Schwinger and Thirring models at finite chemical potential and temperature

Ramón F. Alvarez-Estrada*
Departamento de Física Teórica, Universidad Complutense, 28040, Madrid, Spain

Angel Gómez Nicola†
Theoretical Physics Group, Imperial College, Prince Consort Road, London SW7 2BZ, United Kingdom
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The imaginary time generating functional Z for the massless Schwinger model at nonzero chemical potential μ and temperature T is studied in a torus with spatial length L. The lack of Hermiticity of the Dirac operator gives rise to a nontrivial μ- and T-dependent phase $\mathcal{J}$ in the effective action. When the Dirac operator has no zero modes (trivial sector), we evaluate $\mathcal{J}$, which is a topological contribution, and we find exactly Z, the thermodynamical partition function, the boson propagator and the thermally averaged Polyakov loop. The μ-dependent contribution of the free partition function cancels exactly the nonperturbative one from $\mathcal{J}$, for $L\to\infty$, yielding a zero charge density for the system, which bosonizes at nonzero μ. The boson mass is $e/\sqrt{\pi}$, independent of T and μ, which is also the inverse correlation length between two opposite charges. Both the boson propagator and the Polyakov loop acquire a new T- and μ-dependent term at $L\to\infty$. The imaginary time generating functional for the massless Thirring model at nonzero T and μ is obtained exactly in terms of the above solution of the Schwinger model in the trivial sector. For this model, the μ dependences of the thermodynamical partition function, the total fermion number density and the fermion two-point correlation function are obtained. The phase $\mathcal{J}$ displayed here leads to our new results and allows us to complement nontrivially previous studies on those models. [S0556-2821(98)00704-8]

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I. INTRODUCTION

The Schwinger model is QED in $1+1$ space-time dimensions [1]. Although it is a toy model, it shares many interesting physical properties with more realistic theories such as QCD or the electroweak theory. It is perhaps the simplest example in which gauge invariance does not necessarily imply a massless gauge boson, analogously to the Higgs phenomenon. Other interesting properties of the model are dynamical mass generation, chiral symmetry breaking, and confinement. The model with massless fermions was shown to be exactly solvable in a vacuum (that is, without thermal effects) a long time ago [1,2]. It is equivalent to a theory describing a free boson with mass $e/\sqrt{\pi}$ (bosonization), which is physically a fermion-antifermion bound state (confinement). The finite mass implies a finite correlation length, which physically corresponds to charge screening, long range forces being absent. On the other hand, chiral symmetry is broken through the chiral anomaly [3] rather than spontaneously, since Coleman's theorem [4] prevents any continuous symmetry from being spontaneously broken in two dimensions. When a mass parameter for the fermions is included, the model is no longer solvable but it is still possible to analyze exactly some of the above properties, such as fermion confinement [5].

In the last ten years, there has been a renewed interest in the study of the Schwinger model including statistical mechanics features. The model was also solved and, in particular, the two-point correlation functions and the partition function were obtained at finite temperature T and zero chemical potential μ in [6] when the Dirac operator has no zero modes. Its thermodynamics can be expressed in terms of those of a free boson of mass $e/\sqrt{\pi}$ and free fermions; i.e., bosonization also takes place at finite temperature. A more complete study of the Schwinger model on a torus, which naturally incorporates temperature effects, still at $\mu=0$, has been performed in [7,8]. More specifically, in [7] the model was treated with an arbitrary number of zero modes and the two-point fermion correlation function was calculated, whereas in [8] higher correlation functions were obtained. In [9], the correlation functions have also been studied for nonzero μ. In a recent work [10], the problem of charge screening at finite temperature (with $\mu=0$) in the Schwinger model was analyzed, in connection with the spontaneous breaking of the discrete Z symmetry, which corresponds to the freedom of choosing gauge fields in the Euclidean time direction with any winding number around S^1. These nontrivial gauge transformations will play an essential role in our analysis with a nonzero chemical potential.

On the other hand, we recall that the Thirring model, which describes massless fermions in $1+1$ dimensions with a quartic self-interaction, can also be explicitly solved in a vacuum ($T=\mu=0$) [11,12]. We also recall that the generating functional for the Thirring model at finite T and $\mu=0$ has been obtained in terms of the fermionic one (with an external electromagnetic source) for the Schwinger model [6]. The Thirring model at nonvanishing T and μ has been analyzed in [13] for real time and in [14] in the torus.

In this paper, we shall study, first, the Schwinger model in a medium at thermodynamical equilibrium, by introducing

*Email address: ralvarez@eucmax.sim.ucm.es
†On leave of absence from Departamento de Física Teórica, Universidad Complutense, 28040, Madrid, Spain. Email address: a.gomez@ic.ac.uk

both the temperature and the fermion chemical potential μ. By considering a nonzero μ, we are able to study the system when there is a finite net fermion charge density (in the free case, μ is just the Fermi energy). Some of the questions that naturally arise are (i) whether one can provide simple solutions for the Schwinger model with a nonzero μ, which extend previous studies nontrivially, (ii) whether bosonization takes place at finite fermion charge density and, if so, which is the boson mass, (iii) which is the net fermionic charge of the resulting system, that is, whether the fermions are still confined to neutral mesons, and (iv) how the chemical potential affects charge screening. We shall try to give answers to all of these questions. The second aim of this work is to provide an exact solution for the Thirring model at $T\neq 0$ and $\mu\neq 0$ in the imaginary time formalism, in terms of the corresponding one for the Schwinger model, to compare with previous findings by other authors, using different methods, as a search for consistency and to get some new results.

The plan of this paper is as follows. In Sec. II, we shall deal with the generating functional Z of the Schwinger model at nonzero T and μ, analyzing several important items: the fermionic generating functional Z_f with an external electromagnetic field, the role of the zero modes, the determinant of the Dirac operator, etc., by following steps similar to those in [7]. The lack of Hermiticity of the Dirac operator and a nontrivial phase factor $\mathcal{J}$ will be genuine and crucial features of the $\mu\neq 0$ case. They both will make necessary an extension of the methods developed in [7]. From Sec. III onwards, we shall restrict ourselves to the trivial sector, which is the only relevant one, in order to study the thermodynamics of the system. We shall get Z, Z_f, and $\mathcal{J}$ by using functional methods, generalizing what was done for $\mu=0$ in [6] and deriving the proper extension of the point-splitting regularization when $\mu\neq 0$. Section IV is devoted to (a) several physical results for the Schwinger model, the fermion charge density, the thermodynamical partition function, the boson propagator in the trivial sector, the Polyakov loop (the order parameter of the confining symmetry), and the screening length, and (b) the consistency of our methods. The tasks of obtaining an explicit solution and new results for the Thirring model at nonzero T and μ are undertaken in Sec. V. Section VI contains the conclusions and some discussions. Several results pertaining to the zero modes in the Schwinger model at nonzero T and μ are collected in the Appendix.

II. GENERATING FUNCTIONAL AT FINITE TEMPERATURE AND DENSITY

Our starting point will be the generating functional for the Schwinger model in the imaginary time formalism of thermal field theory [15,16]. We shall work in Euclidean two-dimensional space- (imaginary) time. In principle, we shall keep the length of the system L finite, by imposing suitable boundary conditions in the spatial direction (see below). Thus, one properly defines the spectrum of the Dirac operator and avoids infrared divergences [7,8,14]. At the end of the calculations we shall take the $L\to\infty$ limit. The finite density effects will be implemented by including a chemical potential μ associated to the conservation of the total electric charge (or the number of electrons minus that of positrons). Let $A=(A^\mu)=(A^0,A^1)$ be the electromagnetic potential. Then, the generating functional reads

$$Z[J,\xi,\bar{\xi}]=N(\beta)\int_{periodic}\mathcal{D}A\exp\left[\int_T d^2x(\Gamma[A]+JA)\right]\times Z_f[A,\xi,\bar{\xi}],$$

$$\Gamma[A]=-\frac{1}{2}E^2-\frac{1}{2\alpha}(\partial_\mu A^\mu)^2, \tag{1}$$

where the fermionic generating functional is

$$Z_f[A,\xi,\bar{\xi}]=\int_{antiperiodic}\mathcal{D}\bar{\psi}\mathcal{D}\psi\exp\left[\int_T d^2x(-\bar{\psi}\not{D}(A;\mu)\psi+\bar{\xi}\psi+\bar{\psi}\xi)\right] \tag{2}$$

and the Dirac operator is given by

$$\not{D}(A;\mu)=\not{\partial}-ie\not{A}-\mu\gamma^0. \tag{3}$$

In the above equations, $N(\beta)$ is a temperature-dependent normalization constant, $\beta=1/T$, T being the temperature, $\int_T$ is the integral over the Euclidean two-dimensional torus $[0,\beta]\times[0,L]$, and e is the electric charge, which has dimensions of energy. The fermionic and bosonic external sources are $\bar{\xi}$, ξ and J, respectively. The electric field is $E=F_{01}=\partial_0A_1-\partial_1A_0$ and α is the covariant gauge-fixing parameter. It is important to remark here that the above covariant gauge fixing does not fix the gauge completely on the torus. There is still some residual gauge arbitrariness related to global gauge transformations, which we shall deal with later. The Faddeev-Popov determinant has been absorbed in the measure in Eq. (1), as it plays no dynamical role. Our conventions for the Euclidean Dirac matrices ($\{\gamma_\mu,\gamma_\nu\}=\delta_{\mu\nu}$) are: $\gamma^0=\gamma_0$, $\gamma^1=\gamma_1$ and $\gamma^5=-i\gamma^0\gamma^1$ are the Pauli matrices.

The electromagnetic field and the bosonic external source are periodic in Euclidean time with period β whereas the fermionic fields and sources are antiperiodic. An alternative approach, which we shall not follow here, would have been to take $\not{D}(A;0)$ in Eq. (2), with fermions satisfying the boundary condition $\psi(x^0+\beta,x^1)=-\exp(\beta\mu)\psi(x^0,x^1)$. Concerning the spatial boundary conditions, they cannot be chosen as periodic, in general (after the above choice for the temporal ones), as the Dirac operator may have zero modes on the torus (to avoid duplications, we refer to [7] for a justification). Without loss of generality, we shall choose A_μ so that $A_\mu(x^0,x^1+L)-A_\mu(x^0,x^1)=\partial_\mu(-\Phi x^0/e\beta)$ and hence

$$\psi(x^0,x^1+L)=\exp\left(-i\frac{\Phi}{\beta}x^0\right)\psi(x^0,x^1),$$

$$\bar{\psi}(x^0,x^1+L)=\bar{\psi}(x^0,x^1)\exp\left(i\frac{\Phi}{\beta}x^0\right), \tag{4}$$

with Φ the total flux of the electric field over the torus,

$$\Phi = e\int_T d^2x E(x) = 2\pi(n_+ - n_-), \tag{5}$$

where $n_\pm$ are the number of zero modes with positive and negative chirality. The relation (5) follows directly from the axial anomaly [7,17]. We shall introduce $k = n_+ + n_-$, the total number of zero modes. The gauge sector with $k=0$ will be referred to as the trivial sector. For later use, we recall the following factorization property of the Dirac operator:

$$\not{D}(A;\mu) = \exp(x^0\mu)\not{D}(A;0)\exp(-x^0\mu). \tag{6}$$

A. General structure of $Z_f[A,\xi,\bar{\xi}]$ with zero modes

The contribution of the zero modes to the generating functional has to be analyzed carefully, in order to properly define the functional determinant of the Dirac operator. For that purpose, we shall follow the same steps as in [7]. However, there is an important distinctive feature of the $\mu \neq 0$ case, namely, that $i\not{D}[A;\mu]$ is non-Hermitian. Hence, the set of eigenfunctions of $i\not{D}$ is no longer an orthonormal basis in which the spinor fields could be expanded. To avoid this difficulty we shall expand the spinors in the basis of the Hermitian operators $\not{D}^\dagger\not{D}$ and $\not{D}\not{D}^\dagger$. This will allow us to separate the zero mode contribution up to a phase factor. We shall discuss below this factor and its relevance to the calculation. First, let us consider the set of eigenfunctions of the Hermitian operators,

$$H(A;\mu)\phi_n = [\not{D}^\dagger(A;\mu)\not{D}(A;\mu)]\phi_n = \mu_n\phi_n,$$

$$\bar{H}(A;\mu)\varphi_n = [\not{D}(A;\mu)\not{D}^\dagger(A;\mu)]\varphi_n = \mu_n\varphi_n. \tag{7}$$

The operators H and $\bar{H}$ have the same eigenvalues $\mu_n \geq 0$ (for $\mu_n > 0$, $\not{D}\phi_n$ is an eigenstate of $\bar{H}$) and the zero modes of H ($\bar{H}$) are the same as those of $\not{D}$ ($\not{D}^\dagger$). In addition, since the anomaly (5) is μ independent (for general results on the independence of anomalies on thermal effects, see [18]), $n_+ - n_-$ is the same for both $\not{D}(A;\mu)$ and $\not{D}(A;0)$. As we shall see in Sec. II C, all zero modes have always the same chirality. Therefore, the number k of zero modes is the same for H, $\bar{H}$, and $\not{D}(A;0)$.

