w}{2}\arctan A+2\int_0^1 dy\,\frac{\ln y}{2y^2(1+w)-2y(1+w)+1}+\frac{2}{\sqrt{1-w^2}}\int_0^1\frac{dy}{y}\arctan Ay$$

$$-\frac{1}{1-w}\int_0^1 dy\,\frac{\ln(1+A^2y^2)}{1+A^2y^2}-\frac{1}{\sqrt{1-w^2}}\ln\frac{1+w}{2}\arctan\frac{1}{A}+2\int_0^1 dy\,\frac{\ln y}{2y^2(1-w)-2y(1-w)+1}+\frac{2}{\sqrt{1-w^2}}\int_0^1\frac{dy}{y}\arctan\frac{y}{A}$$

$$-\frac{1}{1+w}\int_0^1 dy\frac{\ln\left(1+\frac{y^2}{A^2}\right)}{1+\frac{y^2}{A^2}}\Bigg\}, \tag{C8}$$

with $A=[(1+w)/(1-w)]^{1/2}$.

The integrals involving ''primed'' arguments are identical to the corresponding ones with ''tilde'' arguments, after the substitution $y\to 1-y$.

To evaluate the remaining integrals we first return to Minkowski space ($a \overset{Mink}{\to} -ia$ with $a\equiv[(\cosh\gamma+1)/(\cosh\gamma-1)]^{1/2}$) and then use the expressions

$$\int_0^1 dx\frac{\ln(1-ax)}{1+ax}=-\frac{1}{a}\left(\frac{\pi^2}{12}-\frac{\ln^2 2}{2}\right)+\frac{1}{a}\left[\ln(1-a)\ln\frac{1+a}{2}+\mathrm{Li}_2\left(\frac{1-a}{2}\right)\right], \tag{C9}$$

$$\int_0^1 dx\,\frac{\ln x}{1-ax}=-\frac{1}{a}\mathrm{Li}_2(a), \tag{C10}$$

where $\mathrm{Li}_2(z)$ denotes the so-called double-logarithmic or Spence function defined according to

$$\mathrm{Li}_2(z)=\sum_{k=1}^{\infty}\frac{z^k}{k^2}. \tag{C11}$$

Using the well-known properties

$$\mathrm{Li}_2(z^2)=2[\mathrm{Li}_2(z)+\mathrm{Li}_2(-z)],\quad \mathrm{Li}_2(1)=-\frac{\pi^2}{6}, \tag{C12}$$

$$\mathrm{Li}_2(z)+\mathrm{Li}_2\left(\frac{1}{z}\right)=2\mathrm{Li}_2(1)+\frac{1}{2}\ln^2|z|,\quad \mathrm{Li}_2(z)+\mathrm{Li}_2(1-z)=\mathrm{Li}_2(1)+\ln|z|\ln|1-z|, \tag{C13}$$

we finally arrive at

$$E_{exch}=\exp\Bigg\{\frac{ig^2}{2\pi\beta}\ln\frac{T}{|x_\perp|}+\frac{g^2}{4\pi^2\beta}\bigg[\frac{7}{2}\pi^2+8\mathrm{Li}_2(1-a)-2\mathrm{Li}_2(1-a^2)-2\mathrm{Li}_2\left(\frac{1-a}{2}\right)+2\mathrm{Li}_2\left(\frac{1}{2}-\frac{1}{2a}\right)$$

$$-2\ln a[1+\ln 2+\ln(a-1)]+4\ln a\ln(a+1)\bigg]-\frac{ig^2}{4\pi\beta}[3\ln 2-4\ln a+\ln(a-1)]\Bigg\}, \tag{C14}$$

with the conventions $\sqrt{-1}=i$, $\ln x=i\pi+\ln|x|$, for $x<0$.

To obtain the final result we mention that $\cosh\gamma/a(\cosh\gamma-1)=a\cosh\gamma/(1+\cosh\gamma)$. Using the definition of the center of mass energy,

$$s=(p_1+p_2)^2=2m^2+2m^2\cosh\gamma, \tag{C15}$$

with the fermions practically on mass shell, we determine

$$\cosh\gamma=\frac{s}{2m^2}-1 \tag{C16}$$

and so in the limit $s \gg m^2$ we obtain

$$a \equiv \left(\frac{w+1}{w-1} \right)^{1/2} = 1 + \frac{2m^2}{s}. \tag{C17}$$

This result allows us to give the asymptotic expression of E_{exch} in the above-mentioned limit.

Taking into account that the contribution of the Spence functions is at least of order m^2/s we are led to Eq. (49) of the text.

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