PHYSICAL REVIEW D VOLUME 57, NUMBER 6 15 MARCH 1998

Touching random surfaces, two-dimensional quantum gravity, and noncritical string theory

Oleg Andreev*
Department of Physics and Astronomy, University of Southern California, Los Angeles, California 90089-0484
(Received 21 October 1997; published 6 February 1998)

A set of physical operators which are responsible for touching interactions in the framework of $c<1$ unitary conformal matter coupled to 2D quantum gravity is found. As a special case, the noncritical bosonic strings are considered. Some analogies with four-dimensional quantum gravity are also discussed, e.g., creation-annihilation operators for baby universes and the Coleman mechanism for the cosmological constant. [S0556-2821(98)02606-X]

PACS number(s): 11.25.Pm, 04.60.−m, 11.25.Hf

I. INTRODUCTION

In the last decade there has been much progress in understanding string theory in two dimensions as well as 2D quantum gravity (see, e.g., [1] and references therein). Of course, for most physical applications one needs to consider much more complicated models; however, many principal issues in string theory and quantum gravity are still not understood, and the hope is that the two-dimensional theory will serve as a useful toy model, in which some of these issues may be addressed. For instance, a renormalization group (RG) approach developed for matrix models by Brézin and Zinn-Justin [2] can be used to formulate a large-N renormalization group in a new M(atrix) theory [3]. Another example of this is topological fluctuations in spacetime that produce baby universes. They were intensively discussed in a framework of four-dimensional quantum gravity in relation with a theory of the cosmological constant and loss of quantum coherence [4]. Recently, it was proposed by David [5] that such fluctuations could lead to a scenario for the so-called $c=1$ barrier in two dimensions [6]. The work discussed in this paper was influenced by David's paper.

David begins with the renormalization group analysis of matrix models, with a new coupling constant that governs the dynamics of touching surfaces, i.e., surfaces which are allowed to touch each other at isolated points. In matrix models trace-squared terms are responsible for touching. For example, the one-matrix model with such interaction is given by [7]

$$Z=\int \mathcal{D}\Phi\ \exp\left[-N\ \mathrm{tr}\left(\frac{\Phi^2}{2}-g\ \frac{\Phi^4}{4}\right)-\frac{x}{2}\left(\mathrm{tr}\ \frac{\Phi^2}{2}\right)^2\right]. \tag{1.1}$$

It is known that the model is solvable. Its phase diagram looks like that in Fig. 1.

The point C at $x=x_c$ corresponds to a critical behavior with the string exponent (string susceptibility) $\gamma=\frac{1}{3}$ [7]. On the other hand, critical lines $x<x_c$ and $x>x_c$ are characterized by the string exponents $\gamma=-\frac{1}{2}$ and $\gamma=\frac{1}{2}$, respectively. The first is described in terms of $c=0$ matter coupled to 2D gravity (pure gravity). As to the second, it is a branched polymer critical line. It should be noted that the multicritical point C appears due to fine-tuned touching interactions. At the same time, touching is not very important for the pure gravity phase. In fact, the above picture is valid for $c\leqslant 1$ models too.

It is well known that the scaling for the critical lines with $\gamma<0$, associated with the conventional matrix models (no trace-squared terms), is described in terms of the Liouville effective action[1] [8]

$$S_{\mathrm{eff}}=\frac{1}{2\pi}\int d^2z\left(\partial\phi\bar{\partial}\phi-\frac{1}{4}\ Q\sqrt{\hat{g}}\hat{R}\phi+t_0\sqrt{\hat{g}}e^{\alpha_+\phi}\right), \tag{1.2}$$

where

$$\alpha_+=\frac{1}{2\sqrt{3}}\left(\sqrt{1-c}-\sqrt{25-c}\right),\quad Q=\sqrt{\frac{25-c}{3}}.$$

t_0 is the renormalized cosmological constant. In the above we also assume that the unitary conformal matter has the central charge c. The string exponent is given by

$$\gamma=\frac{Q}{\alpha_+}+2. \tag{1.3}$$

Klebanov and co-workers [9] argued that the scaling for the multicritical points, associated with the modified matrix models, is also described in terms of the Liouville-type action, but with a negatively dressed Liouville potential (cosmological term): namely,

$$\bar{S}_{\mathrm{eff}}=\frac{1}{2\pi}\int d^2z\left(\partial\phi\bar{\partial}\phi-\frac{1}{4}\ Q\sqrt{\hat{g}}\hat{R}\phi+\bar{t}_0\sqrt{\hat{g}}e^{\alpha_-\phi}\right), \tag{1.4}$$

where

$$\alpha_-=-\frac{1}{2\sqrt{3}}\left(\sqrt{1-c}+\sqrt{25-c}\right).$$

*On leave from the Landau Institute for Theoretical Physics, Moscow.

[1]Here and in the subsequent we restrict ourselves to the spherical topology. We also omit kinetic terms for matter in effective actions.

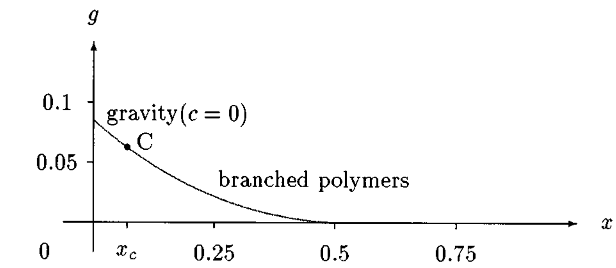

FIG. 1. Phase diagram of the one-matrix model.

It provides the string exponent

$$\bar{\gamma}=\frac{Q}{\alpha_-}+2. \tag{1.5}$$

Comparing to Eq. (1.3), one finds

$$\bar{\gamma}=\frac{\gamma}{\gamma-1}, \tag{1.6}$$

which is in agreement with the matrix model results[2] [7].

However, the missing point of the continuum formulation sketched above is ''touching'' operators, i.e., local operators which are responsible for the touching interactions. Our purpose is to show that the touching interactions can be reproduced in the continuum (Liouville) formulation too. At first sight, it seems naive that a network of touching surfaces is approximated by a surface with insertions of local operators as indicated in Fig. 2. At the present time it is not known whether the situation may be taken under control. Good motivations for this are the reproduction of the string exponents via the Liouville action and the rather special structure of surfaces when they touch each other at isolated points, i.e., locally. So we are bound to learn something if we succeed.

Before continuing our discussion of the touching operators, we will make a detour and recall some basic results on 2D gravity coupled to $c\leqslant 1$ matter.

First, let us summarize notations for a matter sector. It is convenient to bosonize it as

$$S_m=\frac{1}{2\pi}\int d^2z\left(\partial X\bar{\partial}X+i\,\frac{1}{2}\,\alpha_0\sqrt{\hat{g}}\hat{R}X\right), \tag{1.7}$$

where $\alpha_0=\sqrt{(1-c)/12}$. In this language the primary field of the conformal dimension $\Delta^{(0)}$ is represented as the exponent of the free field $X(z,\bar{z})$:

$$V_\alpha(z,\bar{z})=e^{i\alpha X(z,\bar{z})}, \tag{1.8}$$

where $\Delta^{(0)}_\alpha=\frac{1}{2}\alpha(\alpha-2\alpha_0)$.

Dotsenko-Fateev models [10] arise at

$$\alpha=\alpha_{n,m}=\frac{1-n}{2}\,\alpha_-^m+\frac{1-m}{2}\,\alpha_+^m, \tag{1.9}$$

[2]The same relation was also found in multiple spins on dynamical triangulations. The interested reader is referred to lectures of Ambjørn for details [1].

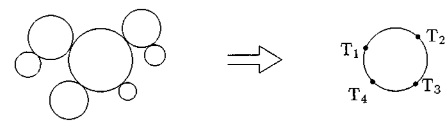

FIG. 2. Approximation of a network of touching surfaces by a single surface with insertions of local operators T_i.

with integers n, m and

$$\alpha_\pm^m=\frac{1}{2\sqrt{3}}\,(\sqrt{1-c}\pm\sqrt{25-c}). \tag{1.10}$$

The corresponding primary fields are given by

$$V_{n,m}(z,\bar{z})=e^{i\alpha_{n,m}X(z,\bar{z})}. \tag{1.11}$$

Their conformal dimensions are written as

$$\Delta^{(0)}_{n,m}=\frac{1}{8}\,[(n\alpha_-^m+m\alpha_+^m)^2-(\alpha_-^m+\alpha_+^m)^2]. \tag{1.12}$$

Minimal models [11] are defined by $(\alpha_+^m)^2=2q/p$, with the coprime integers q and p. These models are very special because of the basic grid of the primary fields:

$$1\leqslant n\leqslant q-1,\quad 1\leqslant m\leqslant p-1.$$

Moreover, for conformal theories with $c<1$, there is a famous result of Friedan, Qiu, and Shenker that the only unitary conformal theories with $c<1$ are the unitary series of the minimal models [12]. They correspond to $q=p+1$ and have the central charge $c=1-6/p(p+1)$ with $p=2,3,\ldots$.