At this point let us expand the spinor fields ψ and $\bar{\psi}$ as

$$\psi(x) = \sum_{p=1}^{k}\alpha_p\phi_p + \sum_{q=k+1}^{\infty}\beta_q\phi_q,$$

$$\bar{\psi}(x) = \sum_{p=1}^{k}\bar{\alpha}_p\varphi_p^\dagger + \sum_{q=k+1}^{\infty}\bar{\beta}_q\varphi_q^\dagger, \tag{8}$$

with

$$\varphi_q = \frac{1}{\sqrt{\mu_q}}\not{D}\phi_q, \quad q = k+1,\ldots,\infty, \tag{9}$$

ϕ_p (φ_p) being the zero modes of H ($\bar{H}$). In this basis, we have $(\varphi_q,\varphi_{q'}) = (\phi_q,\phi_{q'}) = \delta_{qq'}$, where $(\chi,\psi) = \int_T d^2x\chi^\dagger\psi$ is the scalar product on the torus. We get, for the fermionic action,

$$\int_T d^2x\bar{\psi}\not{D}(A;\mu)\psi = \sum_{q=k+1}^{\infty}\sqrt{\mu_q}\bar{\beta}_q\beta_q. \tag{10}$$

Then, the action is diagonal in this basis and, by doing the integration over the Grassmanian variables α_p, the contribution of the zero modes can be factorized. A crucial point should be noticed here. As the spinors ψ and $\bar{\psi}$ are expanded in different basis, the Jacobian of the change of basis from $\mathcal{D}\psi$ to $\Pi_{p,q}d\alpha_p d\beta_q$ is not the inverse of that from $\mathcal{D}\bar{\psi}$ to $\Pi_{p,q}d\bar{\alpha}_p d\bar{\beta}_q$. This fact was already noted by Fujikawa [19] in the context of anomalies with non-Hermitian Dirac operators. Since both changes of variables are formally unitary, when doing them simultaneously we are left with some phase factor $\exp[i\mathcal{J}(A;\mu)]$. Notice that $\mathcal{J}(A;0) = 0$ since then $H = \bar{H} = -\not{D}^2(A;0)$. Also, in principle, the phase factor is different for every k sector, a feature to be recalled by means of a superscript (k). Thus, performing the Gaussian Grassman integrals over $d\bar{\alpha}d\alpha d\bar{\beta}d\beta$ we get

$$\begin{aligned} Z_f[A,\xi,\bar{\xi}] = {} & \exp[i\mathcal{J}^{(k)}(A;\mu)] \\ & \times\exp\left[-i\int_T d^2x d^2y\,\bar{\xi}(x)G(x,y,eA;\mu)\xi(y)\right] \\ & \times\prod_{p=1}^{k}\int_T d^2x d^2y\,\bar{\xi}(x)\phi_p(x)\varphi_p^\dagger(y)\xi(y) \\ & \times\sqrt{\det{}'H(A;\mu)}, \end{aligned} \tag{11}$$

where $\det'$ is the functional determinant when the zero modes are omitted (or factored out) and $G(x,y,eA) = \sum_{q=k+1}^{\infty}(1/\sqrt{\mu_q})\phi_q(x)\varphi_q^\dagger(y)$ is the exact fermionic two-point function, satisfying the differential equation

$$\not{D}(A;\mu)G(x,y,eA;\mu) = \delta^{(2)}(x-y) - \sum_{p=1}^{k}\varphi_p(x)\varphi_p^\dagger(y). \tag{12}$$

The second term on the right-hand side of the above equation is the projector onto the $\bar{H}$ zero mode subspace. For simplicity, we have omitted a superscript (k) in both $G(x,y,eA;\mu)$ and $\det'H(A;\mu)$. From Eq. (11) we see that the zero mode contribution can be factorized in this basis, in which we obtain the contribution of $|\det'\not{D}| = (\det'H)^{1/2}$. However, we have still to clarify which is the role of the phase factor $\exp(i\mathcal{J})$. This will be carried out in the next sections. Let us now recall how to obtain different quantities of physical interest from Eq. (11). If we are interested in the thermodynamics of the Schwinger model, the relevant quantity is the partition function $Z = Z(0,0,0)$, so that, from Eq. (11), only the trivial sector contributes:

$$\begin{aligned} Z(0,0,0) = N\int_{periodic,0}\mathcal{D}A\exp\Bigg[& i\mathcal{J}^{(0)}(A;\mu) \\ & + \int_T d^2x\Gamma[A]\Bigg]\sqrt{\det H(A;\mu)}. \end{aligned} \tag{13}$$

We can obtain thermodynamic observables, such as the free

energy and the particle charge density, by differentiating Z with respect to the temperature and the chemical potential, respectively, thereby generalizing for finite charge density the study carried out in [6]. The subscript 0 in the functional integral above indicates that only the trivial sector contributes. Then, in the trivial sector $\mathcal{J}$ can be identified with the phase of the fermionic determinant. We can also calculate the average fermion charge density $\rho \equiv L^{-1}\int_0^L dx^1 \langle \overline{\psi}\gamma^0\psi\rangle$ in terms of the two-point Green function. A remark is in order here: The equations of motion for the A field imply $\partial_1 E(x) = -ie\,\overline{\psi}\gamma^0\psi$, which is Gauss' law. If the latter is imposed as a quantum constraint on physical states [20], then it would imply $\rho=0$ with the boundary conditions chosen. As is customary [21], one may consider that an external compensating (say, ion) charge ρ_{ex} is present to ensure charge neutrality and hence Gauss' law holds for the total charge $\rho_{tot}=\rho+\rho_{ex}=0$. Alternatively, one may consider an open system that exchanges particles with a reservoir ensuring charge neutrality. With this in mind, we shall make no further reference to ρ_{ex} and concentrate only on the fermion charge density ρ for the electron-photon system. Therefore, Eq. (11) yields

$$\rho = \frac{1}{L}\int_0^L dx^1 \langle \overline{\psi}\gamma^0\psi\rangle = \frac{1}{\beta L}\frac{\partial}{\partial\mu}\log Z = -\frac{1}{L}\int_0^L dx^1 \frac{1}{Z}\frac{\delta}{\delta\xi}\gamma^0\frac{\delta}{\delta\overline{\xi}}Z[0,\xi,\overline{\xi}]\Big|_{\xi=\overline{\xi}=0} = \frac{i}{Z}\frac{1}{L}\int_0^L dx^1 \Bigg\{ \int_{periodic,0}\mathcal{D}A \exp\Bigg[i\mathcal{J}^{(0)}(A;\mu) + \int_T d^2x\,\Gamma[A]\Bigg]\sqrt{\det H(A;\mu)}\ \mathrm{tr}[\gamma^0 G(x,x)] + i\int_{periodic,1}\mathcal{D}A \times \exp\Bigg[i\mathcal{J}^{(1)}(A;\mu) + \int_T d^2x\,\Gamma[A]\Bigg]\sqrt{\det' H(A;\mu)}\,\varphi_1^\dagger(x)\gamma^0\phi_1(x)\Bigg\} \tag{14}$$

and the $k>1$ contributions vanish. We shall show in Sec. II C that the zero modes of H and $\overline{H}$ have all the same chirality, given by $\mathrm{sgn}(\Phi)$. Hence, φ_1 and ϕ_1 are both eigenstates of γ^5 with the same eigenvalue and therefore the second piece in the above equation vanishes. Then, only the trivial sector contributes to the fermion number density.

We also remark that the property $\{\gamma_5, G(x,y)\}=0$, which is not difficult to prove with the above definitions, implies, as in the case of finite T but vanishing μ [7], that the chiral condensate $\langle \overline{\psi}P_\pm\psi\rangle$, with $P_\pm=(1\pm\gamma^5)/2$, does not depend on $G(x,y)$. However, from Eq. (11) we see that it will contain the phase factor $\exp[i\mathcal{J}^{(1)}(A;\mu)]$ (see the Appendix).

B. Imaginary part of the effective action

As $i\not{D} \neq (i\not{D})^\dagger$, we have found the extra factor $\mathcal{J}(A;\mu)$, which is the source-independent piece of the phase of the generating functional. We shall analyze here its physical interpretation, at least in the trivial sector. The general form of $\mathcal{J}(A;\mu)$ can be inferred from the symmetry transformation properties of the phase of the different quantities obtained from Z_f after switching off the external sources. For instance, in the trivial sector, the object of interest is the effective action $Z_f[A,0,0]$. Now, recall that the μ-dependent term in the Dirac action is odd under the operation of charge conjugation C, since it is the number of particles minus the number of antiparticles operator. The rest of the Dirac action is even under C, so that C acts on Z_f by replacing $\not{D}\rightarrow -\not{D}^\dagger$ or, in other words, $Z_f[A^C,0,0]=Z_f^*[A,0,0]$ and, therefore, the phase of the effective action is odd under C, while the modulus is even. This is analogous to the case of the QCD effective chiral Lagrangian, when the symmetry under consideration is spatial parity (P), the phase of the effective action being, then, the Wess-Zumino-Witten term [22]. In our case, $\mathcal{J}^{(0)}(A;\mu)$ should contain only C-odd combinations of the gauge field. As P is a symmetry of the effective action, $\mathcal{J}$ should be CP odd. In addition, it is not difficult to check that the μ term does not generate any anomaly in the gauge current, so that imposing local gauge invariance (see comments below), the only term which fulfills such symmetry requirements is of the form

$$\mathcal{J}^{(0)}(A;\mu) = \widetilde{F}(T,\mu,L)\int_T d^2x A_0(x), \tag{15}$$

where $\widetilde{F}(T,\mu,L)$ is a function, undetermined so far (to be found explicitly later), such that $\widetilde{F}(T,\mu,L)=-\widetilde{F}(T,-\mu,L)$ since, by changing simultaneously $\mu\rightarrow-\mu$ and particle by antiparticle, the theory remains unchanged. There is another point that is worth noticing here. Recall that in the torus the gauge transformations $g(x^0,x^1): S^1\times S^1\rightarrow U(1)$ are parametrized by $\mathbb{Z}\times\mathbb{Z}$, corresponding to the two winding numbers (n,m) around the two circles. For any Φ, the most general gauge transformation $A_\mu\rightarrow A_\mu+\partial_\mu\alpha$, which keeps $\partial_\mu A^\mu$ fixed (so that $\partial_\mu\partial^\mu\alpha=0$) and leaves unchanged the boundary conditions (in space and time) for both fermion and gauge fields is $\alpha(x_0,x_1)=(2\pi n x^0)/\beta+(2\pi m x^1)/L$, up to an additive constant. Different choices of (n,m) correspond to nontrivial, homotopically disconnected, gauge transformations. But then we note that the integral in Eq. (15) is precisely equal to n when A is a pure gauge field. Hence, Eq. (15) changes by $n\widetilde{F}$ when we perform a gauge transformation g labeled by (n,m) and then it is not gauge invariant under nontrivial gauge transformations. In this sense it is a topological term. Therefore, we are imposing local gauge invariance but still allowing a noninvariant topological term dependent on the chemical potential. This as-

sumption is partially motivated by previous works in which μ-dependent topological effective actions were obtained [23,24]. In the next section, we shall get $\widetilde{F}$ explicitly, through another method, thereby justifying our assumption.

