Time dependent variational calculations for the quantum fluctuations of the $\lambda\phi^4$ model

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(Received 25 September 1997; published 18 February 1998)

We work out a time dependent variational approximation with an original approach. The results in one dimensional space in the symmetric and asymmetric phases of the potential at zero temperature are shown. We find analytical solutions to the equations of motion of the quantum fluctuations in the free case in one and three dimensional space for a special kind of initial condition, with special attention to the ultraviolet divergences. Quantum fluctuations are characterized by a renormalized mass which represents a sort of ''mean field.'' With the method we use it is possible to consider spatially dependent configurations for the quantum fluctuations and for the condensate and time dependent situations simulating a dynamical phase transition. To succeed the numerical temporal evolution from any initial condition, a generalized density matrix is defined which obeys a Liouville–von Neunmann–type equation. Its eigenvectors are numerically evolved on a lattice and several numerical solutions are exhibited. We also show the evolution of some other interesting variables with a timelike energy density and particle number of the system. [S0556-2821(98)04106-X]

PACS number(s): 11.10.Ef; 11.15.Tk

I. INTRODUCTION

The study of the time evolution of a system from a given initial condition in quantum field theory is a difficult task concerning very different problems. For bosonic systems there are several examples. In inflation models one has to consider a scalar field, in a given state in a potential, which evolves towards the vacuum of the system. During such time a negative pressure phase occurs and the Universe experiments an enormous expansion (in most of the models it is exponential) at the end of which the field oscillates and produces particles with subsequent reheating and then standard big bang theory proceeds. Quantum fluctuations of the bosonic field are considered as the origin of the background radiation fluctuations and inhomogeneities of energy density, which allowed the formation of large structures in the Universe [1–3]. An interesting task in this domain is the search for analytical solutions [4]. It has been observed a significant influence to the field dynamical evolution when one takes into account quantum fluctuations in a nonperturbative way [5–10]. At the hadronic level, in the context of heavy ions collisions, the evolution of a system of interacting particles (like a ''gas of pions'' which multiplicity is usually quite high in those experiments), given an initial condition, is relevant for the description of several observables [11,12]. Also, at atomic level, several dynamical effects may be studied, with a nonperturbative formalism.

The $\lambda\phi^4$ model has been extensively studied in order to develop suitable methods for more realistic theories like gauge theories [13,14] and chiral models. It is usually considered for inflation studies in spite of its intriguing characteristics like ''triviality'' [15,16].

In the static formalism one only has exact solutions for the free case and the most developed approach to solve the interacting one is perturbation theory which only works well for very small coupling constants as it occurs in electrodynamics. Also in this approach one has a systematic and direct way of dealing with divergences, i.e., one knows exactly how to renormalize. However, nonperturbative effects are important in many cases, like for spontaneous symmetry breaking and bound states. It is worth remembering that in time dependent situations the effective potential formalism is not appropriated [17].

The time dependent variational method has been well studied and can be viewed as an extension of the static case, which corresponds to a summation of ''cactus'' type diagrams for the energy [18]. It takes into account more nonlinearities than perturbation theory. This method allows to consider the problem of initial conditions in a direct and systematic way although several difficulties arise [19]. First, the minimization of the energy (action) for the static (time dependent) case makes too hard to obtain an improved (and thus more realistic) wave function. Also, one only usually knows how to do the calculation with Gaussian wave functionals, which is the exact solution to the free case. Any way it offers an alternative powerful approach and it has already been extensively studied with a trial Gaussian functional and it has shown good results compared to perturbation theory [7,21,9,10,20]. In particular, the time-dependent Gaussian approach has been applied to the quantum mechanical case in [7] where a comparison between the Gaussian approximation with exact solutions has been done, explaining its features and shortcomings in that case. The nonequilibrium quantum electrodynamics case (and also $\lambda\phi^4$) has been worked out using the closed path time method in [21].

The aim of this article is to work out a new method to carry out the temporal evolution of the quantum fluctuations of $\lambda\phi^4$ given an initial condition. This approach corresponds to the time dependent Hartree Bogoliubov (TDHB) method which is equivalent to the Gaussian approach and it becomes exact in the free case and in the limit of large N of a model invariant by transformations of the group $O(N)$. This limit is equivalent to the $N=1$ case with a scale transformation [6]. The article is presented as follows. In Sec. II we present the Gaussian approach and obtain the equations of motion in

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three dimensions, to be general. In Sec. III the analytical solution for the free case in one and three spatial dimensions are presented for an initial condition in the limit of small deviations from the vacuum of the theory. In Sec. IV the TDHB formalism is developed defining a generalized density matrix and the equations of motion entirely equivalent to the former ones are obtained. This section and the numerical results presented in the following one are the most relevant and original results of this paper. The system is placed in a one dimensional lattice and several cases of temporal evolution are performed in symmetric and asymmetric phases. Examples of other observables like particle number and energy density are presented. These results are shown in Sec. V and the conclusions are discussed in Sec. VI.

II. TIME DEPENDENT GAUSSIAN APPROXIMATION TO $\lambda\phi^4$ MODEL

The Lagrangian for a scalar field ϕ with bare mass m_0^2 and coupling constant b is

$$\mathcal{L}(\mathbf{x})=\frac{1}{2}\left\{\partial_\mu\phi(\mathbf{x})\partial^\mu\phi(\mathbf{x})-m_0^2\phi(\mathbf{x})^2-\frac{b}{12}\phi(\mathbf{x})^4\right\}. \quad (1)$$

The corresponding Hamiltonian reads

$$H=\frac{1}{2}\left(\pi^2(\mathbf{x})+(\nabla\phi)^2+m_0^2\phi^2(\mathbf{x})+\frac{b}{12}\phi^4(\mathbf{x})\right), \quad (2)$$

where the action of operators ϕ and π in functional Schrödinger representation over wave functional $\Psi[\phi(\mathbf{x})]=\langle\phi(\mathbf{x})|\Psi[\phi]\rangle$ is

$$\begin{aligned}\hat{\phi}|\Psi[\phi(\mathbf{x})]\rangle&=\phi(\mathbf{x})|\Psi[\phi(\mathbf{x})]\rangle,\\ \hat{\pi}|\Psi[\phi(\mathbf{x})]\rangle&=-i\delta/\delta\phi(\mathbf{x})|\Psi[\phi(\mathbf{x})]\rangle.\end{aligned} \quad (3)$$

For variational calculations in the Schrödinger picture the wave functional evolves like the Schrödinger equation

$$i\frac{\partial}{\partial t}\Psi[\phi(\mathbf{x})]=H\Psi[\phi(\mathbf{x})]. \quad (4)$$

This is, thus, a noncovariant formalism suitable for time dependent problems.

