Upper bound for entropy in asymptotically de Sitter space-time

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We investigate the nature of asymptotically de Sitter space-times containing a black hole. We show that if the matter fields satisfy the dominant energy condition and cosmic censorship holds in the considered space-time, the area of the cosmological event horizon for an observer approaching a future timelike infinity does not decrease; i.e., the second law is satisfied. We also show under the same conditions that the total area of the black hole and the cosmological event horizon, a quarter of which is the total Bekenstein-Hawking entropy, is less than $12\pi/\Lambda$, where Λ is the cosmological constant. The physical implications are also discussed. [S0556-2821(98)03606-6]

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I. INTRODUCTION

There has been interest in space-times with a positive cosmological constant Λ. Recent cosmological observations suggest the existence of Λ in our universe [1]. Also, it is widely believed that inflation took place in the early stages of our universe, where the vacuum energy of a scalar field (inflaton) plays the roll of Λ. Most regions in such a space-time are expected to expand as in de Sitter space-time. Some regions, however, will gravitationally collapse to form black holes if the inhomogeneity of the initial matter distribution is large. Then there will be observers who have two types of event horizons, a black hole event horizon (BEH) and a cosmological event horizon (CEH), just like the observers who approach the future timelike infinity in Schwarzschild–de Sitter space time. Throughout this paper we shall focus on the event horizons for such observers.

Gibbons and Hawking [2] studied the thermodynamic property [3] of event horizons in asymptotically de sitter space-times. In particular, they found that an observer feels thermal radiation coming from the CEH and that the entropy S_C of the CEH is equal to one quarter of its area as in the case of a BEH. Thus, the areas of the event horizons can be interpreted as the entropies, or lack of information of the observer about the regions which he cannot see.

In classical general relativity, there have been a number of studies on the nature of BEHs in the asymptotically de Sitter space-time. Hayward, Shiromizu, and Nakao [4] and Shiromizu, Nakao, Kodama, and Maeda [5] showed that the area of a BEH in the asymptotically de Sitter space-time cannot decrease and has an upper bound $4\pi/\Lambda$ if weak cosmic censorship (WCC) [6] holds. This means that black holes cannot collide with each other if the total area of them exceeds the upper bound.

Davies [7] investigated a CEH in Robertson-Walker mod-

els with Λ and a perfect fluid satisfying the dominant energy condition and showed that the area of the cosmological horizon cannot decrease. From this result, one may expect that in generic asymptotically de Sitter space-times the area of a CEH cannot decrease as in the case of a BEH.

Boucher, Gibbons, and Horowitz [8] showed that the area of the CEH is bounded from the above by $12\pi/\Lambda$ on a regular time-symmetric hypersurface. Shiromizu, Nakao, Kodama, and Maeda [5] also obtained the same conclusion on a maximal hypersurface. However, one cannot say that the same conclusion holds for CEHs in a general nonstationary asymptotically de Sitter space-time, because it is highly nontrivial whether a foliation by such hypersurfaces exists and covers the relevant portion of the space-time.

WCC is assumed in the proof of the above results as well as in the case of a BEH. An example of Schwarzschild–de Sitter space-time shows the significance of this assumption, and also suggests a close relation among the area of the CEH, the WCC and positivity of the *gravitational energy* (mass). Figure 1 shows the mass parameter m as a function of the area A of the event horizon and Figs. 2(a) and 2(b)
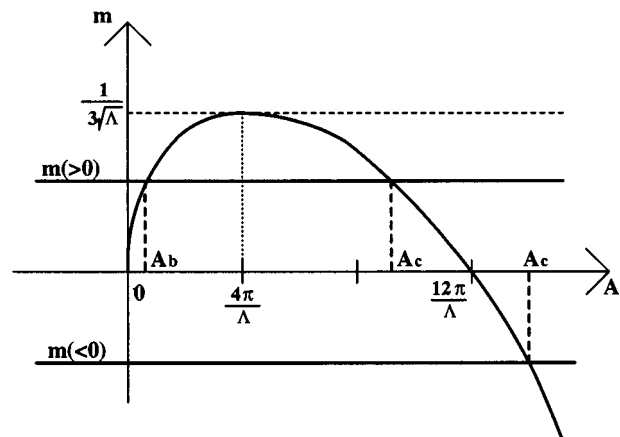

FIG. 1. The mass parameter m of a Schwarzschild–de Sitter solution for a fixed Λ is related to the area A of event horizons as $m = (A/16\pi)^{1/2}(1 - \Lambda A/12\pi)$. A_b, A_c are the areas of a BEH and a CEH, respectively.