C. Instanton decomposition

Following [7], we shall decompose the gauge field into an instanton part $\widetilde{A}_\mu$ and a local fluctuation ϕ as

$$A_\mu = \widetilde{A}_\mu - \epsilon_{\mu\nu}\partial_\nu \phi, \tag{16}$$

with $\phi(x)$ periodic in both space-time directions. Note that $\widetilde{A}_\mu$ yields a constant electric field $\widetilde{E} = \Phi/(eL\beta)$ and, hence, $E = \widetilde{E} + \Delta\phi$. By following steps similar to those in [7], we separate first the contribution of $\widetilde{A}$ and ϕ in $\sqrt{\det' H}$. Using that $\not{D}(A;0) = \exp(e\gamma^5\phi)\not{D}(\widetilde{A};0)\exp(e\gamma^5\phi)$ [7] and Eq. (6), it follows immediately that

$$\not{D}(A;\mu) = \exp(e\gamma^5\phi)\not{D}(\widetilde{A};\mu)\exp(e\gamma^5\phi). \tag{17}$$

Notice that the operator $H(A;\mu)$ in Eq. (7) can be cast as

$$H(A;\mu) = -\nabla_\nu \nabla_\nu (A,\mu) - eE\gamma^5,$$
$$\nabla_\nu = \partial_\nu - ieA_\nu + i\mu\gamma^5\delta_{\nu 1}, \tag{18}$$

and the operator $\overline{H}$ is obtained from H by changing $\mu \to -\mu$. Hence, $H(\widetilde{A};\mu) = -\nabla_\nu\nabla_\nu(\widetilde{A};\mu) - \gamma^5\Phi/L\beta$, so that, as $-\nabla_\nu\nabla_\nu$ in Eq. (18) is a positive operator, all the zero modes of $H(\widetilde{A};\mu)$ have the same chirality, equal to the sign of Φ (recall that $[\gamma^5, H] = 0$). On the other hand, from Eq. (17), we get a zero mode of $\not{D}(A;\mu)$ by multiplying a zero mode of $\not{D}(\widetilde{A};\mu)$ by $\exp(-e\gamma^5\phi) = \exp[-e\ \mathrm{sgn}(\Phi)\phi]$, which is in turn a zero mode of $H(A;\mu)$. We can apply exactly the same argument to $\overline{H}$. Then, the zero modes of H and $\overline{H}$ in Eq. (7) have both the same chirality, which is equal to $\mathrm{sgn}(\Phi)$. This was already used in Sec. II A, in order to omit the one zero mode contribution to the particle density.

It is possible to separate the contribution of $\det' H(\widetilde{A};\mu)$ in $\det' H(A;\mu)$, for arbitrary k. We have sketched the derivation in the Appendix, the general formula, for any k, being given in Eq. (A2). From that expression, we read the usual induced mass term for the boson field, with mass $m = e/\sqrt{\pi}$, which is independent of both the temperature and the chemical potential, thereby generalizing the result for $\mu = 0$ previously derived in [6,7,28]. Let us quote here the result for the trivial sector $k = 0$:

$$\det H(A;\mu) = \det H(\widetilde{A};\mu)\exp\left(\frac{e^2}{\pi}\int_T d^2x\,\phi(x)\Delta\phi(x)\right). \tag{19}$$

D. Determinant of the instanton operator

In order to complete the analysis in the previous section, one should still study the spectrum of $H(\widetilde{A};\mu)$, which will be the purpose of the present section. First, by following [7], we shall decompose the field $\widetilde{A}$ as follows:

$$\widetilde{A}_0 = -\frac{\Phi}{eL\beta}x_1 + \frac{2\pi}{\beta}h_0 + \partial_0\lambda,$$
$$\widetilde{A}_1 = \frac{2\pi}{L}h_1 + \partial_1\lambda, \tag{20}$$

which is the Hodge decomposition of the gauge field in the torus. The contributions proportional to h_0 and h_1 are the so-called harmonic parts and are essential to correctly quantize the model [7,14]. Notice that under a nontrivial gauge transformation (n,m) of the type commented on in Sec. II B, the h fields above are the only ones changing and they do so as $h_0 \to h_0 + n$ and $h_1 \to h_1 + m$, even for $\Phi = 0$. The λ-dependent terms in the last two equations are pure gauge contributions with λ periodic in x^0 and x_1, which will not play any physical role. For instance, with the covariant choice $\partial_\mu A^\mu = 0$, λ is just a constant and that term does not appear in Eq. (20). Let us consider first the case $\Phi = 0$, which is the only relevant one for the partition function.

Since γ^5 commutes with $\widetilde{H}$, we choose the eigenfunctions of $H(\widetilde{A};\mu)$ as states of definite chirality, that is,

$$\Psi^+ = \begin{pmatrix} \phi^+ \\ 0 \end{pmatrix}, \quad \Psi^- = \begin{pmatrix} 0 \\ \phi^- \end{pmatrix}, \tag{21}$$

with $\gamma^5\Psi^\pm = \pm\Psi^\pm$, since $\gamma^5 = \mathrm{diag}(1,-1)$. Then, for $\Phi = 0$ we have to solve $H^\pm(\widetilde{A};\mu)\phi^\pm = \lambda^\pm\phi^\pm$ with

$$H^\pm(\widetilde{A};\mu) = -(\partial_0 - i\overline{h_0})^2 - (\partial_1 - i\overline{h_1} \pm i\mu)^2 \tag{22}$$

and (anti) periodic boundary conditions in the (time) space direction. In the above equation we have introduced $\overline{h_0} = 2\pi e h_0/\beta$ and $\overline{h_1} = 2\pi e h_1/L$. The eigenfunctions are plane waves and the corresponding eigenvalues are

$$\lambda^\pm_{nk} = \left(\frac{2\pi}{\beta}\right)^2\left[n + \frac{1}{2} - eh_0\right]^2 + \left[\frac{2\pi}{L}(k - eh_1) \pm \mu\right]^2, \tag{23}$$

with n,k integers. Notice that the chemical potential breaks the chiral degeneracy which was originally present in the $\mu = 0$ case. Now, by using $\log\det H = \mathrm{Tr}\log H$ and Eq. (23), we get

$$\log\det H(\widetilde{A};\mu) = \frac{1}{2}\sum_{n,k=-\infty}^{\infty}\sum_{\pm\pm}\log[(\omega_n)^2 + (\omega_k - \overline{h_1} \pm \mu \pm i\overline{h_0})^2], \tag{24}$$

with $\omega_n = (2n+1)\pi/\beta$ and $\omega_k = 2\pi k/L$. Now, let us add and subtract $\log\beta^2\Sigma_{n,k}$ to the right-hand side of the above expression. This procedure will give rise to a T-dependent infinite constant, which, in turn, will be absorbed, as customarily, in the normalization constant $N(\beta)$ [6,15,16]. We can perform the summation over n in the above equation with the help of the two formulas [16,25]

$$\log[(2n+1)^2\pi^2+\beta^2(\omega_k\pm\alpha)^2]$$
$$=\int_1^{\beta^2(\omega\pm\alpha)^2}\frac{d\theta}{\theta^2+(2n+1)^2\pi^2}+\log[1+(2n+1)^2\pi^2],$$
$$\sum_{n=-\infty}^{\infty}\frac{1}{(2n+1)^2+\theta^2}=\frac{1}{\theta}\left(\frac{1}{2}-\frac{1}{e^\theta+1}\right). \tag{25}$$

In so doing, we obtain, finally,

$$\log\det H(\widetilde{A};\mu)=\sum_{k=-\infty}^{\infty}\left\{2\beta(\omega_k-\overline{h}_1)+\sum_{\pm\pm}\log[1+\exp -\beta(\omega_k-\overline{h}_1\pm\mu\pm\overline{h}_0)]\right\} \tag{26}$$

up to an irrelevant T- and μ-independent constant. In order to obtain the full partition function, we have to multiply the above expression (which for $e=0$ reproduces the partition function for free fermions at finite density) by $\exp(i\mathcal{J}^{(0)})$ in Eq. (13) and by the ϕ-dependent contribution in Eq. (19) and, then, integrate over the gauge fields (ϕ,h_0,h_1). As a consequence of the decomposition of the gauge field chosen here, the phase factor only depends on h_0, so that

$$\int_T d^2x A_0(x)=-\frac{\Phi L}{2e}+2\pi h_0 L, \tag{27}$$

since ϕ is periodic in the space direction. Then, the ϕ contribution to the partition function in Eq. (19) gives the partition function of a free massive boson [6]. All the dependence on the chemical potential is included in the (h_0,h_1) part, as given in Eqs. (26) and (27). However, we have still to determine the value of $\widetilde{F}(T,\mu,L)$ in Eq. (15). Before undertaking that task, and for completeness, let us analyze the spectrum of $H(\widetilde{A};\mu)$ when $\Phi\neq 0$. In this case, we have

$$H^{\pm}(\widetilde{A};\mu)=-\left(\partial_0-i\overline{h}_0+i\frac{\Phi}{L\beta}x_1\right)^2-(\partial_1-i\overline{h}_1\pm i\mu)^2\mp e\widetilde{E}, \tag{28}$$

together with the boundary conditions in the spatial direction given in Eq. (4). This eigenvalue problem is solved in the Appendix. From the result found there, we remark here that the μ dependence, when $\Phi\neq 0$, appears only in the states, while the determinant in Eq. (A7) depends on the temperature T but not on μ. As the norm of the zero modes in Eq. (A8) is also μ independent, then, if we go to Eqs. (A2) and (11), we realize that the dependence of the fermionic generating functional on μ, when $\Phi\neq 0$, is encoded in the determinants of the matrices M,N in Eq. (A2), which follow immediately from the spectrum found for $\alpha=0$ in this section. Besides, there are μ dependences in both $G(x,y,eA;\mu)$ and $\mathcal{J}^{(k)}(A;\mu)$.