In the Gaussian approximation at zero temperature Ψ is parametrized by

$$\Psi[\phi(\mathbf{x})]=N\exp\left\{-\frac{1}{4}\int d\mathbf{x}d\mathbf{y}\delta\phi(\mathbf{x})(G^{-1}(\mathbf{x},\mathbf{y})+i\Sigma(\mathbf{x},\mathbf{y}))\delta\phi(\mathbf{y})+i\int d\mathbf{x}\overline{\pi}(\mathbf{x})\delta\phi(\mathbf{x})\right\}, \quad (5)$$

where $\delta\phi(\mathbf{x},t)=\phi(\mathbf{x})-\overline{\phi}(\mathbf{x},t)$; the normalization is N, the variational parameters are the condensate $\overline{\phi}(\mathbf{x},t)=\langle\Psi|\phi|\Psi\rangle$ and its conjugated variable $\overline{\pi}(\mathbf{x},t)=\langle\Psi|\pi|\Psi\rangle$; quantum fluctuations represented by the width of the Gaussian $G(\mathbf{x},\mathbf{y},t)=\langle\Psi|\phi(\mathbf{x})\phi(\mathbf{y})|\Psi\rangle$ and its conjugate variable $\Sigma(\mathbf{x},\mathbf{y},t)$.

In variational time dependent calculations we have to choose an action to be minimized in order to obtain the equations of motion. We take the well known action from Jackiw and Kerman [22]

$$I=\int dt\langle\Psi|i\frac{\partial}{\partial t}-\hat{H}|\Psi\rangle. \quad (6)$$

In order to calculate it, we take the mean value of an operator $\hat{O}$ given by

$$\langle\Psi|\hat{O}|\Psi\rangle=\int\mathcal{D}[\phi]\Psi^*\hat{O}\Psi. \quad (7)$$

The mean value of the action after the Gaussian integrations with the trial functional (5) is

$$\begin{aligned}I=\int dt\Bigg\{&\int d^3y\dot{\Sigma}(\mathbf{x},\mathbf{y})G(\mathbf{y},\mathbf{z})+\overline{\phi}(\mathbf{x})\dot{\overline{\pi}}(\mathbf{x})-\frac{1}{8}G^{-1}(\mathbf{x},\mathbf{x})+\\&-2\int d^3yd^3z\Sigma(\mathbf{x},\mathbf{y})G(\mathbf{y},\mathbf{z})\Sigma(\mathbf{z},\mathbf{x})+\\&-\frac{1}{2}[\Delta G(\mathbf{x},\mathbf{x})+(\nabla\overline{\phi}(\mathbf{x}))^2+m_0^2G(\mathbf{x},\mathbf{x})+m_0^2\overline{\phi^2}(\mathbf{x})+]\\&-\frac{b}{8}\left(\int d^3yG(\mathbf{x},\mathbf{y})G(\mathbf{y},\mathbf{x})+\frac{b}{3}\overline{\phi^4}(\mathbf{x})+2\overline{\phi^2}(\mathbf{x})G(\mathbf{x},\mathbf{x})\right)\Bigg\}.\end{aligned} \quad (8)$$

Variations with respect to the variational parameters and their conjugate yield the following equations of motion (repeated spatial index means integration over that variable):

$$\begin{aligned}&\frac{\delta I}{\delta\Sigma(\mathbf{x},\mathbf{y})}\rightarrow\partial_tG(\mathbf{x},\mathbf{y})=2[G(\mathbf{x},\mathbf{z})\Sigma(\mathbf{z},\mathbf{y})+\Sigma(\mathbf{x},\mathbf{z})G(\mathbf{z},\mathbf{y})],\\&\frac{\delta I}{\delta G(\mathbf{x},\mathbf{y})}\rightarrow\partial_t\Sigma(\mathbf{x},\mathbf{y})=\left(2\Sigma(\mathbf{x},\mathbf{z})\Sigma(\mathbf{z},\mathbf{y})-\frac{1}{8}G^{-1}(\mathbf{x},\mathbf{z})G^{-1}(\mathbf{z},\mathbf{y})\right)+\left(\frac{\Gamma(\mathbf{x},\mathbf{y})}{2}+\frac{b}{2}\overline{\phi}(\mathbf{x})^2\right),\\&\frac{\delta I}{\delta\overline{\pi}(\mathbf{x})}\rightarrow\partial_t\overline{\phi}(\mathbf{x})=-\overline{\pi}(\mathbf{x}),\\&\frac{\delta I}{\delta\overline{\phi}(\mathbf{x})}\rightarrow\partial_t\overline{\pi}(\mathbf{x})=\Gamma(\mathbf{x},\mathbf{y})\overline{\phi}(\mathbf{y})+\frac{b}{6}\overline{\phi^2}(\mathbf{x}),\end{aligned} \quad (9)$$

where $\Gamma(\mathbf{x},\mathbf{y})=-\Delta+[m_0^2+(b/2)G(\mathbf{x},\mathbf{x})]\delta(\mathbf{x}-\mathbf{y})$. In this approximation the interaction term $b\phi^4$ becomes quadratic: i.e., it contributes to a self consistent mass, as we will see explicitly.

For a translational invariant system, we perform the Fourier transformation

$$\overline{\phi}(\mathbf{x},t)=\frac{1}{(2\pi)^3}\int d\mathbf{k}\overline{\phi}(k,t)\exp(i\mathbf{k}(\mathbf{x})),$$

$$\overline{\pi}(\mathbf{x},t)=\frac{1}{(2\pi)^3}\int d\mathbf{k}\overline{\pi}(k,t)\exp(i\mathbf{k}(\mathbf{x})),$$

$$G(\mathbf{x},\mathbf{y})=\frac{1}{(2\pi)^3}\int d\mathbf{k}G(k,k')\exp(i\mathbf{k}\cdot\mathbf{x}-i\mathbf{k}'\cdot\mathbf{y}),\tag{10}$$

$$\Sigma(\mathbf{x},\mathbf{y})=\frac{1}{(2\pi)^3}\int d\mathbf{k}\Sigma(k,k')\exp(i\mathbf{k}\cdot\mathbf{x}-i\mathbf{k}'\cdot\mathbf{y}).$$

We can then eliminate the variables $\overline{\pi}$ and Σ. The equations in the asymmetric phase become

$$\ddot{G}_{kk'}-\frac{\dot{G}^2_{kk'}}{2}G^{-1}_{kk'}-\frac{1}{2}G^{-1}_{kk'}+2\left(k^2+m_0^2+\frac{b}{2}G(x,x)+\frac{b}{2}\overline{\phi}\right)G_{kk'}=0,$$

$$\ddot{\overline{\phi}}_k+\left(k^2+m_0^2+\frac{b}{6}\overline{\phi}^2+\frac{b}{2}G(x,x)\right)\overline{\phi}_k=0,\tag{11}$$

with $G_{\mathbf{kk}'}=\langle\mathbf{k}|G(\mathbf{x},\mathbf{x})|\mathbf{k}+\mathbf{q}\rangle$. These equations were generalized for the out of equilibrium (nonzero temperature) using different methods in [21,20,10].

For the Gaussian ansatz we use, in the symmetric phase, ($\overline{\phi}=0$), one needs only two initial conditions for the temporal evolution of these equations, $G(t=0)$ and $\dot{G}(t=0)$, which is proportional to its imaginary part: Σ. The evolution of $\overline{\phi}$ only occurs for a nonzero initial condition $\overline{\phi}$, since the two phases of the potential are, in principle at this level of knowledge, disconnected.