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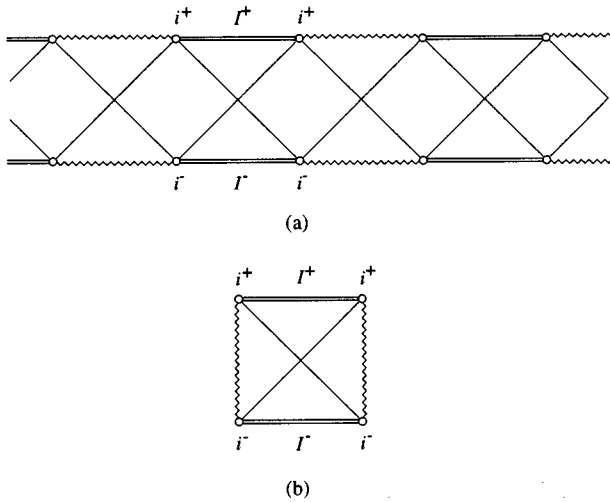FIG. 2. Penrose diagrams of Schwarzschild–de Sitter space-times with mass parameters (a) $m > 0$ and (b) $m < 0$, respectively.

show the Penrose diagrams for the cases of $m > 0$ and $m < 0$, respectively. One easily finds that if the WCC holds ($m > 0$) the area A_C of the CEH is bounded from the above by $12\pi/\Lambda$. Indeed, one finds that the *total* area of the BEH and the CEH has an upper bound $12\pi/\Lambda$. On the other hand, if the WCC is violated ($m < 0$) A_C is not bounded.

In this paper, we show the area theorem that the area of the CEH in an asymptotically de Sitter space-time containing a black hole cannot decrease so that the second law of thermodynamics is satisfied, and the total area of BEH and CEH is less than $12\pi/\Lambda$, hence total Bekenstein-Hawking entropy is less than $3\pi/\Lambda$, if the space-time satisfies the WCC and the energy conditions. To this end, we define a quasilocal energy in a space-time with Λ and its monotonicity and positivity. Very roughly speaking, our analysis is a generalization of the argument of the previous paragraph to general asymptotically de Sitter space-times which are neither stationary nor spherically symmetric. We follow the notation of Ref. [9] and use the units $c = G = \hbar = k_B = 1$.

II. ASYMPTOTICALLY de SITTER SPACE-TIME AND THE AREA LAW FOR A COSMOLOGICAL EVENT HORIZON

In this section we shall show the area theorem (theorem 1) for a CEH in an asymptotically de Sitter space-time. As a precise definition of an asymptotically de Sitter space-time satisfying the WCC, we assume space-time (M, g) to be *strongly asymptotically predictable from the partial Cauchy surface Σ and de Sitter in the future* [5], and just call it *asymptotically de Sitter*. In what follows, causal relationships are considered in a larger manifold $(\widetilde{M}, \widetilde{g})$ in which (M, g) is conformally embedded. Note that the future conformal infinity $\mathcal{I}^+$ of M is a spacelike hypersurface in $\widetilde{M}$ [10].

We shall consider asymptotically de Sitter space-times containing a black hole and an observer whose world line λ has a future endpoint at the ''future timelike infinity.'' Then $\dot{J}^-(\lambda)$ consists of two components, the BEH and the CEH for the observer [2]. As the BEH can be defined by $\dot{J}^-(\mathcal{I}^+)$,

the CEH can be also defined in terms of $\mathcal{I}^+$. Namely, we define the *cosmological event horizon* (CEH) to be the past Cauchy horizon $H^-(\mathcal{I}^+)$ of the future infinity.

In general, the topology of $\mathcal{I}^+$ is not determined. However, it seems reasonable to suppose that $\mathcal{I}^+$ is diffeomorphic to $S^2 \times (0,1)$ if the topology of the BEH is S^2. In analogy to weakly asymptotically simple, empty, and future asymptotically predictable space-time (see proposition 9.2.3 of Ref. [9]), we also assume that there is a continuous onto map $\alpha: (0, \infty) \times \Sigma \rightarrow D^+(\Sigma) - \Sigma$ satisfying the following. (1) For each $t \in (0, \infty)$, $\alpha_t := \alpha(t, \cdot)$, and restriction of α on $(0, t) \times \alpha_t^{-1}(\Sigma_t - \mathcal{I}^+)$ are homeomorphisms, where $\Sigma_t := \alpha(\{t\} \times \Sigma)$; (2) for each $t \in (0, \infty)$, Σ_t is a Cauchy surface for $D(\Sigma)$ such that (a) $\Sigma_{t2} - \mathcal{I}^+ \subseteq I^+(\Sigma_{t1} - \mathcal{I}^+)$ when $t_2 > t_1$ and (b) the edge of $\Sigma_t - \mathcal{I}^+$ in $\widetilde{M}$ is a spacelike two-sphere in $\mathcal{I}^+$. We define $W_t := \Sigma_t \cap \mathcal{I}^+$. We have $W_{t1} \subset W_{t2}$ for $t_2 > t_1$ and $\cup_{t \in (0, \infty)} W_t = \mathcal{I}^+$. We also present a lemma on the topology of a CEH.

Lemma 1. (Each component of) any sufficiently nice cut of the cosmological event horizon $H^-(\mathcal{I}^+)$ is a topological two-sphere.