III. GENERATING FUNCTIONAL IN THE TRIVIAL SECTOR

In this section, we shall use functional methods in order to calculate the generating functional in the trivial sector. By employing these methods, we shall also obtain the phase factor $\mathcal{J}^{(0)}(A;\mu)$. This will allow us to get the full fermion charge density and the partition function, as well as to establish the consistency with the results of the previous section. We recall that the covariant gauge fixing is not complete on the torus. We still have the freedom of performing a nontrivial gauge transformation $(h_0,h_1)\to(h_0+n,h_1+m)$, with the h fields in Eq. (20) and n,m integers, corresponding to loops that wind n times around the temporal direction and m times around the spatial one. It is clear that n and m are not fixed by $\partial^\mu A_\mu=0$. Throughout this section, though, we shall work with the covariant gauge fixing, ignoring this residual gauge arbitrariness. The latter has also been treated in a situation related to the one analyzed here, but not quite identical with it: specifically, for (real time) QED on a spatial circle, at zero temperature and chemical potential [26]. We have to bear in mind that in Eq. (1), we are integrating the gauge field over all possible values of the fields h_μ, that is, $h_\mu\in\mathbb{R}$. Fixing the gauge for those fields would consist in restricting them to a range $[0,1)$ [7], since they change by an integer under a global gauge transformation. Then, if the effective action is globally gauge invariant, and unaffected by the residual gauge arbitrariness, the difference between integrating over all h or restricting them to a $[0,1)$ interval is an infinite constant independent of T and μ, so that arbitrariness cannot affect physical observables such as the free energy or the particle density. For $e=0$, the action depends on derivatives of the electromagnetic field, and the covariant gauge fixing, even if not complete, suffices to get a well-defined propagator, unaffected by that arbitrariness. In general, when e is nonvanishing, $Z[J,\xi,\overline{\xi}]$ is also unaffected by the arbitrariness, after having integrated over all fields. However, for a given A_μ, both $\det H(\widetilde{A};\mu)$ and the fermionic generating functional Z_f may be subject to it, even if $\mu=0$. In particular, when $\mu\neq 0$, we have seen in Sec. II B that Z_f is not globally gauge invariant, due to the induced topological term in the phase $\mathcal{J}$, which changes when $h_0\to h_0+n$. Thus, if we restrict h_0 to a $[0,1)$ interval, the result for the observables would depend on our choice and then it is consistent to let $h_0\in\mathbb{R}$. We shall come again to this point at the end of Sec. IV A, where we shall perform the integration over the h fields explicitly, using the results derived in Sec. II. It is not difficult to check that all the formal functional manipulations that we shall carry out in this section, except those related to $L[A]$, are also unaffected by the residual gauge arbitrariness.

Thus, as a first step, let us rewrite the generating functional in Eq. (1) for the trivial sector, with the aid of standard functional techniques, as

$$Z[J,\xi,\overline{\xi}]=Z_{EM}Z_F\exp\left[-ie\int_T d^2x\frac{\delta}{\delta\xi(x)}\gamma^\nu\frac{\delta}{\delta J^\nu(x)}\frac{\delta}{\delta\overline{\xi}(x)}\right]\times\exp\left(\int_T d^2x d^2y\left[\frac{1}{2}J_\mu(x)D^{\mu\nu}(x-y)J_\nu(y)-i\overline{\xi}(x)S(x,y;\mu)\xi(y)\right]\right), \tag{29}$$

where Z_{EM} and Z_F [$Z_F(T,\mu,L)=Z_F$ for short] are the free

boson and fermion partition functions and $D^{\mu\nu}(x-y)$ and $S(x,y;\mu)$ are the free gauge boson and fermion propagators, respectively:

$$D_{\mu\nu}(x-y)=\frac{1}{\beta L}\sum_{n,k=-\infty}^{\infty} e^{i\omega(x-y)}\frac{1}{\omega^2}\left[\delta_{\mu\nu}+(\alpha-1)\frac{\omega_\mu\omega_\nu}{\omega^2}\right],$$

$$S(x,y;\mu)=-\frac{i}{\beta L}\sum_{n,k=-\infty}^{\infty} e^{i\omega(x-y)}\frac{1}{\gamma^0(\omega_n+i\mu)+\gamma^1\omega_k}, \tag{30}$$

where $\omega=(\omega_n,\omega_k)$, with $\omega_k=2\pi k/L$, $\omega_n=2\pi n/\beta$ in the bosonic propagator and $\omega_n=(2n+1)\pi/\beta$ in the fermionic one. Now, we shall make use of some known functional differentiation formulae [12] and, in particular of

$$\exp\left[-i\int d^2x d^2y \frac{\delta}{\delta\xi(x)}A(x,y)\frac{\delta}{\delta\bar{\xi}(y)}\right]\times\exp\left(i\int d^2x d^2y\,\bar{\xi}(x)B(x,y)\xi(y)\right)$$
$$=\exp\left[i\int d^2x d^2y\,\bar{\xi}(x)\bar{B}(x,y)\xi(y)+L\right]. \tag{31}$$

Here, $A(x,y)$ and $B(x,y)$ are arbitrary functions, to be regarded as the kernels of the operators A and B, respectively, $\bar{B}=B(1+AB)^{-1}$ and $L=-\mathrm{Tr}\,\log[1+AB]^{-1}$, Tr indicating the trace over functional and Dirac spaces. Thus, one finds

$$Z[J,\xi,\bar{\xi}]$$
$$=Z_{EM}Z_F\exp\left[\frac{1}{2}\int_T d^2x d^2y J_\mu(x)D^{\mu\nu}(x-y)J_\nu(y)\right]\times\exp\left[-\frac{1}{2}\int_T d^2x d^2y\frac{\delta}{\delta A_\mu(x)}D^{\mu\nu}(x-y)\frac{\delta}{\delta A_\mu(y)}\right]$$
$$\times\exp\left\{-i\int_T d^2x d^2y\,\bar{\xi}(x)G(x,y,ieA;\mu)\xi(y)+L[A]\right\}, \tag{32}$$

with $A_\mu(x)\equiv -i\int_T d^2y D_{\mu\nu}(x-y)J_\nu(y)$, after having performed the functional differentiations, which appears to leave no trace of the residual gauge arbitrariness in $Z[J,\xi,\bar{\xi}]$. The so-called closed fermion loop functional $L[A]$ can be written formally as

$$L[A]=\mathrm{tr}_D\int_0^e de'\int_T d^2x \not{A}(x)G(x,x,ie'A;\mu), \tag{33}$$

where tr_D denotes the Dirac trace. We recall that $G(x,y,eA;\mu)$ is the two-point function, which, in the trivial sector, satisfies Eq. (12) with $k=0$, that is, with its second term on the right-hand side omitted.

In order to get a well-defined expression for the generating functional, we need, first, to regularize $L[A]$ in Eq. (33). For that purpose, we shall appeal to the point-splitting regularization [12]. Therefore, we should deal with the following limit: $\lim_{x\to y}G(x,y,eA;\mu)$. In Minkowski space-time, the limit should be taken by keeping the points x and y relatively space like, in order to maintain causality [12]. As we are working in Euclidean space-time, we shall not impose, in principle, such a restriction. We shall comment below on the different ways of taking the limit. Before that, and generalizing [12], we shall derive the point-splitting regularization prescription in our present case with nonzero μ. We start with the formal definition of the gauge current in the presence of an external background field A_μ:

$$\langle j_\mu(x)\rangle_f[eA]=\langle\bar{\psi}(x)\gamma_\mu\psi(x)\rangle_f[eA]=i\lim_{x\to y}\mathrm{tr}_D\gamma_\mu G(x,y,eA;\mu), \tag{34}$$

where $\langle O\rangle_f=\int\mathcal{D}\bar{\psi}\mathcal{D}\psi O\exp[-\int\bar{\psi}\not{D}\psi]$. We shall obtain the regularized version of the right-hand side of the above equation as follows: We shall demand that such a regularized gauge current be conserved and gauge invariant. Notice that, under a gauge transformation $A_\mu\to A_\mu-\partial_\mu\Lambda$, $G(x,y,eA;\mu)$ changes as

$$G(x,y,eA;\mu)\to G(x,y,eA;\mu)\exp\{ie[\Lambda(x)-\Lambda(y)]\}. \tag{35}$$

Based upon this, it is easy to show that the product

$$G(x,y,eA;\mu)\exp\left[-ie\int_x^y d\xi^\sigma A_\sigma(\xi)\right] \tag{36}$$

is gauge invariant. However, if we use Eq. (36) in Eq. (34), calculate the divergence of the current j_μ so defined, and use $\not{D}(A;\mu)G=\delta^{(2)}(x-y)$, we find that such a divergence does not vanish for $\mu\neq 0$. To ensure that the current is divergenceless, we have to add an extra μ-dependent term, which leads to the regularized gauge current

$$\langle j_\mu(x)\rangle_f^{reg}[eA]=i\lim_{x\to y}\mathrm{tr}_D\gamma_\mu G(x,y,eA;\mu)\exp\left[-ie\int_x^y d\xi^\sigma A_\sigma(\xi)\right]\times\exp[-\mu(x^0-y^0)], \tag{37}$$

which is, indeed, gauge invariant and divergenceless. Note that Euclidean covariance is broken since the system is in a thermal bath. We are now ready to define the regularized fermion closed loop as

$$L^{reg}[A]=-i\,\mathrm{tr}_D\int_0^e de'\int_T d^2x A_\mu(x)\langle j_\mu(x)\rangle_f^{reg}[ie'A]$$
$$=\mathrm{tr}_D\int_0^e de'\int_T d^2x A_\mu(x)\lim_{x\to y}\mathrm{tr}_D\gamma_\mu\times G(x,y,ie'A;\mu)\exp\left[e'\int_x^y d\xi^\sigma A_\sigma(\xi)\right]\times\exp[-\mu(x^0-y^0)]. \tag{38}$$

The limit $x\to y$ has to be taken in a symmetric way, regarding (x,y) [12]. In order to calculate the fermion closed loop in Eq. (38), we shall consider an ansatz for the exact Green function similar to that in [6]:

$$G(x,y,eA;\mu)=\exp[-ie[\chi(x)-\chi(y)]S(x,y;\mu). \quad (39)$$

It is not difficult to check that, with the above ansatz, $G(x,y)$ is a solution of $\not{D}G(x,y)=\delta^{(2)}(x-y)$, provided that $\chi(x)$ is a solution of $\not{\partial}_x\chi(x)=-\not{A}(x)$. In turn, the solution is

$$\chi(x)=-\int_T d^2y\Delta(x-y)\not{\partial}_y\not{A}(y),$$

$$\Delta(x-y)=-\frac{1}{\beta L}\sum_{n,k=-\infty}^{\infty}e^{i\omega(x-y)}\frac{1}{\omega_n^2+\omega_k^2}, \quad (40)$$

where $\omega=(\omega_n,\omega_k)$ with $\omega_n=2\pi n/\beta$, $\omega_k=2\pi k/L$. So $G(x,y,eA;\mu)$ appears to be unaffected by the residual gauge arbitrariness.

Hence, from the above equations, we have to find the behavior of $S(x,y;\mu)$ when $x\to y$. For that purpose, we apply the standard formulas

$$\frac{1}{\beta}\sum_{n=-\infty}^{\infty}f(\omega_n+i\mu)=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega f(\omega)+\oint_C d\omega f(\omega)$$
$$-\frac{1}{2\pi}\sum_{\pm}\int_{-\infty\mp i\epsilon+i\mu}^{\infty\mp i\epsilon+i\mu}d\omega f(\omega)\frac{1}{e^{\pm i\beta\omega}+1},$$

$$\frac{1}{\beta}\sum_{k=-\infty}^{\infty}f(\omega_k)=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\omega f(\omega)$$
$$+\frac{1}{2\pi}\sum_{\pm}\int_{-\infty\mp i\epsilon}^{\infty\mp i\epsilon}d\omega f(\omega)\frac{1}{e^{\pm iL\omega}-1}, \quad (41)$$

where $\epsilon\to 0^+$, $\omega_n=(2n+1)\pi i/\beta$, $\omega_k=2\pi k/L$, and C is the rectangular contour in the complex ω plane running through the points $(+\infty,-\infty,-\infty+i\mu,+\infty+i\mu)$. We apply Eqs. (41) to $S(x,y;\mu)$ in Eqs. (30) and retain only the dominant contributions in $x\to y$. The complex plane integrals along the lines $(-\infty\mp i\epsilon,+\infty\mp i\epsilon)$ and $(-\infty\mp i\epsilon+i\mu,+\infty\mp i\epsilon+i\mu)$ are performed by forming a closed contour, by means of an infinite arc above (below), corresponding to the $+$ $(-)$ sign in Eqs. (41), and applying the residue theorem. We get

$$\lim_{x\to y}S(x,y;\mu)=e^{(x^0-y^0)\mu}\left[\frac{1}{2\pi}\frac{(x-y)_\mu\gamma^\mu}{(x-y)^2}+i\gamma^0F(T,\mu,L)\right.$$
$$\left.+\mathcal{O}(x-y)^2\right], \quad (42)$$

where

$$F(T,\mu,L)=\frac{1}{2L}\sum_{k=-\infty}^{+\infty}\left[\frac{1}{e^{\beta(\omega_k+\mu)}+1}-\frac{1}{e^{\beta(\omega_k-\mu)}+1}\right], \quad (43)$$

with $\omega_k=2\pi k/L$. We recall that Eq. (42) reproduces the $T=\mu=0$ result given in [12] and the $\mu=0$, $T\neq 0$ one $[F(\mu=0)=0]$ in [6].