In Minkowski space there is no exact analytical solution but some numerical works have been done. The choice of initial conditions is entirely subordinate to the approximation, in the sense that were it not Gaussian one could have to consider three conditions instead of two [6]. The analysis of these equations in [9,10] show that initial conditions (for homogeneous G and $\overline{\phi}$) are crucial for the time interval in which the system evolves towards the minimum and for the speed of the field evolution. Our approach, however, is different as we will see in Sec. IV.

A. Static case considerations

The vacuum state is obtained when $\Sigma=\dot{G}=\dot{\overline{\phi}}=0$, and, in function of the renormalized mass at the minimum, μ^2, the fluctuations are described by

$$G_0(\mathbf{x},\mathbf{y})=\langle\mathbf{x}|\frac{1}{\sqrt{-\Delta+\mu^2}}|\mathbf{y}\rangle,\tag{12}$$

where μ^2 in the vacuum is given by the self consistent GAP equation

$$\mu^2=m_0^2+\frac{b}{2}\mathrm{Tr}G_0(x,x)+\frac{b}{2}\overline{\phi}^2.\tag{13}$$

This equation defines the symmetric phase, with $\overline{\phi}=0$, and asymmetric one with $\overline{\phi}^2=-6m_0^2/b-3\langle\mathbf{x}|G|\mathbf{x}\rangle$. Part of these solutions have been studied in three dimensions in [16], as well as the renormalization procedure. We see that given the ''renormalized mass,'' μ^2, and the coupling constant, b, the bare mass is defined by the vacuum of the chosen phase.

In one spatial dimension, which corresponds to the case we will analyze numerically, the GAP equation can be made finite by renormalizing the mass. We must eliminate the divergence of the kernel $G_0(\mathbf{x},\mathbf{y};\mu^2)$ by fixing a mass scale, $\overline{\mu^2}$, from $G_1(\mathbf{x},\mathbf{y};\overline{\mu^2})$. This yields the same result of the renormalization of the equations of movement and it results in the following GAP equation:

$$\mu^2=\overline{\mu^2}(\overline{\phi}=0)+\frac{b}{2}\overline{\phi}^2-\frac{3b}{4}\log\left(\frac{\mu^2}{\overline{\mu^2}}\right).\tag{14}$$

As we will see, any deviation of the physical mass from μ^2 introduces dynamical effects. More details on renormalization in one or three dimensional spaces with different prescriptions can be found in [23,6,16,15].

III. SMALL DEVIATIONS AROUND THE SYMMETRIC MINIMUM

In the symmetric phase, we will consider small deviations around the minimum by linearizing the equations of motion. The initial condition is thus given by $G(x,t=0)=G_0+\delta G(x,t=0)$, where G_0 is the value in the vacuum, and δG is the deviation from the minimum. In momentum space keeping only the linear terms in δG of (11) we obtain

$$\delta\ddot{G}_{kk'}=-\delta G_{kk'}(\sqrt{k^2+\mu^2}+\sqrt{(k+q)^2+\mu^2})^2-\frac{b}{4}\left(\frac{1}{\sqrt{k^2+\mu^2}}+\frac{1}{\sqrt{(k+q)^2+\mu^2}}\right)\sum_p \delta G_{pp+q}.\tag{15}$$

One is left to the problem analogous to that of an infinite number of coupled harmonic oscillators. We will consider a plane wave prescription to δG in the homogeneous case ($q=0$)

$$\delta G_k(t)=\delta G_k(t=0)e^{-i\Omega t}.\tag{16}$$

It is important to remark that we are dealing with nonrenormalized equations of motion and this may make possible the determination of time dependent divergences.

The free case solution for Eq. (15) in d dimensions is

$$\delta G(\mathbf{x},\mathbf{y};t)=-\int\frac{d^dk}{(2\pi)^d}\delta G_{k,k'}(t=0)\cos[M(k,q)t],\tag{17}$$

where $M(k,q)=(\sqrt{k^2+\mu^2}+\sqrt{(k+q)^2+\mu^2})$.

Since the trial form of a Gaussian for the system remains valid during the temporal evolution we associate the devia-

tion from the minimum of the initial condition (G_0) to the small change of the mass, as Eq. (12) suggests. In this uniform case, $q=0$, we consider $\delta m^2=m^2-\mu^2$ and then $\delta G(t=0)=(\mu^2-m^2)/2(\sqrt{k^2+\mu^2})^{3/2}$. The explicit solution for the time evolution in d dimensions reads

$$\delta G(t)=(\mu^2-m^2)\int \frac{d^dk}{(2\pi)^d}\frac{\cos(2\sqrt{k^2+\mu^2}t)}{4(k^2+\mu^2)^{3/2}}. \quad (18)$$

We will refer to these solutions (18) as semianalytical solutions and they will be useful to verify the numerical calculation. For a space of dimension $d=3$ dimensions, Eq. (18) keeps an ultraviolet divergence.

When $t=0$ the integral is very easy and in one dimensional space it is given by

$$\delta G(t=0)=\frac{\mu^2-m^2}{8\pi\mu^2}. \quad (19)$$

In one spatial dimension, the integral (18), at any t, with a change of variables, ($y^2=k^2+\mu^2$) can be written as

$$\delta G(t)=(\mu^2-m^2)\int_\mu^\infty \frac{dy}{(4\pi)}\frac{\cos(2yt)}{y^2(y^2-\mu^2)^{1/2}}. \quad (20)$$

We notice from Eq. (18) that there is no ultraviolet divergences and we find as a solution the following expression [24]:

$$\langle \mathbf{x}|\,\delta G|\mathbf{x}\rangle=\frac{\mu^2-m^2}{8\pi}\left[\frac{1}{2}\mu^{-2}B\left(\frac{1}{2},1\right)\,{}_1F_2\left(-\frac{1}{4};0,\frac{1}{2};-\mu^2t^2\right)+\frac{\sqrt{\pi}}{2}t^2\Gamma\left(\frac{3}{2}\right)\,{}_1F_2\left(\frac{1}{2};2,\frac{3}{2};-\mu^2t^2\right)\right], \quad (21)$$

where F are the generalized hypergeometric functions [24]; $\Gamma(b)$ are gamma functions and $B(a,b)$ is the beta function

$$\Gamma(a)=\int_0^\infty t^{z-1}e^{-t}dt,$$
$$B(a,b)=\int_0^1 t^{a-1}(1-t)^{b-1}dt. \quad (22)$$

The hypergeometric functions can be expanded in series [24]:

$${}_1F_2(a;b,c;z)=\sum_{k=0}^\infty \frac{(a)_k}{(b)_k(c)_k}\frac{z^k}{k!}, \quad (23)$$

where $(a)_k=\Gamma(a+k)/\Gamma(a)$. The important thing here is to know that this series converges for $|z|<\infty$ [24], since the numerical integration of expression (20) offers a very safe way to obtain $\delta G(t)$.

Although in this paper we are not analyzing the case of three-dimensional space we will show the solution of Eq. (18) because it has an interesting feature. In three dimensions the solution of (18) with an integration by parts can be written as

$$\langle \mathbf{x}|\,\delta G|\mathbf{x}\rangle=-\frac{m^2-\mu^2}{8\pi^2}\left(\frac{\pi}{2}N_0(2\mu t)+\mu^2\int_\mu^\infty dy\frac{\cos(2yt)}{y^2(y^2-\mu^2)^{1/2}}\right), \quad (24)$$

where N_0 is the zero order Bessel function and the second term corresponds to the one dimensional integral case unless for a multiplicative constant. For $t=0$ we see one divergence in $N_0(t=0)$ which might be regularized and it is related the initial condition or static case, it is absent in one dimensional space. A detailed study of the time dependent case in 3 spatial dimensions will be shown elsewhere.