Proof. Since $D^-(\mathcal{I}^+) \cap M$ is a future set in M, its boundary in M, i.e., the CEH, must be a C^{1-} embedded submanifold of M (see proposition 6.3.1 of Ref. [9]). Moreover, $\mathrm{int}D^-(\mathcal{I}^+)$ is simply connected because it is homeomorphic to $\mathcal{I}^+ \times \mathbf{R}$ and $\mathcal{I}^+$ is simply connected. Thus the conclusion follows.
□

We use the following lemma, which is shown in Ref. [9], to prove Lemma 3.

Lemma 2. Let Σ be a partial Cauchy surface. For any $p \in D^-(\Sigma)$, $J^+(p) \cap D^-(\Sigma)$ is compact.

Lemma 3. $D^-(\mathcal{I}^+) = \cup_{t \in (0, \infty)} D^-(W_t)$.

Proof. Let us define a continuous function $\mathcal{I}^+ \ni p \mapsto t \in (0, \infty)$ defined by $p \in \mathrm{edge}(W_t)$. Because lemma 2 implies that for any $p \in D^-(\mathcal{I}^+)$, $J^+(p) \cap \mathcal{I}^+$ is compact in $\widetilde{M}$, there exists a maximum value for the function above. So there is a $t \in (0, \infty)$ such that $W_t \supseteq J^+(p) \cap \mathcal{I}^+$ and hence $p \in D^-(W_t)$. Thus we have $D^-(\mathcal{I}^+) \subseteq \cup_{n \in N} D^-(W_n) \subseteq \cup_{t \in (0, \infty)} D^-(W_t)$. It follows from $D^-(\mathcal{I}^+) \supseteq D^-(W_t)$ for each $t \in (0, \infty)$ that $D^-(\mathcal{I}^+) \supseteq \cup_{t \in (0, \infty)} D^-(W_t)$.
□

In the next step we will prove lemma 5 by using the following limit curve lemma [11].

Lemma 4 (limit curve lemma). Let $\gamma_n : (-\infty, \infty) \rightarrow M$ be a sequence of inextendible nonspacelike curves (parametrized by the arc length in g_R which is a complete Riemannian metric). Suppose that $p \in M$ is an accumulation point of the sequence $\{\gamma_n(0)\}$. Then there exists an inextendible nonspacelike curve γ such that $\gamma(0) = p$, and a subsequence $\{\gamma_m\}$ which converges to γ uniformly (with respect to g_R) on compact subsets of $\mathbf{R}$.

Lemma 5. For any generator λ of $H^-(\mathcal{I}^+)$, parametrized with respect to the g_R arc length, there exists a sequence $\{\lambda_n\}$ of null geodesics in $D^-(\mathcal{I}^+)$, parametrized with respect to the g_R arc length, such that (1) $\{\lambda_n\}$ converges uniformly to λ with respect to h on compact subsets of $\mathbf{R}$, and (2) each λ_n generates an achronal set.

Proof. Let p be a point of λ which is not the endpoint. Any neighborhood U of p contains a point of $D^-(\mathcal{I}^+)$. It follows from lemma 3 that there exist a $n_0 \in \mathbf{N}$ such that

$U \cap D^-(W_n) \neq \emptyset$ hence $U \cap H^-(W_n) \neq \emptyset$ for all $n \geq n_0$. Then one can construct a sequence $\{p_n\}$ such that $p_n \in H^-(W_{m_n})$ and $p_n \to p$, where (W_{m_n}) is a subsequence of $\{W_n\}$. Letting λ_n be the generator of $D^-(W_{m_n})$ through p_n, one has from lemma 4 that there exists a inextendible nonspacelike C^0 curve γ through p such that $\{\lambda_n\}$ converges to γ uniformly on compact subsets of $\mathbf{R}$. However, because $\{\lambda_n\}$ can have its accumulation points only on $H^-(\mathcal{I}^+)$, γ must lie on $H^-(\mathcal{I}^+)$. Since γ is a nonspacelike curve through p and is lying on $H^-(\mathcal{I}^+)$, it must coincide with λ.

$\square$

Finally we present the following area theorem of the CEH.

Theorem 1 (area law for a CEH). In an asymptotically de Sitter space-time with a piecewise smooth CEH satisfying the weak energy condition $A(H^-(\mathcal{I}^+) \cap \Sigma_{t_2}) \geq A(H^-(\mathcal{I}^+) \cap \Sigma_{t_1})$ for $t_2 > t_1$, where A ($\mathcal{S}$) denotes the area of a two-surface $\mathcal{S}$.

Proof. Piecewise smoothness of the CEH implies that there are a finite number of pairwise disjoint smooth submanifolds U_i's such that the CEH is $\cup_i \bar{U}_i$. It suffices to show that the expansion $\theta \geq 0$ on each $p \in \text{int } U_i$ because each U_i is foliated by future inextendible null geodesic generators. For any point $p \in \text{int } U_i$ for some i there is an open set $V \ni p$ diffeomorphic to $\mathcal{S} \times \mathbf{R}$, where $\mathcal{S}$ is a locally spacelike two-surface containing p with compact closure. By lemma 4 and compactness of $\bar{\mathcal{S}}$ there is a sequence of diffeomorphisms $\phi_n : V \to V_n \subset H^-(W_n)$ such that (1) each $\phi_n(\mathcal{S})$ is spacelike, (2) each ϕ_n preserves the foliations by null geodesic generators, and (3) $\phi_n(B)$ converges uniformly to V on compact subsets of $\mathcal{S} \times \mathbf{R}$. Suppose that the expansion θ of future-directed null geodesic generators of the CEH were negative at p. Then by the continuity of θ there would be some n such that the expansion θ_n of generators of V_n would be negative at $\phi_n(p)$. From the weak energy condition the generator from $\phi_n(p)$, since it is future complete, would have a conjugate point of $\phi_n(\mathcal{S})$ (see proposition 4.4.6 of Ref. [9]). This contradicts the achronality of $H^-(W_n)$.