Physical states in 2D gravity coupled to $c\leqslant 1$ matter were studied in the framework of the Becchi-Rouet-Stora-Tyutin (BRST) quantization [13]. There an important role is played by the BRST operator

$$Q_{\mathrm{BRST}}=\oint dz\;c(z)\left(T_m(z)+T_L(z)+\frac{1}{2}\,T_{\mathrm{gh}}(z)\right),$$

where $T_m(z)$, $T_L(z)$, and $T_{\mathrm{gh}}(z)$ are the stress energy tensors for matter, Liouville, and ghost sectors, respectively. The physical states (operators $\mathcal{O}$) are defined as the cohomology classes of this BRST operator. In this work we will mainly focus on the physical operators without ghost excitations, i.e., the tachyon and discrete states [14]. It is convenient to use a representation for such states when a matter sector is bosonized in a way as we sketched earlier. The tachyon-type states are given by

$$\mathcal{T}^\pm_{n,m}=\int d^2z V_{n,m}(z,\bar{z})e^{\beta^\pm(\Delta^{(0)}_{n,m})\phi(z,\bar{z})}, \tag{1.13}$$

$$\beta^\pm(\Delta^{(0)}_{n,m})=\frac{1}{2\sqrt{3}}\,(\pm\sqrt{1-c+24\Delta^{(0)}_{n,m}}-\sqrt{25-c}). \tag{1.14}$$

Since in the case of interest $\Delta^{(0)}_{n+p+1,m+p}=\Delta^{(0)}_{n,m}$, n is restricted to a range $1\leqslant n\leqslant p+1$. Thus one has the matter primaries $V_{n,m}$ not only inside the basic grid, but also outside it. The discrete states appear as the border case operators $n = p+1$ or $m=0 \bmod p$ (see, e.g., [15] and Appendix B). Notice that there are two independent Liouville exponents $\beta^{\pm}$, corresponding to two choices of dressing. From this point of view the scalings for the critical lines with $\gamma<0$ and multicritical points are described by the effective actions with the positively and negatively dressed Liouville potentials, respectively.

The outline of the paper is as follows. In Secs. II A and II B we describe touching interactions in the continuum. We not only reproduce the known matrix model results, but find rather amusing new ones. Moreover, analogies with four-dimensional quantum gravity appear. Section III will present the conclusions and directions for future work. In two appendixes we give some technical details which are relevant for our discussion of the touching operators.

II. TOUCHING INTERACTIONS IN THE CONTINUUM

A. $c<1$ models

Let us now show how touching interactions appear in the continuum formulation. To do this, it is useful to begin with a geometrical analysis.

1. Geometry

First of all, we turn to a geometrical interpretation of operators contained in the effective actions (1.2) and (1.4). It is well known that $\mathcal{P} \equiv \mathcal{T}^{\pm}_{1,1} = \int d^2z \sqrt{\hat{g}} e^{\alpha_{\pm}\phi}$ are called the puncture operators. A motivation for this is that an insertion of such operator into the path integral fixes a point on a Riemann surface. Such fixing corresponds to what in the theory of Riemann surfaces is called a puncture (see, e.g., [16] and references therein). This can be formulated in terms of the partition functions. Regarding $Z=\langle 1\rangle$ as the partition function of an original surface, the partition function for the punctured surface is $Z_{\text{punc}}=\langle\mathcal{P}\rangle$. Note that this definition of the puncture operator differs from the one used in [17], namely, $\int d^2z \sqrt{\hat{g}} V e^{(-Q/2)\phi}$. They can only coincide at $c=1$ which is special because $\alpha_+ = \alpha_- = -Q/2 = -\sqrt{2}$.

Let us now look more specifically at touching interactions. Heuristically, the idea is that a network of touching spheres includes both the main surface (parent) as well as the pinched spheres attached to the parent (see Fig. 2). It is well known that a surface attached to the parent by a wormhole (tiny neck) is called a baby universe [4]. However, in the context of two-dimensional gravity, the notion is simplified. A sphere attached to the parent is usually called a baby universe [1]. In our case we also have pinched spheres attached to the parent. After this is understood, it immediately comes to mind to introduce a new notion. By analogy with the baby universe, we define a k-branched baby universe as the $(k-1)$-pinched sphere attached to the parent by a tiny neck.[3] Here we identify the standard baby universe with the 1-branched baby universe.

[3]A motivation for such name is the structure of the attached surface. It allows one to label the baby universes by the integer number k.

The distribution of the baby universes on a surface was analyzed in [18] via dynamical triangulations. It was shown that the average number of minimum neck baby universes (whose neck thickness is of order of the ultraviolet cutoff) of area B on a closed genus g surface of area A scales as

$$N_A(B) \propto A^{3-\gamma(g)}(A-B)^{\gamma(g)-2}B^{\gamma-2}, \qquad (2.1)$$

where $\gamma(g)=\gamma(1-g)+2g$.

We want now to repeat the analysis of Ref. [18] in order to find the average number of the minimum neck k-branched baby universes of area B on a closed genus g surface of area A. Note that the derivation is sufficiently generic, and so one can apply it for both the critical lines and multicritical points (conventional and modified matrix models). We claim that

$$N_A(k,B) \propto A^{3-\Gamma(g)}(A-B)^{\Gamma(g)-2}B^{k\Gamma-2}, \qquad (2.2)$$

where $\Gamma=\gamma,\bar{\gamma}$ and $N_A(1,B)\equiv N_A(B)$. The only fact needed to get Eq. (2.2) is that the partition function for the k-pinched sphere of area A scales as $Z_k(A)\propto A^{(k+1)\Gamma-3}$. It can be found repeatedly, reducing to the 1-pinched sphere via a sewing procedure. In the last case it is simply obtained by sewing two spheres with punctures.

It follows from the statement (2.2) that the average number of the k-branched baby universes on the surface should scale as

$$N_A(k)=\int dB N_A(k,B) \propto A^{k\Gamma}. \qquad (2.3)$$

Suppose that the k-branched baby universes can be reproduced by a local operator. This means that its normalized one-point correlation function should scale as

$$\langle\langle a_k^\dagger\rangle\rangle_A = \frac{\langle a_k^\dagger\rangle_A}{\langle 1\rangle_A} \propto A^{k\Gamma}. \qquad (2.4)$$

Here the symbol $\langle\ \rangle_A$ denotes the correlation functions computed using the actions (1.2) and (1.4) at fixed area A. Here $a_k^\dagger = \mathcal{A}_k^\dagger, \bar{\mathcal{A}}_k^\dagger$, where $\mathcal{A},\bar{\mathcal{A}}$ correspond to the conventional and modified matrix models, respectively.

On the other hand, this implies [6]

$$\langle\langle a_k^\dagger\rangle\rangle_A \propto A^{1-\Delta_k^{\text{KPZ}}}. \qquad (2.5)$$

As a result, one finds that the Knizhnik-Polyakov-Zamolodchikov (KPZ) scaling dimension of $a_k^\dagger$ is given by

$$\Delta_k^{\text{KPZ}} = 1-k\Gamma. \qquad (2.6)$$

It seems natural from physical point of view to call the $a_k^\dagger$'s as the creation operators as it was done in four dimensions [4]. Then it immediately comes to mind to define the annihilation operators. A possible way to do this is to make use of two-point correlation functions. Let a_k be the annihilation operators. Then two-point functions obey

$$\langle\langle a_k^\dagger a_k\rangle\rangle_A \propto O(1). \qquad (2.7)$$

This allows one to find the KPZ scaling dimension of the operator a_k. It is given by

$$\Delta_k^{\rm KPZ}=1+k\Gamma. \tag{2.8}$$

It should be stressed that a difference from the four-dimensional case is that we define the a_k's via a scalar product and not the standard commutation relations.