IV. NUMERICAL METHOD: TDHB IN SYMMETRIC PHASE

From here on we change sometimes the notation in order to place the system on a lattice, so, instead of continuous space ($\mathbf{x},\mathbf{y}$) we may have discrete parameters (i,j).

Gaussian approximation corresponds to a mean field approximation where the two or more field coupling terms are decomposed in terms of two point Green's function $\langle \mathbf{x}|G|\mathbf{y}\rangle$, and this is manifested in the self consistent mass (13). This kind of approach can be established in terms of a generalized density matrix R in the frame of TD Hartree Bogoliubov formalism. This will be done in the following in order to perform numerical calculations.

One defines the R matrix as [25]

$$R(x,y)=\begin{pmatrix}\rho(x,y) & \kappa(x,y)\\ -\kappa^*(x,y) & -\rho^*(x,y)\end{pmatrix}, \quad (25)$$

where $\rho(\mathbf{x},\mathbf{y})=\frac{1}{2}\langle a(\mathbf{x})a^\dagger(\mathbf{y})+a^\dagger(\mathbf{y})a(\mathbf{x})\rangle$, is Hermitian and $\kappa(\mathbf{x},\mathbf{y})=-\langle a(\mathbf{x})a(\mathbf{y})\rangle$ is symmetric; they are respectively the one and two particle density matrices. These two quantities are directly related to the variational parameters G and Σ by the creation and annihilation operators, which, written for a lattice of d with mesh size Δx, are

$$a(j)=\frac{1}{\sqrt{2}}\{\phi(j)(\Delta x)^{(d-1)/2}+i\pi(j)(\Delta x)^{(1+d)/2}\}, \quad (26)$$

$$a^\dagger(j)=\frac{1}{\sqrt{2}}\{\phi(j)(\Delta x)^{(d-1)/2}-i\pi(j)(\Delta x)^{(1+d)/2}\}. \quad (27)$$

The mean values of the fields allow one to relate, in the symmetric phase, to all these variables as

$$G(i,j)=\langle\phi(i)\phi(j)\rangle=\frac{1}{(\Delta x)^{d-1}}\mathrm{Re}(\rho(i,j)-\kappa(i,j)),$$
$$F(i,j)=\langle\Pi(i)\Pi(j)\rangle=\frac{1}{(\Delta x)^{d+1}}\mathrm{Re}(\rho(i,j)+\kappa(i,j)), \quad (28)$$

where $F(i,j)=G(i,j)^{-1}/4+4\Sigma(i,k)G(k,l)\Sigma(l,i)$.

The density matrix $R(\mathbf{x},\mathbf{y})$ at zero temperature is composed by pure states and has a sympletic structure which will be conserved in the temporal evolution. This basically means that $\tau R\tau = R^t$ (R^t is the transposed of R), where we used the ''sympletic metric''

$$\tau = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{29}$$

In the case of pure states the following relation is also satisfied:

$$R^2 = 1. \tag{30}$$

This relation guarantees a number of conserved quantities (finite on the lattice) at each time step in the temporal evolution, since $(d/dt)R^2 = 0$.

The variation of the total energy, $E = \int d^d z \langle H(\mathbf{z})\rangle$, produces the Hartree-Bogoliubov energy $H(\mathbf{x},\mathbf{y})$ which is defined by [25]

$$\frac{1}{2} H_{ij}(\mathbf{x},\mathbf{y}) = \frac{\delta E}{\delta R_{ji}(\mathbf{y},\mathbf{x})}. \tag{31}$$

On a Cartesian lattice of d dimensions with mesh size Δx, we obtain

$$H_{i,j} = \begin{pmatrix} W_{i,j} & D_{i,j} \\ -D_{i,j} & -W_{i,j} \end{pmatrix}, \tag{32}$$

where we have defined the matrices

$$\begin{aligned} W_{ij} &= \frac{1}{2}(\Gamma'_{ij}(\Delta x)^{1-d} + (\Delta x)^{-1-d}\delta_{ij}), \\ D_{ij} &= \frac{1}{2}(\Gamma'_{ij}(\Delta x)^{1-d} - (\Delta x)^{-1-d}\delta_{ij}), \end{aligned} \tag{33}$$

where $\Gamma'_{ij} = -\Delta_{ij} + (m_0^2 + bG_{ii}/2)\delta_{ij} = -\Delta_{ij} + m^2\delta_{ij}$.

With these quantities we can write the equations of motion under the form of Liouville–von Neunmann *entirely equivalent to the equations* (9) in the symmetric phase, respecting expression (30):

$$i\dot{R}_{ij} = [H_{ik}, R_{kj}]. \tag{34}$$

This complete equivalence was verified explicitly by substituting $H_{i,j}$ and $R_{i.j}$ written in terms of $G_{i,j}$ and $\Sigma_{i,j}$ in the above equation.

Consequently we can consider the time evolution of the system by diagonalizing the generalized density matrix R and performing the temporal evolution of its eigenvectors (u_n and v_n), which have the sympletic normalization, in the lattice: $\Sigma_i \lambda_n \{|u_n(i)|^2 - |v_n(i)|^2\} = 1$, consistent with expression (30).

A new evolution equation can now be written as

$$i\partial_t \begin{pmatrix} u_n(i) \\ v_n(i) \end{pmatrix} = H(i,j) \begin{pmatrix} u_n(j) \\ v_n(j) \end{pmatrix}. \tag{35}$$

Thus, with initial conditions for G and Σ we diagonalize the density matrix $R(t=0)$, whose eigenvectors (u_n and v_n) are evolved in time and related to ρ and κ by

$$\begin{aligned} \rho_{ij} &= \sum_{n>0} \lambda_n (u_n(i)u_n^*(j) + v_n^*(i)v_n^*(j)), \\ \kappa_{ij} &= -\sum_{n>0} \lambda_n (u_n(i)u_n^*(j) + v_n^*(i)v_n^*(j)). \end{aligned} \tag{36}$$

These variables may be expressed in terms of G and F at each time step by Eq. (28) and so on.

Since H depends on the evolved eigenvectors we utilize a self consistent method which require interactions in each time step. We have verified the independence of the evolution sense with the inverse evolution operation $t \to -t$, as well as the independence of the results with relation to a variation of the time step Δt.

Now we are lead, for example, in the symmetric phase ($\bar{\phi} = 0$) to give the initial conditions. We choose one configuration for the fluctuation G at $t=0$, and its ''velocity'' (which will be taken to be zero along this paper, $\Sigma \sim \dot{G} = 0$). As we have seen the two point Green's function has a ''natural parametrization'' corresponding to its static form

$$G(x,y,t) = \langle x | \frac{1}{\sqrt{-\Delta + m^2}} | y \rangle \quad \text{with} \quad \Sigma(x,y,t=0) = 0. \tag{37}$$

If we put $m^2 = \mu^2$ we get the vacuum. The renormalized mass becomes thus the ''new'' variational parameter $m^2 = m(t)^2$, and it is possible to consider several spatial and temporal different configurations $m^2 = m^2(\mathbf{x},t)$.[1] The bare mass m_0^2 (for both phases) in the energy equation (32), is given by the GAP equation (13), determining the required vacuum state. However, these initial conditions correspond to the particular case of the Gaussian approximation: it is possible to take more general prescriptions to which one needs to consider three conditions [6].