$\square$

Corollary. If the assumptions of theorem 1 hold and every future incomplete null geodesic terminates in a strong curvature singularity of Królak [12], then every generator of the CEH is future complete.

Proof. From the proof of theorem 1, the expansion of each null geodesic generator cannot be negative. This contradicts the condition of strong curvature singularity.

$\square$

III. QUASILOCAL ENERGY IN SPACE-TIMES WITH Λ

Here we will define the quasilocal energy $E(\mathcal{S})$ in space-time with Λ and examine its monotonicity and positivity, which we will use to show the existence of an upper bound for entropy (theorem 2) in Sec. IV. Let us introduce Hayward's double null formalism [13], namely, smooth foliations of null three-hypersurfaces labeled by $\xi_\pm$ such that each intersection of two hypersurfaces of constant $\xi_\pm$ is a closed spacelike two-surface. We have the evolution vector $u_\pm = \partial/\partial \xi_\pm$, the normal one-forms $n_\pm = -d\xi_\pm$, the metric $h = g + e^{-f}(n_+ n_- + n_- n_+)$ induced on the two-surface, the projection $\perp$ on the two-surface, the shift vectors $r_\pm = \perp u_\pm$, and the null normal vectors $l_\pm = u_\pm - r_\pm$. The ex-

pansion $\theta_\pm$, the shear $\sigma^\pm$, and the twist ω on a two-surface are defined as

$$\theta_\pm = \frac{1}{2} h^{-1} : \mathcal{L}_\pm h, \tag{3.1}$$

$$\sigma^\pm = \mathcal{L}_\pm h - \theta_\pm h, \tag{3.2}$$

$$\omega = \frac{1}{2} e^f h \cdot [l_+, l_-], \tag{3.3}$$

where $\mathcal{L}_\pm$ represents the Lie derivatives along the vector fields $l_\pm$, and dots and colons denote single and double contractions, respectively. The quasilocal energy is defined in each embedded spatial two-surface $\mathcal{S}$ as

$$E(\mathcal{S}) := \frac{1}{8\pi} \sqrt{\frac{A}{16\pi}} \int_{\mathcal{S}} \mu \left(\mathcal{R} + e^f \theta_+ \theta_- - \frac{2\Lambda}{3} \right), \tag{3.4}$$

where A, $\mathcal{R}$, and μ represent the total area of $\mathcal{S}$, the Ricci scalar on $\mathcal{S}$, and the area two-form on $\mathcal{S}$, respectively. This is the Hawking energy with last term added in the integrand. Physically, $E(\mathcal{S})$ is the gravitational energy subtracted by the energy due to the cosmological constant Λ, so that it is considered as the energy of the matter fields. In Schwarzschild-de Sitter space-time $E(\mathcal{S})$ coincides with the mass parameter m. In spherically symmetric space-times with dust, $E(\mathcal{S})$ coincides with the mass function [14]. In space-time without Λ our quasilocal energy $E(\mathcal{S})$ reduces to the Hawking energy.

The Einstein equations are given by

$$e^{-f} \mathcal{L}_\pm (e^f \theta_\pm) + \frac{1}{2} \theta_\pm^2 + \frac{1}{4} \|\sigma_\pm\|^2 = -8\pi \phi_\pm, \tag{3.5}$$

$$\mathcal{L}_\pm \theta_\pm + \theta_+ \theta_- + e^{-f} \left[\frac{1}{2} \mathcal{R} - \left| \frac{1}{2} \mathcal{D}f \pm \omega \right|^2 \right.$$
$$\left. + \mathcal{D} \cdot \left(\frac{1}{2} \mathcal{D}f \pm \omega \right) \right] = 8\pi \rho + e^{-f} \Lambda, \tag{3.6}$$

where $\phi_\pm = T(l_\pm, l_\pm)$ and $\rho = T(l_+, l_-)$ for the energy tensor T, and $\mathcal{D}$ is the covariant derivative with respect to h.