Of course, the annihilation operators can be defined by a geometrical analysis too. Let us give an example. Consider the case where a surface is the 1-pinched sphere; more complicated cases can be treated in a similar way. The number of the 1-pinched spheres of area A scales as $A^{2\Gamma-3}$. On the other hand, the number of degenerate 1-pinched spheres of the same area scales as $A^{\Gamma-3}$. The latter assumes that the 1-pinched sphere degenerates into the sphere. It is clear because the baby universe vanishes. From the above statements, it follows that the average number of the degenerate 1-pinched spheres scales as $A^{\Gamma-3}/A^{2\Gamma-3}=A^{\Gamma}$. This means that the normalized one-point function of the annihilation operator a_1 should scales as A^{Γ}. As a result, we recover its KPZ scaling dimension $\Delta_1^{\rm KPZ}=1+\Gamma$.

2. Detailed examination of operators

Let local operators which are responsible for the k-branched baby universes belong to the physical operators of 2D gravity coupled to conformal matter. It is known that such operators are characterized by the KPZ scaling dimension [6]. In the conformal gauge this dimension is completely defined by the ϕ zero mode [8]. So for the operator $\mathcal{O}_k$ with the Liouville exponent β_k, $\mathcal{O}_k\propto e^{\beta_k\phi_0}$, the KPZ scaling dimension is given by $\Delta_k^{\rm KPZ}=1-\beta_k/\alpha$, if the Liouville potential is $e^{\alpha\phi}$. In the above ϕ_0 is the zero mode of ϕ.

For the critical lines with $\gamma<0$ (conventional matrix models), we use the above statements as well as Eqs (2.6) and (2.8) in order to find the Liouville exponents of the creation and annihilation operators

$$\mathcal{A}_k^{\dagger}\propto e^{k(\alpha_+-\alpha_-)\phi_0},\quad \mathcal{A}_k\propto e^{k(\alpha_--\alpha_+)\phi_0}. \tag{2.9}$$

For the multicritical points (modified matrix models), similar calculations lead to

$$\overline{\mathcal{A}}_k^{\dagger}\propto e^{k(\alpha_--\alpha_+)\phi_0},\quad \overline{\mathcal{A}}_k\propto e^{k(\alpha_+-\alpha_-)\phi_0}. \tag{2.10}$$

It is interesting to note that all exponents vanish at $c=1$, which leads to $\Delta^{\rm KPZ}|_{c=1}=1$.

Let us now consider the partition function, taking into account contributions from the branched baby universes. It is known that such configurations are present in the path integral over metrics. They correspond to singular world-sheet metrics. The partition function is given by

$$Z_{\rm pinched}=\sum_{k=0}^{+\infty} w_k Z_k, \tag{2.11}$$

where Z_k is a contribution of the k-pinched sphere. w_k is a weight factor of each contribution. Suppose that $Z_{\rm pinched}$ is described by the actions (1.2) and (1.4) perturbed by the creation and annihilation operators[4]

$$S'_{\rm eff}=S_{\rm eff}+\sum_{k=1}^{+\infty} t_k\mathcal{A}_k+t_k^{\dagger}\mathcal{A}_k^{\dagger}, \tag{2.12}$$

$$\overline{S'}_{\rm eff}=\overline{S}_{\rm eff}+\sum_{k=1}^{+\infty} \overline{t_k}\overline{\mathcal{A}_k}+\overline{t_k^{\dagger}}\overline{\mathcal{A}_k^{\dagger}}. \tag{2.13}$$

Under this assumption, the gravitational dimensions of the coupling constants obey[5]

$$\dim t_k>0,\quad \dim t_k^{\dagger}<0,\quad \dim \overline{t_k}<0,\quad \dim \overline{t_k^{\dagger}}>0,$$

from which it follows that the actions (2.12) and (2.13) are not renormalizable. However, if we define the theory as

$$S'_{\rm eff}=S_{\rm eff}+\sum_{l=1}^{+\infty} t_k\mathcal{A}_k, \tag{2.14}$$

$$\overline{S'}_{\rm eff}=\overline{S}_{\rm eff}+\sum_{k=1}^{+\infty} \overline{t_k^{\dagger}}\overline{\mathcal{A}_k^{\dagger}}, \tag{2.15}$$

then the actions are renormalizable. In other words, $\mathcal{A}_k^{\dagger}$ and $\overline{\mathcal{A}}_k$ are the irrelevant operators that disappear in the IR limit. Treating the actions such leads us to a conclusion that the baby universes can be neglected for the critical lines with $\gamma<0$, but they are relevant for the multicritical points. This fact has been noted previously in the framework of the RG approach to matrix models [5].

It is interesting to note that all couplings ($t_k^{\dagger},t_k,\overline{t_k^{\dagger}},\overline{t_k}$) automatically become marginal at $c=1$. This is in accordance with the conjecture on their role in the $c=1$ barrier.

Finally, let us discuss a relation with the result of David [5]. To do this we must remember the definition of the scaling dimension in the framework of the renormalization group approach [2]. It is defined by

$$\Delta_x^{\rm RG}=\frac{2\beta(\Delta_x^{(0)})}{Q}, \tag{2.16}$$

where Q and $\beta(\Delta_x^{(0)})$ are the background charge and Liouville exponent.

Combining this with Eqs. (2.9) and (2.10), we learn that for the operators $\mathcal{A}_1^{\dagger}$, $\overline{\mathcal{A}_1^{\dagger}}$ the scaling dimensions are simply

$$\Delta_1^{\rm RG}=2\sqrt{\frac{1-c}{25-c}},\quad \overline{\Delta}_1^{\rm RG}=-2\sqrt{\frac{1-c}{25-c}}, \tag{2.17}$$

which are the formulas derived in Ref. [5].

3. Examination revisited

Up to now our discussion has not been sensitive to a detailed structure of the physical operators $\mathcal{O}_k$. Suppose now that the touching operators are the tachyon-type operators $\mathcal{T}_{n,m}^{\pm}$; i.e., they are given by the exponents of the free fields as in Eq. (1.13). Intuitively, this comes about because these operators are somewhat descendant from the puncture operators $\mathcal{T}_{1,1}^{\pm}$. Indeed, the dimensions of the operators $\mathcal{A}_1^{\dagger}$, $\overline{\mathcal{A}_1^{\dagger}}$

[4]In fact, the sums are finite [see Eq. (2.20) below].

[5]This dimension is equal to $1-\Delta^{\rm KPZ}$.

were obtained from the dimensions of $\mathcal{T}^{\pm}_{1,1}$ via a sewing procedure [5]. On the other hand, the tachyon operators are the simplest physical operators in the theory and, moreover, they are moduli of the theory; so it is natural to start by looking for the touching operators among them. We will fill this gap in our determination of the touching operators in Appendix A.

Accepting the above assumption, an interesting conclusion which we can draw is that $\mathcal{A}^{\dagger}_k=\overline{\mathcal{A}}_k$ and $\mathcal{A}_k=\overline{\mathcal{A}}^{\dagger}_k$. Indeed, if the Liouville exponents are given by Eqs. (2.9) and (2.10), then it follows from Eq. (1.13) that for the creation and annihilation operators, we get

$$\mathcal{A}^{\dagger}_k=\overline{\mathcal{A}}_k=\mathcal{T}^{+}_{2k-1,2k+1}=\int d^2z V_{2k-1,2k+1}(z,\overline{z})e^{k(\alpha_+-\alpha_-)\phi(z,\overline{z})}, \tag{2.18}$$

$$\mathcal{A}_k=\overline{\mathcal{A}}^{\dagger}_k=\mathcal{T}^{+}_{2k+1,2k-1}=\int d^2z V_{2k+1,2k-1}(z,\overline{z})e^{k(\alpha_--\alpha_+)\phi(z,\overline{z})}. \tag{2.19}$$

In particular, the operators introduced by David are simply $\mathcal{T}^{+}_{1,3}$ and $\mathcal{T}^{+}_{3,1}$.

It is interesting to note that the theory has a finite number of creation-annihilation operators for the branched baby universes, which means finite sums in Eqs. (2.12) and (2.13). In fact, since Eq. (1.13) n belongs to the range $1\leqslant n\leqslant p+1$, the largest value of k is given by[6]

$$\max k=\begin{cases}\left[\frac{p}{2}\right]+1 & \text{for } \mathcal{A}^{\dagger}_k \text{ and } \overline{\mathcal{A}}_k,\\ \left[\frac{p}{2}\right] & \text{for } \mathcal{A}_k \text{ and } \overline{\mathcal{A}}^{\dagger}_k.\end{cases} \tag{2.20}$$

It is clear that it is dependent on the matter central charge. In other words, the shape of world sheets is determined by matter residing on them.