Thus there are two free parameters: the physical mass μ^2 and the coupling constant. In 3 spatial dimensions b is constrained by the running coupling constant equation [16]. If we take into account some inflation models and primordial energy density inhomogeneities in 3 dimensions its values (b) for most of the inflation models are severely constrained to the order of 10^{-10} [2]. This makes the field to keep nearly the free case dynamics. In lattice calculation, in three spatial dimensions, the value of $b=6$ is usually adopted.

An interesting variable to regard is the energy density since we will examine a nonhomogeneous configuration. On the lattice, the energy density [the mean value of (17) in the symmetric case] can be written as

[1] In fact, a description only dependent on m and $\dot{m}$ may be considered with the elimination of the other degrees of freedom by integration in momentum space $\mathbf{k}$ of movement equations and effective action.

$$\langle \mathcal{H} \rangle (i,i) = \frac{1}{2\Delta^2 x} \mathrm{Re}(\rho(i,i) + \kappa(i,i)) + \frac{1}{2} \sum_j \overline{\Gamma}(j,i)\, \mathrm{Re}(\rho(j,i) - \kappa(j,i)) \frac{1}{\Delta x}, \quad (38)$$

where $\overline{\Gamma}(j,i) = -\Delta_{j,i} + (m_0^2 + bG_{j,i}/4)\delta_{j,i}$. This expression allows us to plot the energy density in the lattice and its trace gives the total energy which is conserved in all cases showed below.

Another useful variable is the particle number $[\hat{N}(i,i) = a^{\dagger}(i)a(i)]$, whose mean value in a general lattice of dimension (d), in the symmetric case, is given by

$$\langle N \rangle (i,i) = \frac{1}{2} G(i,i)(\Delta x)^{d-1} + \frac{1}{2} F(i,i)(\Delta x)^{d+1} + [\Sigma(i,j)G(j,i) - G(i,j)\Sigma(j,i)](\Delta x)^d - \frac{1}{2}\delta^d_{i,j}(\Delta x)^d. \quad (39)$$

This expression can be rewritten in terms only of the $\rho(i,j)$ variable as

$$\langle N \rangle (i,i) = \mathrm{Re}(\rho(i,i)) - \mathrm{Im}(\rho(i,i)) - \frac{1}{2}\delta^d(i,j)(\Delta x)^d. \quad (40)$$

V. RESULTS ON A LATTICE OF 1 + 1 DIMENSIONS

In this section we show some numerical results on a lattice of one spatial dimension of 5 fm with a mesh size of 0.1 fm. We have considered between 6 and 14 interactions in each time step for the energy H and eigenvectors determination, self consistently with a time step, chosen in such a way to guarantee the stability of the evolution, given by $\Delta t = 0.03$ fm. The physical mass has been chosen as $\mu = 100$ MeV. We have considered periodic boundary conditions for the Green function $G(i,j)$.

For an initial condition in the vacuum $m^2 = \mu^2$ and $\Sigma = 0$, the system remains in such state, as expected, with conserved total energy, independently of time step value. The spatial configuration of $G(i,i)$ is shown in Fig. 1, remembering that it is a symmetric and diagonal matrix.

Then we consider an homogeneous initial condition corresponding to a small deviation of the mass from the vacuum value: $m(t=0) = 0.9\mu$ (with $\mu = 100$ MeV), in the free case ($b=0$). In Fig. 2 we show the temporal evolution of the deviation of the width with relation to its vacuum value, $\delta G(i,i;t) = G(i,i;t) - G_0(i,i)$. This numerical solution (continuous line) is compared to the semianalytical solution of Eq. (18). The system oscillates around the state of minimum indefinitely since there is no dissipation. Total energy is conserved and results are independent of the considered values of the mesh size, as well as in all following cases.

The cases for the coupling constant equal to $b = 6$ fm^{-2} and 30 fm^{-2} are considered for $\delta G(t)$ and exhibited respectively in Figs. 3 and 4, with an initial condition equal to that

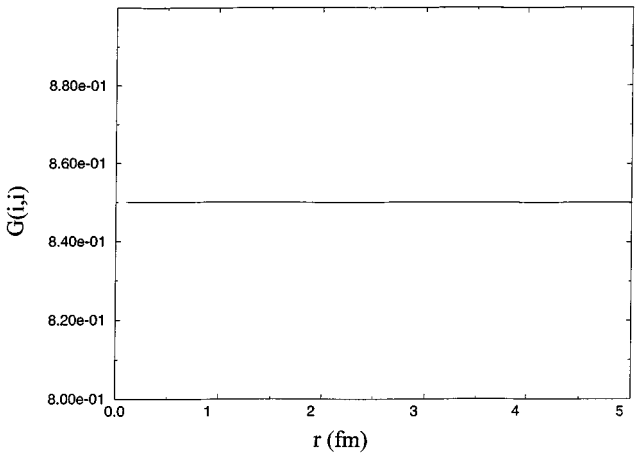

FIG. 1. Spatial configuration for $G(i,i)$, with $m = 100$ MeV, with relation to the points of the lattice.

of Fig. 2. Frequency oscillation changes a lot depending on the coupling constant but the amplitude depends exclusively on the initial condition. The particle number corresponding to the case of Fig. 2 (for $b = 6$ fm^{-2}) is shown in Fig. 5. We observe that this variable is not conserved and there are two aspects to be considered here. First, as it can be seen by the figure, the number of particles at the beginning and at the end of the temporal evolution are not exact integer numbers. This shows that the considered states are not eigenstates of this quantum number, whose symmetry is broken by the approximation. On the other hand it is obvious from the expression (39) that the number of particles at zero temperature depends strongly on the relevant variables of the system, namely G and Σ, in such a way to vary with temporal evolution. However an initial condition with a definite particle number can be chosen, and moreover, the t-dependent eigenstates can be projected into states with definite particle number [26]. This procedure would be of great interest when considering cases with internal symmetries gauge or chiral theories, given that as we see the variational

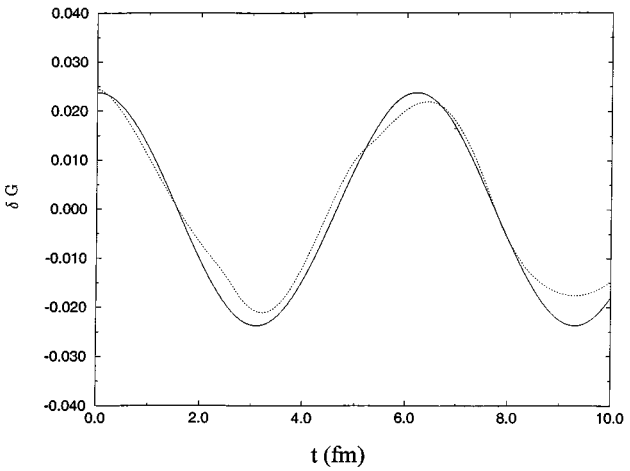

FIG. 2. Deviation with relation to the vacuum state $\delta G(t)$ in function of time for the numerical solution (solid line) and the semianalytical solution (dotted line) in the free case with an initial condition given by $m(t=0) = 0.9\mu$ and the physical mass $\mu = 100$ MeV.