Let us examine the monotonicity of $E(\mathcal{S})$ on an outgoing null hypersurface $\xi_- = \text{const}$ (the monotonicity on an ingoing null hypersurface $\xi_+ = \text{const}$ or on a spacelike hypersurface can be argued similarly). The derivative of the energy $E(\mathcal{S})$ along the outgoing direction l_+ is

$$8\pi \mathcal{L}_+ E = \sqrt{\frac{A}{16\pi}} \left\{ \frac{1}{2A} \int_{\mathcal{S}} \mu \theta_+ \int_{\mathcal{S}} \mu (\mathcal{R} + e^f \theta_+ \theta_-) \right.$$
$$- \int_{\mathcal{S}} \mu \theta_- \left(\frac{1}{4} \|\sigma_+\|^2 + 8\pi \phi_+ \right)$$
$$- \int_{\mathcal{S}} \mu \theta_+ \left[\frac{1}{2} \mathcal{R} + \frac{1}{2} e^f \theta_+ \theta_- + \mathcal{D} \cdot \left(\frac{1}{2} \mathcal{D}f + \omega \right) \right.$$
$$\left. \left. - \left| \frac{1}{2} \mathcal{D}f + \omega \right|^2 - 8\pi e^f \rho \right] \right\}. \tag{3.7}$$

We assume that the matter fields satisfy the dominant energy conditions $\phi_+ \geq 0$ and $\rho \geq 0$, and take a foliation of the hypersurface $\xi_- = \text{const}$ by spatial two-surfaces $\mathcal{S}$. The energy $E(\mathcal{S})$ is nondecreasing in the outgoing null direction ($\theta_+ \geq 0, \theta_- \leq 0$), $\mathcal{L}_+ E \geq 0$, if

$$\langle \theta_+ \rangle \langle F \rangle \geq \langle \theta_+ F \rangle \tag{3.8}$$

on each $\mathcal{S}$, where

$$F := \mathcal{R} + e^f \theta_+ \theta_- + 2\mathcal{D} \cdot \left(\frac{1}{2} \mathcal{D}f + \omega \right), \tag{3.9}$$

$$\langle \cdot \rangle := \frac{\|_{\mathcal{S}} \mu \cdot}{\|_{\mathcal{S}} \mu}. \tag{3.10}$$

We remark that each term of F except the third term is invariant under rescaling of the outgoing null normal l_+. An example of the foliations satisfying Eq. (3.8) is one with $F = \text{const}$, which we can take by the rescaling of l_+. Another example is the uniformly expanding foliation [15].

IV. UPPER BOUND FOR THE AREA

In this section we will show that the total area of the BEH and the CEH is bounded in asymptotically de Sitter space-times (theorem 2). We define the apparent horizons according to Hayward [16]. A *marginal surface* is a spatial two-surface $\mathcal{S}$ on which $\theta_+ = 0$ or $\theta_- = 0$. A *black hole apparent horizon* (BAH) is the closure $\overline{T_B}$ of a hypersurface T_B foliated by marginal surfaces on which $\theta_+ = 0$, $\theta_- < 0$ and $\mathcal{L}_- \theta_+ < 0$. A *cosmological apparent horizon* (CAH) is $\overline{T_C}$ foliated by marginal surfaces on which $\theta_- = 0$, $\theta_+ > 0$ and $\mathcal{L}_+ \theta_- > 0$. Here the coordinates $\xi_\pm$ are taken so that they are constant on each of the above spatial two-surfaces.

Hayward, Shiromizu, and Nakao [4] showed that the area of a BAH has an upper bound $4\pi/\Lambda$. They also showed that the area of a BEH is less than $4\pi/\Lambda$ by implicitly assuming the existence of the limit two-surface $\mathcal{S}$ of the BEH, though its physical meaning is not clear (see the Appendix). Instead, one can reach the same conclusion under a physically reasonable condition; strongly future asymptotically predictability (or WCO) in an ''extended'' sense [17]. It states that singularities are hidden inside not only a BEH but also a BAH. More precisely, the closure of the domain of dependence of a partial Cauchy surface contains not only $\mathcal{I}^+$ and the BEH but also the outermost part of the BAH, i.e., (i) there exist $t > 0$ and a subset T'_B of T_B, foliated by marginal surfaces, such that $H^-(\overline{T'_B}) \cap J^+(\Sigma_t) \supseteq \dot{J}^-(\mathcal{I}^+) \cap J^+(\Sigma_t)$ and $[I^-(\overline{T'_B}) \cap \mathcal{I}^+(\Sigma_t)] \subseteq D^+(\Sigma_t)$. We give the following proposition, whose proof we will give in Appendix.

Proposition 1. In an asymptotically de Sitter space-time satisfying condition (i) above and the weak energy condition, the area of a black hole event horizon (BEH) *is less than* $4\pi/\Lambda$.

Now we will show that the total area of the BEH and CEH has an upper bound $12\pi/\Lambda$ by making use of proposition 1. We require the following conditions. (ii) There exists $t_0 \geq 0$ such that the cross section of $\dot{J}^-(\mathcal{I}^+) \cap \Sigma_t (t \geq t_0)$ is smooth one connected component and the topology is S^2; (iii) there exists a marginal surface $\mathcal{S}_t$ with $\theta_- = 0$ whose

topology is S^2 in each $\Sigma_t (t \geq t_0)$ and surrounds $\dot{J}^-(\mathcal{I}^+) \cap \Sigma_t$; (iv) $I^-(T_C) \cap J^+(\Sigma_{t_0}) = (I^-(\mathcal{I}^+) - \overline{\mathcal{D}^-(\mathcal{I}^+)}) \cap J^+(\Sigma_0)$; (v) any null geodesic generator of BEH is future complete; (vi) matter fields satisfy the dominant energy condition. [This implies that matter field satisfies the weak energy condition (see, e.g., Ref. [9]).]; and (vii) There exists a foliation satisfying Eq. (3.8) on each outgoing null hyperspace $\dot{J}^-(\mathcal{S}_t)$ inside the CAH. Condition (iv) is similar to condition (i) above.