So the two phases (critical lines with $\gamma<0$ and multicritical points) differ not only by a branch of gravitational dressing for the puncture operators (cosmological terms), but also by different roles of the same operators: creation (annihilation) of the branched baby universes in one case and their annihilation (creation) in the other.

In order to take the assumption that the touching operators are the tachyon-type ones into account completely, it is advantageous to go in a slightly different way. Instead of using the geometrical point of view, we will follow renormalization group arguments and look for perturbations which become marginal at $c=1$.

Let us perturb the continuum theory, so that the effective actions (1.2) and (1.4) become

$$S'_{\text{eff}}=S_{\text{eff}}+\sum_{m=-\infty}^{+\infty}\sum_{n=1}^{p+1} t_{n,m}\mathcal{T}_{n,m}, \tag{2.21}$$

$$\overline{S}'_{\text{eff}}=\overline{S}_{\text{eff}}+\sum_{\overline{m}=-\infty}^{+\infty}\sum_{\overline{n}=1}^{p+1} \overline{t}_{\overline{n},\overline{m}}\mathcal{T}_{\overline{n},\overline{m}}. \tag{2.22}$$

Here $t_{n,m}$, $\overline{t}_{\overline{n},\overline{m}}$ are renormalized couplings. $\mathcal{T}_{n,m}$, $\mathcal{T}_{\overline{n},\overline{m}}$ denote the tachyon-type operators defined in Eq. (1.13).

Since the transition occurs at $c=1$, the gravitational dimensions of the couplings obey

$$\dim t_{n,m}|_{c=1}=\dim \overline{t}_{\overline{n},\overline{m}}|_{c=1}=0.$$

These conditions are equivalent to

$$\beta^{\pm}(\Delta^{(0)}_{n,m})|_{c=1}=\sqrt{\frac{1}{2p(p+1)}}\times(\pm|np-m(p+1)|-2p-1)|_{p=\infty}=0. \tag{2.23}$$

There are two solutions of equation $\beta^{+}(\Delta^{(0)}_{n,m})|_{c=1}=0$ in the range $1\leqslant n\leqslant p+1$: namely,

$$n=m\pm 2, \tag{2.24}$$

while the equation $\beta^{-}(\Delta^{(0)}_{n,m})|_{c=1}=0$ has no solutions in this range. By substituting Eq. (2.24) into $\beta^{+}(\Delta^{(0)}_{n,m})$, we easily find the Liouville exponents

$$\beta^{+}(\Delta^{(0)}_{n,n\pm 2})=\frac{1\pm n}{2}(\alpha_+-\alpha_-), \tag{2.25}$$

where

$$\alpha_+=-\sqrt{\frac{2p}{p+1}},\quad \alpha_-=-\sqrt{\frac{2(p+1)}{p}}.$$

The main new novelty of the above calculation is the appearance of operators with n even. As we have seen, the operators with n odd arise from the idea on the role of the touching interactions in the $c=1$ barrier. From the geometrical point of view, they correspond to the creation-annihilation operators for the branched baby universes. Now we would like to complement the discussion by including the operators $\mathcal{T}^{+}_{n,n\pm 2}$ with even[7] n. In general, this issue is not completely understood. Here we can only speculate. The idea is heuristically that both string exponents of surfaces with boundaries and gravitational scaling dimensions of the operators for n even have $\Gamma/2$ as a unit of ‘‘measurement.’’ It is natural therefore to relate these operators with holes on a surface. To illustrate this, consider a geometry in which a surface is made by pinching the hemisphere at a point on a boundary, as shown in Fig. 3 on the left. Such a surface is reproduced by gluing the sphere to a point on the boundary of the hemisphere. It is easy to find the area dependence of the partition function for this case. It is given by $Z_{1/2}(A)\propto A^{3/2\overline{\gamma}-3}$. On the other hand, this scaling is recovered by inserting the operator $\mathcal{T}^{+}_{2,0}$ into the path integral for the

[6] $[a]$ means the integer part of a.

[7] It should be noted that $n=-m=1$ is special because $\beta^{+}(\Delta^{(0)}_{1,-1})\equiv 0$. As a result, the matter field is given by screening operators.

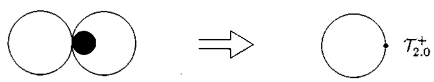

FIG. 3. Approximation of the pinched hemisphere by the sphere with an insertion of the local operator $\mathcal{T}^{+}_{2,0}$.

sphere, as in Fig. 3 on the right. This stimulates one to introduce the notion of a banged baby universe as the hemisphere attached to the parent by a point on the boundary and interpret $\mathcal{T}^{+}_{2,0}$ as the creation operator for the banged baby universe. Since we restrict ourselves to the spherical topology, we leave the detailed analysis of these operators for future study.

It is also not difficult to recognize the discrete state in $\mathcal{T}^{+}_{2,0}$ [14]. This can be done using a linear map[8]

$$X=\frac{Q}{2\sqrt{2}}\,\mathbf{X}-\frac{i\alpha_0}{\sqrt{2}}\,\boldsymbol{\phi},\qquad \phi=\frac{i\alpha_0}{\sqrt{2}}\,\mathbf{X}+\frac{Q}{2\sqrt{2}}\,\boldsymbol{\phi}. \tag{2.26}$$

Under this map one gets an effective $c=1$ matter dressed by gravity. In terms of the new variables, the operator $\mathcal{T}^{+}_{2,0}$ becomes

$$\mathcal{T}^{+}_{2,0}=\int d^2z\; e^{i\sqrt{2}\mathbf{X}(z,\bar{z})}. \tag{2.27}$$

The holomorphic (antiholomorphic) part of the integrand in Eq. (2.27) is the highest weight state of a spin-1 su(2) multiplet.

At this point, it is necessary to make a remark. One of the important statements about the discrete states was the following notice by Polyakov [19]. The discrete states correspond to the contributions of singular world-sheet metrics, pinched spheres in the models under discussion, in the path integral over metrics. From our discussion of this issue, we have seen that there is, however, an important new feature that we must now clarify. We claim that for the unitary $c<1$ models, in addition to the conventional discrete states, there are a set of states $\mathcal{T}^{+}_{n,n\pm 2}$ which are also relevant. Moreover, they are dominant. From the algebraic point of view, the latter correspond to fractional values of the su(2) spin.

4. Consequences

Now we can easily read off some interesting conclusions. One of the first important observations is the following observation about a structure of the partition function Z_{pinched}. According to Eq. (2.20), there are no creation operators for the branched baby universes with k larger than $\max k$. It means that higher pinched spheres are obtained by attaching two or more creation operators to the parent. The effective action underlying such picture is given by

$$\overline{S}'_{\text{eff}}=\overline{S}_{\text{eff}}+\sum_{k=1}^{[p/2]}\overline{t^{\dagger}_k}\,\overline{\mathcal{A}^{\dagger}_k}. \tag{2.28}$$

It should be stressed that this restriction is completely due to unitarity of the $c<1$ matter.

Next, let us go on to look more carefully at the cases of interest. For the critical lines with $\gamma<0$, we find

$$Z_{k+1}/Z_k\to 0 \quad \text{under } \; t_0\to 0, \tag{2.29}$$

where $Z_k=\langle\mathcal{A}^{\dagger}_k\rangle$. So the leading contribution to Z_{pinched} comes from the sphere, the next from the pinched sphere, etc. We believe that this fact allows one to interpret this phase as the weak coupling regime for the touching interactions. This time they can contribute to subleading orders only. Formally, the most relevant operator is $\mathcal{A}^{\dagger}_1$. This is also in harmony with an idea of David that one should be able to catch the effects of touching in this phase via this operator.