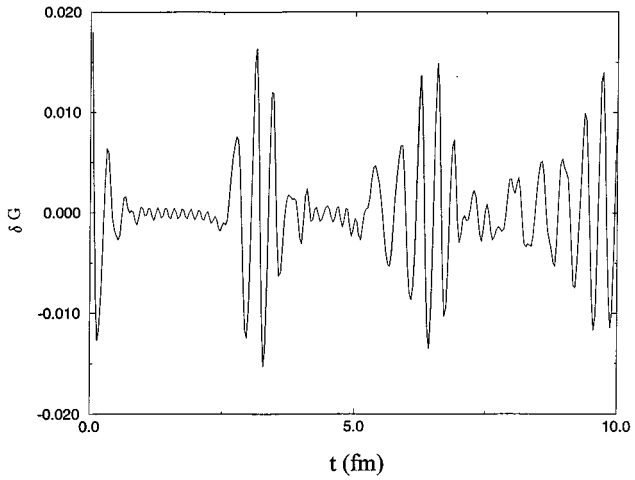

FIG. 3. Deviation $\delta G(t)$ in function of time for an initial condition $m=0.9\mu$, mass $\mu=100$ MeV and coupling $b=6$ fm^{-2}.

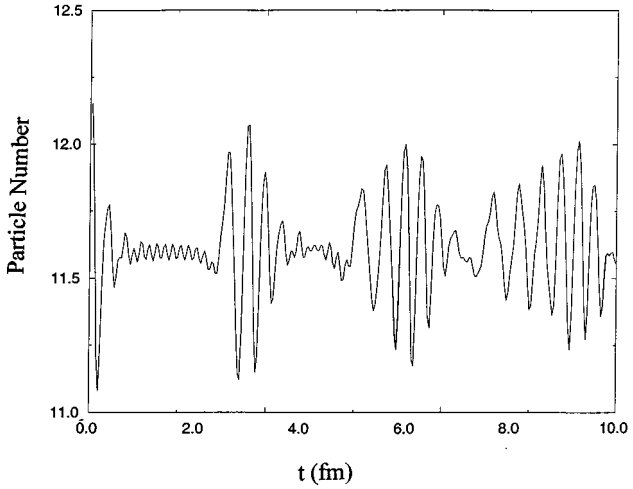

FIG. 5. Particle number as defined in the text for the case of the conditions shown in Fig. 2.

approach may break some symmetry of the system, as it actually does [13].

Another case of great general interest as initial condition is the bubble configuration as shown in Fig. 6. It is a zero mass bubble measured by the size $A=5$ fm centered at the lattice point, $x_c=2.5$ fm, inside the solution of symmetric vacuum as the expression:

$$m^2=\mu^2\tanh\left(\frac{x_i-x_c}{A}\right). \tag{41}$$

The configurations of the function $\langle x|G|x\rangle$ and the energy density respectively for the times $t=0,0.33,3.3,5.0,5.7$ fm and $t=0,3.3,5.0,10$ fm are shown in Figs. 7 and 8. The evolution of the mean value on the lattice $GM=\Sigma_i^N G(i)/N$ and the value of one point of the lattice $G(25)$ is shown in Fig. 9. The energy is distributed among the degrees of freedom of the lattice and the system still oscillates around the state of minimum. Again, a mesh size variation of the lattice was done with a corresponding variation of the parameters (b and m_0) in such a way to guarantee the independence of the results with respect to the mesh size.

In Fig. 10 we show the evolution of the Green's function $G(i,i;t)$ for a time dependent mass. The initial condition corresponds to the vacuum of the symmetric phase for $\mu=100$ MeV, which yields, in the symmetric phase, by the GAP equation, $m_{0s}^2=-2.474621$ fm^{-2} with $b=6$ fm^{-2}, according to Eq. (13). When $t\geqslant 3$ fm the evolution is determined by the asymmetric phase $m_{0a}^2=-2.478473$ fm^{-2} coupled with the condensate, for which we have chosen to have an initial condition at $t=3$ fm equal to $\bar{\phi}(t=3\ \text{fm})=1.1\bar{\phi}_0$, where $\bar{\phi}_0=\sqrt{3\mu^2/b}$ corresponds to the minimum of the potential. This time dependence of m_0^2 simulates a phase transition. The Green's function tends to oscillate around the asymmetric vacuum point. This value of the condensate can be compared to the minimum in a tree approximation where $\bar{\phi}_0=\sqrt{-6m_0^2/b}$. Then, in Fig. 11 we show the particle number for this last case of phase transition. There is particle creation, which ''dissipates'' the width $G(i,t)$. Finally in Fig. 12, the evolution of the deviation of the con-

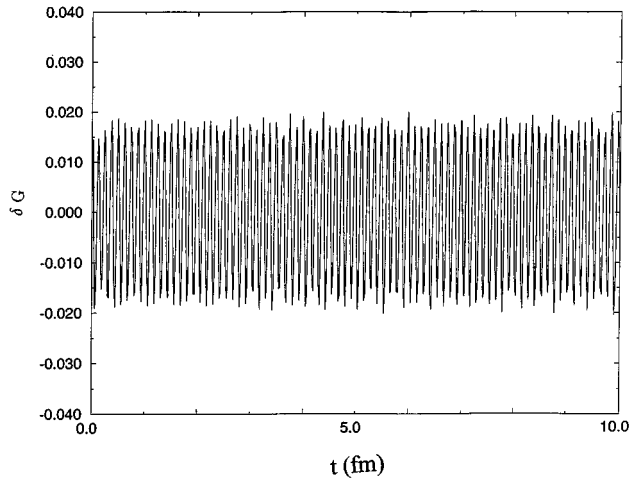

FIG. 4. Deviation $\delta G(t)$ in function of time ($\times$ 0.1 fm) for an initial condition $m=0.9\mu$, mass $\mu=100$ MeV and coupling $b=30$ fm^{-2}.

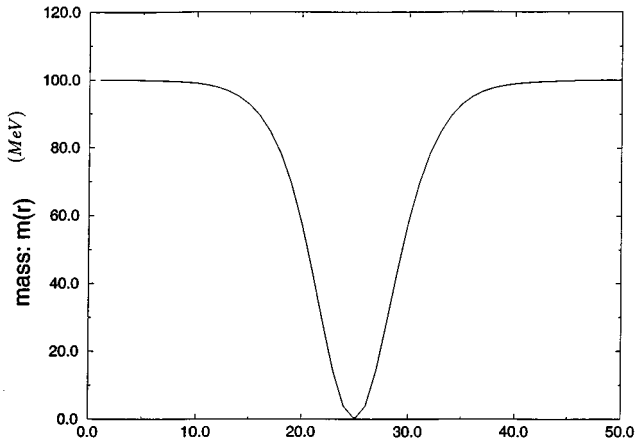

FIG. 6. Initial condition corresponding to a bubble of zero mass $m^2=\mu^2[\tanh((r-2.5)/5)]^2$ for a physical mass of $\mu=100$ MeV.