Lemma 6. For an arbitrary small positive value ϵ_1 there is an acausal hypersurface $\Sigma_{t_1}(t_1 > t_0)$ such that for any closed spacelike two-surface $\mathcal{S}_B$ of $\dot{J}^-(\mathcal{I}^+) \cap J^+(\Sigma_{t_1})$ the quasilocal energy $E(\mathcal{S}_B)$ satisfies

$$E(\mathcal{S}_B) \geq \frac{1}{8\pi} \sqrt{\frac{A(\mathcal{S}_B)}{16\pi}} \left(8\pi - \epsilon_1 - \frac{2\Lambda}{3} A(\mathcal{S}_B) \right) > 0. \tag{4.1}$$

Proof. Consider each null geodesic generator l_+ of the BEH. By conditions (v), (vi), and the Raychaudhuri equation (3.5), $\lim_{\xi \to \infty} \theta_+ = 0$ is satisfied, where ξ is an affine parameter of l_+. $\lim_{\xi \to \infty} \int_{\mathcal{S}} \mu e^f \theta_+ \theta_- = 0$ is also satisfied because the area of a BEH has an upper bound. Therefore there is a Σ_{t_1} such that for any closed spacelike two-surface $\mathcal{S}_B$ of $\dot{J}^-(\mathcal{I}^+) \cap J^+(\Sigma_{t_1})$, $\int_{\mathcal{S}} \mu e^f \theta_+ \theta_-$ is larger than $-\epsilon_1$, where ϵ_1 is an arbitrary small positive value. From Eq. (3.4) and proposition 1 one can get the desired result by using the Gauss-Bonnet theorem and condition (ii). $\square$

Lemma 7. $J^-(T_C) \cap J^+(\Sigma_{t_0}) = [\cup_t J^-(\mathcal{S}_t)] \cap J^+(\Sigma_{t_0})$.

Proof. For any point $p \in J^-(T_C) \cap J^+(\Sigma_{t_0})$ there is a point $q \in J^+(p) \cap T_C$. Then there is $\mathcal{S}_t \ni q$ so that $p \in J^-(\mathcal{S}_t)$. $\square$

Theorem 2. If an asymptotically de Sitter space-time satisfies the conditions (i)–(vii) above, $A_B := \lim_{\xi_+ \to \infty} A(\mathcal{S}_B)$ and $A_C := \lim_{\xi_- \to \infty} A(\mathcal{S}_C)$ satisfy

$$A_B + A_C + \sqrt{A_B A_C} \leq \frac{12\pi}{\Lambda}. \tag{4.2}$$

Remark. In particular, the area A_C of the CEH is less than $12\pi/\Lambda$.

Proof. For any closed spacelike two-surface $\mathcal{S}_C$ of $H^-(\mathcal{I}^+) \cap D^+(\Sigma)$ there exists a partial Cauchy surface $\Sigma_{\mathcal{S}_C}$ containing $\mathcal{S}_C$. Consider a sequence of marginal surfaces $\mathcal{S}_n$ ($n \in \mathbb{N}$) defined above and define N_n^+ and N_n^- as the null hypersurfaces generated by the future-directed outgoing and ingoing null geodesic generators of $\dot{J}^-(\mathcal{S}_n)$, respectively. Denote the spacelike two-surface $N_n^- \cap \Sigma_{\mathcal{S}_C}$ by $\mathcal{K}_n$. From condition (iv) and lemma 7 it follows that $\lim_{n \to \infty} A(\mathcal{K}_n) = A(\mathcal{S}_C)$. The expansion θ_- of l_- is non-negative in the future direction between $\mathcal{S}_n$ and $\mathcal{K}_n$ because $\mathcal{L}_-(e^f \theta_-) \leq 0$ there, as implied by the Raychaudhuri equation (3.5) of l_- and condition (vi), by $\theta_- = 0$ on each $\mathcal{S}_n$. Thus, as in the proof of proposition 1, there exists n_1 for an arbitrary small positive value ϵ_2 such that for all $n > n_1$

$$A(\mathcal{S}_C) - \epsilon_2 \leq A(\mathcal{S}_n) \qquad (4.3)$$

is satisfied.