Now let us turn to the multicritical points. In contrast to the previous case, $\overline{\mathcal{A}^{\dagger}_k}$ with $k=[p/2]$ is proved to be the most relevant operator in Eq. (2.28).[9] From the geometrical point of view, it means that most branched baby universes are dominant. As a result, the expansion of Z_{pinched} in powers of $\overline{t^{\dagger}_k}$, as may follow from the action (2.28), is not valid. So we no longer have the weak coupling regime for the touching interactions. Instead of this, we interpret this phase as the strong coupling regime for the touching interactions. At this point, it is necessary to discuss a relation with the David scenario where the touching interactions were taken into account by the baby universes, i.e., $\overline{\mathcal{A}^{\dagger}_1}$. In our consideration of this issue, we have seen that the most relevant operator is $\overline{\mathcal{A}^{\dagger}_{[p/2]}}$. The latter means that the David picture is valid at least for the pure gravity $(p=2)$ and Ising $(p=3)$ models. However, for $p\geqslant 4$ this cannot be the whole story, for the reason that the branched baby universes come into the game and, moreover, they are dominant.

Finally, let us note that the conclusion by Klebanov [9] that the scaling limits of the conventional matrix models (critical lines with $\gamma<0$) and modified matrix models (multicritical points) differ due to the branches of gravitational dressing for the Liouville potential can be extended. According to our discussion, these scaling limits correspond to different phases of the touching interactions, namely, weak and strong coupling regimes.

B. Strings

We now turn to the problem of shedding some light on touching interactions for $c=1$ models. It is well known that such models are noncritical bosonic strings or, equivalently, two-dimensional critical strings [1]. Thus we will try to analyze the effects of singular world-sheet metrics (pinched spheres) in the Polyakov path integral. In doing so, we will

[8]In fact, the map defined in Eq. (2.26) is a Lorentz boost in a two-dimensional Minkowski space with coordinates (X,ϕ). We refer to [1,19] for more details.

[9]It is straightforward to get this result in the framework of the RG approach [2]. Looking at the scaling dimensions $\Delta^{\text{RG}}_k=-(k+1)/(2p+1)$, we see that $\overline{t^{\dagger}_{[p/2]}}$ is the most relevant. However, it is less relevant in comparison with the cosmological constant $\overline{t_0}$.

not follow the geometrical analysis of Sec. II A 1. Instead of this, we look for the limit $p\to\infty$.

1. $p\to\infty$ limit

One of the novelties that appears at $c=1$ is that the string exponents defined in Eqs. (1.3) and (1.5) vanish. As a result, direct use of the geometrical point of view fails. Moreover, scaling violations for the phase associated with the conventional matrix models are also a serious obstacle on this way. In finding touching interactions for $c=1$ models, it seems sensible to take as a starting point the model of Sec. II A for arbitrary p and then define the limit $p\to\infty$. To see what really happens, consider the effective actions. The Liouville exponents $\alpha_\pm$ will be $\underline{-}\sqrt{2}$. One can imagine that the effective actions $S_{\rm eff}$ and $\overline{S}_{\rm eff}$ coincide, but it is not true. As Polchinski pointed out [20], the Liouville potential for $S_{\rm eff}$ is given by $\phi e^{-\sqrt{2}\phi}$, which leads to the scaling violations. On the other hand, there are no scaling violations for the phase associated with the modified matrix models, and so the potential for $\overline{S}_{\rm eff}$ is simply $e^{-\sqrt{2}\phi}$ [9]. Thus one has, for the effective actions (1.2) and (1.4) at $c=1$,

$$S_{\rm eff}=\frac{1}{2\pi}\int d^2z\left(\partial\phi\bar{\partial}\phi-\frac{1}{\sqrt{2}}\sqrt{\hat{g}}\hat{R}\phi+t_0\sqrt{\hat{g}}\phi e^{-\sqrt{2}\phi}\right), \tag{2.30}$$

$$\overline{S}_{\rm eff}=\frac{1}{2\pi}\int d^2z\left(\partial\phi\bar{\partial}\phi-\frac{1}{\sqrt{2}}\sqrt{\hat{g}}\hat{R}\phi+\overline{t_0}\sqrt{\hat{g}}e^{-\sqrt{2}\phi}\right), \tag{2.31}$$

with the background charge $Q=2\sqrt{2}$.

Now we come to the analysis of the actions (2.12) and (2.13). Obviously, under the limit $p\to\infty$ these actions are given by

$$S'_{\rm eff}=S_{\rm eff}+\sum_{k=1}^{\infty} t_k\mathcal{A}_k+t_k^\dagger\mathcal{A}_k^\dagger, \tag{2.32}$$

$$\overline{S'}_{\rm eff}=\overline{S}_{\rm eff}+\sum_{k=1}^{\infty} \overline{t_k}\overline{\mathcal{A}}_k+\overline{t_k^\dagger}\overline{\mathcal{A}}_k^\dagger, \tag{2.33}$$

with the operators

$$\overline{\mathcal{A}}_k=\mathcal{A}_k^\dagger=\int d^2z\; e^{-i\sqrt{2}X(z,\bar{z})},\quad \mathcal{A}_k=\overline{\mathcal{A}_k^\dagger}=\int d^2z\; e^{i\sqrt{2}X(z,\bar{z})}. \tag{2.34}$$

There is an interesting observation related to the vanishing of the string exponents that the operators are independent of k. In other words, one cannot distinguish the branched baby universes at $c=1$. Instead of this, there are collective potentials for the touching interactions with the following effective couplings:

$$t=\sum_{k=1}^{\infty} t_k,\quad t^\dagger=\sum_{k=1}^{\infty} t_k^\dagger,\quad \overline{t}=\sum_{k=1}^{\infty}\overline{t_k},\quad \overline{t^\dagger}=\sum_{k=1}^{\infty}\overline{t_k^\dagger}.$$

The actions (2.32) and (2.33) are rewritten as

$$S'_{\rm eff}=S_{\rm eff}+t\int d^2z\; e^{i\sqrt{2}X(z,\bar{z})}+t^\dagger\int d^2z\; e^{-i\sqrt{2}X(z,\bar{z})}, \tag{2.35}$$

$$\overline{S'}_{\rm eff}=\overline{S}_{\rm eff}+\overline{t}\int d^2z\; e^{-i\sqrt{2}X(z,\bar{z})}+\overline{t^\dagger}\int d^2z\; e^{i\sqrt{2}X(z,\bar{z})}. \tag{2.36}$$

Thus we have generalized the touching operators to $c=1$ models. At this point a few comments are in order.

(i) It is interesting to note that the holomorphic (antiholomorphic) parts of the touching operators for the string models are none other than the screening operators of the $c=1$ conformal field theory (matter sector in the particular case at hand). It is well known that they represent the raising and lowering operators of the su(2) algebra and generate the multiplets of the discrete states (see Appendix B for details). From this point of view our introduction of the annihilation operators seems plausible. However, such operators do not lead to the standard Heisenberg algebra, as happens in the framework of four-dimensional quantum gravity, but su(2).

(ii) According to our discussion in Sec. II A 4, the weak and strong coupling regimes for the touching interactions are associated with the conventional and modified matrix models for $c<1$. At $c=1$ relations which are similar to Eq. (2.29) are not valid anymore. Instead of them, we have $Z_{k+1}/Z_k\sim 1$, which indicates the presence of a boundary between these phases. However, this boundary looks singular because one does not get into the same theory under the $p\to\infty$ limit.

(iii) If one makes use of a perturbation of the actions (2.35) and (2.36) according to which the creation and annihilation operators are involved with the same effective coupling constant, a result will be the sine-Gordon model coupled to 2D gravity. Thus the sine-Gordon model coupled to 2D gravity is an appropriate framework to take into account effects of singular world-sheet metrics in the Polyakov path integral for the noncritical strings. Unfortunately, one knows very little about integrable models in the presence of quantum gravity. Some issues have been discussed in [1,21,22].

2. Cosmological constant and touching interactions

There is a serious problem in quantum gravity related to the vanishing of the cosmological constant. Several different proposals are known to solve it. One of them is based on the idea of uncontrollable emissions of tiny baby universes. It was intensively discussed in the framework of a four-dimensional case (see, e.g., [4]).

Let us now try a two-dimensional case. It is well known that the cosmological constant is being renormalized in a singular way as[10]

$$\frac{\overline{t_0}}{\Gamma[0]}=\overline{\mathbf{t}}_0, \tag{2.37}$$

[10]In the literature on the $c=1$ models, the cosmological term (puncture operator) is usually chosen as $e^{-\sqrt{2}\phi}$, which in our language corresponds to the multicritical points.

where $\Gamma[x]$ is the gamma function. The origin of this multiplicative renormalization is, of course, the short distance divergences. In calculating amplitudes one needs to perform multiple integrals. There are some prescriptions to do this. One of them is an analytic continuation. Shifting the exponents of the integrals, one brings them into a standard Dotsenko-Fateev form. Next, the integrals are computed by an analytic continuation.