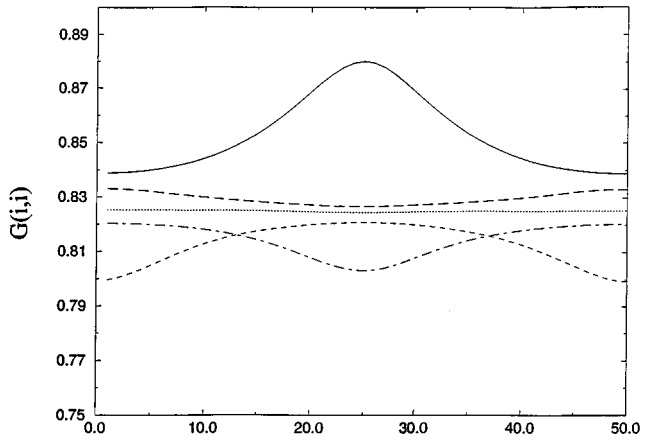

FIG. 7. Temporal evolution of bubble configuration $G(r,t)$ for an initial condition given in Fig. 6 and coupling constant $b=6$ fm^{-2}. Curves correspond to configurations at t= 0 (solid line), 10 (long-dashed, short-dashed), 100 (dotted), 150 (short-dashed), 170 (long-dashed) Δt, with $\Delta t=0.03$ fm.

densate $\delta\overline{\phi}(t)=\overline{\phi}(t)-\overline{\phi}_0$ with relation to its value in the minimum is presented. The field oscillates around the vacuum value. A self consistent treatment of this problem with the evolution of $\overline{\phi}$ in three spatial dimension is under work. Results are in very good qualitative agreement with those obtained by the authors of [9,10].

VI. CONCLUSIONS

In this paper we have worked out a new method to perform temporal evolution of quantum fluctuations applied to $\lambda\phi^4$ model in the Schrödinger representation. First, however, we have found the solutions for the free equations of motion in 1 and 3 dimensional space in the special case of small deviations from the vacuum as initial condition and concluded that, concerning ultra violet divergences, the three dimensional case presents only one divergence in the initial condition (equivalent to a static case divergence) which is absent in the one dimensional case.

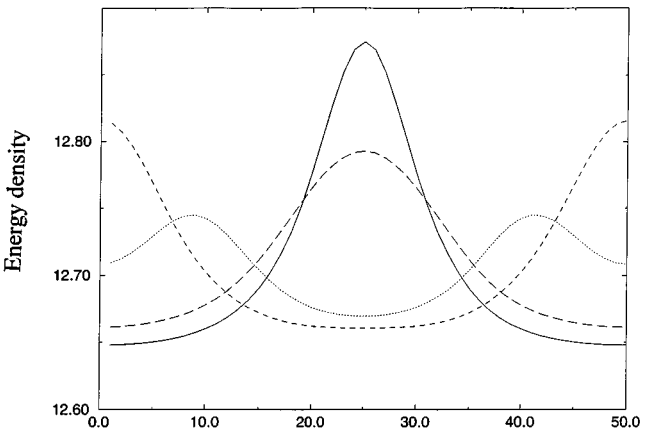

FIG. 8. Temporal evolution of energy density $\rho(t,r)$ fm^{-1} corresponding to the evolution of $G(r,t)$ shown in Fig. 5. Curves correspond to times $t=0$ (solid line), $t=100$ (dotted line), $t=150$ (dashed line) and $t=300$ (long dashed line) Δt.

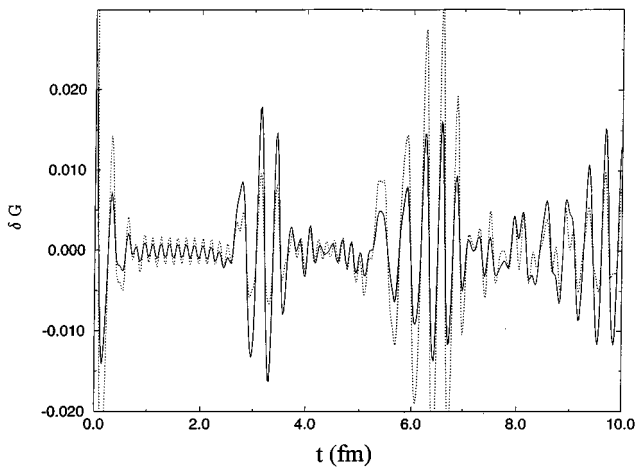

FIG. 9. Deviation $\delta G(t)$ in function of time for central point of the lattice (continue line) and for the mean deviation of all the points ($\delta\overline{G}$) (dotted line) for the initial condition of Fig. 6.

Then we have shown the numerical method that consists on a mean field approximation for the quantum fluctuations in which a generalized density matrix obeys a Liouville–von Neumann–type movement equations. These equations are exactly equal to those equations obtained by a variational method with a Gaussian trial wave functional in the symmetric phase. We can on the other hand couple these equations to the equation of the condensate ($\overline{\phi}$) in the asymmetric phase. The equations written in that form allow a simple way to perform time evolution of the density matrix eigenvectors corresponding to the evolution of variational parameters respecting sympletic structure of the equations of motion. This means in particular a number of conserved quantities in the lattice equal to the number of points of this lattice. If we were not in lattice but in the continuum, there would be an infinite number of conserved quantities. In one dimensional space several initial configurations were given and the re-

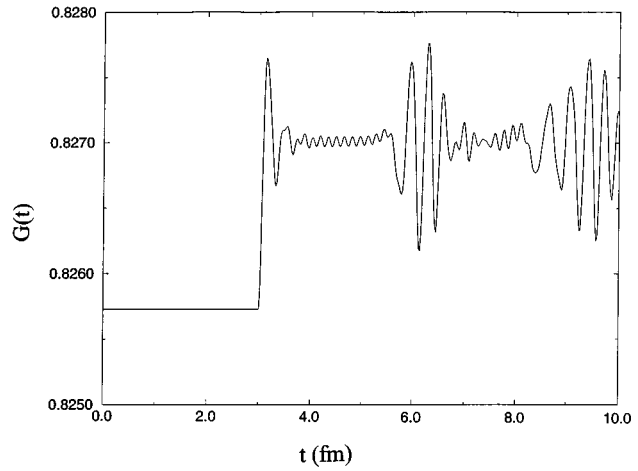

FIG. 10. Deviation $\delta G(t)$ in function of time for an initial condition in the symmetrical vacuum, with $\mu=100$ MeV, until $t=3$ fm. From this time on, the evolution is determined by the asymmetric phase with the condensate $\overline{\phi}(t=3\ \text{fm})=1.1\overline{\phi}_0$. Coupling constant was taken to be $b=6$ fm^{-2}.