Next, we shall replace both Eqs. (40) and the limit (42) into Eqs. (39) and (38). We have taken the $x\to y$ limit in two different ways and established that the same result is arrived at. We have taken, first, $x^0-y^0\to 0$, $x^1-y^1\to 0$ with $(x^1-y^1)/(x^0-y^0)=1$ and, second, the Minkowski causal choice (see [12]) $x^0=y^0$ and $(x^1-y^1)\to 0$. Anyway, what it is important to note here is that the exponential μ dependence in Eq. (42) is exactly cancelled with that in the regulator in Eq. (38). Then, no matter how we take the $x\to y$ limit, we always get a term $\mathrm{tr}\int\not{A}(x)\gamma^0F$ in L^{reg}. The possible divergence in L^{reg} arising from the first piece in Eq. (42) is absent since we have taken the limit symmetrically. Thus, finally we arrive at

$$L^{reg}[A]=\frac{1}{2}\int_T d^2xd^2yA_\alpha(x)\Pi^{\alpha\beta}(x-y)A_\beta(y)$$
$$+2eF(T,\mu,L)\int_T d^2xA_0(x), \quad (44)$$

with

$$\Pi_{\alpha\beta}(x-y)=\frac{1}{\beta L}\frac{e^2}{\pi}\sum_{n,k=-\infty}^{+\infty}e^{i\omega(x-y)}\left[\delta_{\alpha\beta}-\frac{\omega_\alpha\omega_\beta}{\omega^2}\right], \quad (45)$$

where $\omega_n=2\pi n/\beta$ and $\omega_k=2\pi k/L$. So $L^{reg}[A]$ is, in principle, affected by the residual gauge arbitrariness. The functional differentiations with respect to $A(x)$ in Eq. (32) can be performed by employing the following formula, which is valid for any linear operators P and Q [12]:

$$\exp\left[-\frac{i}{2}\int\frac{\delta}{\delta A}P\frac{\delta}{\delta A}\right]\exp\left[\frac{i}{2}\int AQA+i\int fA\right]$$
$$=\exp\left[\frac{i}{2}\int A\bar{P}A+i\int A(1-QP)^{-1}f\right.$$
$$\left.+\frac{1}{2}\,\mathrm{tr}\,\log(1-QP)^{-1}+\frac{i}{2}\int fQ(1-QP)^{-1}f\right], \quad (46)$$

where $\bar{P}=P(1-QP)^{-1}$ and we have omitted, for simplicity, all the space-time dependences. We shall concentrate on the bosonic generating functional $Z[J,0,0]$, as we have already analyzed the dependence on the fermionic sources in the previous section, up to the phase factor (to be calculated below). Upon applying Eq. (46) in Eq. (32), $L[A]$ being given in Eq. (44), we get

$$Z[J,0,0]=Z_{EM}Z_F\exp\left[\frac{1}{2}\int_T d^2x d^2y (J-iG)_\mu(x)\mathbb{D}^{(0)}_{\mu\nu}(x-y)\right.$$
$$\left.\times(J-iG)_\nu(y)\right]\exp\left[\frac{1}{2}\,\mathrm{Tr}\,\log(1+\Pi D)^{-1}\right], \quad (47)$$

where $G_0=2eF(T,\mu,L)$, $G_1=0$. In turn, $\mathbb{D}^{(0)}_{\mu\nu}$ is the exact boson propagator at $\mu=0$, which can be expressed, formally, as $\mathbb{D}^{(0)}=D(1+\Pi D)^{-1}$, D and Π being given in Eqs. (30) and (45), respectively. Its explicit representation is

$$\mathbb{D}^{(0)}_{\mu\nu}(x-y)=\frac{1}{\beta L}\sum_{n,k=-\infty}^{\infty} e^{i\omega(x-y)}\left[\frac{1}{\omega^2+m^2}\left(\delta_{\mu\nu}-\frac{\omega_\mu\omega_\nu}{\omega^2}\right)\right.$$
$$\left.+\alpha\frac{\omega_\mu\omega_\nu}{(\omega^2)^2}\right], \quad (48)$$

where $m^2=e^2/\pi$ is the induced boson mass squared.

In the following section, we shall obtain, from Eqs. (44) and (47), the complete form of the fermionic generating functional in the trivial sector, including the phase factor, the exact boson propagator at $\mu\neq0$, and the partition function, and check the results with the method used in the previous section.

IV. PHYSICAL RESULTS

A. Charge density and the partition function

By setting $J=0$ in the expression (47) we get the partition function

$$Z(T,\mu)=\frac{Z_F(T,\mu)}{Z_F(T,\mu=0)}Z(T,\mu=0)\exp\left[-\frac{1}{2}[2eF(T,\mu,L)]^2\right.$$
$$\left.\times\int_T d^2x d^2y\,\mathbb{D}^{(0)}_{00}(x-y)\right]. \quad (49)$$

However, the above integrals of $\mathbb{D}^{(0)}_{00}$ are ambiguous, in the sense that changes in the order in which such integrals are done yield different results. For instance, let us perform, first, the spatial integrals: That would force us to set $k=0$ in Eq. (48), which would give rise, after doing the temporal integral, to an infinite and gauge-dependent result. Conversely, by changing the order of the integrals, let us perform, first, the one over the (imaginary) time. In so doing, we arrive at a finite and gauge-independent answer. The latter prescription seems to be a natural and physically reasonable choice. In addition, as we shall see, it is consistent with the results obtained in the previous section. If we adopt this prescription in Eq. (49), we get

$$Z(T,\mu)=\frac{Z_F(T,\mu)}{Z_F(T,\mu=0)}Z(T,\mu=0)$$
$$\times\exp[-2\beta L\,\pi F^2(T,\mu,L)]. \quad (50)$$

This is a genuine nonperturbative result, since the argument of the exponential in the term that corrects the free fermionic partition function is independent of the electric charge. As we shall see, such a nonperturbative behavior comes directly from the topological structure discussed in Sec. II, namely, from the phase $\mathcal{J}[A;\mu]$ of the fermionic generating functional. At this point, we recall the result obtained in [6] for $Z(T,\mu=0)$. The latter partition function was proved to factorize into the product of that for a free fermionic field times that for a free massive boson field with mass m, divided by the one for a free massless field (finite temperature bosonization). In our present case, with $\mu\neq0$, we may wonder if such a factorization actually takes place as well and, if so, whether the whole system may have a net fermionic charge or not. Let us consider, first, the free fermionic partition function $Z_F(T,\mu)=Z_F(T,\mu,L)$:

$$\log Z_F(T,\mu,L)=\sum_{k=-\infty}^{+\infty}\{\log[1+e^{-\beta(\omega_k-\mu)}]$$
$$+\log[1+e^{-\beta(\omega_k+\mu)}]\}. \quad (51)$$

The net free fermion charge density is

$$\frac{1}{\beta L}\frac{\partial}{\partial\mu}\log Z_F(T,\mu,L)=-2F(T,\mu,L), \quad (52)$$

F being given in Eq. (43). Thus, from Eqs. (52) and (50) we have

$$Z(T,\mu)=Z(T,\mu=0,L)\exp\left\{-2\beta L\left[\pi F^2(T,\mu,L)\right.\right.$$
$$\left.\left.+\int_0^\mu d\mu' F(T,\mu',L)\right]\right\}=Z(T,\mu,L), \quad (53)$$

which is our final result for the partition function at $\mu\neq0$ for finite L. Now, let us analyze the behavior of the function F in the limit $L\to\infty$. In such a limit, the sum over k becomes a trivial integral, which yields

$$F(T,\mu,L\to\infty)=-\frac{\mu}{2\pi}. \quad (54)$$

Thus, in the $L\to\infty$ limit, the μ dependence in Eq. (53) exactly cancels out. We get, for the full partition function,

$$Z(T,\mu,L\to\infty)=Z(T,\mu=0,L\to\infty)$$
$$=Z_{EM}Z_F(T,\mu=0,L\to\infty)$$
$$\times\exp\left[\frac{1}{2}\,\mathrm{Tr}\,\log(1+\Pi D)^{-1}\right], \quad (55)$$

which is only T dependent. Its explicit expression can be found in Sec. IV of [6]. That is, in the $L\to\infty$ limit the system bosonizes as well, and the only effective degrees of freedom are those of a massive boson field, the net fermionic charge of the system being $\rho=(\beta L)^{-1}\partial\log Z/\partial\mu=0$. We make the following remarks. (i) This is a nonperturbative effect (which has been established due to the fact that the Schwinger model at finite chemical potential and temperature can, still, be solved exactly). (ii) It is characteristic of two dimensions. Recall, for instance, that in perturbative four-dimensional QED in the infinite volume limit, the free energy $\log Z$ is μ dependent and hence $\rho\neq0$ [21]. (iii) The result (55) holds in

the $L\to\infty$ limit: If we keep L finite, F is no longer given by Eq. (54) but it acquires further corrections and the study of the counterpart of Eq. (55) will not be attempted here.

We now turn to the issue of the C-violating phase factor, addressed previously in Sec. II. Thus, we consider the fermionic generating functional $Z_f[A,\xi,\bar{\xi}]$, treating now A as an external background field. We apply Eq. (31) and we arrive at an expression analogous to Eq. (32), but now taking $J=0$, omitting the derivatives with respect to A, and replacing $e\to -ie$. By recalling the regularized fermion closed loop functional obtained in Eq. (38), we get

$$
\begin{aligned}
&Z_f[A,\xi,\bar{\xi}] \\
&= Z_F Z_{EM} \exp\left[-i\int_T d^2x d^2y\, \bar{\xi}(x) G(x,y,eA;\mu)\xi(y)\right] \\
&\times \exp\left[-\frac{1}{2}\int_T d^2x d^2y A_\alpha(x)\Pi^{\alpha\beta}(x-y)A_\beta(y)\right] \\
&\times \exp\left[-2ieF(T,\mu,L)\int_T d^2x A_0(x)\right]. \qquad (56)
\end{aligned}
$$

The comparison with Eq. (11) in the trivial sector leads to identify

$$
\mathcal{J}^{(0)}(A,\mu) = -2eF(T,\mu,L)\int_T d^2x A_0(x), \qquad (57)
$$

which has the form that we had anticipated in Eq. (15), based upon the C symmetry properties of the phase of the fermionic determinant, and so $\widetilde{F}=-2eF$.