We are going to find the multiplicative renormalization of the touching couplings. In order to do this, we follow a similar procedure as it was used to derive Eq. (2.37). The calculation for this case (see Appendix B) leads to the result

$$\overline{t\Gamma}[0]=\overline{\mathbf{t}},\quad \overline{t\Gamma}[0]=\overline{\mathbf{t}^{\dagger}}. \tag{2.38}$$

We see that the bare cosmological constant and touching couplings are renormalized in different ways; namely, the cosmological constant goes to "zero," but the touching interaction couplings go to "infinity." Here an analogy with the four-dimensional case appears again because such behavior reminds one of Coleman's idea, that touching interactions (wormholes) have the effect of making the cosmological constant vanish [4]. Although it looks in many ways attractive, we have to stress its speculative character. It rests on the multiplicative renormalization argument only, and so further work is needed to prove it strictly.

III. CONCLUSIONS AND REMARKS

First, let us say a few words about the results.

In this work we have found a set of the physical operators which are responsible for the touching interactions in the framework of $c<1$ unitary conformal matter coupled to 2D quantum gravity. It turned out that one can interpret the critical lines with $\gamma<0$ (conventional matrix models) and multicritical points (modified matrix models) as different phases for touching, namely, the weak and strong coupling regimes. Next, we defined the touching operators for the noncritical bosonic strings. It shows that if the creation and annihilation operators are involved with the same effective coupling constant, then the sine-Gordon model coupled to 2D gravity is an appropriate framework to take into account effects of singular world-sheet metrics in the Polyakov path integral. Some analogies with the four-dimensional case are also discussed, e.g., the creation-annihilation operators for the baby universes and Coleman mechanism for the cosmological constant.

Let us conclude by mentioning a few open problems together with interesting features of the touching interactions in the continuum.

(i) Of course, the most important open problem is to understand the touching interactions in the critical strings or how to take into account effects of singular world-sheet metrics in the Polyakov path integral. Unfortunately it is unknown in general how to realize this program. Our analysis of Sec. II essentially rests on the Liouville mode ϕ, and so any attempt to use it for critical strings will fail.

(ii) In order to calculate the multiplicative renormalizations of the coupling constants, we found special correlators of the discrete states. This seems strange because it is possible to find them directly from the action (2.36). However, by calculating correlators we solve one more problem which is formulated as the deformation of the operator product (OP) algebra of the discrete states by the presence of nonvanishing cosmological and touching coupling constants. Although a special solution is known [15,23], the problem is still open. Some progress in this direction has already been done [24].

(iii) The operator $\mathcal{T}^{+}_{3,1}$ is special because it interpolates between matrix models [25]. In the simplest case it describes the flow from Ising $(p=3)$ to pure gravity $(p=2)$. We offer a qualitative physical interpretation of such a transition based on our geometrical picture. First, let us recall that the shape of world sheets depends on the central charge of matter residing on them; namely, higher pinched world sheets correspond to higher central charges. Next note that $\mathcal{T}^{+}_{3,1}$ is nothing but the annihilation operator for the baby universes in the framework of the conventional matrix models, and so it smooths a shape that leads to a proper reducing of the central charge. As a result, one has the flow from Ising to pure gravity. On the other hand, it is the creation operator in the context of the modified matrix models, and so it wrinkles a shape that increases the central charge. This time there is the flow from pure gravity to Ising. Of course, these conclusions are heuristic and further work is needed to make them more rigorous.

ACKNOWLEDGMENTS

I am indebted to I. Bars, J.-L. Gervais, V. Kazakov, and R. Metsaev for useful discussions, and especially F. David for his comments on an earlier draft of the paper. I am also grateful to G. Lopes Cardoso for reading the manuscript. This research was supported in part by NRC grant GAC021197 and by the Russian Basic Research Foundation under grant 96-02-16507.

APPENDIX A

In discussing touching interactions, we assumed in Sec. II A 3 that local operators which are responsible for the branched baby universes are the tachyon-type physical operators. In the present appendix, we will analyze some aspects of this story in somewhat more depth.

To begin with, we review some facts about the BRST formalism [13,26]. The physical states are the cohomology classes of the BRST operator Q_{BRST} whose explicit form is given in Sec. I. These classes are labeled by the ghost number G. The tachyon and discrete operators appear at ghost number 2. So the operators (1.13) are rewritten as

$$\mathcal{T}^{\pm}_{n,m}(z,\bar{z})=c(z)\overline{c(\bar{z})}V_{n,m}(z,\bar{z})e^{\beta^{\pm}(\Delta^{(0)}_{n,m})\phi(z,\bar{z})}. \tag{A1}$$

Such a class was intensively discussed in Sec. II A 3.

As for the new BRST classes, the first nontrivial example appears at $G=0$. These operators are denoted as $\mathcal{O}_{j,m}\overline{\mathcal{O}}_{j,m}$. It is well known that the holomorphic (antiholomorphic) operators $\mathcal{O}_{j,m}$ ($\overline{\mathcal{O}}_{j,m}$) generate the chiral (antichiral) ground ring [27], namely, $\mathcal{O}_{j_1,m_1}\mathcal{O}_{j_2,m_2}=\mathcal{O}_{j_1+j_2,m_1+m_2}$. This allows one to determine the explicit form of an arbitrary operator from the first few, which are given by[11]

$$\mathcal{O}_{1/2,1/2}=\left(cb+\frac{i}{\sqrt{2}}\,\partial\mathbf{X}-\frac{1}{\sqrt{2}}\,\partial\boldsymbol{\phi}\right)\exp\left(\frac{i}{\sqrt{2}}\,\mathbf{X}+\frac{1}{\sqrt{2}}\,\boldsymbol{\phi}\right), \tag{A2}$$

$$\mathcal{O}_{1/2,-1/2}=\left(cb-\frac{i}{\sqrt{2}}\,\partial\mathbf{X}-\frac{1}{\sqrt{2}}\,\partial\boldsymbol{\phi}\right)\exp\left(-\frac{i}{\sqrt{2}}\,\mathbf{X}+\frac{1}{\sqrt{2}}\,\boldsymbol{\phi}\right), \tag{A3}$$

where $(\mathbf{X},\boldsymbol{\phi})$ refer to the effective $c=1$ matter dressed by gravity. In order to translate these operators into the $c<1$ theory, one can use a linear map

$$\mathbf{X}=\frac{Q}{2\sqrt{2}}\,X+\frac{i\alpha_0}{\sqrt{2}}\,\phi,\quad \boldsymbol{\phi}=-\frac{i\alpha_0}{\sqrt{2}}\,X+\frac{Q}{2\sqrt{2}}\,\phi, \tag{A4}$$

which is inverse to Eq. (2.26). However, we do not need to do this. It is easy to understand that the operators $\mathcal{O}_{j,m}\overline{\mathcal{O}}_{j,m}$ are not responsible for the branched baby universe. Indeed, they have nonzero Liouville exponents at $c=1$, and so they cannot be written as in Eqs. (2.9) and (2.10).

Up to now we have discussed only a part of the BRST cohomology. Another part is recovered by the operator $a+\bar{a}$ [26], where

$$a=c\left(-i\alpha_0\partial X+\frac{Q}{2}\,\partial\phi\right)+2\partial c. \tag{A5}$$

It is BRST invariant. So applying $a+\bar{a}$ to $\mathcal{T}^{\pm}_{n,m}$ and $\mathcal{O}_{j,m}\overline{\mathcal{O}}_{j,m}$, one can form the new families of BRST-invariant (physical) operators[12] $(a+\bar{a})\mathcal{T}^{\pm}_{n,m}$ with $G=3$ and $(a+\bar{a})\mathcal{O}_{j,m}\overline{\mathcal{O}}_{j,m}$ with $G=1$. Obviously, they have the same Liouville exponents as $\mathcal{T}^{\pm}_{n,m}$ and $\mathcal{O}_{j,m}\overline{\mathcal{O}}_{j,m}$. Because of this reason, the operators $(a+\bar{a})\mathcal{O}_{j,m}\overline{\mathcal{O}}_{j,m}$ are not appropriate for a role of the touching operators. As for the $(a+\bar{a})\mathcal{T}^{\pm}_{n,m}$'s, as their Liouville exponents are fitted to Eqs. (2.9) and (2.10) at $n=m\pm 2$, they may be responsible for the branched baby universes. So there is a puzzle here. Before continuing our discussion of this puzzle, we wish to complete the review of the BRST cohomology classes.