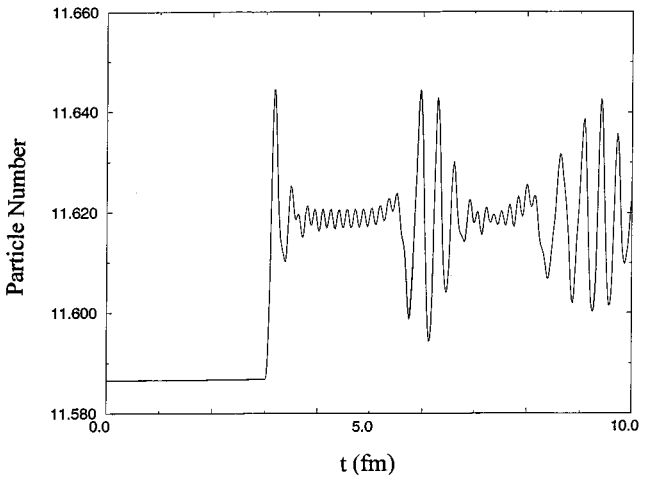

FIG. 11. Particle number for the case of Fig. 10.

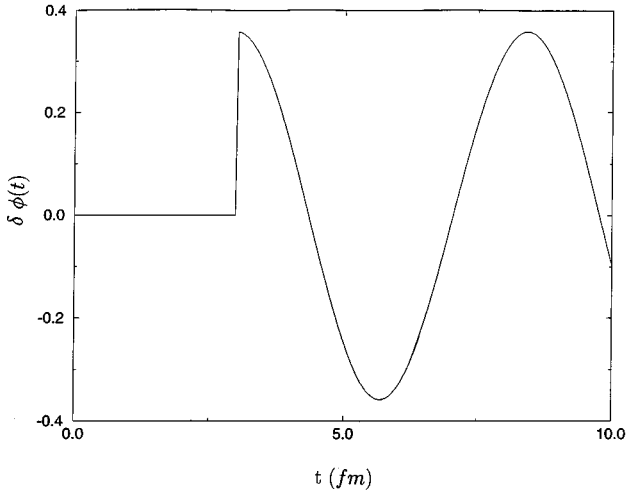

FIG. 12. Temporal evolution of the deviation of the condensate with relation to its vacuum value for the case of Fig. 10.

spective evolution shown for the quantum fluctuations deviations from the vacuum. For the free case we were able to compare the semianalytical solution with the numerical one being the curves very close to each other. The field always oscillates around the vacuum state (in the symmetric or asymmetric phase), being the energy distributed among the degrees of freedom of the system: quantum fluctuations, condensate ($\overline{\phi}$) and kinetic contributions, on the lattice. This occurs specially in the case of nonhomogeneous configuration, like a bubble of zero mass in the vacuum. The results concerning homogeneous configuration are in very good qualitative agreement with those obtained by [9,10] in three dimensional space with a different method. The influence of b was shown by considering $b=0$, 6 and 30 fm^{-2}, and for higher coupling constants, higher is the frequency oscillation. The amplitude of the oscillation depends on the initial condition, as it happens in simple quantum mechanics with an analogous potential. Time dependence of energy density and particle number observables were also shown, remembering that we have considered periodic boundary conditions.

One interesting problem we are left is the application of this method to other more realistic models, like to pions in heavy ion collisions, considering the observations of the last section, but in three dimensional space. As stated above this kind of ''mean field'' may break symmetry properties like those associated to particle number conservation and angular momenta [25]. The same occurs for the theories where some symmetry plays an important role like in gauge or chiral symmetry models. An extension of the used numerical method in three dimensional space for nonequilibrium cases or finite temperature is thus of high interest and is under work.

ACKNOWLEDGMENTS

The author is very grateful to D. Vautherin and H. Flocard for interesting and useful discussions. F.L.B. wishes to thank the financial support given by FAPESP and by CAPES (Brazil) and he also would like to thank the hospitality of the Division de Physique Theorique de l'IPN d'Orsay, France.

[1] A. Guth, Phys. Rev. D **23**, 347 (1981).
[2] *Current Topics in Astrofundamental Physics: The Early Universe*, Proceedings, edited by N. Sanchez and A. Zichichi (NATO Advanced Study Institute) (Kluwer Academic, Dordrecht, 1995).
[3] A. Linde, *Particle Physics and Inflationary Cosmology* (Harwood Academic, Chur, Switzerland, 1990).
[4] F. L. Braghin, C. Martin, and D. Vautherin, Phys. Lett. B **348**, 343 (1995).
[5] A. H. Guth and S. Y. Pi, Phys. Rev. D **32**, 1899 (1985).
[6] F. Cooper and E. Mottola, Phys. Rev. D **36**, 3114 (1987).
[7] F. Cooper, S.-Y. Pi, and P. N. Stancioff, Phys. Rev. D **34**, 3831 (1986).
[8] F. L. Braghin, thesis, Université de Paris XI, 1995; Proceedings of the ''Rencontres des Jeunes Chercheurs,'' Super Besse, France, SPF, 1994, edited by A. Falvard and P. Gay (LPC, Clermont-FD, 1994).
[9] D. Boyanovsky and H. de Vega, Phys. Rev. D **49**, 2769 (1994).
[10] D. Boyanovsky, H. J. de Vega, R. Holman, D.-S. Lee, and A. Singh, Phys. Rev. D **51**, 4419 (1995).
[11] J. P. Blaizot and A. Kryzywicki, Phys. Rev. D **50**, 442 (1994).
[12] U. Ornik, F. Pottag, and R. Weiner, Phys. Rev. Lett. **63**, 2641 (1989).
[13] A. K. Kerman and D. Vautherin, Ann. Phys. (N.Y.) **192**, 408 (1989).
[14] I. I. Kogan and A. Kovner, Phys. Rev. D **52**, 3719 (1995).
[15] A. K. Kerman and C-Y. Lin, Ann. Phys. (N.Y.) **241**, 185 (1995).
[16] A. K. Kerman, C. Martin, and D. Vautherin, Phys. Rev. D **47**, 632 (1993).

[17] J. Weinberg and A. Wu, Phys. Rev. D **36**, 2474 (1987).
[18] T. Barnes and G. I. Ghandhour, Phys. Rev. D **22**, 924 (1980).
[19] R. P. Feynman, in *Proceedings of the International Workshop on Variational Calculations in Quantum Field Theory*, Wangerooge, West Germany, 1987, edited by L. Polley and D. E. L. Pottinger (World Scientific, Singapore, 1988).
[20] O. Éboli, R. Jackiw, and So Young Pi, Phys. Rev. D **37**, 3557 (1988).
[21] F. Cooper, S. Habib, Y. Kluger, E. Mottola, J. P. Paz, and P. R. Anderson, Phys. Rev. D **50**, 2848 (1994).
[22] R. Jackiw and A. K. Kerman, Phys. Lett. **71A**, 158 (1979).
[23] P. M. Stevenson, Phys. Rev. D **32**, 1389 (1985).
[24] A. P. Prudnikov, Yu. A. Brychkov, and O. I. Marichev, *Integrals and Series* (Gordon and Breach Sciences, Amsterdam, 1986).
[25] J. P. Blaizot and G. Ripka, *Quantum Theory of Finite Systems* (MIT Press, Cambridge, MA, 1986).
[26] N. Onishi and T. Horibata, Prog. Theor. Phys. **64**, 1650 (1980); H. C. Flocard, Notes de cours Méthodes Microcospiques en Physique Nucléaire, D.E.A. Champs, Particules, Matière, Paris XI, Orsay (unpublished).