We shall now provide an interesting check of consistency. On the basis of Eqs. (13), (19), (26), and (27), we can calculate the partition function through the method developed in Sec. II. We shall do so in the $L\to\infty$ limit. Let us first differentiate with respect to μ in Eq. (26) and then take the $L\to\infty$ limit by replacing ω_k by continuous ω and Σ_k by $(L/2\pi)\int d\omega$. The resulting integral can be done using

$$
\int_{-\infty}^{+\infty} d\omega \left[\frac{1}{1+e^{\beta(\omega-a+b)}} - \frac{1}{1+e^{\beta(\omega-a-b)}}\right] = -2b, \qquad (58)
$$

which is convergent when both pieces of the integrand are added together. Then, in the $L\to\infty$ limit we obtain

$$
\frac{\partial}{\partial\mu}\log\,\det H(\widetilde{A};\mu) = \frac{2\beta L\mu}{\pi} \qquad (59)
$$

and, hence,

$$
\sqrt{\det H}(\widetilde{A};\mu,T) = \sqrt{\det H}(\widetilde{A};\mu=0,T)\exp\left(L\beta\frac{\mu^2}{2\pi}\right). \qquad (60)
$$

We now insert the above result in Eq. (19) and, then, in Eq. (11), for the trivial sector, and we compare to Eq. (56). We see that, before integrating out the photon field, the results obtained with both methods are consistent with each other if $L\to\infty$, since we get the same μ dependence, namely, the one coming from the phase factor (the explicit form of which has been obtained with the second method) times that of the free fermionic partition function, as it appears in Eq. (53) when $F\to -\mu/2\pi$. To complete the check, we shall see that the integration over the gauge boson fields also gives the same result with both methods in the $L\to\infty$ limit, namely, that in Eq. (55). From our previous discussion, it follows that the only part that depends on μ in the full partition function is the integral over the (h_0,h_1) fields, the integrand of which is the exponential of Eq. (27) for $\Phi=0$ times $(-2ieF)$, multiplied by $\sqrt{\det H(\widetilde{A})}$. From Eq. (60), it follows that all the h dependence of $\det H$ is contained in the $\mu=0$ part. In order to extract the dependence on h_0, we differentiate again in Eq. (26). Using again Eq. (58) we get, in the $L\to\infty$ limit,

$$
\begin{aligned}
&\frac{\partial}{\partial h_0}\log\,\det H(\widetilde{A};\mu=0;h_0,h_1) \\
&\quad = 2ie\sum_{s=\pm}\int_{-\infty}^{+\infty} d\omega \frac{s}{1+e^{\beta(\omega-h_1-ish_0)}} = -\frac{8\pi L e^2 h_0}{\beta}
\end{aligned} \qquad (61)
$$

which turns out to be independent of h_1. Thus, from Eqs. (60) and (61), we derive

$$
\begin{aligned}
\sqrt{\det H}(\widetilde{A};\mu,T;h_0,h_1) &= \sqrt{\det H}(\widetilde{A};\mu=0,T;0,h_1) \\
&\times \exp\left(L\beta\frac{\mu^2}{2\pi} - \frac{2\pi L e^2 h_0^2}{\beta}\right).
\end{aligned} \qquad (62)
$$

At this stage, we have to integrate over the fields (h_0,h_1). As commented at the beginning of Sec. III, it is necessary to integrate over $h_0\in\mathbb{R}$, to achieve consistency. This is reinforced by the topological term (57) in the phase of the fermionic generating functional. Thus, from Eqs. (27) and (62), it follows that the relevant factor carrying the μ dependence is

$$
\begin{aligned}
&e\int_{-\infty}^{+\infty} dh_0 \exp\left(-2i\mu h_0 L - \frac{2\pi L h_0^2}{\beta}\right) \\
&\quad = e\sqrt{\frac{\beta}{2L}}\exp\left(-\frac{\mu^2 L\beta}{2\pi}\right).
\end{aligned} \qquad (63)
$$

It turns out that the exponential μ dependence in Eq. (63) cancels that in Eq. (62), and, hence, we arrive, again at the result (55), obtained with our previous method.

B. Boson propagator, the screening length, and the Polyakov loop

The exact boson propagator, which results after the integration of both fermion and gauge fields, can be obtained, in the trivial sector, just by differentiating Eq. (47) twice with respect to the external sources and setting $J=0$. We find

$$\mathbb{D}^{(\mu)}_{\alpha\beta}(x-y)=\mathbb{D}^{(0)}_{\alpha\beta}(x-y)-(2eF)^2\int_T d^2u\,d^2z\,\mathbb{D}^{(0)}_{\alpha 0}(x-u)\times\mathbb{D}^{(0)}_{\beta 0}(y-z), \tag{64}$$

$\mathbb{D}^{(0)}_{\alpha\beta}(x-y)$ being given in Eq. (48). The above expression is, again, formal, in the sense that we have to specify the order in which the spatial and temporal integrals should be done in the second piece on the right-hand side. If we adopt the same prescription as that in Sec. IV A (that is, by doing the integrals in the order that gives a finite answer), we arrive in momentum space at

$$\mathbb{D}^{(\mu)}_{00}(\omega_n,\omega_k)=\mathbb{D}^{(0)}_{00}(\omega_n,\omega_k)-\delta_{n0}\delta_{k0}\beta L\frac{4\pi^4F^2(T,\mu,L)}{e^2},$$

$$\mathbb{D}^{(\mu)}_{0i}(\omega_n,\omega_k)=\mathbb{D}^{(\mu)}_{i0}(\omega_n,\omega_k)=\mathbb{D}^{(0)}_{0i}(\omega_n,\omega_k),$$

$$\mathbb{D}^{(\mu)}_{ij}(\omega_n,\omega_k)=\mathbb{D}^{(0)}_{ij}(\omega_n,\omega_k), \tag{65}$$

$\mathbb{D}^{(0)}_{00}(\omega_n,\omega_k)$ being given by the expression between square brackets in Eq. (48). In the $L\to\infty$ limit, ω_k is replaced by a continuous variable ω and $L\,\delta_{k0}F^2$ goes to $(\mu^2/4\pi^2)\,\delta(\omega)$, in Eqs. (65). As a check of consistency, we notice that the propagator (65) (which is μ dependent) satisfies the gauge invariance condition $\omega^\mu\omega^\nu\,\mathbb{D}_{\mu\nu}=\alpha$, with α the gauge-fixing parameter. For finite L, the only Euclidean pole of the propagator (65) remains at $\omega^2=-m^2=-e^2/\pi$, which defines the mass of the boson and is independent of μ. However, in the $L\to\infty$ limit, a new pole appears at $\omega^2=0$ due to the contribution of $\delta(\omega)$ in the second (μ-dependent) term. To clarify what this pole means physically, let us calculate first the inverse correlation length squared M^2 in the $L\to\infty$ limit, through the usual definition

$$M^2=\mathcal{D}_{00}(\omega_n=0,\omega\to 0), \tag{66}$$

where $\mathcal{D}_{\alpha\beta}(\omega_n,\omega)$ is the boson self-energy, which is defined in momentum space through

$$\mathcal{D}_{\alpha\beta}(\omega_n,\omega)=[\mathbb{D}^{-1}]^{(\mu)}_{\alpha\beta}(\omega_n,\omega)-[D^{-1}]_{\alpha\beta}(\omega_n,\omega). \tag{67}$$

Accordingly, from Eq. (66) and the propagators in Eqs. (30) and (65), we obtain

$$M^2(T,\mu)=0. \tag{68}$$

That is, the mass M^2 vanishes in the $L\to\infty$ limit. This is indeed a consistent result if we recall the connection between the screening length and the equation of state of the system given in [16]:

$$M^2(T,\mu)=e^2\frac{\partial}{\partial\mu}\rho(T,\mu), \tag{69}$$

ρ being the total fermion charge density. Hence Eqs. (68) and (69) are consistent in the $L\to\infty$ limit: We got a total zero charge density, while M^2 tends to zero. However, were M^2 to be interpreted as a vanishing screening mass, then the system would be in a confined phase and the Z symmetry would be restored, which does not occur for $\mu=0$ and $T\neq 0$ in the $L\to\infty$ limit [10]. We shall outline below a correct understanding of the screening mass. The order parameter of this symmetry is the thermal average of the Polyakov loop

$$P_{\tilde{e}}(x^1)\equiv\exp\left[i\tilde{e}\int_0^\beta d\tau A_0(\tau,x^1)\right] \tag{70}$$

for an arbitrary charge $\tilde{e}$ and a spatial point x^1. In addition, the correlator of two Polyakov loops with charges $\tilde{e}$ and $-\tilde{e}$ and spatial points x^1 and u^1 measures, at large spatial separation, the effective potential between both charges. Thus, in order to clarify the role of the two different masses m and M, let us calculate the Polyakov loop and the correlator for $\mu\neq 0$ in the $L\to\infty$ limit. First, we shall use the functional methods developed in Sec. III. After the obvious relabelling of variables, the insertion of $P_{\tilde{e}}(x^1)$ in the functional integral (1) amounts formally to replacing $J(y)\to iP(y)$ with $P_0(y)=\tilde{e}\,\delta(y^1-x^1)$ and $P_1(y)=0$. Then, recalling our result (47), we get

$$\langle P_{\tilde{e}}(x^1)\rangle=\exp\left[-\frac{1}{2}\int d^2y\,d^2z\,\mathbb{D}^{(0)}_{00}(y-z)P_0(y)[P_0(z)-2G_0(z)]\right]=\exp\left[-\frac{\beta\tilde{e}^2}{2m}-\beta\mu\frac{\tilde{e}}{e}\right], \tag{71}$$

where in the last step we have first followed our convention of performing the time integration before the spatial one and then we have taken the limit $L\to\infty$, so as to transform spatial sums into integrals. The result (71) can also be obtained using the decomposition (16) and (20) for the gauge field. As we saw in previous sections, all the μ dependence is included in the h_0-dependent part. Then, since

$$\int_0^\beta d\tau A_0(\tau,x^1)=2\pi h_0-\partial_x^1\int_0^\beta d\tau\phi(\tau,x^1), \tag{72}$$

all we have to do is to insert a piece $\exp(2\pi i h_0\tilde{e}/e)$ in the integrand in Eq. (63), in the $L\to\infty$ limit. It is straightforward to get again the μ dependence of the Polyakov loop in Eq. (71) consistently, once we integrate over $h_0\in\mathbb{R}$.

We see that $\langle P_{\tilde{e}}(x^1)\rangle$ never vanishes for any value of T and μ, so that the Z symmetry is never restored for $L\to\infty$. Our result extends that obtained in [10] for $\mu=0$. Then, there should exist a nonvanishing screening mass, which, consistent with our result for the propagator, should be $m=e/\sqrt{\pi}$, independent of μ. This is confirmed by calculating the correlator of two Polyakov loops. Following the same steps as for $\langle P\rangle$ we obtain with both methods that $\langle P_{\tilde{e}}(x^1)P_{-\tilde{e}}(u^1)\rangle$ is independent of μ and

$$\lim_{x^1-u^1\to\infty}\langle P_{\tilde{e}}(x^1)P_{-\tilde{e}}(u^1)\rangle=\exp\left[-\frac{\beta\tilde{e}^2}{2m}\right]; \tag{73}$$

that is, m is the screening mass between the two opposite charges. It seems clear that the definition (66), leading to Eq. (68), does not give the right answer for the screening mass between two opposite charges and cannot be used to infer any conclusion about the breaking of the confinement or Z

symmetry. However, it is the consistent equation to use if we want to get the equation of state of the system through Eq. (69).

Notice that we have restricted ourselves to the trivial sector. The boson propagator could receive additional contributions from sectors with $k \geq 1$, apart from that in Eq. (64). However, we have seen that ρ only receives contributions from the trivial sector, so that, Eq. (68) is valid for any sector if Eq. (69) holds. In addition, it is enough to restrict ourselves to the trivial sector to calculate correlators of Polyakov loops [10]. Hence, we expect the above results for the screening mass to remain valid for $\Phi \neq 0$.