Given a state with ghost number G and Liouville exponent β, the two-point function on the sphere defines a dual state with ghost number $6-G$ and Liouville exponent $-Q-\beta$ [26]. One immediately see that, with the Liouville exponents as defined in Eq. (1.14), $\beta^-(\Delta^{(0)})=-Q-\beta^+(\Delta^{(0)})$. So this definition provides a pairing between the positively and negatively dressed states. Note that 6 comes from the ghost zero modes on the sphere, while $-Q$ appears from the Liouville background charge.[13] Using such a procedure, it is possible to find two new BRST cohomologies classes $\mathcal{P}_{n,m}\overline{\mathcal{P}}_{n,m}$ and $(a+\bar{a})\mathcal{P}_{n,m}\overline{\mathcal{P}}_{n,m}$ at ghost numbers 4 and 5, respectively [26]. Since $\mathcal{P}_{n,m}\overline{\mathcal{P}}_{n,m}$ are dual to $\mathcal{T}^{+}_{n,m}$, it implies that they are the negatively dressed states with the Liouville exponents $\beta^-(\Delta^{(0)}_{n,m})$. At these values of the exponents, it is impossible to satisfy Eqs. (2.9) and (2.10). This follows from the fact that $k(\alpha_+-\alpha_-)|_{c=1}=0$, while $\beta^-(\Delta^{(0)}_{n,m})$ never vanishes at $c=1$ for $1\leqslant n\leqslant p+1$. So the operators $\mathcal{P}_{n,m}\overline{\mathcal{P}}_{n,m}$ are not appropriate for the touching operators. For essentially this reason the operators $(a+\bar{a})\mathcal{P}_{n,m}\overline{\mathcal{P}}_{n,m}$ are also rejected. However, it is not the whole story about the BRST cohomology. Witten and Zwiebach found that there exist BRST-invariant operators which cannot be written as products of the holomorphic and antiholomorphic operators [26]. If $\mathcal{Y}^{\pm}_{n,m}$ denotes the holomorphic part of the operator $\mathcal{T}^{\pm}_{n,m}$ defined in Eq. (A1), then the rest of the BRST cohomology is given by

$$\mathcal{Y}^{+}_{n,m}\overline{\mathcal{O}}_{\bar{n},\bar{m}},\quad \mathcal{O}_{n,m}\overline{\mathcal{Y}}^{\pm}_{\bar{n},\bar{m}},\quad \mathcal{Y}^{-}_{n,m}\overline{\mathcal{P}}_{\bar{n},\bar{m}},\quad \mathcal{P}_{n,m}\overline{\mathcal{Y}}^{-}_{\bar{n},\bar{m}}, \tag{A6}$$

and their products with $(a+\bar{a})$. It is well known that in tensoring together holomorphic and antiholomorphic operators (left- and right-moving states), one should restrict oneself to operators of equal ''left'' and ''right'' Liouville exponents. This allows one to reject these operators by the same arguments as it was done for $\mathcal{O}_{j,m}\overline{\mathcal{O}}_{j,m}$, $\mathcal{Y}^{-}_{n,m}\overline{\mathcal{Y}}^{-}_{n,m}$, and $\mathcal{P}_{n,m}\overline{\mathcal{P}}_{n,m}$ and their products with the operator $(a+\bar{a})$ in above.

Summarizing, we have two classes of the BRST-invariant operators which may formally be the touching operators, namely, $\mathcal{T}^{+}_{n,n\pm 2}$ and $(a+\bar{a})\mathcal{T}^{+}_{n,n\pm 2}$. It remains to make our choice. Before doing it, let us discuss two points.

First, let us recall what we want. Our goal is to describe a network of touching surfaces by a single surface (parent) with insertions of local operators. Moreover, we would like to have a field theory description, i.e., an effective action whose terms are responsible for pinched spheres attached to the parent.

Next, let us turn to moduli. We recall that the moduli are operators that can be added to the action of the conformal field theory. In the particular case at hand, they come from spin-0 operators of ghost number 2 [26]. For the operators $\mathcal{T}^{+}_{n,n\pm 2}(z,\bar{z})$ defined in Eq. (A1) the corresponding moduli are $V_{n,n\pm 2}(z,\bar{z})e^{\beta^+(\Delta^{(0)}_{n,n\pm 2})\phi(z,\bar{z})}$; i.e., they are the integrands of the tachyon-type operators (1.13). It is clear that this is precisely what we need. Thus the touching operators are given by $\mathcal{T}^{+}_{n,n\pm 2}$.

APPENDIX B

The purpose of this appendix is to compute the multiplicative renormalization of the touching couplings. It turns out

[11]Note that $\mathcal{O}_{0,0}\equiv 1$.

[12]To be precise, $a\mathcal{O}(0)=\oint_{C_0}(dz/z)a(z)\mathcal{O}(0)$; the contour $\mathcal{C}_0$ surrounds 0.

[13]Notice that the dual states arising under factorization of correlation functions are not the ones defined via the two-point functions on the sphere, but states differing from them by $b_0-\bar{b}_0$. It leads to ghost number $5-G$.

that it is easy to find it by computing correlators of the discrete states of the $c=1$ models.

To begin with, let us recall how the discrete states appear in the theory. Taking the limit $p\to\infty$, one has, for the matter sector [see Eqs. (1.9) and (1.10)],

$$\alpha_+^m=-\alpha_-^m=\sqrt{2},\quad \alpha_{n,m}=\sqrt{2}j,\quad j\equiv\frac{n-m}{2}. \tag{B1}$$

In addition, the primaries (1.11) are rewritten as $V_{j,\pm j}(z,\overline{z})=e^{\pm i\sqrt{2}jX(z,\overline{z})}$.

It is well known that the theory has $\widehat{\mathrm{su}}(2)\oplus\widehat{\mathrm{su}}(2)$ as the symmetry algebra. The holomorphic currents are

$$H^{\pm}(z)=e^{\pm i\sqrt{2}X(z)},\quad H^0=\frac{i}{\sqrt{2}}\,\partial X(z). \tag{B2}$$

Obviously, their zero modes $H^a=\oint dz\, H^a(z)$ generate the su(2) algebra.[14] $H^{\pm}$ also play a role of the screening operators of the $c=1$ conformal field theory.

It was realized a long time ago [28] that the primary fields form tensor products of su(2) multiplets (holomorphic and antiholomorphic)

$$V_{j,m}(z,\overline{z})=\mathcal{N}_0(j,m)(H^-\overline{H}^-)^{j-m}V_{j,j}(z,\overline{z}), \tag{B3}$$

$$\mathcal{N}_0(j,m)=\frac{(j+m)!}{(2j)!(j-m)!},\quad j=0,\tfrac{1}{2},1,..., \tag{B4}$$

such that only $V_{j,\pm j}$ are the tachyon-type primary fields defined in Eq. (1.11). As to the others, they are ''discrete primaries.''

Now let us couple the $V_{j,m}$'s to gravity. It can be done directly, using the formulas (1.13) and (1.14). As a result, one gets

$$\mathcal{T}^{\pm}_{j,m}=\mathcal{N}_1(j,m)\int d^2z\; V_{j,m}(z,\overline{z})e^{\beta^{\pm}(j)\phi(z,\overline{z})},$$

$$\beta^{\pm}(j)=\sqrt{2}(-1\pm j). \tag{B5}$$

Here the normalization factors $\mathcal{N}_1(j\cdot m)=(2j)!(j+m)!(j-m)!$ are introduced to have the following OP algebra of the integrands:

$$\mathcal{T}^{+}_{j_1,m_1}(z,\overline{z})\mathcal{T}^{+}_{j_2,m_2}(0)=\frac{1}{|z|^2}\,(j_1m_2-j_2m_1)\mathcal{T}^{+}_{j_1+j_2-1,m_1+m_2}(0), \tag{B6}$$

[14]We use the normalization $\oint_{C_0}dz/z=1$ and omit $2(\pi)$ when it is irrelevant in the context of the present work.

with a vanishing value of the cosmological constant as well as touching couplings [27,29].