Consider outgoing null hypersurfaces $\hat{N}_n^+$: $= N_n^+ \cap J^+(\Sigma_{t_1}) \cap J^+(\Sigma_{\mathcal{S}_C})$. From the condition (iv) and lemma 7, for any neighborhood U of $\dot{J}(\mathcal{I}^+) \cap J^+(\Sigma_{t_1}) \cap J^+(\Sigma_{\mathcal{S}_C})$, there is $n_2 > n_1$ such that for $n > n_2$ each $\hat{N}_n^+$ intersect U. For $n > n_2$, take spacelike two-surfaces $\mathcal{Q}_n$ in $\hat{N}_n^+ \cap U$. The sequence $\{\mathcal{Q}_n\}$ converges to a spacelike two-surface $\mathcal{S}_B$ of $\dot{J}^-(\mathcal{I}^+) \cap J^+(\Sigma_{t_1}) \cap J^+(\Sigma_{\mathcal{S}_C})$. By the continuity of $E(\mathcal{Q}_n)$, for an arbitrary small $\epsilon_3 > 0$, there is $n_3 > n_2$ such that $E(\mathcal{S}_B) - \epsilon_3 \leq E(\mathcal{Q}_n)$ for each $n > n_3$. By condition (vii), the energy $E(\mathcal{S})$ is nondecreasing from $\mathcal{Q}_n$ to $\mathcal{S}_n$ on $\hat{N}_n^+$. Thus $E(\mathcal{S}_B) - \epsilon_3 \leq E(\mathcal{S}_n)$ for each $n > n_3$. By lemma 6 and Eq. (3.4) for $\mathcal{S}_n$, this inequality can be rewritten as

$$A(\mathcal{S}_B) + A(\mathcal{S}_n) \leq \frac{12\pi}{\Lambda} - \sqrt{A(\mathcal{S}_B)A(\mathcal{S}_n)} + O(\epsilon_1) + O(\epsilon_3).$$
$$(4.4)$$

Since $\mathcal{S}_C$ is an arbitrary two-surface of $H^-(\mathcal{I}^+)$, one gets the desired result by taking limit $\epsilon_1, \epsilon_3 \to 0$.

□

V. CONCLUSIONS AND DISCUSSION

We have shown in theorem 1 that in asymptotically de Sitter space-time the area $A(\mathcal{S}_C)$ of the CEH is nondecreasing if the WCC and the weak energy condition hold. This means that the area law of event horizons holds not only for a BEH but also for a CEH, hence, it also applies to the total area of event horizons (total Bekenstein-Hawking entropy, i.e., a quarter of the total area of the BEH and the CEH). Next we have shown in theorem 2 that the final values of the areas satisfy $A_B + A_C + \sqrt{A_B A_C} \leq 12\pi/\Lambda$. This means that the final values of entropies $S_B : A_B/4$ of the BEH and $S_C := A_C/4$ of the CEH, satisfy

$$S_B + S_C + \sqrt{S_B S_C} \leq 3\pi/\Lambda. \qquad (5.1)$$

In particular, the total entropy is bounded from the above by $3\pi/\Lambda$ in asymptotically de Sitter space-time. We note that the inequality in theorem 2 is stronger than the previous result and conjecture which state that $A(\mathcal{S}_B) \leq 4\pi/\Lambda$ and $A(\mathcal{S}_C) \leq 12\pi/\Lambda$.

As discussed in Ref. [2], a BEH is unstable against Hawking radiation, while a CEH is stable. Physically this suggests that all asymptotically de Sitter space-times approach de Sitter space-time. This is consistent with the inequality (5.1) which states that for a fixed Λ the total entropy attains its maximum in de sitter space-time, although the quantum effects were not taken into account in the derivation of the inequality. This curious correspondence suggests that the inequality is another law of EH thermodynamics in asymptotically de Sitter space-times.

It is of interest to pursue connections of the present result with the cosmic no hair conjecture [2,18]. Here we consider a weaker version of the conjecture which states that a space-time with Λ has a future asymptotic region rather than being recollapsing, i.e., the space-time is future asymptotically de

Sitter. Since the areas of the EHs have a universal bound (i.e., are bounded by numbers which depend only on Λ) and the areas are expected to become larger when matter falls into them, one expects that the amount of matter which falls into the EHs have a universal bound. So, in the collapse of an isolated object, if most of the matter falls into either the BEH or the CEH, i.e., if there are no heavy shelllike ''stars'' surrounding the black hole, one can expect that the total initial energy of the matter should be bounded by a number which only depends on Λ. This may provide a criterion for the existence of the future asymptotic region of space-time, that is, a criterion for the validity of the cosmic no hair conjecture.

To solve the problems above, it is very important to know the property of the total entropy S_T of the universe, i.e., the sum of the entropy of the EHs and that of the matter between the EHs. We conjecture that in asymptotically de sitter space-time S_T is nondecreasing, i.e., the generalized second law of thermodynamics holds, and also S_T is bounded.

APPENDIX: PROOF OF PROPOSITION 1

In their proof of the theorem of the upper bound for the area of the BEH in Ref. [4] Hayward, Shiromizu, and Nakao implicitly assumed that there is a limited two-surface of the BEH on which quantities such as $\mathcal{L}_- \theta_+$ are continuous, i.e., independent of how one approaches the ''timelike infinity'' i^+. However, this is physically not very well motivated and is highly nontrivial in general. Here, we shall drop the assumption above and prove a slightly modified version of the theorem.