If the length of the system L is kept finite, the definition (66) for M^2 is no longer valid. Using Eq. (69) instead would give rise to a nonzero and μ-dependent M^2, directly from Eq. (53). Remember that for finite L there is no pole of the propagator at $\omega^2 = 0$. However, it is not clear whether Eq. (69) remains valid for finite L. On the other hand, for finite L and $\mu = 0$, the screening mass is only different from zero if $\tilde{e}/e \in \mathbb{Z}$ [10]. A more rigorous analysis of the finite length corrections for $\mu \neq 0$ is beyond the scope of this work.

V. THIRRING MODEL AT FINITE T AND μ

A. Generating functional

We shall consider a system of many massless fermions (and antifermions) in one dimension (inside a finite interval of length L) at equilibrium at absolute temperature T and chemical potential μ. By assumption, their dynamics is described by the Thirring quartic Lagrangian. Let $\xi, \bar{\xi}$ be fermionic external sources. Then, in the imaginary time formalism, the generating functional of the system reads now

$$Z[\xi,\bar{\xi}] = N(\beta,\mu) \int_{antiperiodic} \mathcal{D}\bar{\psi}\mathcal{D}\psi \exp\left[\int_T d^2x \left[-\bar{\psi}(\partial - \mu\gamma^0)\psi + \bar{\xi}\psi + \bar{\psi}\xi - \frac{g^2}{2}(\bar{\psi}\gamma^\nu\psi)(\bar{\psi}\gamma_\nu\psi) \right] \right], \tag{74}$$

where g is the coupling constant. The partition function is $Z[0,0]$. One can also cast $Z[\xi,\bar{\xi}]$ as follows:

$$Z[\xi,\bar{\xi}] = \exp\left[-g \int_T d^2x \frac{\delta}{\delta\xi(x)} \gamma_\nu \frac{\delta}{\delta J_\nu(x)} \frac{\delta}{\delta\bar{\xi}(x)} \right] \times Z_1[\xi,\bar{\xi};J]|_{J=0}, \tag{75}$$

$$Z_1[\xi,\bar{\xi};J] = NZ_F \exp\left[\frac{1}{2} \int_T d^2x' d^2y' J_\alpha(x') \times K^{\alpha\beta}(x'-y') J_\beta(y') \right] \times \exp\left[\int_T d^2x'' d^2y'' \bar{\xi}(x'') S(x'',y'';\mu) \xi(y'') \right], \tag{76}$$

$$K^{\alpha\beta}(x'-y') = \frac{1}{\beta L} \sum_{n,k=-\infty}^{+\infty} e^{i\omega(x'-y')} \left[\delta^{\alpha\beta} - f(\omega^2) \frac{\omega^\alpha \omega^\beta}{\omega^2} \right], \tag{77}$$

with S in Eqs. (30), Z_F the free fermionic partition function, and $J = (J_0, J_1)$ a boson source, to be set equal to zero after all functional differentiations with respect to it have been performed in Eq. (75) and ω in Eq. (77) is the same as in the gauge boson free propagator in Eqs. (30). We have introduced an arbitrary function $f(\omega^2)$, by virtue of the fact that the current $\bar{\psi}\gamma^\nu\psi$ in the actual Thirring model is conserved. A proof of Eqs. (75) and (76) follows readily through steps similar to those in [6,12]. At this stage, using standard techniques [12], one can rewrite Eqs. (75) and (76) as

$$Z[\xi,\bar{\xi}] = \exp\left[-\frac{1}{2} \int_T d^2x' d^2y' \frac{\delta}{\delta A^*_\alpha(x')} K^{\alpha\beta}(x'-y') \times \frac{\delta}{\delta A^*_\beta(y')} \right] Z_f[A^*_\nu,\xi,\bar{\xi}]|_{A^*=0}, \tag{78}$$

$$Z_f[A^*_\nu,\xi,\bar{\xi}] = NZ_F \times \exp\left[-ig \int_T d^2x \frac{\delta}{\delta\xi(x)} A^*_\nu(x) \frac{\delta}{\delta\bar{\xi}(x)} \right] \times \exp\left[\int_T d^2x'' d^2y'' \bar{\xi}(x'') S(x'',y'';\mu) \xi(y'') \right], \tag{79}$$

$$A^*_\nu(x) = -i \int_T d^2z K_{\nu\beta}(x-z) J^\beta(z). \tag{80}$$

Again, one sets $A^* = 0$ in the above equations, after having carried out all functional differentiations. Standard functional techniques allow us now to establish that $Z_f[A^*_\nu,\xi,\bar{\xi}]$, as given in Eq. (79), also coincides with the right-hand side of Eq. (2) [when due care is taken of the normalization factor $N(\beta,\mu)$] provided that, in the latter, one replaces e by g and A by A^*. Let us concentrate on A^* belonging to the trivial sector (see comments below). Then, by recalling the developments in Sec. III, one finds, immediately,

$$Z_f[A^*_\nu,\xi,\bar{\xi}] = Z_F \exp\left[-i \int_T d^2x \int_T d^2y \bar{\xi}(x) G(x,y,gA^*;\mu) \xi(y) + L[A^*] \right], \tag{81}$$

where $G(x,y,gA^*;\mu)$ and $L[A^*]$ are now given by the right-hand sides of Eqs. (39), (40), and (44) (with the same S, Δ, Π, and F), when one replaces e,A by g,A^*, respectively. Thus, we have provided the solution for the Thirring model at finite T,μ in terms of the fermionic generating

functional for the Schwinger model, without zero modes ($\Phi=0$). The formal use of Eqs. (75)–(77) and (78)–(80) (which can be checked upon comparing the corresponding expansions into powers of g) is justified if we restrict $Z_f[A^*_\nu,\xi,\overline{\xi}]$ to the trivial sector of the Schwinger model, as implemented through Eq. (81). In this regard, it is interesting to note that, upon using Eq. (80) in Eq. (5) and taking into account that the propagator $K_{\alpha\beta}$ in Eq. (77) satisfies periodic boundary conditions in both x^0 and x^1 directions, one immediately finds $\Phi[A^*]=0$, which confirms that $Z_f[A^*,\xi,\overline{\xi}]$ should be restricted to the trivial sector when use is made of Eqs. (78)–(80) and, hence, the consistency of Eq. (81). On the other hand, this appears also to be consistent with the idea that, in the end, we are going to set $A^*=0$ and then we can take the vector field A^*_ν as a configuration in the trivial sector, that is, topologically connected with $A^*=0$. It is unknown to us whether the Thirring model may have other solutions [besides that given in Eqs. (78) and (81)].

B. Thermodynamical partition function and fermion correlation function

The thermodynamical partition function becomes, upon applying Eq. (46) to Eqs. (78) and (81),

$$Z[0,0]=Z_F\exp\left\{\frac{1}{2}\,\mathrm{Tr}\,\log(1+\Pi K)^{-1}-\frac{1}{2}[2gF(T,\mu,L)]^2\times\int_T d^2x d^2y[K(1+\Pi K)^{-1}]^{00}(x-y)\right\},\tag{82}$$

where in momentum space we have

$$[(1+\Pi K)^{-1}]^{\alpha\beta}(x-y)=\frac{1}{\beta L}\sum_{n,k=-\infty}^{+\infty}e^{i\omega(x-y)}\left[\delta^{\alpha\beta}-\frac{g^2/\pi}{1+g^2/\pi}\left(\delta^{\alpha\beta}-\frac{\omega^\alpha\omega^\beta}{\omega^2}\right)\right],$$

$$[K(1+\Pi K)^{-1}]^{\alpha\beta}(x-y)=\frac{1}{\beta L}\sum_{n,k=-\infty}^{+\infty}e^{i\omega(x-y)}\left[\frac{\delta^{\alpha\beta}}{1+g^2/\pi}-\frac{\omega^\alpha\omega^\beta}{\omega^2}\left(f(\omega^2)-\frac{g^2/\pi}{1+g^2/\pi}\right)\right].\tag{83}$$

Using Eq. (54), Eq. (82) yields readily, for $L\to\infty$,

$$Z[0,0]=Z_F(T,\mu)\exp\left\{La\log\frac{1}{1+g^2/\pi}\right\}\exp\left\{-\beta L\frac{g^2\mu^2}{2\pi^2}\times\left[\frac{1}{1+g^2/\pi}-b\left(f(0)-\frac{g^2/\pi}{1+g^2/\pi}\right)\right]\right\},\tag{84}$$

where

$$a=\frac{1}{2}\sum_{n=-\infty}^{+\infty}\int_{-\infty}^{+\infty}\frac{dk}{2\pi},$$

$$b=\lim_{L\to\infty}\frac{1}{(\beta L)^2}\int_T d^2x d^2y\sum_{n,k=-\infty}^{+\infty}e^{i\omega(x-y)}\frac{\omega^0\omega^0}{\omega^2}.\tag{85}$$

Notice that the first exponential on the right-hand side of Eq. (84) is independent of β, μ and then it is irrelevant as far as the thermodynamics of the model is concerned. On the other hand, the constant b (which is independent of both T and μ) could in principle give rise to a dependence on $f(0)$. Again, the very definition of b is ambiguous. If we agree to evaluate the (imaginary time) integrals over x^0 and y^0 in Eqs. (85) before the spatial ones, then $b=0$ and, hence, the independence of the partition function of $f(\omega^2)$ follows, which is a welcome result. Another independent reason to favor this prescription is that it is the same as that leading from Eq. (49) to Eq. (50). Now, by recalling the expression for the free charge density in Eq. (52), we have from Eq. (84) that the total fermion number density of the system reads, in the $L\to\infty$ limit,

$$\rho=\frac{\mu}{\pi+g^2}.\tag{86}$$

Hence we obtain that the Thirring model at finite T and μ is no longer a free fermion gas, but the fermion density acquires a correction in g^2, as it stands in Eq. (86). It differs from the result in [13] in which only the free contribution to the fermion density remains. It is clear from our analysis starting from the Schwinger model that the correction to the free gas comes entirely from the topological contribution depending on the F function. This contribution only depends on the harmonic field h_0 in the decomposition of the gauge field. The calculation in [13] was done in real time formalism, in which this term is not present, whereas in [14] the model is solved in the torus. As it is emphasized in [7,8,14], the toroidal compactification is very useful to deal with infrared divergences and the harmonic parts of the gauge field are essential to correctly quantize the model. It is the most natural choice when using the imaginary time formalism, as we have done in this work. On the other hand, if we evaluate the pressure of the system, which follows directly by taking the logarithm of the partition function in Eq. (84), our result (not quoted for brevity) agrees with [14], which provides a check of consistency between our methods and those used in that work.

Finally, we shall give the exact fermion correlation function for the Thirring model at nonzero T and μ,

$$G(x,y)=\left.\frac{\delta^2\log Z[\xi,\overline{\xi}]}{\delta\overline{\xi}(x)\,\delta\xi(y)}\right|_{\xi=\overline{\xi}=0}=\Theta(x,y)S(x,y),$$

$$\Theta(x,y)=\exp\left\{-g^2\frac{1}{\beta L}\sum_{n,k=-\infty}^{+\infty}\frac{1}{\omega^2}[1-e^{i\omega(x-y)}]\times\left[f(\omega^2)-\frac{g^2/\pi}{1+g^2/\pi}\right]+2gF(T,\mu,L)c\right\},\tag{87}$$

where we have used again Eq. (46) into Eqs. (78) and (81) and performed the functional differentiation. In turn, c is given by the formal expression

$$c=\sum_{n,k=-\infty}^{+\infty}[e^{-i\omega x}-e^{-i\omega y}]\frac{\delta_{0,n}\delta_{0,k}}{\omega^2}\left[-\frac{\not{\omega}\gamma^0}{1+g^2/\pi}+\omega_0\left(f(\omega^2)-\frac{g^2/\pi}{1+g^2/\pi}\right)\right], \tag{88}$$

which, again, turns out to be ambiguous. Like we did with the same ambiguities before, let us evaluate the summation over n in Eq. (88) (which is reminiscent of imaginary time integrations) before the spatial summation and let $L\to\infty$. Then one gets

$$c=\frac{i}{1+g^2/\pi}(x_1-y_1)\gamma^1\gamma^0. \tag{89}$$

VI. CONCLUSIONS AND DISCUSSION

The main new results obtained in this work are the following.