In order to find the multiplicative renormalizations of couplings, let us compute a few terms on the right-hand side of Eq. (B6) due to the presence of the nonvanishing cosmological and touching coupling constants.[15] The coefficient at $\mathcal{T}^{\pm}_{j_3,-m_3}$ is given by

$$\langle\mathcal{T}^{+}_{j_1,m_1}(0)\mathcal{T}^{+}_{j_2,m_2}(1)\widetilde{\mathcal{T}}^{-}_{j_3,m_3}(\infty)\rangle, \tag{B7}$$

with a conjugate operator defined as

$$\widetilde{\mathcal{T}}^{-}_{j,m}(z,\overline{z})=\widetilde{\mathcal{N}}_1(j,m)(H^+\overline{H}^+)^{j+m}V_{j,-j}(z,\overline{z})e^{-\sqrt{2}(1+j)\phi(z,\overline{z})},$$

$$\widetilde{\mathcal{N}}_1(j,m)=[(2j)!(j+m)!]^{-2}.$$

To find it, one can expand $e^{-\overline{S'_{\mathrm{eff}}}}$ in powers of $\overline{t_0}$, $\overline{t}$, and $\overline{t^{\dagger}}$ and interpret the resulting terms as correlation functions in the free theory.

As a warmup, let us reproduce the multiplicative renormalization of the cosmological constant. Following Dotsenko [15], set $m_1=j_1$, $m_2=j_3-j_2$, and $m_3=-j_3$. It is clear that the normalization factors do not lead to $\Gamma[0]$, and so we drop them. The contribution of the matter sector is given by

$$\Gamma^2[j_1+j_2-j_3+1]\prod_{i=1}^{j_1+j_2-j_3}\frac{\Gamma^2[i]\Gamma^2[2j_3+i]}{\Gamma^2[2j_1+1-i]\Gamma^2[2j_2+1-i]}+O(\overline{t\,t^{\dagger}}). \tag{B8}$$

It also does not lead to $\Gamma[0]$ (at least in the leading order of $\overline{t\,t^{\dagger}}$). On the other hand, the Liouville sector contributes

$$\left(\frac{\overline{t_0}}{\Gamma[0]}\right)^{j_1+j_2-j_3-1}\prod_{i=1}^{j_1+j_2-j_3-1}\frac{\Gamma^2[2j_1-i]\Gamma^2[2j_2-i]}{\Gamma^2[1+i]\Gamma^2[2j_3+1+i]}. \tag{B9}$$

This expression shows that one has the multiplicative renormalization (2.37) for the cosmological constant. Note that such a computation is an old story [15]. The only novelty is the contributions of the touching operators in Eq. (B8). However, they can be neglected.

Now let us turn to the touching couplings. In contrast to the previous case, set $m_1=j_1$, $m_2=j_2$, $m_3=-j_3$, and, moreover, $j_3=j_1+j_2-1$. The normalization factors do not give $\Gamma[0]$, and we drop them again. The Liouville correlator is trivial. So the only contribution is due to the matter sector. It is given by

[15]The deformation of this algebra only by the nonvanishing cosmological constant was found in [15,23].

$$\frac{\Gamma^2[2j_3+1]}{\Gamma^2[2j_1]\Gamma^2[2j_2]}\sum_{k=0}^{\infty}\frac{\Gamma[k+1]\Gamma[k+2]}{\Gamma[2k+2]} \times(\overline{t}\Gamma[0])^{k+1}(\overline{t^{\dagger}}\Gamma[0])^{k}. \qquad \text{(B10)}$$

The result (B10) is obtained by using Dotsenko-Fateev multiple 2D integrals [10]. Some further transformations of the resulting products have been done to simplify the final expression.

A conclusion which we can draw from this calculation is that the multiplicative renormalizations of the touching couplings are given by Eq. (2.38).

[1] A. M. Polyakov, in Les Houches, 1988; L. Alvarez-Gaume and C. Gomez, Lectures at the Trieste Spring School, 1991; I. R. Klebanov, Lectures at the Trieste Spring School, 1991; D. Kutasov, Lectures at the Trieste Spring School, 1991; F. David, in Les Houches, 1992; J. Ambjørn, in Les Houches, 1994.

[2] E. Brézin and J. Zinn-Justin, Phys. Lett. B **288**, 54 (1992).

[3] M. R. Douglas, ''*D*-branes and Matrix Theory in Curved Space,'' presented at Strings '97, hep-th/9707228.

[4] S. Coleman, Nucl. Phys. **B307**, 867 (1988); **B310**, 643 (1988); S. B. Giddings and A. Strominger, *ibid.* **B306**, 890 (1988); **B307**, 854 (1988).

[5] F. David, Nucl. Phys. **B487**, 633 (1997).

[6] V. G. Knizhnik, A. M. Polyakov, and A. B. Zamolodchikov, Mod. Phys. Lett. A **3**, 819 (1988).

[7] S. R. Das, A. Dar, A. M. Sengupta, and S. R. Wadia, Mod. Phys. Lett. A **5**, 1041 (1990); G. Korchemsky, *ibid.* **7**, 3081 (1992); Phys. Lett. B **296**, 323 (1992); S. Gubser and I. R. Klebanov, *ibid.* **340**, 35 (1994).

[8] F. David, Mod. Phys. Lett. A **3**, 819 (1988); J. Distler and H. Kawai, Nucl. Phys. **B321**, 509 (1989).

[9] I. R. Klebanov, Phys. Rev. D **51**, 1836 (1995); J. L. Barbón, K. Demeterfi, I. R. Klebanov, and C. Schmidhuber, Nucl. Phys. **B440**, 189 (1995).

[10] Vl. S. Dotsenko and V. A. Fateev, Nucl. Phys. **B240 [FS12]**, 312 (1984); **B251 [FS13]**, 691 (1985).

[11] A. A. Belavin, A. M. Polyakov, and A. B. Zamolodchikov, Nucl. Phys. **B241**, 333 (1984).

[12] D. Friedan, Z. Qiu, and S. Shenker, Phys. Rev. Lett. **52**, 1575 (1984).

[13] B. Lian and G. Zuckerman, Phys. Lett. B **254**, 417 (1991); **266**, 21 (1991); P. Bouwknegt, J. McCarthy, and K. Pilch, Commun. Math. Phys. **145**, 541 (1992).

[14] D. J. Gross, I. R. Klebanov, and M. J. Newman, Nucl. Phys. **B350**, 621 (1991); D. J. Gross and I. R. Klebanov, *ibid.* **B352**, 671 (1991); A. M. Polyakov, Mod. Phys. Lett. A **6**, 635 (1991).

[15] Vl. S. Dotsenko, Theor. Math. Phys. **92**, 938 (1992); Mod. Phys. Lett. A **7**, 2505 (1992).

[16] E. Witten, Nucl. Phys. **B340**, 281 (1990).

[17] N. Seiberg, in *Random Surfaces and Quantum Gravity*, edited by O. Alvarez, E. Marinari, and P. Windey (Plenum, New York, 1990).

[18] S. Jain and S. D. Mathur, Phys. Lett. B **286**, 239 (1992).

[19] A. M. Polyakov, Lectures at the Jerusalem Winter School, 1991; A few projects in string theory, in Les Houches, 1992.

[20] J. Polchinski, Nucl. Phys. **B346**, 253 (1990).

[21] G. Moore, ''Gravitational Phase Transitions and the Sine-Gordon Model,'' Report No. YCT-P1-92, hep-th/9203061.

[22] A. Bilal, presented at the Institut d'Etudes Scientifiques de Cargèse, 1995, hep-th/9601129.

[23] M. Li, Nucl. Phys. **B382**, 242 (1992); S. Kachru, Mod. Phys. Lett. A **7**, 1419 (1992); J. L. F. Barbón, Int. J. Mod. Phys. A **7**, 7579 (1992).

[24] O. Andreev (work in progress).

[25] P. Di Francesco and D. Kutasov, Nucl. Phys. **B342**, 589 (1990).

[26] E. Witten and B. Zwiebach, Nucl. Phys. **B377**, 55 (1992).

[27] E. Witten, Nucl. Phys. **B373**, 187 (1992).

[28] J. Goldstone (unpublished); V. G. Kac, in *Group Theoretical Methods in Physics*, edited by W. Beiglbock *et al.* (Springer-Verlag, Berlin, 1979); G. Segal, Commun. Math. Phys. **80**, 301 (1981).

[29] I. R. Klebanov and A. M. Polyakov, Mod. Phys. Lett. A **6**, 3273 (1991).