Proof of proposition 1. It is enough to show that the area of a BEH in $J^+(\Sigma)$ has an upper bound because the area does not decrease in the future direction as shown in Ref. [5]. Let us consider a sequence of marginal surfaces $\mathcal{S}_n$ with $\theta_+ = 0$ on the BAH T_B and take a sequence of subsets T_n of T_B' such that $T_{n-1} \subset T_n$, $\text{edge}(T_n) = \mathcal{S}_n$, $\cup_{n \in N} T_n = T_B'$. Consider spacelike two-surfaces $\mathcal{T} : \dot{J} \theta(\mathcal{I}^+) \cap \Sigma_t$ and $\mathcal{T}_n : H^-(T_n) \cap \Sigma_t$ for some (sufficiently large) fixed t. We can observe $D^-(T_B') = \cup_{n \in N} D^-(T_n)$, by replacing $\mathcal{I}^+$ in lemma 3 with T_B'. This together with condition (i) implies that the sequence $\mathcal{T}_n$ converges to $\mathcal{T}$. The expansion θ_+ of each null geodesic generator l_+ of $H^-(T_n)$ is non-negative in the future direction between $\mathcal{T}_n$ and $\mathcal{S}_n$ because $\mathcal{L}_+(e^f \theta_+) \leq 0$ there, as implied by the Raychaudhuri equation (3.5) and the weak energy condition, and by $\theta_+ = 0$ on each $\mathcal{S}_n$. Thus,

$$A(\mathcal{T}_n) \leq A(\mathcal{S}_n) \leq \frac{4\pi}{\Lambda}, \qquad (A1)$$

where the second inequality is obtained by integrating Eq. (3.6) multiplied by e^f on marginal surface $\mathcal{S}_n$ and using the Gauss-Bonnet theorem [4]. Since the sequence $\mathcal{T}_n$ converges to $\mathcal{T}$, for arbitrary small $\epsilon > 0$ there exists a $n_0 \in \mathbf{N}$ such that $A(\mathcal{T}_n)$ with $n > n_0$ satisfies

$$A(\mathcal{T}) - \epsilon \leq A(\mathcal{T}_n). \tag{A2}$$

From inequalities (A1) and (A2) we have

$$A(\mathcal{T}) - \epsilon \leq \frac{4\pi}{\Lambda}. \tag{A3}$$

Since this holds for any ϵ we have

$$A(\mathcal{T}) \leq \frac{4\pi}{\Lambda}. \tag{A4}$$

$\square$

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[1] M. Fukugita, F. Takahara, K. Yamashita, and Y. Yoshii, Astrophys. J. Lett. **361**, L1 (1990); T. Totani, K. Sato, and Y. Yoshii, Astrophys. J. **460**, 303 (1996); G. Efstathiou, W. J. Sutherland, and S. J. Maddox, Nature (London) **348**, 705 (1990); J. A. Peacock and S. J. Dodds, Mon. Not. R. Astron. Soc. **267**, 1020 (1994).

[2] G. W. Gibbons and S. W. Hawking, Phys. Rev. D **15**, 2738 (1977).

[3] J. Beckenstein, Phys. Rev. D **7**, 2333 (1973); **9**, 3292 (1974); S. W. Hawking, Commun. Math. Phys. **43**, 199 (1975); Phys. Rev. D **13**, 191 (1976).

[4] S. A. Hayward, T. Shiromizu, and K. Nakao, Phys. Rev. D **49**, 5080 (1994).

[5] T. Shiromizu, K. Nakao, H. Kodama, and K. Maeda, Phys. Rev. D **47**, R3099 (1993).

[6] R. Penrose, Riv. Nuovo Cimento **1**, 252 (1969).

[7] P. C. W. Davies, Class. Quantum Grav. **4**, L225 (1987).

[8] W. Boucher, G. W. Gibbons, and G. Horowitz, Phys. Rev. D **30**, 2447 (1984).

[9] S. W. Hawking and G. F. R. Ellis, *The Large Scale Structure of Space-time* (Cambridge University Press, Cambridge, England, 1973).

[10] R. Penrose and W. Rindler, *Spinors and Spacetime* (Cambridge University Press, Cambridge, England, 1986), Vol. 2.

[11] J. K. Beem, P. E. Ehrlich, and K. L. Easley, *Global Lorentzian Geometry,* 2nd ed. (Dekker, New York, 1996).

[12] A. Królak, J. Math. Phys. **28**, 2685 (1987).

[13] S. A. Hayward, Phys. Rev. D **49**, 831 (1994).

[14] K. Nakao, gr-qc/9507022.

[15] S. A. Hayward, Class. Quantum Grav. **11**, 3037 (1994).

[16] S. A. Hayward, Phys. Rev. D **49**, 6467 (1994).

[17] W. Israel, Phys. Rev. Lett. **57**, 397 (1986).

[18] S. W. Hawking and I. G. Moss, Phys. Lett. **110B**, 35 (1982).