(1) In the imaginary time formalism, the fermionic generating functional Z_f with an external electromagnetic field and the full generating functional Z for the Schwinger model have been explicitly obtained for any spatial length L in the trivial sector (in which the Dirac operator has no zero modes).

(2) The work previously done in [7] at finite T but $\mu=0$, in which the model was formulated in a two-dimensional torus for an arbitrary number of zero modes, can be extended when both T and μ are nonzero. Such an extension has to be worked out carefully due to some nontrivial peculiarities of the $\mu\neq 0$ case. Technically, the main distinctive feature is the lack of Hermiticity of the Dirac operator. This implies a nonvanishing phase factor $\mathcal{J}^{(k)}$ for the fermion determinant in the sector with k zero modes. Using functional methods we have evaluated this term for $k=0$ (the trivial sector), which plays a crucial role in the solutions for the Schwinger and Thirring models presented here and in the physical features thereof. That phase depends on T and μ, is linear in the zeroth component of the electromagnetic potential A_0 (in agreement with charge conjugation symmetry arguments), and vanishes if $\mu=0$ for any T,A_0. Furthermore, this term is topological, in the sense that it changes only under nontrivial gauge transformations, with nonzero winding number around the S^1 parametrizing the Euclidean time. In terms of the Hodge decomposition of the gauge field in the torus, it only depends on the harmonic part. The existence of topological μ-induced effective actions seems to be a common feature of different models [23,24].

(3) For the Schwinger model we have calculated the thermodynamical partition function. The topological phase factor in the effective action gives rise to a nonperturbative contribution that in the $L\to\infty$ limit exactly cancels the μ dependence contained in the free fermionic partition function. Then, at $L\to\infty$, the partition function is independent of μ and hence the total charge density of the system is zero. In other words, the system bosonizes even though it could have a net fermionic charge density at nonzero μ. The partition function factorizes, as in [6] (into that of free fermions at $\mu=0$ times a factor, which is the ratio of that of massive bosons, divided by that of free massless bosons), the mass of the boson being $m=e/\sqrt{\pi}$, independent of T and μ. The exact boson propagator has an additional μ-dependent piece for any L. At $L\to\infty$ this new piece gives rise to a vanishing inverse correlation length squared M^2, which is interpreted through the relationship between M^2 and the first derivative of the charge density given in [16]. However, by calculating for $\mu\neq 0$ the thermal average of the Polyakov loop (which is μ dependent) and its correlator (μ independent), we have shown that the Z symmetry is broken for any T and μ (deconfinement) and that the screening mass between two opposite charges is the mass m of the boson. A study of what happens regarding the above-mentioned cancellation in the thermodynamical partition function, when L is kept finite, lies outside the scope of this work. Our computations of the thermodynamical partition function through two different methods yielding the same result establish the consistency of our approach.

(4) Several important features of the solution of the Schwinger model for $\mu\neq 0$ in the sectors with zero modes are summarized, as they are closely related to the analysis in the trivial sector. Namely, we have given the general structure of the fermion determinant, solving the spectrum of the Dirac operator for an instanton configuration when $\mu\neq 0$. The chemical potential breaks the chiral degeneracy of the spectrum. We remark that the correlation functions for $\mu\neq 0$ have been analyzed in [9], although in that paper a different approach based on bosonization is used and the harmonic part of the gauge field (and hence the contribution of the phase factors) is not considered. The analysis of the phase factors and the fermionic two-point function when there are zero modes lies beyond the scope of this work.

(5) In the imaginary time formalism as well, the generating functional for the massless Thirring model at finite T,μ is constructed in terms of the fermionic generating functional Z_f for the Schwinger model, previously found in this work. We have justified that it is enough to restrict ourselves to the trivial sector for Z_f. The thermodynamical partition function, the total fermion number density, and the fermion correlation function have been computed for nonvanishing μ and T. A distinctive feature is that all of them depend nontrivially on μ, as a consequence of the nontrivial phase $\mathcal{J}^{(0)}$ of the Schwinger model. Our result for the pressure agrees with [14], which shows that our different approach is consistent. Our total fermion density differs from [13], where it was obtained, using real time formalism, that the model is equivalent to that of free fermions. The origin of that discrepancy is that in [13] the harmonic pieces of the vector field are not considered.

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APPENDIX: THE FERMION FUNCTIONAL DETERMINANT FOR $\Phi \neq 0$

We shall outline here the calculation of $\det' H(A;\mu)$ in Sec. II C, following steps similar to those in [7], with suitable generalizations for our present case. First, we shall relate $\det' H(A;\mu)$ with $\det' H(\widetilde{A};\mu)$. For that purpose, we define $\mathbb{D}_\alpha$ replacing $e \to e\alpha$ in the arguments of the two exponentials in Eq. (17). Then, the corresponding $H_\alpha = \mathbb{D}_\alpha^\dagger \mathbb{D}_\alpha$ interpolates between $H(A)$ and $H(\widetilde{A})$ when α varies from 0 to 1, and so on for $\overline{H}_\alpha = \mathbb{D}_\alpha \mathbb{D}_\alpha^\dagger$. The operator H_α can be cast as in Eqs. (18), but now with $A_\mu = \widetilde{A}_\mu - \alpha \epsilon_{\mu\nu}\partial_\nu \phi$, and the operator $\overline{H}_\alpha$ is obtained from H_α by changing $\mu \to -\mu$. By using ζ regularization [27], we have

$$\log \det' H_\alpha = -\frac{d}{ds}\zeta_H(s;\alpha)\bigg|_{s=0},$$

$$\zeta_H(s;\alpha) = \sum_{q=k+1}^{\infty} \mu_q^{-s}(\alpha), \tag{A1}$$

$\mu_q(\alpha)$ denoting, generically, the nonvanishing eigenvalues of H_α. As in Sec. II A, we choose the eigenstates of $\overline{H}_\alpha$ as $\varphi_q^{(\alpha)} = (\mathbb{D}_\alpha \phi_q^{(\alpha)})/\sqrt{\mu_q(\alpha)}$, where $\phi_q^{(\alpha)}$ are the eigenstates of H_α for $\mu_q(\alpha) \neq 0$. Now we use the Feynman-Hellmann formula $\dot{\mu}_q(\alpha) = (\phi_q^{(\alpha)}, \dot{H}_\alpha \phi_q^{(\alpha)})$, where the overdot indicates derivation with respect to α, and the Seeley–de Witt expansion [27] for H_α. Then, following similar steps as in [7] we can write the derivative of $\log \det' H_\alpha$ in Eqs. (A1) with respect to α, in terms of $E_\alpha = \widetilde{E} + \alpha \Delta\phi$, $\phi(x)$, and the zero modes $\phi_p^{(\alpha)}$ and $\varphi_p^{(\alpha)}$ of H_α and $\overline{H}_\alpha$. The latter are related to the zero modes of H and $\overline{H}$ simply by multiplying by $\exp(-e\alpha\gamma_5)$. Then, the integral in α can be done and we obtain

$$\det' H(A;\mu) = \det' H(\widetilde{A};\mu) \det\{N^{(1)}[N^{(0)}]^{-1}\} \times \det\{M^{(1)}[M^{(0)}]^{-1}\} \exp\left(\frac{2e^2}{\pi} \int_T d^2x \, \phi(x) \times \left[\widetilde{E} + \frac{1}{2}\Delta\phi(x) \right] \right), \tag{A2}$$

where the elements of the matrices $N^{(\alpha)}$ and $M^{(\alpha)}$ are $N^{(\alpha)}_{pp'} = \int_T d^2x \, \phi_p^{(\alpha)\dagger} \phi_{p'}^{(\alpha)}$ and $M^{(\alpha)}_{pp'} = \int_T d^2x \, \varphi_p^{(\alpha)\dagger} \varphi_{p'}^{(\alpha)}$.

Second, we are going to calculate the spectrum of $H(\widetilde{A};\mu)$ in Eq. (28), with the boundary conditions discussed in the text. Like in the $\mu = 0$ case [7], we shall try eigenfunctions bearing the form

$$\phi_{n,m}^{\pm} = e^{(2n+1)\pi i/\beta} e^{(i\overline{h}_1 \mp i\mu)x_1} \xi_{n,m}^{\pm}(x_1). \tag{A3}$$

By plugging Eq. (A3) into the eigenvalue equation, we arrive at an harmonic oscillator eigenvalue problem, which can be solved in the standard fashion. However, the functions $\phi^{\pm}$ in Eq. (A3) do not satisfy the right boundary conditions. It is not difficult to see that

$$\hat{\phi}_{n,m}^{\pm}(x^0,x^1) = \sum_{j=-\infty}^{\infty} e^{(2\pi i e h_1 \mp i\mu L)j} \phi_{n+jk',m}^{\pm}(x^0,x^1), \tag{A4}$$

with $k' \equiv n_+ - n_- = \Phi/2\pi$, are the correct eigenfunctions, which do satisfy the right boundary conditions in Eqs. (4). We have used that the $\xi(x_1)$ functions in Eq. (A3) depend on x^1 and n only through the combination $y = x^1 + 2\pi L(n + 1/2 - eh_0)/\Phi$.

The usual harmonic oscillator quantization condition for the $\xi_{n,m}^{\pm}$ states reads, in this case,

$$\frac{\lambda^{\pm} L\beta}{|\Phi|} \pm \mathrm{sgn}\Phi = 2m+1, \tag{A5}$$

m being an integer, with $m \geq 0$. Then, the eigenvalues for $\Phi \neq 0$ are independent of μ, and they are $\lambda = 0$ with degeneracy k, and $\lambda_m = 2m|\Phi|/L\beta$ with degeneracy $2k$. As we had anticipated, the zero modes appear with only one chirality, equal to the sign of Φ. The eigenfunctions are those in Eqs. (A4) and (A3) with

$$\xi_{n,m}^{\pm} = H_m\left[\sqrt{\frac{|\Phi|}{L\beta}} y \right] \exp\left[-\frac{|\Phi|}{2L\beta} y^2 \right], \tag{A6}$$

where H_m are the Hermite polynomials. Once we know the eigenvalues, we can calculate the determinant using again ζ-function regularization and we get

$$\det' H(\widetilde{A};\mu) = \exp\left[-\frac{d}{ds}\zeta_H(s)\bigg|_{s=0} \right] = \left(\frac{\pi L\beta}{|\Phi|} \right)^k, \tag{A7}$$

independent of μ. If we concentrate only on the zero modes, that is, the (n,m) eigenstates in Eq. (A4) with $m = 0$, it turns out that they are already orthogonal and that their norms are independent of μ:

$$||\phi_{n,0}||^2 = \left(\frac{\pi L\beta^3}{|\Phi|} \right)^{1/2}. \tag{A8}$$

As was commented in the text, once we know the spectrum of the Dirac operator for $\Phi \neq 0$ we could calculate the chiral condensates, which do not depend on the Green function $G(x,y,eA;\mu)$, up to the phase factor. In particular, we have

$$\langle \overline{\psi}(x) P_\pm \psi(x) \rangle_f = \exp[i\mathcal{J}^{(1)}(A;\mu)] \sqrt{\det' H(A;\mu)} \, \varphi_1^\dagger(x) \times P_\pm \phi_1(x), \tag{A9}$$

where the superscript 1 indicates the sector with only one zero mode.

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